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Non-uniform stabilization of control systems

IASSON KARAFYLLIS
Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780, Athens, Greece

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A version of non-uniform in time robust global asymptotic stability is proposed and enables us to derive: (1) sufficient conditions for the stabilization of uncertain nonlinear triangular time-varying control systems; (2) sufficient conditions for the solution of the partial-state global stabilization problem for autonomous systems. The results are obtained via the method of integrator backstepping and are generalizations of the existing corresponding results in the literature.

Keywords: integrator backstepping; time-varying feedback; time-varying systems.

1. Introduction

The notion of non-uniform in time Robust Global Asymptotic Stability (RGAS) has been proved to be fruitful for the solution of several problems in Control Theory (see (Karafyllis & Tsinias, 2002a,b; Tsinias & Karafyllis, 1999; Tsinias, 2000) for applications to tracking problems and to the robust stabilization of uncertain systems that cannot be stabilized by continuous static time-invariant feedback and (Karafyllis & Tsinias, 2001) for the extension of the notion of Input-to-State Stability (ISS) to the time-varying case). It is shown in Karafyllis & Tsinias (2001, 2002b) that, even for autonomous systems for which uniform in time asymptotic stabilization by a continuous static feedback is not feasible, it is possible to exhibit non-uniform in time asymptotic stabilization by means of a smooth time-varying feedback.

In this paper our interest is focused on uncertain nonlinear time-varying triangular systems. In order to find sufficient conditions for the robust stabilization of such systems, we first strengthen the notion of Robust Global Asymptotic Stability (RGAS) given in Karafyllis & Tsinias (2001), by introducing the notion of ϕ-RGAS in such a way that it allows the estimation of the rate of convergence to the equilibrium point. Roughly speaking, for the system

$$\begin{aligned} &\dot{x} = f(t, x, d) \\ &x \in \Re^n, t \geqslant 0, d \in D \end{aligned} \tag{1.1}$$

where $D \subset \Re^m$ is a compact set and $f(t, 0, d) = 0$ for all $(t, d) \in \Re^+ \times D$, we say that $0 \in \Re^n$ is ϕ-RGAS if it is in general non-uniformly in time RGAS and particularly, there exists a smooth function $\phi : \Re^+ \to [1, +\infty)$ such that every solution of (1.1) satisfies the following property:

$$\lim_{t \to +\infty} \phi^p(t)|x(t)| = 0, \quad \forall p \geqslant 0 \tag{1.2}$$

In Section 3 we develop the main tool for the integrator backstepping method that it is used in this paper, and in Section 4 this tool is used for the following triangular system:

$$\begin{aligned} &\dot{x}_i = f_i(t,\theta,x_1,\dots,x_i) + g_i(t,\theta,x_1,\dots,x_i)x_{i+1} \quad i=1,\dots,n \\ &u := x_{n+1} \\ &x = (x_1,\dots,x_n)^T \in \Re^n, t \geqslant 0, u \in \Re \end{aligned} \tag{1.3}$$

where the uncertainty $\theta = \theta(t)$ is any measurable function taking values in a compact set $\Omega \subset \Re^l$. We obtain a set of sufficient conditions (Proposition 4.1) for the robust global asymptotic stabilization of (1.3), which is a direct generalization of the corresponding set of sufficient conditions given in the literature for the autonomous case (Jiang *et al.*, 1994; Tsinias, 1996).

The problem of the stabilization by means of partial-state time-varying feedback is also addressed (Proposition 4.2). Specifically, we study systems of the form

$$\begin{aligned} &\dot{z} = f_0(t,\theta,z,x_1) \\ &\dot{x}_i = f_i(t,\theta,z,x_1,\dots,x_i) + g_i(t,\theta,z,x_1,\dots,x_i)x_{i+1} \quad i=1,\dots,n \\ &u := x_{n+1} \end{aligned} \tag{1.4}$$

where $z \in \Re^m$, $x = (x_1,\dots,x_n)^T \in \Re^n$, $u \in \Re$, $\theta = \theta(t)$ is any measurable function taking values in a compact set $\Omega \subset \Re^l$, and we obtain sufficient conditions for the robust global asymptotic stabilization of (1.4) by means of a partial state smooth time-varying feedback of the form $u = k(t,x)$.

Using these results, we next study the applications of time-varying feedback to autonomous control systems. In Section 5, the following two applications of time-varying feedback to autonomous control systems are studied:

(1) We prove that for every function $\phi(\cdot)$ there exists a smooth time-varying feedback of the form $u = k(t,x)$, such that $0 \in \Re^n$ is ϕ-RGAS for the system

$$\begin{aligned} &\dot{x}_i = f_i(\theta,x_1,\dots,x_i) + g_i(\theta,x_1,\dots,x_i)x_{i+1} \quad i=1,\dots,n \\ &u := x_{n+1} \end{aligned} \tag{1.5}$$

where $x = (x_1,\dots,x_n)^T \in \Re^n$, $u \in \Re$, $\theta = \theta(t)$ is any measurable function taking values in a compact set $\Omega \subset \Re^l$ (Corollary 5.1). Roughly speaking, this means that we can design a smooth time-varying feedback so that the solutions of (1.5) converge to the equilibrium point as 'fast' as desired. We emphasize that this feature cannot be accomplished by the use of locally Lipschitz time-invariant feedback.

(2) The stabilization of autonomous systems by means of partial-state smooth time-varying feedback (Theorem 5.4). Specifically, consider the system

$$\dot{z} = f_0(z,x,u) \tag{1.6a}$$

$$\dot{x}_i = f_i(\theta,x_1,\dots,x_i) + g_i(\theta,x_1,\dots,x_i)x_{i+1} \quad i=1,\dots,n$$

$$u := x_{n+1} \tag{1.6b}$$

where $z \in \Re^m$, $x = (x_1,\dots,x_n)^T \in \Re^n$, $u \in \Re$, $\theta = \theta(t)$ is any measurable function taking values in a compact set $\Omega \subset \Re^l$, f_0, f_i and g_i are continuous with respect to $\theta \in \Omega$ and locally Lipschitz with respect to (z,x,u), uniformly in $\theta \in \Omega$, with $f_0(0,0,0) = 0$, $f_i(\theta,0,\dots,0) = 0$ for all $\theta \in \Omega$, for $i = 1,\dots,n$.

System (1.6) can be regarded as the cascade connection of two subsystems. Sufficient conditions for the global asymptotic stability of the equilibrium point for a cascade connection of two independent subsystems were given in Jiang *et al.* (1994) and recently in Panteley & Loria (1998). We provide sufficient conditions for the existence of a smooth partial-state time-varying feedback of the form $u = k(t, x)$, such that $0 \in \Re^m \times \Re^n$ is GAS. This is achieved in Theorem 5.4 and to this end we are using the Lyapunov characterization of forward completeness given in Angeli & Sontag (1999). We guarantee the existence of such a feedback, under the hypotheses:

(i) Subsystem (1.6a) is forward complete with (x, u) as input.
(ii) $0 \in \Re^m$ is GAS for the ‘unforced’ subsystem $\dot{z} = f_0(z, 0, 0)$ (0-GAS property).

This is a generalization of the existing results since forward completeness and 0-GAS is weaker than ISS (or even iISS as shown in Angeli *et al.*, 2000).

Notation

* By $C^j(A)(C^j(A; \Omega))$, where $j \geqslant 0$ is a non-negative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of order j on A.
* By $B_r(\overline{B}_r)$, where $r > 0$, we denote the open (closed) ball of radius r in $\Re^n$, centred at $0 \in \Re^n$.
* For definitions of classes K, K_∞, KL see [8].
* By $D^+ f(t)$ we denote the upper right-hand side Dini derivative of the scalar function f, i.e. $D^+ f(t) = \limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h}$.
* We denote by M_D the class of measurable functions $d : \Re^+ \to D$.

2. Definitions and preliminary technical results

In this section we give the notion of ϕ-RGAS for time-varying systems and we present definitions and technical lemmas that play a key role in proving the main results of the paper. Their proofs can be found in the Appendix.

DEFINITION 2.1 We denote by K^+ the class of non-decreasing C^∞ functions $\phi : \Re^+ \to \Re$ with $\phi(0) \geqslant 1$, and we denote by $K^* \subset K^+$, the class of C^∞ functions that belong to K^+ and satisfy $\lim_{t \to +\infty} \frac{\dot{\phi}(t)}{\phi^r(t)} = 0$, for some $r \geqslant 1$.

For example the functions $\phi(t) = 1$, $\phi(t) = 1 + t$, $\phi(t) = \exp(t)$ all belong to the class K^*. The following lemma states some of the properties of these classes of functions.

LEMMA 2.2 For every $p \in K^*$, $q \in K^*$ and for all constants $M \geqslant 1$ and $a \geqslant 0$, it holds that the functions $p(\cdot) + q(\cdot)$, $p(\cdot)q(\cdot)$ and $Mp^a(\cdot)$ are of class K^* as well. Furthermore, for every function ϕ of class K^+, there exists a function $\tilde{\phi}$ of class K^*, such that $\phi(t) \leqslant \tilde{\phi}(t)$ for all $t \geqslant 0$.

We next give the notion of ϕ-RGAS, which directly extends the notion of RGAS presented in Karafyllis & Tsinias (2001). This notion is introduced in such a manner that we can have an estimate of the rate of convergence of the solution to the equilibrium point. Consider the system (1.1), where D is a compact subset of $\Re^m$ and the vector field $f : \Re^+ \times \Re^n \times D \to \Re^n$ satisfies the following conditions:

(1) The function $f(t, x, d)$ is measurable in t, for all $(x, d) \in \Re^n \times D$.
(2) The function $f(t, x, d)$ is continuous in d, for all $(t, x) \in \Re^+ \times \Re^n$.
(3) The function $f(t, x, d)$ is locally Lipschitz with respect to x, uniformly in $d \in D$, in the sense that for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^n$, there exists a constant $L \geqslant 0$ such that

$$\begin{aligned} &|f(t, x, d) - f(t, y, d)| \leqslant L|x - y| \\ &\forall t \in I,\ \forall (x, y) \in S \times S,\ \forall d \in D \end{aligned} \tag{2.1}$$

with $f(t, 0, d) = 0$, for all $(t, d) \in \Re^+ \times D$.

Let us denote by $x(t, t_0, x_0; d) = x(t)$ the unique solution of (1.1) at time t that corresponds to input $d \in M_D$ with initial condition $x(t_0) = x_0$ (see Fillipov, 1988).

DEFINITION 2.3 Let ϕ be a function of the class K^+.

- We say that $0 \in \Re^n$ is ϕ-Robustly Globally Stable (ϕ-RGS), if for every $T \geqslant 0$, $p \geqslant 0$ and $\varepsilon > 0$, it holds that $\sup\{\phi^p(t)|x(t)| : d \in M_D, t \geqslant t_0, |x_0| \leqslant \varepsilon, t_0 \in [0, T]\} < +\infty$ and there exists a $\delta = \delta(\varepsilon, T, p) > 0$ such that $\phi^p(t)|x(t)| \leqslant \varepsilon$, for all $|x_0| \leqslant \delta$, $t \geqslant t_0$, $t_0 \in [0, T]$ and input $d \in M_D$.
- $0 \in \Re^n$ is called a ϕ-Robust Global Attractor (ϕ-RGA), if for every $R > 0$, $\varepsilon > 0$, $p \geqslant 0$ and $T \geqslant 0$, there is a $\tau := \tau(\varepsilon, T, R, p) \geqslant 0$ such that $\phi^p(t)|x(t)| \leqslant \varepsilon$ for all $x_0 \in \overline{B}_R$, $t \geqslant t_0 + \tau$, $t_0 \in [0, T]$ and input $d \in M_D$.
- $0 \in \Re^n$ is called ϕ-Robustly Globally Asymptotically Stable (ϕ-RGAS), if it is ϕ-RGS and ϕ-RGA. If $0 \in \Re^n$ is ϕ-RGAS for $\phi(t) := 1$ then we simply write that $0 \in \Re^n$ is RGAS.

In Tsinias & Karafyllis (1999) we required $t^N|x(t)| \leqslant \varepsilon$ for every integer $N \geqslant 0$ in the definition of the L_t–Global Asymptotic Stability. It is clear that the present definition includes this case with $\phi(t) = 1+t \in K^*$. The following lemma clarifies the consequences of the notion of ϕ-RGAS and provides estimates of the solutions.

LEMMA 2.4 Suppose that $0 \in \Re^n$ is ϕ-RGAS, for system (1.1) with $d \in D$ as input. Then the following statements hold:

(i) For every function $\tilde{\phi} \in K^+$ that satisfies $\tilde{\phi}(t) \leqslant \phi(t)$ for all $t \geqslant 0$, $0 \in \Re^n$ is $\tilde{\phi}$-RGAS for system (1.1). Particularly, $0 \in \Re^n$ is RGAS.
(ii) For every pair of constants $R \geqslant 1$ and $p \geqslant 0$, $0 \in \Re^n$ is $\tilde{\phi}$-RGAS, for system (1.1), where $\tilde{\phi}(t) = R\phi^p(t)$.
(iii) For every $p \geqslant 0$, there exist functions $\sigma(\cdot) \in KL$ and $\beta(\cdot) \in K^+$, such that the following estimate holds for the solution of (1.1):

$$|x(t)| \leqslant \frac{1}{\phi^p(t)}\sigma\left(\beta(t_0)|x_0|, t - t_0\right),\ \forall t \geqslant t_0,\ \forall d \in M_D. \tag{2.2}$$

For the construction of a smooth time-varying feedback, we need to introduce the following class of convex functions.

DEFINITION 2.5 We say that $a : \Re^+ \to \Re^+$ belongs to K_{con} if:

(1) $a \in C^1(\Re^+) \cap K$.
(2) The function $\frac{da}{ds}(s)$ is non-decreasing.

The following technical lemmas state some important properties of class K_{con} that are used in the subsequent sections of this paper.

LEMMA 2.6 The following statements hold:

(i) $K_{con} \subset K_\infty$.
(ii) If a, β belong to K_{con} then $a + \beta, a\beta, ao\beta$ also belong to K_{con}.
(iii) For every $a \in C^0(\Re^+)$ there exists a constant $R > 0$ and a function $\beta \in K_{con}$ such that $a(s) \leqslant R + \beta(s), \forall s \geqslant 0$.
(iv) If $a(\cdot) \in K_{con}$ the following properties hold:

$$\lambda a(s) \leqslant a(\lambda s),\ \forall s \geqslant 0,\ \forall \lambda \geqslant 1 \tag{2.3a}$$

$$a(s_1) + a(s_2) \leqslant a(s_1 + s_2) \leqslant a(2s_1) + a(2s_2),\ \forall (s_1, s_2) \in \left(\Re^+\right)^2. \tag{2.3b}$$

LEMMA 2.7 Consider the vector field $f \in C^0(\Omega \times \Re^n; \Re^m)$, where $\Omega \subset \Re^l$ is a compact set, which is locally Lipschitz with respect to $x \in \Re^n$, uniformly with respect to $\theta \in \Omega$ and satisfies $f(\theta, 0) = 0$, for all $\theta \in \Omega$. Then there exists $a \in K_{con}$ such that:

$$|f(\theta, x)| \leqslant a(|x|),\ \forall (\theta, x) \in \Omega \times \Re^n. \tag{2.4}$$

LEMMA 2.8 For every $a \in K_{con}$, there exists an odd function $\beta \in C^\infty(\Re)$, functions $\gamma \in K_{con}$, $\tilde{a} \in K_{con} \cap C^\infty(\Re^+)$ and a constant $R > 0$, with the following properties:

$$a(s) \leqslant \tilde{a}(s),\ \forall s \geqslant 0 \tag{2.5}$$

$$a(s) \leqslant \beta(s) \leqslant \gamma(s),\ \forall s \geqslant 0 \tag{2.6}$$

$$\left|\frac{d\beta}{ds}(s)\right| \leqslant R + \gamma(s),\ \forall s \geqslant 0. \tag{2.7}$$

The next lemma shows a fundamental property of forward complete time-varying systems. It shows that the 'reachable set' contains a closed ball of positive radius at all times. This fact is going to be used in Section 3 of the paper.

LEMMA 2.9 Consider system (1.1) and suppose that there exists a function $\rho(\cdot) \in K_{con}$ and a function $\phi(\cdot) \in K^+$ such that

$$|f(t, x, d)| \leqslant \rho(\phi(t)|x|),\ \forall (t, x, d) \in \Re^+ \times \Re^n \times D. \tag{2.8}$$

Suppose, furthermore, that for all $r \geqslant 0$, $t_0 \geqslant 0$ and $t \geqslant t_0$ we have

$$\sup\{|x(t)|; |x_0| \leqslant r, d \in M_D\} < +\infty. \tag{2.9}$$

Then it holds that

$$\overline{B}_{\beta(t,t_0,r)} \subseteq \{x(t); |x_0| \leqslant r, d \in M_D\} \tag{2.10}$$

where $\beta(t, t_0, r)$ is the unique solution of initial value problem:

$$\begin{aligned} &\dot{w} = -\rho(\phi(t)w) \\ &w \in \Re, w(t_0) = r \geqslant 0\,. \end{aligned} \tag{2.11}$$

3. Adding a time-varying integrator

The following technical lemma is the basic tool in the integrator backstepping method that we intend to use. Notice that in the time-varying case, there are many technical difficulties to obtain such a result, concerning the rate of convergence of the solution to the equilibrium point, as well as the issue of whether the dynamics converge to zero or not. Most of the technical assumptions introduced below are automatically satisfied in the autonomous case.

LEMMA 3.1 Consider the system

$$\dot{x} = F(t, \theta, x, y) \tag{3.1a}$$

$$\dot{y} = f(t, \theta, x, y) + g(t, \theta, x, y)u$$
$$x \in \Re^n, y \in \Re, u \in \Re, t \geqslant 0, \theta \in \Omega \tag{3.1b}$$

where $\Omega \subset \Re^l$ is a compact set, with $F(t, \theta, 0, 0) = 0$, $f(t, \theta, 0, 0) = 0$, for all $(t, \theta) \in \Re^+ \times \Omega$ and F, f, g are measurable with respect to t, continuous with respect to θ, and locally Lipschitz with respect to (x, y) uniformly in $\theta \in \Omega$. Suppose that there exists $\phi \in K^*$ such that the following hold:

(H1) There exists a function $\gamma \in K_\infty$, being locally Lipschitz on $\Re^+$, a $C^j (j \geqslant 1)$ mapping $k : \Re^+ \times \Re^n \to \Re$ with $k(\cdot, 0) = 0$, a constant $\mu \geqslant 0$, such that $0 \in \Re^n$ is ϕ-RGAS for

$$\dot{x} = F\left(t, \theta, x, k(t, x) + d\frac{\gamma(|x|)}{\phi^\mu(t)}\right) \tag{3.2}$$

with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input.

(H2) There exists a function $p \in K_{\text{con}}$ such that

$$s \leqslant p(\gamma(s)), \ \forall s \geqslant 0. \tag{3.3}$$

(H3) There exists $a \in K_{\text{con}}$ such that the following inequalities hold for all $(t, \theta, x, y) \in \Re^+ \times \Omega \times \Re^n \times \Re$:

$$|F(t, \theta, x, y)| \leqslant a(\phi(t)|(x, y)| \tag{3.4}$$

$$|k(t, x) + \left|\frac{\partial k}{\partial t}(t, x)\right| \leqslant a(\phi(t)|x|) \tag{3.5}$$

$$\left|\frac{\partial k}{\partial t}(t, x)\right| \leqslant \phi(t) + a(\phi(t)|x|). \tag{3.6}$$

(H4) There exist constants $K > 0$ and $\delta \geqslant 0$ such that the following inequalities hold for all $(t, \theta, x, y) \in \Re^+ \times \Omega \times \Re^n \times \Re$:

$$\frac{1}{K\phi^\delta(t)} \leqslant g(t, \theta, x, y) \leqslant \phi(t) + a(\phi(t)|(x, y)|) \tag{3.7}$$

$$|f(t, \theta, x, y) \leqslant a(\phi(t)|(x, y)|). \tag{3.8}$$

Then for every locally Lipschitz function $\Gamma(\cdot) \in K_\infty$, there exists a C^∞ mapping $\overline{k} : \Re^+ \times \Re \to \Re$ with $\overline{k}(\cdot, 0) = 0$ and constants $M \geqslant 1$, $q \geqslant 1$, such that $0 \in \Re^{n+1}$ is $\tilde{\phi}$-RGAS for

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = \tilde{F}(t, \theta, \tilde{x}, \overline{k}(t, y - k(t, x)) + d\Gamma(|\tilde{x}|)) \tag{3.9}$$

with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input, where $\tilde{x} := (x, y)$, $\tilde{F}(t, \theta, \tilde{x}, u) := \begin{pmatrix} F(t, \theta, x, y) \\ f(t, \theta, x, y) + g(t, \theta, x, y)u \end{pmatrix}$ and

$$\tilde{\phi}(t) := M\phi^q(t). \tag{3.10}$$

Furthermore, there exists a function $\tilde{a}(\cdot) \in K_{\mathrm{con}}$, such that hypothesis (H3) is satisfied with $\tilde{F}(\cdot)$, $\tilde{x}$, $\tilde{k}(t, \tilde{x}) := \overline{k}(t, y - k(t, x))$, $\tilde{a}(\cdot)$ and $\tilde{\phi}(\cdot)$ instead of $F(\cdot)$, x, $k(t, x)$, $a(\cdot)$ and $\phi(\cdot)$, respectively. When $\phi(\cdot)$ is bounded then the mapping $\overline{k}$ can be chosen to be independent of t.

Proof. Let $\Phi(t, t_0, x_0; (\theta, d))$ denote the solution of (3.2) initiated from $x_0 \in \Re^n$ at time $t_0 \geqslant 0$ and corresponding to input $(\theta, d) \in M_D$ and $\tilde{x}(t) = (x(t), y(t))$ denote the solution of (3.9) initiated from $\tilde{x}_0 \in \Re^{n+1}$ at time $t_0 \geqslant 0$ and corresponding to input $(\theta, d) \in M_D$. The proof is based on the following observations:

(i) By property (ii) of Lemma 2.4 and definition (3.10), it suffices to show that $0 \in \Re^{n+1}$ is ϕ-RGAS for (3.9) with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input.

(ii) In order to prove that $0 \in \Re^{n+1}$ is ϕ-RGAS for (3.9) with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input, it suffices to show that there exists a function $G(\cdot) \in K_{\mathrm{con}}$ and a C^0 function $E : \Re^+ \times \Re^+ \to \Re^+$ with $E(t, \cdot) \in K_\infty$ for all $t \geqslant 0$ and $E(\cdot, s)$ being non-decreasing, in such a way that the following inequality holds for all $t \geqslant t_0$, $s \geqslant 0$:

$$\begin{aligned}&\sup\{|\tilde{x}(t)|; |\tilde{x}_0| \leqslant s, (\theta, d) \in M_D\} \leqslant \\ &G(\phi(t) \sup\{|\Phi(t, t_0, x_0; (\theta, d))|; |x_0| \leqslant E(t_0, s), (\theta, d) \in M_D\}).\end{aligned} \tag{3.11}$$

Indeed, notice that by virtue of (3.11), property (iv) of Lemma 2.6 and the facts that $\phi(t) \geqslant 1$ and $E(\cdot, s)$ is non-decreasing, we have for all $q \geqslant 0$, $T \geqslant 0$, $s \geqslant 0$ and $h \geqslant 0$:

$$\begin{aligned}&\sup\{\phi^q(t_0 + h)|\tilde{x}(t_0 + h)|; |\tilde{x}_0| \leqslant s, t_0 \in [0, T], (\theta, d) \in M_D\} \leqslant \\ &G\left(\phi^{q+1}(t_0 + h) \sup\{|\Phi(t_0 + h, t_0, x_0; (\theta, d))|; |x_0| \leqslant E(T, s), t_0 \in [0, T], (\theta, d) \in M_D\}\right).\end{aligned} \tag{3.12}$$

Furthermore, by virtue of (iii) of Lemma 2.4 and the fact that $0 \in \Re^n$ is ϕ-RGAS for (3.2), it follows that there exist functions $\sigma(\cdot) \in KL$ and $\beta(\cdot) \in K^+$, such that the following estimate holds for the solution of (3.2):

$$\phi^{q+1}(t)|\Phi(t, t_0, x_0; (\theta, d))| \leqslant \sigma(\beta(t_0)|x_0|, t - t_0), \ \forall t \geqslant t_0, \ \forall (\theta, d) \in M_D. \tag{3.13}$$

Combining (3.12) with (3.13) we obtain the following estimate, which holds for all $h, s, T, q \geqslant 0$:

$$\sup\left\{\phi^q(t_0 + h)|\tilde{x}(t_0 + h)|; |\tilde{x}_0| \leqslant s, t_0 \in [0, T], (\theta, d) \in M_D\right\} \leqslant G(\sigma(\beta(T)E(T, s), h)),$$

which implies that $0 \in \Re^{n+1}$ is ϕ-RGAS for (3.9) with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input.

Since the proof is long and technical, we divide it into two parts.

First part: Construction of Feedback. Given a locally Lipschitz function $\Gamma(\cdot) \in K_\infty$, we construct the feedback law $\tilde{k}(t, \tilde{x}) := \overline{k}(t, y - k(t, x))$, where $\overline{k}(\cdot) \in C^\infty(\Re^+ \times \Re)$ with $\overline{k}(\cdot, 0) = 0$, such that the analogue of (H3) is satisfied.

Second part: Stability Analysis. Exploiting the properties of the constructed feedback and Lemma 2.9, we prove that (3.11) holds for appropriate functions $G(\cdot) \in K_{con}$ and a $E(\cdot) \in C^0(\Re^+ \times \Re^+)$. The methodology used is entirely different from the methodology used in Tsinias & Karafyllis (1999) for the case $\phi(t) = 1 + t$.

Without loss of generality, we may assume that the function $p(\cdot)$, involved in (3.3), is of class $K_{con} \cap C^\infty(\Re^+)$. Indeed, this follows from Lemma 2.8, which guarantees the existence of a function $\tilde{p}(\cdot) \in K_{con} \cap C^\infty(\Re^+)$ that satisfies $p(s) \leqslant \tilde{p}(s)$, for all $s \geqslant 0$. Consequently, if $p(\cdot)$ is not of class $K_{con} \cap C^\infty(\Re^+)$, we can replace it by $\tilde{p}(\cdot)$. Similarly, without loss of generality, we may assume that $\gamma \in K_{con} \cap C^\infty(\Re^+)$, because if this is not the case then we can replace $p(\cdot) \in K_{con} \cap C^\infty(\Re^+)$ in (3.3) by $\overline{p}(s) := p(s) + s$ and $\gamma(s)$ by $\overline{\gamma}(s) := \overline{p}^{-1}(s) \leqslant \gamma(s)$, which belongs to $K_\infty \cap C^\infty(\Re^+)$.

First part: Construction of Feedback. In this part of proof we use repeatedly inequalities (2.3a), (2.3b) of Lemma 2.6 for functions of class K_{con}, as well as the fact that $\phi(t) \geqslant 1$ for all $t \geqslant 0$. Notice that, by application of Lemma 2.7, to the even extensions of γ and $\Gamma(\cdot)$ (which are locally Lipschitz), there exist functions $\tilde{\gamma}(\cdot) \in K_{con}$ and $\tilde{\Gamma}(\cdot) \in K_{con}$ such that

$$\gamma(s) \leqslant \tilde{\gamma}(s), \ \forall s \geqslant 0 \tag{3.14a}$$

$$\Gamma(s) \leqslant \tilde{\Gamma}(s), \ \forall s \geqslant 0. \tag{3.14b}$$

Define

$$u := \overline{k}(t, z) + d\Gamma(|\tilde{x}|) \tag{3.15a}$$

$$z := y - k(t, x) \tag{3.15b}$$

where $\overline{k}(\cdot)$ is yet to be selected. We get from (3.1) and (3.15a), (3.15b):

$$\begin{aligned} \dot{z} \;=\; & f(t, \theta, x, k(t, x) + z) + dg(t, \theta, x, k(t, x) + z)\Gamma(|\tilde{x}|) - \frac{\partial k}{\partial t}(t, x) \\ & - \frac{\partial k}{\partial t}(t, x)F(t, \theta, x, k(t, x) + z) + g(t, \theta, x, k(t, x) + z)\overline{k}(t, z). \end{aligned} \tag{3.15c}$$

Moreover, by (3.3), (3.5), (3.14a) and (3.15b), there exist functions $G_i(\cdot) \in K_{\text{con}}$ ($i = 1, \ldots, 3$) such that

$$|\tilde{x}| \leqslant G_1\left(\phi^{\mu+1}(t)|z|\right), \text{ when } \gamma(|x|) \leqslant 2\phi^{\mu}(t)|z| \tag{3.16a}$$

$$|\tilde{x}| \leqslant G_2\left(\phi(t)|x|\right), \text{ when } \phi^{\mu}(t)|z| \leqslant \gamma(|x|) \tag{3.16b}$$

$$|z| \leqslant G_3\left(\phi(t)|\tilde{x}|\right), \ \forall (t, \tilde{x}) \in \Re^+ \times \Re^{n+1} \tag{3.16c}$$

where $\mu \geqslant 0$ is the constant involved in (3.2). Furthermore, property (iii) of Lemma 2.6 implies the existence of a function $G_4(\cdot) \in K_{\text{con}}$ and a constant $R_1 > 0$ such that

$$0 \leqslant \max_{0 \leqslant \xi \leqslant s} \left\{ \frac{\mathrm{d}\gamma}{\mathrm{d}s}(\xi) \right\} \leqslant R_1 + G_4(s), \ \forall s \geqslant 0. \tag{3.16d}$$

It follows from (3.4), (3.5), (3.6), (3.7), (3.8), (3.14b), (3.15c), (3.16a) and (3.16d) that there exists a constant $\nu \geqslant 2$ and functions $a_i(\cdot) \in K_{\text{con}}$, $i = 1, 2$, that satisfy the following inequalities:

$$\frac{\mathrm{d}}{\mathrm{d}t}|z(t)| \leqslant a_1(\phi^{\mu+\nu}(t)|z|) + \operatorname{sgn}(z) g(t, \theta, x, k(t, x) + z)\bar{k}(t, z),$$
$$\text{for } 2\phi^{\mu}(t)|z| \geqslant \gamma(|x|) \text{ and } z \neq 0 \tag{3.17}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(|x(t)|)\right| \leqslant a_2(\phi^{\mu+\nu}(t)|z|), \quad \text{for } 2\phi^{\mu}(t)|z| \geqslant \gamma(|x|) > 0 \tag{3.18}$$

Since $\phi \in K^*$, there exist constants $K' \geqslant 0$ and $r \geqslant 1$ with

$$0 \leqslant \dot{\phi}(t) \leqslant K'\phi^r(t), \ \forall t \geqslant 0. \tag{3.19}$$

Inequalities (3.17), (3.18) in conjunction with (3.19) imply

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi^{\mu}(t)|z(t)| - \gamma(|x(t)|)\right) \leqslant\ & K'\mu\phi^{\sigma}(t)|z| + a_1(\phi^{\sigma}(t)|z|) + a_2(\phi^{\sigma}(t)|z|) \\ & + \phi^{\mu}(t)\operatorname{sgn}(z) g(t, \theta, x, k(t, x) + z)\bar{k}(t, z)\end{aligned}$$
$$\text{for } 2\phi^{\mu}(t)|z| \geqslant \gamma(|x|) > 0 \tag{3.20a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi^{\mu}(t)|z(t)|\right) \leqslant\ & K'\mu\phi^{\sigma}(t)|z| + a_1(\phi^{\sigma}(t)|z|) \\ & + \phi^{\mu}(t)\operatorname{sgn}(z) g(t, \theta, x, k(t, x) + z)\bar{k}(t, z)\end{aligned}$$
$$\text{for } 2\phi^{\mu}(t)|z| \geqslant \gamma(|x|) \text{ and } z \neq 0 \tag{3.20b}$$

where

$$\sigma := 2\mu + \nu + r - 1. \tag{3.21}$$

We define the function $a_3(\cdot) \in K_{\text{con}}$:

$$a_3(s) := a_1(s) + a_2(s) + K'\mu s + (R_1 + G_4(s))\rho(s) \tag{3.22}$$

$$\rho(s) := a\,(s + a(s) + \tilde{\gamma}(s)) \tag{3.23}$$

where $\tilde{\gamma}(\cdot) \in K_{\rm con}$, $R_1 > 0$ and $G_4(\cdot) \in K_{\rm con}$ are defined in (3.14a) and (3.16d), respectively. Notice that, by virtue of (3.4), (3.5), (3.14a) and definition (3.23), we have

$$\left| F\left(t, \theta, x, k(t,x) + d\frac{\gamma(|x|)}{\phi^{\mu}(t)}\right)\right| \leqslant \rho(\phi^2(t)|x|). \tag{3.24}$$

Furthermore, by virtue of Lemma 2.8, there exists an odd C^{∞} function $\psi(\cdot)$, a function $G_5(\cdot) \in K_{\rm con}$ and a constant $R_2 > 0$, with the following properties:

$$a_3(s) \leqslant \psi(s) \leqslant G_5(s), \ \ \forall s \geqslant 0 \tag{3.25a}$$

$$\left|\frac{d\psi}{ds}(s)\right| \leqslant R_2 + G_5(s), \ \ \forall s \geqslant 0. \tag{3.25b}$$

We also define

$$\overline{k}(t,z) := -\psi\left((1+K)\phi^{\sigma+\delta}(t) z\right) \tag{3.26}$$

where $K > 0$ and $\delta \geqslant 0$ are the constants involved in (3.7). It is clear that the mapping $\tilde{k}(t,x,y) = \overline{k}(t, y - k(t,x))$ is of class $C^j(\Re^+ \times \Re^{n+1})$. Moreover, inequalities (3.5), (3.6), (3.19) in conjunction with (3.25a), (3.25b) imply that there exists a function $\tilde{a}(\cdot) \in K_{\rm con}$ and constants $q \geqslant 1$, $M \geqslant 1$, such that (H3) is satisfied with $\tilde{x}$, $\tilde{k}(\cdot)$, $\tilde{a}(\cdot)$ and $\tilde{\phi}(\cdot)$ instead of x, $k(\cdot)$, $a(\cdot)$ and $\phi(\cdot)$, respectively, where $\tilde{\phi}(\cdot)$ is defined in (3.10) and is of class K^* by virtue of (iii) of Lemma 2.2. When $\phi(\cdot)$ is bounded we may select for $\tilde{R} := \sup_{t \geqslant 0} \phi(t)$:

$$\overline{k}(t,z) := \overline{k}(z) = -\psi\left((1+K)\tilde{R}^{\sigma+\delta} z\right). \tag{3.26$'$}$$

The major property of the constructed feedback is the following inequality, which is a consequence of (3.7), (3.16d), (3.22), (3.25a), (3.26) and the fact that the function $\psi(\cdot)$ is odd:

$$\begin{aligned} \operatorname{sgn}(z)\phi^{\mu}(t) g(t,\theta,x,k(t,x)+z)\overline{k}(t,z) \leqslant{} & -K'\mu\phi^{\sigma}(t)|z| - a_1(\phi^{\sigma}(t)|z|) - a_2(\phi^{\sigma}(t)|z|) \\ & - \frac{d\gamma}{ds}\left(\phi^{\mu}(t)|z|\right)\rho\left(\phi^{2+\mu}(t)|z|\right). \end{aligned} \tag{3.27}$$

Notice that by virtue of inequalities (3.20a), (3.20b) and (3.27), it follows that:

$$\frac{d}{dt}\left(\phi^{\mu}(t)|z(t)| - \gamma(|x(t)|)\right) \leqslant 0, \quad \text{when } 2\phi^{\mu}(t)|z| \geqslant \gamma(|x|) > 0 \tag{3.28a}$$

$$\frac{d}{dt}\left(\phi^{\mu}(t)|z(t)|\right) \leqslant -\frac{d\gamma}{ds}\left(\phi^{\mu}(t)|z|\right)\rho\left(\phi^{2+\mu}(t)|z|\right), \quad \text{when } 2\phi^{\mu}(t)|z| \geqslant \gamma(|x|) \text{ and } z \neq 0. \tag{3.28b}$$

Second part: Stability Analysis. We define:

$$L_t := \left\{(x, y) \in \Re^n \times \Re : \phi^\mu(t)|y - k(t, x)| \leqslant \gamma(|x|)\right\}. \tag{3.29}$$

Notice that by virtue of (3.28a) and definitions (3.15b), (3.29) of z, L_t, respectively, it follows that the region L_t is positively invariant (the case $|x| = 0$ implies $z = 0$, which is the equilibrium position of (3.9)). As long as the trajectory of the solution of (3.9) remains outside L_t we obtain using (3.15b), (3.28b) and (3.29) that

$$D^+\left(\phi^\mu(t)|z(t)|\right) \leqslant -\frac{\mathrm{d}\gamma}{\mathrm{d}s}(\phi^\mu(t)|z(t)|)\rho\left(\phi^{2+\mu}(t)|z(t)|\right). \tag{3.30}$$

Let $\beta(t, t_0, r)$ denote the unique solution of the following initial value problem:

$$\begin{aligned}&\dot{w} = -\rho\left(\phi^2(t)w\right)\\ &w \in \Re,\, w(t_0) = r \geqslant 0.\end{aligned} \tag{3.31}$$

Indeed, by virtue of inequality (3.24) and Lemma 2.9, we guarantee that

$$\overline{B}_{\beta(t,t_0,r)} \subseteq \{\Phi(t, t_0, x_0; (\theta, d)), |x_0| \leqslant r, (\theta, d) \in M_D\} \tag{3.32}$$

where $\Phi(t, t_0, x_0; (\theta, d))$ denotes the solution of (3.2) initiated from $x_0 \in \Re^n$ at time $t_0 \geqslant 0$ and corresponding to input $(\theta, d) \in M_D$. Furthermore, differential inequality (3.30) implies (by virtue of the comparison principle in Khalil, 1996):

$$\begin{aligned}&\phi^\mu(t)|z(t)| \leqslant \gamma(\beta(t, t_0, \gamma^{-1}(\phi^\mu(t_0)|z(t_0)|)))\\ &\text{as long as the trajectory of (3.9) remains outside } L_t.\end{aligned} \tag{3.33}$$

Thus by (3.14a), (3.16a), (3.16c) and (3.33) we obtain the estimate for the solution of (3.9):

$$\begin{aligned}&|\tilde{x}(t)| \leqslant G_1\left(\phi(t)\tilde{\gamma}\left(\beta(t, t_0, \gamma^{-1}(\phi^\mu(t_0)G_3(\phi(t_0)|\tilde{x}(t_0)|)))\right)\right)\\ &\text{as long as the trajectory of (3.9) remains outside } L_t.\end{aligned} \tag{3.34}$$

Moreover, by virtue of (3.32), (3.33) and the facts that $G_3(\cdot)$, $\tilde{\gamma}(\cdot) \in K_{\mathrm{con}}$, $\phi(t) \geqslant 1$ and using property (iv) of Lemma 2.6, we obtain that the following estimate holds for $G_6 := G_1 o \tilde{\gamma}$:

$$\begin{aligned}&\sup\{|\tilde{x}(t)|; |\tilde{x}_0| \leqslant s, (\theta, d) \in M_D\}\\ &\leqslant G_6\left(\phi(t)\sup\left\{|\Phi(t, t_0, x_0; (\theta, d))|; |x_0| \leqslant \gamma^{-1}\left(G_3(\phi^{\mu+1}(t_0)s)\right), (\theta, d) \in M_D\right\}\right)\\ &\text{as long as the trajectory of (3.9) remains outside } L_t.\end{aligned} \tag{3.35}$$

Let us denote by $T \leqslant +\infty$ the first time that the solution is entering L_T. Notice that for all $t \geqslant T$, by positive invariance of L_t, the solution remains inside L_t. Moreover, there exists an input $(\tilde{\theta}, \tilde{d}) \in M_D$ such that component $x(t)$ of the solution $\tilde{x}(t)$ of (3.9) satisfies $x(t) \equiv \Phi(t, T, x(T); (\tilde{\theta}, \tilde{d}))$, for all $t \geqslant T$.

Let $t \geqslant t_0$ be arbitrary. We distinguish the cases:

(a) $T = t_0$. In this case we have $|x(t)| \leqslant \sup\{|\Phi(t, t_0, x_0; (\theta, d))|; (\theta, d) \in M_D\}$ and consequently using (3.16b) we obtain the estimate:

$$\begin{aligned} &\sup\{|\tilde{x}(t)|; |\tilde{x}_0| \leqslant s, (\theta, d) \in M_D\} \leqslant \\ &G_2\left(\phi(t) \sup\{|\Phi(t, t_0, x_0; (\theta, d))|; |x_0| \leqslant s, (\theta, d) \in M_D\}\right). \end{aligned} \tag{3.36}$$

(b) $T \notin [t_0, t]$. In this case estimate (3.35) holds.

(c) $t_0 < T \leqslant t$. In this case we have:

$$|x(t)| \leqslant \sup\{|\Phi(t, T, x(T); (\theta, d))|; T \in [t_0, t], (\theta, d) \in M_D\}$$

By definition (3.29) and continuity of the solution we also have for the case (c): $\gamma(|x(T)|) = \phi^{\mu}(T)|z(T)|$. Moreover, estimate (3.33) implies $\phi^{\mu}(T)|z(T)| \leqslant \gamma(\beta(T, t_0, \gamma^{-1}(\phi^{\mu}(t_0)|z_0|)))$. These estimates in conjunction with (3.16b), (3.16c) give

$$\sup\{|\tilde{x}(t)|; |\tilde{x}_0| \leqslant s, (\theta, d) \in M_D\} \leqslant G_2\big(\phi(t) \sup I\big) \tag{3.37a}$$

$$\begin{aligned} I := \Big\{&|\Phi(t, T, x(T); (\theta, d))|; T \in [t_0, t], (\theta, d) \in M_D, \\ &|x(T)| \leqslant \beta\left(T, t_0, \gamma^{-1}\left(\phi^{\mu}(t_0) G_3(\phi(t_0)s)\right)\right)\Big\}. \end{aligned} \tag{3.37b}$$

Inclusion (3.32) in conjunction with estimate (3.37) and the fact that $G_3(\cdot) \in K_{\text{con}}$ and $\phi(t) \geqslant 1$, shows that the following estimate holds for case (c):

$$\begin{aligned} &\sup\{|\tilde{x}(t)|; |\tilde{x}_0| \leqslant s, (\theta, d) \in M_D\} \\ &\leqslant G_2\big(\phi(t) \sup\big\{|\Phi(t, t_0, x_0; (\theta, d))|; |x_0| \leqslant \gamma^{-1}\left(G_3(\phi^{\mu+1}(t_0)s)\right), (\theta, d) \in M_D\big\}\big). \end{aligned} \tag{3.38}$$

Combining estimates (3.35), (3.36) and (3.38) for the cases (a), (b) and (c), respectively, we obtain the desired inequality (3.11) for

$$G(s) := 2G_2(s) + G_6(s) \tag{3.39a}$$

$$E(t_0, s) := \max\left\{s, \gamma^{-1}(G_3(\phi^{\mu+1}(t_0)s))\right\}. \tag{3.39b}$$

Indeed, by virtue of property (ii) of Lemma 2.6, we have $G(\cdot) \in K_{\text{con}}$. The proof is complete. □

The following lemma provides a sufficient condition for hypothesis (H2). In fact, Lemma 3.2 shows that hypothesis (H2) is the analogue of the hypothesis of local exponential stability made in Jiang *et al.* (1994), Tsinias (1996) for the autonomous case.

LEMMA 3.2 Suppose that $\gamma \in K_{\infty}$ is a locally Lipschitz function that satisfies $\gamma(s) \geqslant \Lambda s$, $\forall s \in [0, \eta]$, for some constants $\eta, \Lambda > 0$. Then (H2) is satisfied.

Proof. Clearly, $\gamma^{-1}(s) \leqslant \Lambda^{-1}s$, for all $s \in [0, \gamma(\eta)]$ and consequently the function $g(s) := \sup_{0<u\leqslant s} \frac{\gamma^{-1}(u)}{u}$ is continuous on $(0, +\infty)$, positive and non-decreasing with $g(s) \leqslant \Lambda^{-1}$ for all $s \in (0, \gamma(\eta)]$. Defining $g(0) := \lim_{s\to 0^+} g(s)$, we have $\gamma^{-1}(s) \leqslant g(s)s$, for all $s \geqslant 0$. We also define $h(s) := \int_0^{s+1} g(r)\mathrm{d}r$, which satisfies $h(s) \geqslant g(s)$, for all $s \geqslant 0$ and notice that the function $p(s) := h(s)s$ is of class K_{con}. Combining, we get $\gamma^{-1}(s) \leqslant p(s)$, for all $s \geqslant 0$, which implies (H2). □

4. Applications to time-varying control systems

We are now ready to apply induction using Lemma 3.1 and the method of integrator backstepping for the stabilization of system (1.3).

PROPOSITION 4.1 Consider system (1.3), where $x = (x_1, \dots x_n)^T \in \Re^n$, $\theta \in \Omega \subset \Re^l$, Ω being a compact set, f_i and g_i are measurable with respect to $t \geqslant 0$, continuous with respect to $\theta \in \Omega$ and locally Lipschitz with respect to $(x_1, \dots x_i)$, uniformly in $\theta \in \Omega$, for $i = 1, \dots, n$. Suppose that there exist functions $\phi \in K^*$, $a \in K_{\text{con}}$, constants $\delta \geqslant 0$, $K > 0$, such that the following hold for all $(t, x, \theta) \in \Re^+ \times \Re^n \times \Omega$ and $i = 1, \dots, n$:

$$|f_i(t, \theta, x_1, \dots, x_i)| \leqslant a(\phi(t)|(x_1, \dots, x_i)|) \tag{4.1}$$

$$\frac{1}{K\phi^\delta(t)} \leqslant g_i(t, \theta, x_1, \dots, x_i) \leqslant \phi(t) + a(\phi(t)|(x_1, \dots, x_i)|). \tag{4.2}$$

Then for every locally Lipschitz function $\Gamma(\cdot) \in K_\infty$ there exists a C^∞ mapping $k : \Re^+ \times \Re^n \to \Re$ with $k(\cdot, 0) = 0$, a function $\eta \in K_{\text{con}}$ and a constant $p \geqslant 0$, with

$$|k(t, x)| \leqslant \eta\left(\phi^p(t)|x|\right), \ \forall (t, x) \in \Re^+ \times \Re^n \tag{4.3}$$

such that $0 \in \Re^n$ is ϕ-RGAS for the following system:

$$\begin{aligned} \dot{x}_i &= f_i(t, \theta, x_1, \dots, x_i) + g_i(t, \theta, x_1, \dots, x_i)x_{i+1} \quad i = 1, \dots, n \\ x_{n+1} &= k(t, x) + d\Gamma(|x|) \end{aligned} \tag{4.4}$$

with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input. Moreover, when $\phi(\cdot)$ is bounded then the mapping k can be chosen to be independent of t.

Proof. The proof of the general case is based on Lemma 3.1 and follows by using standard induction arguments like those given in Jiang *et al.* (1994). For reasons of simplicity we consider the case $n = 2$. The general case follows similarly by induction. First we consider the one-dimensional subsystem

$$\begin{aligned} &\dot{x}_1 = f_1(t, \theta, x_1) + g_1(t, \theta, x_1)x_2 \\ &\text{with } x_2 \text{ as input.} \end{aligned} \tag{4.5}$$

Let $M > 0$. We define

$$\tilde{a}(s) := (1 + s)a(s) + (1 + M)s \tag{4.6}$$

which obviously by Lemma 2.6 is of class K_{con}. By Lemma 2.8 there exists an odd function $\psi(\cdot) \in C^\infty(\Re)$, a function $\beta(\cdot) \in K_{\text{con}}$ and a constant $R_1 > 0$ such that

$$\tilde{a}(s) \leqslant \psi(s) \leqslant \beta(s), \ \forall s \geqslant 0 \tag{4.7a}$$

$$\left|\frac{d\psi}{ds}(s)\right| \leqslant R_1 + \beta(s), \ \forall s \geqslant 0. \tag{4.7b}$$

Furthermore, since $\phi(\cdot) \in K^*$, there exists a constant $r \geqslant 1$ such that

$$\lim_{t \to +\infty} \frac{\dot{\phi}(t)}{\phi^r(t)} = 0. \tag{4.7c}$$

We set

$$k_1(t, x_1) := -\psi((1+K)\phi^{1+\delta+r}(t)x_1) \tag{4.8}$$

or for the case of bounded $\phi(\cdot) \in K^*$ with $R_2 := \sup_{t \geqslant 0} \phi(t)$:

$$k_1(t, x_1) := k_1(x_1) = -\psi((1+K)R_2^{1+r+\delta} x_1) \tag{4.8'}$$

where K, δ are the constants involved in (4.2). Obviously $k_1(\cdot)$ is a function of class $C^\infty(\Re^+ \times \Re)$ with $k_1(\cdot, 0) = 0$. It follows from (4.2), (4.6), (4.7a) and definition (4.8), the fact that $\psi(\cdot)$ is odd and $\phi(t) \geqslant 1$ for all $t \geqslant 0$, as well as (iv) of Lemma 2.6, that the following inequality holds for all $(t, \theta, x_1) \in \Re^+ \times \Omega \times \Re$:

$$\begin{aligned} \operatorname{sgn}(x_1) g_1(t, \theta, x_1) k_1(t, x_1) \leqslant -\tilde{a}\left(\phi^{r+1}(t)|x_1|\right) &\leqslant -M\phi^r(t)|x_1| - (1+|x_1|)a(\phi(t)|x_1|) \\ &\quad -\phi(t)|x_1|. \end{aligned} \tag{4.9}$$

Moreover, inequalities (4.7a)–(4.7c) and definition (4.8) imply the existence of constants $R_3 \geqslant 1$, $\sigma \geqslant 1$ and a function $\zeta(\cdot) \in K_{\text{con}}$, such that the following inequalities hold for all $(t, x_1) \in \Re^+ \times \Re$:

$$|k_1(t, x_1)| + \left|\frac{\partial k_1}{\partial t}(t, x_1)\right| \leqslant \zeta\left(\tilde{\phi}(t)|x_1|\right) \tag{4.10a}$$

$$\left|\frac{\partial k_1}{\partial x_1}(t, x_1)\right| \leqslant \tilde{\phi}(t) + \zeta\left(\tilde{\phi}(t)|x_1|\right) \tag{4.10b}$$

$$\tilde{\phi}(t) := R_3 \phi^\sigma(t). \tag{4.10c}$$

We claim that $0 \in \Re$ is ϕ-RGAS for the closed-loop system (4.5) with $x_2 = k_1(t, x_1) + d|x_1|$, $d \in [-1, 1]$. In order to prove this claim, notice that by (4.1), (4.2) and (4.9) we have for all $x_1 \neq 0$, $t \geqslant 0$ and $(\theta, d) \in D = \Omega \times [-1, 1]$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|x_1| &= \operatorname{sgn}(x_1) f(t, \theta, x_1) + \operatorname{sgn}(x_1) g(t, \theta, x_1) k_1(t, x_1) + \operatorname{sgn}(x_1) g(t, \theta, x_1) d|x_1| \\ &\leqslant -M\phi^r(t)|x_1|. \end{aligned}$$

Since $x_1 = 0$ is the equilibrium point of the closed-loop system (4.5) with $x_2 = k_1(t, x_1) + d|x_1|$, $d \in [-1, 1]$ we get

$$D^+|x_1(t)| \leqslant -M\phi^r(t)|x_1(t)|, \text{ for all } t \geqslant t_0. \tag{4.11}$$

The differential inequality (4.11) and the comparison lemma in Khalil (1996) give for all $q \geqslant 0, s \geqslant 0$:

$$\sup\left\{\phi^q(t)|x_1(t)|;\ |x_1(t_0)| \leqslant s, (\theta, d) \in M_D\right\} \leqslant \phi^q(t_0)\exp\left(-\int_{t_0}^{t}\left(M\phi^r(\tau) - q\frac{\dot{\phi}(\tau)}{\phi(\tau)}\right)\mathrm{d}\tau\right)s. \tag{4.12}$$

By (4.7c) it follows that for every $q \geqslant 0$ there exists a finite time $T := T(q) \geqslant 0$, such that $\frac{1}{2}M\phi(t) \geqslant q\frac{\dot{\phi}(t)}{\phi^r(t)}$ for all $t \geqslant T$. This implies $\int_0^{+\infty}\left(M\phi^r(\tau) - q\frac{\dot{\phi}(\tau)}{\phi(\tau)}\right)\mathrm{d}\tau = +\infty$ for all $q \geqslant 0$ and consequently by (4.12), we have that $0 \in \Re$ is ϕ-RGAS for the closed-loop system (4.5) with $x_2 = k_1(t, x_1) + d|x_1|$, $d \in [-1, 1]$. By (ii) of Lemma 2.4 and definition (4.10c) it also follows that $0 \in \Re$ is $\tilde{\phi}$-RGAS for the closed-loop system (4.5) with $x_2 = k_1(t, x_1) + d|x_1|$, $d \in [-1, 1]$.

Consider next the two-dimensional subsystem

$$\begin{aligned}\dot{x}_1 &= f_1(t, \theta, x_1) + g_1(t, \theta, x_1)x_2\\ \dot{x}_2 &= f_2(t, \theta, x_1, x_2) + g_1(t, \theta, x_1, x_2)x_3\end{aligned}$$
with x_3 as input.

We apply Lemma 3.1 for this system. Clearly, by the previous analysis hypothesis (H1) is satisfied for $\tilde{\phi}(\cdot) \in K^*$ as defined by (4.10c), $\gamma(s) := s$ and $\mu = 0$. Hypothesis (H2) is trivially satisfied, while hypotheses (H3) and (H4) are consequences of inequalities (4.1), (4.2), (4.10a), (4.10b). The desired conclusion follows from the application of Lemma 3.1. The proof is complete. □

The following proposition is concerned with the problem of partial-state robust feedback stabilization of uncertain nonlinear time-varying systems. It is the analogue of Corollary 3.4 in [11] and Theorem 2.4 in [12], although here we consider uncertain systems. Its proof follows directly by induction and Lemmas 3.1 and 3.2.

PROPOSITION 4.2 Consider the system (1.4), where $z \in \Re^m$, $x = (x_1, \ldots x_n)^T \in \Re^n$, $\theta \in \Omega \subset \Re^l$, Ω is a compact set, $f_i (i = 0, \ldots, n)$ and $g_i (i = 1, \ldots, n)$ are measurable with respect to $t \geqslant 0$, C^0 with respect to $\theta \in \Omega$ and locally Lipschitz with respect to (z, x), uniformly in $\theta \in \Omega$. Suppose that there exist functions $\phi \in K^*$, $a \in K_{\mathrm{con}}$, constants $\delta \geqslant 0$ and $K > 0$, such that the following hold for all $(t, z, x, \theta) \in \Re^+ \times \Re^m \times \Re^n \times \Omega$ and $i = 1, \ldots, n$:

$$|f_0(t, \theta, z, x_1)| \leqslant a(\phi(t)\left|(z^T, x_1)\right|) \tag{4.13}$$

$$|f_i(t, \theta, z, x_1, \ldots, x_i)| \leqslant a(\phi(t)\left|(z^T, x_1, \ldots, x_i)\right|) \tag{4.14}$$

$$\frac{1}{K\phi^\delta(t)} \leqslant g_i(t, \theta, z, x_1, \ldots, x_i)) \leqslant \phi(t) + a(\phi(t)\left|(z^T, x_1, \ldots, x_i)\right|). \tag{4.15}$$

Suppose, furthermore, that $0 \in \Re^m$ is ϕ-RGAS for the following system

$$\dot{z} = f_0\left(t, \theta, z, d\frac{\gamma(|z|)}{\phi^\mu(t)}\right) \tag{4.16}$$

with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input, where $\mu \geqslant 0$ and $\gamma \in K_\infty$ is a locally Lipschitz function that satisfies the following inequality for certain constants $\eta, \Lambda > 0$:

$$\Lambda s \leqslant \gamma(s), \ \forall s \in [0, \eta]. \tag{4.17}$$

Then for every locally Lipschitz function $\Gamma(\cdot) \in K_\infty$ there exists a C^∞ mapping $k : \Re^+ \times \Re^n \to \Re$ with $k(\cdot, 0) = 0$, such that $0 \in \Re^m \times \Re^n$ is ϕ-RGAS for the following system:

$$\begin{aligned} \dot{z} &= f_0(t, \theta, z, x_1) \\ \dot{x}_i &= f_i(t, \theta, z, x_1, \dots, x_i) + g_i(t, \theta, z, x_1, \dots, x_i)x_{i+1} \quad i = 1, \dots, n \\ x_{n+1} &= k(t, x) + d\Gamma(|(z, x)|) \end{aligned} \tag{4.18}$$

with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input. Moreover, when $\phi(\cdot)$ is bounded then the mapping k can be chosen to be independent of t.

EXAMPLE 4.3 Consider the system:

$$\begin{aligned} &\dot{z} = -2tz + \theta_1(t)\exp(\mu t)x \\ &\dot{x} = \exp(at)\theta_2(t)w(z, x) + u \\ &(z, x) \in \Re^2, t \geqslant 0, u \in \Re, \theta = (\theta_1, \theta_2) \in \overline{B}(0, r) \end{aligned} \tag{4.19}$$

where $a, r, \mu > 0$ are known constants, $w(z, x)$ is a locally Lipschitz function satisfying $w(0, 0) = 0$ and θ denotes the vector of the uncertain parameters of the system. It is immediate to verify that $0 \in \Re$ is e^t-RGAS for the subsystem

$$\dot{z} = -2tz + \theta_1(t)d(t)|z| \tag{4.20}$$

where $|\mathrm{d}(t)| \leqslant 1$. Indeed, for every $(\theta(\cdot), d(\cdot)) \in M_{\overline{B}(0,r)\times[-1,1]}$ the solution of (4.20) satisfies the estimate

$$|z(t)| \leqslant \exp\left\{r(t - t_0) - \left(t^2 - t_0^2\right)\right\}|z_0|.$$

Thus by Proposition 4.2 there exists a C^∞ mapping $k : \Re^+ \times \Re \to \Re$ with $k(\cdot, 0) = 0$ with such that the origin for (4.19) with $u = k(t, x)$ is e^t-RGAS.

5. Applications to autonomous control systems

In this section we study the applications of time-varying feedback laws to autonomous control systems. We first consider the case (1.5). We establish that the use of time-varying feedback can robustly 'accelerate' the rate of convergence of the solution to the equilibrium point. This is shown by the following corollary, which is an immediate consequence of Proposition 4.1 and Lemma 2.7.

COROLLARY 5.1 Consider system (1.5), where $x = (x_1, \dots x_n)^T \in \Re^n$, $\theta \in \Omega \subset \Re^l$, Ω being a compact set, f_i and g_i are continuous with respect to $\theta \in \Omega$ and locally Lipschitz with respect to $(x_1, \dots x_i)$, uniformly in $\theta \in \Omega$, with $f_i(\theta, 0, \dots, 0) = 0$, for all $\theta \in \Omega$, for $i = 1, \dots, n$. Suppose that there exists a constant $K > 0$ such that the following hold for all $(x, \theta) \in \Re^n \times \Omega$ and $i = 1, \dots, n$:

$$\frac{1}{K} \leqslant g_i(\theta, x_1, \dots, x_i). \tag{5.1}$$

Then for every $\phi(\cdot) \in K^*$ and $\Gamma(\cdot) \in K_\infty$ being locally Lipschitz, there exists a C^∞ mapping $k : \Re^+ \times \Re^n \to \Re$ with $k(\cdot, 0) = 0$, a function $\eta(\cdot) \in K_{\text{con}}$ and a constant $p \geqslant 0$, with

$$|k(t, x)| \leqslant \eta\left(\phi^p(t)|x|\right), \ \forall (t, x) \in \Re^+ \times \Re^n \tag{5.2}$$

such that $0 \in \Re^n$ is ϕ-RGAS for the following system with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input:

$$\begin{aligned} \dot{x}_i &= f_i(\theta, x_1, \dots, x_i) + g_i(\theta, x_1, \dots, x_i)x_{i+1} \quad i = 1, \dots, n \\ x_{n+1} &= k(t, x) + d\Gamma(|x|). \end{aligned} \tag{5.3}$$

Proof. Let $\phi(\cdot) \in K^*$. Since each f_i and g_i is locally Lipschitz, uniformly in $\theta \in \Omega$, Lemma 2.7 implies that there exists a function $a \in K_{\text{con}}$, such that the following inequalities hold for all $(t, \theta, x) \in \Re^+ \times \Omega \times \Re^n$ and $i = 1, \dots, n$:

$$|f_i(\theta, x_1, \dots, x_i)| \leqslant a(|(x_1, \dots, x_i)|) \leqslant a(\phi(t)|(x_1, \dots, x_i)|) \tag{5.4a}$$

$$|g_i(\theta, x_1, \dots, x_i) - g_i(\theta, 0, \dots, 0)| \leqslant a(|(x_1, \dots, x_i)|) \leqslant a(\phi(t)|(x_1, \dots, x_i)|). \tag{5.4b}$$

Furthermore, inequality (5.1) gives for $i = 1, \dots, n$:

$$\frac{1}{K\phi(t)} \leqslant \frac{1}{K} \leqslant g_i(\theta, x_1, \dots, x_i). \tag{5.4c}$$

Inequalities (5.4a)–(5.4c) establish that all hypotheses of Proposition 4.1 are fulfilled for $\overline{\phi}(t) = R\phi(t)$, where $R := 1 + \sup_{\theta \in \Omega, i=1,\dots,n} g_i(\theta, 0, \dots, 0)$. Notice that it holds: $\overline{\phi}(t) \geqslant \phi(t)$ for all $t \geqslant 0$. Therefore for every locally Lipschitz function $\Gamma(\cdot) \in K_\infty$, there exists a C^∞ mapping $k : \Re^+ \times \Re^n \to \Re$, a function $\tilde{\eta}(\cdot) \in K_{\text{con}}$, a constant $p \geqslant 0$, with $|k(t, x)| \leqslant \tilde{\eta}(\overline{\phi}^p(t)|x|)$, such that $0 \in \Re^n$ is $\overline{\phi}$-RGAS for (3.63) with $(\theta, d) \in \Omega \times [-1, 1]$ as input. Consequently, by Lemma 2.4 and inequality $\overline{\phi}(t) \geqslant \phi(t)$, it follows that $0 \in \Re^n$ is ϕ-RGAS for (5.3) with $(\theta, d) \in \Omega \times [-1, 1]$ as input. Moreover, (5.2) is satisfied for $\eta(s) := \tilde{\eta}(R^p s)$. □

REMARK 5.2 Notice that the origin for system (1.5) cannot become ϕ-RGAS for $\phi(t) = \exp(t)$ with the application of locally Lipschitz static time-invariant feedback (i.e. the rate of convergence to $0 \in \Re^n$ of the solution of the closed-loop system (1.5) with a static locally Lipschitz time-invariant feedback cannot be 'faster' than the exponential rate).

EXAMPLE 5.3 Consider the two dimensional system

$$\begin{aligned} &\dot{x}_1 = \theta(t)x_1^2 + x_2 \\ &\dot{x}_2 = u \\ &(x_1, x_2) \in \Re^2, u \in \Re \end{aligned} \tag{5.5}$$

where $\theta(\cdot) : \Re^+ \to [-1, 1]$ is an unknown time-varying parameter. Let $\phi(\cdot) \in K^*$ be a function that satisfies $\lim_{t\to+\infty} \frac{\dot{\phi}(t)}{\phi^r(t)} = 0$, for some constant $r \geqslant 1$. Then $0 \in \Re^2$ is ϕ-RGAS for the following system with $(\theta, d) \in [-1, 1]^2$ as input:

$$\begin{aligned} \dot{x}_1 &= \theta(t)x_1^2 + x_2 \\ \dot{x}_2 &= k(t, x_1, x_2) + d(t)|x_1| + d(t)|x_2| \end{aligned} \tag{5.6}$$

where $k(t, x_1, x_2)$ is the C^∞ time-varying feedback law, defined for some $M > 0$ sufficiently large, by the following relation:

$$\begin{aligned} k(t, x_1, x_2) &:= -M\phi^{2r}(t)\left(z + z^3 + z^5\right) \\ z &:= x_2 + 3\phi^r(t)\left(x_1 + x_1^3\right). \end{aligned} \tag{5.7}$$

Notice that for the selection $\phi(t) \equiv 1 \in K^*$, the feedback law defined by (5.7) is actually time-invariant and guarantees uniform global asymptotic stability of the origin for system (5.6).

Next we consider the problem of partial-state feedback stabilization of autonomous systems of the form (1.6). In the literature the usual assumption is that subsystem (1.6a) is ISS with (x, u) as input. Here we intend to relax this hypothesis, by making use of time-varying feedback of the form $u = k(t, x)$.

THEOREM 5.4 Consider the system (1.6) and suppose that there exists a constant $K > 0$, such that the following hold for all $(x, \theta) \in \Re^n \times \Omega$ and $i = 1, \ldots, n$:

$$\frac{1}{K} \leqslant g_i(\theta, x_1, \ldots, x_i). \tag{5.8}$$

Furthermore suppose that the subsystem $\dot{z} = f_0(z, x, u)$ is forward complete with (x, u) as input and that $0 \in \Re^m$ is GAS for the system $\dot{z} = f_0(z, 0, 0)$. Then for every locally Lipschitz function $\Gamma(\cdot) \in K_\infty$ there exists a C^∞ mapping $k : \Re^+ \times \Re^n \to \Re$, with $k(t, 0) = 0$, for all $t \geqslant 0$, such that $0 \in \Re^m \times \Re^n$ is RGAS for the following system with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input:

$$\begin{aligned} \dot{z} &= f_0(z, x, u) \\ \dot{x}_i &= f_i(\theta, x_1, \ldots, x_i) + g_i(\theta, x_1, \ldots, x_i)x_{i+1} \quad i = 1, \ldots, n \\ u &= x_{n+1} = k(t, x) + d\Gamma(|x|). \end{aligned} \tag{5.9}$$

Proof. The proof is divided into three parts.

First part. We obtain estimates for the solution $z(t)$ of the subsystem (1.6a), with $(x, u) \in \Re^n \times \Re$ as input.

Second part. We design a feedback law $k(t, x)$, such that $0 \in \Re^n$ is ϕ-RGAS for the subsystem (1.6b) with $x_{n+1} = k(t, x) + d\Gamma(|x|)$, for an appropriate choice of $\phi \in K^*$.

Third part. We prove that $0 \in \Re^m \times \Re^n$ is RGAS for (5.9).

First part

Since $0 \in \Re^m$ is GAS for the subsystem $\dot{z} = f_0(z, 0, 0)$, then in Angeli *et al.* (2000, Lemma IV.10) and Sontag (1998, Theorem 3), there exists a smooth function $V : \Re^n \to \Re^+$, K_∞ functions $a_1, a_2, \lambda, \delta$, such that for all $(z, x, u) \in \Re^m \times \Re^n \times \Re$ we have

$$a_1(|z|) \leqslant V(z) \leqslant a_2(|z|) \tag{5.10a}$$

$$\frac{\partial V}{\partial z}(z) f_0(z, x, u) \leqslant -V(z) + \lambda(|z|)\delta(|(x, u)|). \tag{5.10b}$$

Furthermore, in Angeli & Sontag (1999, Corollary 2.11), there exists a smooth and proper function $W : \Re^n \to \Re^+$ and functions a_3, a_4, σ of class K_∞ and a constant $R > 0$, such that for all $(z, x, u) \in \Re^m \times \Re^n \times \Re$ we have

$$a_3(|z|) \leqslant W(z) \leqslant a_4(|z|) + R \tag{5.10c}$$

$$\frac{\partial W}{\partial z}(z) f_0(z, x, u) \leqslant W(z) + \sigma(|(x, u)|). \tag{5.10d}$$

Notice that by virtue of (5.10c), (5.10d) the solution $z(t)$ initiated at $z(t_0) = z_0$ of the subsystem (1.6a) satisfies

$$a_3(|z(t)|) \leqslant \exp(t - t_0)a_4(|z_0|) + R) + \int_{t_0}^{t} \exp(t - \tau)\sigma(|(x(\tau), u(\tau))|)\mathrm{d}\tau,\ \forall t \geqslant t_0. \tag{5.11}$$

On the other hand, by (5.10a), (5.10b) and (5.11) we obtain the estimate for $\tilde{\lambda}(s) := \lambda(2a_3^{-1}(2s))$ and for all $\xi \in [t_0, t]$:

$$\begin{aligned} a_1(|z(t)|) &\leqslant \exp(-(t - \xi))a_2(|z(\xi)|) \\ &\quad + \int_{\xi}^{t} \tilde{\lambda}\left(\int_{t_0}^{\tau} \exp(\tau - s)\sigma(|(x(s), u(s))|)\mathrm{d}s\right) \delta(|(x(\tau), u(\tau))|)\mathrm{d}\tau \\ &\quad + \int_{\xi}^{t} \tilde{\lambda}(\exp(\tau - t_0)(a_4(|z_0|) + R))\delta(|(x(\tau), u(\tau))|)\mathrm{d}\tau,\ \forall t \geqslant \xi. \end{aligned} \tag{5.12}$$

Second part

In Sontag (1998, Corollary 10), there exist functions $q_1, q_2, \mu \in K_\infty \cap C^\infty((0, +\infty))$, such that

$$\sigma(rs) \leqslant q_1(r)q_1(s),\ \forall r, s \geqslant 0 \tag{5.13a}$$

$$\delta(rs) \leqslant q_2(r)q_2(s),\ \forall r, s \geqslant 0 \tag{5.13b}$$

$$\tilde{\lambda}(rs) \leqslant \mu(r)\mu(s),\ \forall r, s \geqslant 0. \tag{5.13c}$$

Without loss of generality, we may assume that $q_1^{-1}, q_2^{-1} \in K_\infty \cap C^\infty((0, +\infty))$. Define

$$\tilde{\phi}(t) := 1 + \frac{1}{q_1^{-1}(\exp(-t))} + \frac{1}{q_2^{-1}\left(\frac{\exp(-t)}{\mu(\exp(t))}\right)}. \tag{5.14a}$$

Notice that $\tilde{\phi} \in K^+$. Consequently, by Lemma 2.2, there exists $\phi \in K^*$ such that

$$\phi(t) \geqslant \tilde{\phi}(t),\ \forall t \geqslant 0. \tag{5.14b}$$

Furthermore, by (5.8) and Corollary 5.1, we have that for every locally Lipschitz function $\Gamma(\cdot) \in K_\infty$ there exists a C^∞ mapping $k : \Re^+ \times \Re^n \to \Re$, a function $\eta \in Q$, a constant $p \geqslant 0$, with

$$|k(t,x)| \leqslant \eta\left(\phi^p(t)|x|\right),\ \forall (t,x) \in \Re^+ \times \Re^n \tag{5.15}$$

such that $0 \in \Re^n$ is ϕ-RGAS for the following system with $(\theta, d) \in D := \Omega \times [-1, 1]$ as input:

$$\begin{aligned} \dot{x}_i &= f_i(\theta, x_1, \ldots, x_i) + g_i(\theta, x_1, \ldots, x_i)x_{i+1} \quad i = 1, \ldots, n \\ x_{n+1} &= k(t,x) + d\Gamma(|x|). \end{aligned} \tag{5.16}$$

Third part

Since is ϕ-RGAS for (5.16) with input $(\theta, d) \in M_D$, by Lemma 2.4 it follows that there exist a KL function ζ and a K^+ function β, such that

$$\phi^{p+1}(t)|x(t)| \leqslant \zeta\left(\beta(t_0)|x_0|, t - t_0\right),\ \forall t \geqslant t_0 \tag{5.17}$$

where $p \geqslant 0$ is the constant involved in (5.15). Moreover, applying Lemma 2.7 for the even extension of $\Gamma(\cdot)$ on $\Re$ (which is locally Lipschitz), we find that there exists a function $\tilde{\Gamma}(\cdot) \in K_{\text{con}}$ such that

$$\Gamma(s) \leqslant \tilde{\Gamma}(s),\ \forall s \geqslant 0. \tag{5.18}$$

Inequalities (5.15), (5.17), (5.18) and the fact that $\tilde{\Gamma}$, $\eta \in K_{\text{con}}$, imply that the following estimate holds for all $t \geqslant t_0$:

$$\begin{aligned} &|(x(t), u(t))| \leqslant |x(t)| + |u(t)| = |x(t)| + |k(t, x(t)) + d(t)\Gamma(|x(t)|)| \\ &\leqslant |x(t)| + \eta\left(\phi^p(t)|x(t)|\right) + \tilde{\Gamma}(|x(t)|) \\ &\leqslant \left(1 + \frac{\mathrm{d}\eta}{\mathrm{d}s}\left(\phi^p(t)|x(t)|\right) + \frac{\mathrm{d}\tilde{\Gamma}}{\mathrm{d}s}\left(\phi^p(t)|x(t)|\right)\right)\phi^p(t)|x(t)| \\ &\leqslant \frac{1}{\phi(t)} H\left(\beta(t_0)|x_0|\right)\zeta\left(\beta(t_0)|x_0|, t - t_0\right) \end{aligned} \tag{5.19}$$

where $H(s) := 1 + \frac{\mathrm{d}\eta}{\mathrm{d}s}(\zeta(s,0)) + \frac{\mathrm{d}\tilde{\Gamma}}{\mathrm{d}s}(\zeta(s,0))$ is a continuous, positive and non-decreasing function. By (5.13a) and (5.19) it follows that

$$\int_{t_0}^{\tau} \exp(\tau - s)\sigma(|(x(s), u(s))|)\mathrm{d}s \leqslant \exp(\tau) q_1\left(Z\left(\beta(t_0)|x_0|\right)\right) \int_{t_0}^{\tau} q_1\left(\frac{1}{\phi(s)}\right)\mathrm{d}s,\ \forall \tau \geqslant t_0$$

where $Z(s) := H(s)\zeta(s, 0),\ Z \in K_\infty$. On the other hand (5.14) implies that $q_1\left(\frac{1}{\phi(t)}\right) \leqslant \exp(-t)$, which in conjunction with the latter inequality gives

$$\int_{t_0}^{\tau} \exp(\tau - s)\sigma(|(x(s), u(s))|)\mathrm{d}s \leqslant \exp(\tau) q_1\left(Z\left(\beta(t_0)|x_0|\right)\right),\ \forall \tau \geqslant t_0. \tag{5.20}$$

The above inequality, combined with (5.12), (5.13b), (5.13c) and (5.19) gives for all $\xi \in [t_0, t]$:

$$a_1(|z(t)|) \leqslant \exp(-(t-\xi))a_2(|z(\xi)|) \\ +q_2\left(Z\left(\beta(t_0)|x_0|\right)\right)\left[\mu\left(q_1\left(Z\left(\beta(t_0)|x_0|\right)\right)\right)+\mu\left(a_4(|z_0|)+R\right)\right]I(t,\xi) \tag{5.21a}$$

$$I(t,\xi) := \int_{\xi}^{t} \mu(\exp(\tau))q_2\left(\frac{1}{\phi(\tau)}\right)\mathrm{d}\tau. \tag{5.21b}$$

As previously, using (5.14) we may establish that $\mu(\exp(t))q_2\left(\frac{1}{\phi(t)}\right) \leqslant \exp(-t)$ and thus

$$a_1(|z(t)|) \leqslant \exp(-(t-\xi))a_2(|z(\xi)|) + \\ +\exp(-\xi)q_2\left(Z\left(\beta(t_0)|x_0|\right)\right)\left[\mu\left(q_1\left(Z\left(\beta(t_0)|x_0|\right)\right)\right)\mu\left(a_4(|z_0|)+R\right)\right] \\ \text{for all } \xi \in [t_0,t] \text{ and } t \geqslant \xi. \tag{5.22}$$

The inequality above and (5.17) imply that $0 \in \Re^m \times \Re^n$ is RGAS for (5.9). In order to establish this fact, notice that, by virtue of (5.17) and (5.22), for all $T \geqslant 0$ and $r \geqslant 0$, it holds

$$\sup\left\{|(z(t),x(t))|;\,(\theta,d)\in M_D,\,t\geqslant t_0,\,|(z_0,x_0)|\leqslant r,\,t_0\in[0,T]\right\} \leqslant G(\beta(T)r) \tag{5.23a}$$

$$G(s) := \zeta(s,0)+a_1^{-1}\left(a_2(s)+q_2\left(Z(s)\right)\left[\mu\left(q_1\left(Z(s)\right)\right)+\mu\left(R+a_4(s)\right)\right]\right) \tag{5.23b}$$

where G is a class K_∞ function. This establishes stability. In order to prove attractivity let $\varepsilon > 0$, $r > 0$ and $T \geqslant 0$. There exists a $\tau := \tau(\varepsilon, T, r) \geqslant T$, such that $\exp(-\xi)q_2(Z(\beta(T)r))[\mu(q_1(Z(\beta(T)r))) + \mu(a_4(r) + R)] \leqslant a_1(\varepsilon)$, for all $\xi \geqslant \tau$. It follows from (5.17), (5.22) and (5.23) that for all $t \geqslant \tau$ it holds

$$\sup\left\{|(z(t),x(t))|;\,(\theta,d)\in M_D,\,|(z_0,x_0)|\leqslant r,\,t_0\in[0,T]\right\} \\ \leqslant \zeta(\beta(T)r,t-T)+a_1^{-1}(\exp(-(t-\tau))a_2(G(\beta(T)r))+a_1(\varepsilon)).$$

This proves that $\overline{\lim_{t\to+\infty}} \sup\left\{|(z(t),x(t))|;\,(\theta,d)\in M_D,\,|(z_0,x_0)|\leqslant r,\,t_0\in[0,T]\right\} \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary we have that $\overline{\lim_{t\to+\infty}} \sup\{|(z(t),x(t))|;\,(\theta,d)\in M_D,\,|(z_0,x_0)|\leqslant r,$ $t_0 \in [0,T]\} = 0$. The proof is complete. □

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Appendix

Proof of Lemma 2.2. The implications (i)–(iii) are obvious. We only prove the last statement of the lemma. Define the function

$$\mu(s) := \begin{cases} \phi(\log s) + s & \text{if } s \geqslant 1 \\ (\phi(0)+1)s & \text{if } 0 \leqslant s < 1. \end{cases} \tag{A.1}$$

Clearly $\mu(\cdot) \in K_\infty$ and satisfies

$$\phi(t) \leqslant \mu\left(\mathrm{e}^t\right), \ \forall t \geqslant 0. \tag{A.2}$$

Define the function

$$\rho(s) := \begin{cases} 0 & \text{if } s = 0 \\ \frac{1}{\mu^{-1}\left(\frac{1}{s}\right)} & \text{if } 0 < s. \end{cases} \tag{A.3}$$

Again we have that $\rho(\cdot) \in K_\infty$ and let $\tilde{\rho}(\cdot) \in K_\infty \cap C^\infty((0,+\infty))$ be a function with $\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}s}(s) \geqslant 1$ for all $s > 0$ and $\lim_{s\to 0^+} \frac{\tilde{\rho}(s)}{s} = +\infty$, that satisfies $\tilde{\rho}(s) \geqslant \rho(s)$ for all $s \geqslant 0$. Thus by (A.3) we have

$$\frac{1}{\mu^{-1}\left(\frac{1}{s}\right)} \leqslant \tilde{\rho}(s) \quad \forall s > 0 \quad \Rightarrow \mu\left(\mathrm{e}^t\right) \leqslant \frac{1}{\tilde{\rho}^{-1}\left(\mathrm{e}^{-t}\right)} \ \forall t \geqslant 0. \tag{A.4}$$

Define

$$\tilde{\phi}(t) := \frac{1}{\tilde{\rho}^{-1}\left(e^{-t}\right)}. \tag{A.5}$$

Since $\tilde{\rho}(\cdot) \in K_\infty \cap C^\infty((0, +\infty))$ with $\frac{d\tilde{\rho}}{ds}(s) \geqslant 1$ for all $s > 0$, we easily establish that $\tilde{\phi}(\cdot) \in C^\infty(\Re^+)$ and that $\tilde{\phi}(\cdot)$ is non-decreasing. Furthermore, since $\frac{d\tilde{\rho}}{ds}(s) \geqslant 1$ for all $s > 0$ and $\lim_{s\to 0^+} \frac{\tilde{\rho}(s)}{s} = +\infty$, it follows that $0 \leqslant \frac{d\tilde{\rho}^{-1}}{ds}(s) \leqslant 1$, for all $s \geqslant 0$. This fact and definition (A.5) imply

$$\frac{d\tilde{\phi}}{dt}(t) = \tilde{\phi}^2(t)\frac{d\tilde{\rho}^{-1}}{ds}\left(e^{-t}\right)e^{-t} \leqslant \tilde{\phi}^2(t)e^{-t}.$$

The latter inequality gives $\lim_{t\to+\infty} \frac{1}{\tilde{\phi}^2(t)}\frac{d\tilde{\phi}}{dt}(t) = 0$. Notice that we can easily establish (using the implied inequality $\tilde{\rho}^{-1}(s) \leqslant s$ for all $s \geqslant 0$) that $\tilde{\phi}(t) \geqslant e^t \geqslant 1$, for all $t \geqslant 0$. Consequently, we conclude that $\tilde{\phi}(\cdot) \in K^*$. Moreover, by (A.2), (A.4) and definition (A.5) we get that $\phi(t) \leqslant \tilde{\phi}(t)$, for all $t \geqslant 0$. The proof is complete. □

Proof of Lemma 2.4. Implications (i) and (ii) are immediate consequences of the definition of ϕ-RGAS and the fact that $\phi(t) \geqslant 1$, for all $t \geqslant 0$. We focus on implication (iii). Let $p \geqslant 0$ and consider the time-varying transformation

$$z := \phi^p(t)x. \tag{A.6}$$

Clearly, $z(t)$ satisfies the following system of differential equations:

$$\dot{z} = p\frac{\dot{\phi}(t)}{\phi(t)}z + \phi^p(t) f\left(t, \frac{z}{\phi^p(t)}, d\right)$$
$$z \in \Re^n, t \geqslant 0, d \in D.$$

The fact that $0 \in \Re^n$ is ϕ-RGAS for (1.1) and definition (A.6) imply that $0 \in \Re^n$ is RGAS for the system above with d as input. Furthermore, Proposition 2.2 in Karafyllis & Tsinias (2002b) guarantees the existence of a KL function $\sigma(\cdot)$ and a K^+ function $\tilde{\beta}(\cdot)$ such that

$$|z(t)| \leqslant \sigma\left(\tilde{\beta}(t_0)|z_0|, t - t_0\right), \ \forall t \geqslant t_0. \tag{A.7}$$

The desired (2.2) is a consequence of inequality (A.7), definition (A.6) and the selection $\beta(t) := \tilde{\beta}(t)\phi^p(t)$.

Proof of Lemma 2.6. The statements (i)–(ii) are obvious. We prove statements (iii)–(iv). (iii) Define the function

$$\gamma(s) := s + \sup_{0\leqslant\tau\leqslant s} |a(\tau)|. \tag{A.8}$$

Clearly $\gamma \in C^0(\Re^+)$ and is strictly increasing with

$$a(s) \leqslant \gamma(s), \ \forall s \geqslant 0. \tag{A.9}$$

Let $h : \Re \to \Re^+$ be a C^∞ function with $h(s) = 0$ if $s \notin (0,1)$ and $\int_\Re h(s)\mathrm{d}s = \int_0^1 h(s)\mathrm{d}s = 1$. Define the function

$$\tilde{\gamma}(s) := \int_\Re \gamma(w)h(w-s)\mathrm{d}w = \int_0^1 \gamma(s+w)h(w)\mathrm{d}w. \tag{A.10}$$

Clearly $\tilde{\gamma} \in C^\infty(\Re^+)$, is non-decreasing and satisfies

$$\gamma(s) \leqslant \tilde{\gamma}(s) \leqslant \gamma(s+1), \ \forall s \geqslant 0. \tag{A.11}$$

Let $r > 0$ be an arbitrary constant and define the following functions:

$$\delta(s) := \begin{cases} \dfrac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}(r) + 1 & \text{if } 0 \leqslant s \leqslant r \\ \sup\limits_{r \leqslant \tau \leqslant s} \dfrac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}(\tau) + 1 & \text{if } s > r \end{cases} \tag{A.12}$$

$$\beta(s) = \int_0^s \delta(w)\mathrm{d}w. \tag{A.13}$$

Since by definition (A.12), δ is continuous, non-decreasing and satisfies $\delta(s) \geqslant 1$, $\forall s \geqslant 0$, we have that $\beta \in K_{\mathrm{con}}$. Notice that for $s > r$ we get by (A.12) and (A.13)

$$\beta(s) \geqslant \int_r^s \delta(w)\mathrm{d}w \geqslant \tilde{\gamma}(s) - \tilde{\gamma}(r)$$

and the latter inequality in conjunction with (A.9) and (A.11) implies that $a(s) \leqslant R + \beta(s)$, $\forall s \geqslant 0$, for $R = \tilde{\gamma}(r)$.

(iv) Inequality (2.3a) is a consequence of the inequality $\dot{a}(s) \leqslant \dot{a}(\lambda s)$, which holds for all $\lambda \geqslant 1$ and $s \geqslant 0$. The right-hand side of inequality (2.3b) is a well-known property of the functions of class K. The left-hand side of inequality (2.3b) is a consequence of the inequality $\dot{a}(s_1) \leqslant \dot{a}(s_1 + s_2)$, which holds for all $s_1 \geqslant 0$ and $s_2 \geqslant 0$.

The proof is complete. □

Proof of Lemma 2.7. Since f is locally Lipschitz in x, uniformly in θ, there exist constants $L, r > 0$, such that

$$|f(\theta, x)| \leqslant L|x|, \ \forall \theta \in \Omega, \ \text{for } |x| \leqslant r. \tag{A.14}$$

Consider the function

$$\beta(s) := \sup_{\substack{\theta \in \Omega \\ |x| \leqslant s}} |f(\theta, x)|. \tag{A.15}$$

Clearly, the mapping $\beta(\cdot)$ is continuous and non-decreasing with $\beta(0) = 0$. Furthermore by (A.14) we have

$$\beta(s) \leqslant Ls, \quad \text{for } s \leqslant r. \tag{A.16}$$

Clearly the function $\zeta(s) = \frac{1}{s}\int_s^{2s} \beta(w)\mathrm{d}w$ belongs to the class $K_\infty \cap C^1((0, +\infty))$ and satisfies $\zeta(s) \geqslant \beta(s)$ for all $s \geqslant 0$. Define the functions:

$$\delta(s) := \begin{cases} \dfrac{\mathrm{d}\zeta}{\mathrm{d}s}\left(\dfrac{r}{2}\right) & \text{for } 0 \leqslant s < \dfrac{r}{2} \\ \sup\limits_{\frac{r}{2}\leqslant\tau\leqslant s} \dfrac{\mathrm{d}\zeta}{\mathrm{d}s}(\tau) & \text{for } s \geqslant \dfrac{r}{2} \end{cases} \tag{A.17}$$

$$a(s) := \lambda s + \int_0^s \delta(w)\mathrm{d}w \tag{A.18}$$

$$\lambda := \max\left\{L + 1, \frac{2}{r}\zeta\left(\frac{r}{2}\right)\right\}. \tag{A.19}$$

Since δ is a continuous, non-decreasing function we have that $a \in K_{\mathrm{con}}$. We claim that (2.4) is satisfied. Notice that $a(s) \geqslant \lambda s \geqslant (L + 1)s$ and consequently for all $|x| \leqslant r$ and $\theta \in \Omega$, we obtain using (A.14)

$$|f(\theta, x)| \leqslant L|x| \leqslant a(|x|).$$

On the other hand, for $s \geqslant \frac{r}{2}$, by (A.17) and (A.19) we have that

$$a(s) = \lambda s + \int_0^{\frac{r}{2}} \delta(w)\mathrm{d}w + \int_{\frac{r}{2}}^s \delta(w)\mathrm{d}w \geqslant \lambda\frac{r}{2} + \int_{\frac{r}{2}}^s \frac{\mathrm{d}\zeta}{\mathrm{d}s}(w)\mathrm{d}w \geqslant \zeta(s) \geqslant \beta(s)$$

The proof is complete. □

Proof of Lemma 2.8. Let $h : \Re^+ \to \Re^+$ be a C^∞ function that satisfies $h(s) = 0$, $\forall s \notin (0, 1)$ and $\int_{\Re^+} h(s)\mathrm{d}s = \int_0^1 h(s)\mathrm{d}s = 1$. Define

$$\tilde{a}(s) := \int_{\Re^+} a(u)h(u - s)\mathrm{d}u - \int_{\Re^+} a(u)h(u)\mathrm{d}u. \tag{A.20}$$

Notice that $\tilde{a}(0) = 0$ and $\tilde{a} \in C^\infty(\Re^+)$. Furthermore, we have

$$\tilde{a}(s) = \int_0^1 (a(s + w) - a(w))h(w)\mathrm{d}w. \tag{A.21}$$

Clearly by virtue of (2.3b) of Lemma 2.6 and (A.21), it follows that (2.5) holds. Moreover by (A.21) we get

$$\frac{\mathrm{d}\tilde{a}}{\mathrm{d}s}(s) = \int_0^1 \frac{\mathrm{d}a}{\mathrm{d}s}(s + w)h(w)\mathrm{d}w \tag{A.22}$$

which implies that $\frac{\mathrm{d}\tilde{a}}{\mathrm{d}s}(s)$ is non-decreasing and positive. Therefore $\tilde{a} \in K_{\mathrm{con}} \cap C^\infty(\Re^+)$. Define $\tilde{R} := \frac{\mathrm{d}a}{\mathrm{d}s}(2r)$ for some $r > 0$ and notice that we also have

$$a(s) \leqslant \tilde{R}s, \ \forall s \in [0, 2r]. \tag{A.23}$$

Finally, define

$$\tilde{\beta}(s) := \begin{cases} \tilde{R}s & \text{for } 0 \leqslant s \leqslant r \\ \tilde{R}s + \exp\left(\frac{1}{r} - \frac{1}{s-r}\right)\tilde{a}(s) & \text{for } s > r. \end{cases} \tag{A.24}$$

Clearly $\tilde{\beta} \in C^{\infty}(\Re^{+})$ and its odd extension $\beta(\cdot)$ is also a smooth function on $\Re$. Moreover by (2.5), (A.23) and definition (A.24) we have that $a(s) \leqslant \beta(s)$, $\forall s \geqslant 0$. By virtue of (iii) of Lemma 2.6, there exist $M > 0$ and $\delta \in K_{\rm con}$ such that $\frac{d\tilde{a}}{ds}(s) \leqslant M + \delta(s)$ and since $\sup_{s>0} \frac{1}{s^2} \exp\left(-\frac{1}{s}\right) \leqslant 4$, we can easily verify that (2.6) and (2.7) are satisfied for

$$R := \tilde{R} + M \exp\left(\frac{1}{r}\right)$$

$$\gamma(s) := 5 \exp\left(\frac{1}{r}\right)(\tilde{a}(s) + \delta(s)) + \tilde{R}s.$$

The proof is complete. □

Proof of Lemma 2.9. In order to prove (2.10), let $r \geqslant 0$, $t_0 \geqslant 0$, $t \geqslant t_0$ and notice that for the solution $\beta(t, t_0, r)$ of (2.11) we have

$$\frac{\partial}{\partial \tau}\beta(t-\tau, t_0, r) = \rho(\phi(t-\tau)\beta(t-\tau, t_0, r)), \ \forall \tau \in [0, t-t_0]. \tag{A.25}$$

Let a vector $x \in \overline{B}_{\beta(t,t_0,r)}$ be the initial condition for the problem

$$\begin{aligned} &\frac{d\xi}{d\tau}(\tau) = -f(t-\tau, \xi, d) \\ &\xi(0) = x, d \in D, \tau \geqslant 0. \end{aligned} \tag{A.26}$$

This is the time-reversed system (1.1). It is clear by virtue of (2.8) that the following differential inequality is satisfied:

$$D^{+}|\xi(\tau)| \leqslant \rho(\phi(t-\tau)|\xi(\tau)|), \ \forall \tau \in [0, t]. \tag{A.27}$$

It follows by (A.25), (A.27) and the comparison Lemma in Khalil (1996), that

$$|\xi(\tau)| \leqslant \beta(t-\tau, t_0, r), \ \forall \tau \in [0, t-t_0]. \tag{A.28}$$

Since $\beta(t_0, t_0, r) = r$, inequality (A.28) shows that there exists $x_0 \in \overline{B}_r$ and $d \in M_D$ such that $x(t, t_0, x_0; d) = x$ and it is given by $x_0 = \xi(t - t_0)$ for the particular $d \in M_D$. Since $x \in \overline{B}_{\beta(t,t_0,r)}$ is arbitrary, it follows that (2.10) holds. The proof is complete. □