Brief paper

Delay-robustness of linear predictor feedback without restriction on delay rate[☆]

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A R T I C L E   I N F O

A B S T R A C T

Robustness is established for the predictor feedback for linear time-invariant systems with respect to possibly time-varying perturbations of the input delay, with a constant nominal delay. The prior results have addressed qualitatively constant delay perturbations (robustness of stability in L^2 norm of actuator state) and delay perturbations with restricted rate of change (robustness of stability in H^1 norm of actuator state). The present work provides simple formulas that allow direct and accurate computation of the least upper bound of the magnitude of the delay perturbation for which the exponential stability in supremum norm on the actuator state is preserved. While the prior work has employed Lyapunov–Krasovskii functionals constructed via backstepping, the present work employs a particular form of small-gain analysis. Two cases are considered: the case of measurable (possibly discontinuous) time-varying perturbations and the case of constant perturbations.

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1. Introduction

Linear predictor feedback has been used widely for the stabilization of linear time-invariant systems with constant input delays. Artstein in Artstein (1982) was the first to provide a rigorous extension of the so-called Smith predictor (see Krstic, 2009 and the discussion therein). Many applications and extensions of the linear predictor feedback have appeared in the literature (see for instance Lozano, Castillo, Garcia, & Dzul, 2004, Mazenc, Mondie, & Francisco, 2004, Mirkin & Raskin, 2003, Niculescu, 2001). More recently, research efforts have been focused on nonlinear extensions of predictor-based feedback for nonlinear systems with input delays (see Bekiaris-Liberis & Krstic, 2013, Karafyllis, 2011, Karafyllis & Krstic, 2012; Krstic, 2004, 2008, 2009, 2010a), on the implementation issues of linear predictor feedback (see Zhong & Mirkin, 2002, Zhong, 2004, Zhong, 2010 and references therein) and on different types of linear predictor feedback (see Zhou, Lin, & Duan, 2012).

However, the study of robustness properties of the linear predictor feedback with respect to perturbations of the input delay are rather scarce. To the best of our knowledge, the first robustness study for perturbations of the input delay appeared in Krstic (2008), where Lyapunov techniques were employed. An alternative delay-robustness result for constant delays was presented in Section 5.3 in Krstic (2009). The efforts were continued in Bekiaris-Liberis and Krstic (2013), where Lyapunov functionals were proposed for the robustness study for time-varying delays and perturbations. The results in Bekiaris-Liberis and Krstic (2013) showed that, not only the magnitude but also the rate of change of the delay perturbation may be important for the robustness analysis. The norm on the actuator state in which the stability was studied was L^2 in Krstic (2008) and H^1 in [10, Section 5.3] and Bekiaris-Liberis and Krstic (2013).

In this work, we consider the system:

$$\dot{x}(t) = Ax(t) + Bu(t - r - \varepsilon d(t))$$
$$x(t) \in \Re^n, \ u(t) \in \Re^m, \ d(t) \in [-1, 1], \quad \text{for } t \geq 0, \text{ a.e.} \quad (1.1)$$

where $0 < \varepsilon \leq r$ are constants. The linear predictor feedback is based on the constant nominal value of the delay $r > 0$:

$$u(t) = k \exp(Ar)x(t) + k \int_t^{t+r} \exp(A(t + r - s))$$
$$\times Bu(s - r)ds, \quad \text{for } t \geq 0 \quad (1.2)$$

where $k \in \Re^{m \times n}$ is a constant matrix such that the matrix $(A + Bk)$ is Hurwitz. We show that, provided that $\varepsilon > 0$ is sufficiently small, there exist constants $Q, \sigma > 0$ such that for all $x_0 \in \Re^n$, $u_0 \in C^0([-r - \varepsilon, 0]; \Re^m)$ with $u_0(0) = k \exp(Ar)x_0 + k \int_{-r}^0$

$\exp(-As)Bu_0(s)ds$ the solution $(x(t), u(t)) \in \Re^n \times \Re^m$ of (1.1), (1.2) with initial condition $x(0) = x_0, u(t) = u_0(t)$ for $t \in [-r - \varepsilon, 0]$ satisfies the following exponential stability estimate in the supremum norm of the actuator state:

$$|x(t)| + \max_{t-\varepsilon-r \leq s \leq t}(|u(s)|)$$

$$\leq Q \exp(-\sigma t)\left(|x_0| + \max_{-\varepsilon-r \leq s \leq 0}(|u_0(s)|)\right), \quad \forall t \geq 0 \qquad (1.3)$$

for arbitrary disturbance $d : \Re_+ \to [-1, 1]$ that belongs to one of the following classes:

(1) The perturbation $d : \Re_+ \to [-1, 1]$ is an arbitrary measurable function, i.e., $d \in L^\infty(\Re_+; [-1, 1])$ (Theorem 2.1).
(2) The perturbation $d : \Re_+ \to [-1, 1]$ is constant (Corollary 2.3).

Clearly, (1.3) shows robust global exponential stability for the closed-loop system (1.1), (1.2). The estimation of $\varepsilon > 0$ will be given by explicit inequalities, which are derived by small-gain arguments. The inequalities can be used easily by the control practitioner in order to guarantee the successful application of the linear predictor feedback control strategy.

Notation. Throughout the paper we adopt the following notation:

∗ For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm, by x' its transpose. For a real matrix $A \in \Re^{n \times m}, A' \in \Re^{m \times n}$ denotes its transpose and $|A| := \sup\{|Ax| ; x \in \Re^n, |x| = 1\}$ is its induced norm. $I \in \Re^{n \times n}$ denotes the identity matrix.
∗ $\Re_+$ denotes the set of non-negative real numbers.
∗ Let $I \subseteq \Re$ be an interval and $U \subseteq \Re^m$ be a set. By $L^\infty(I; U)$ we denote the space of measurable and essentially bounded functions $u(\cdot)$ defined on I and taking values in $U \subseteq \Re^m$. By $\sup_{t \in I}|u(t)|$ we denote the essential supremum for a function $u \in L^\infty(I; U)$. For $u \in L^\infty(I; U)$ we denote by $\|u\|$ the essential supremum of u on $I \subseteq \Re$. If $I \subseteq \Re$ is an unbounded interval, then by $L^\infty_{loc}(I; U)$ we denote the space of measurable and locally essentially bounded functions $u(\cdot)$ defined on I and taking values in $U \subseteq \Re^m$. By $C^0(I; U)$ we denote the space of continuous functions $u(\cdot)$ defined on I and taking values in $U \subseteq \Re^m$.

Throughout the paper, all differential equations with right hand sides which are measurable in $t \geq 0$ are required to hold almost everywhere (a.e.). Moreover, all quantities involving point values of measurable functions are assumed to hold almost everywhere. This happens because the measurable functions are equivalent classes of functions (which are insensitive to variations on a measure zero set).

2. Main results

Arbitrary measurable perturbations $d \in L^\infty(\Re_+; [-1, 1])$ of the delay can be considered for system (1.1). Indeed, we notice that, this fact follows from the consideration of system (1.1) with

$$\dot{u}(t) = k\exp(Ar)(Ax(t) + Bu(t - r - \varepsilon d(t)) - Bu(t - r))$$

$$+ kA\int_{-r}^0 \exp(-As)Bu(t + s)ds + kBu(t). \qquad (2.1)$$

Differential equation (2.1) is obtained by formally differentiating (1.2) with respect to $t \geq 0$. System (1.1) with (2.1) is a linear autonomous system described by Retarded Functional Differential Equations with disturbance $d \in L^\infty(\Re_+; [-1, 1])$ and state space $\Re^n \times C^0([-r - \varepsilon, 0]; \Re^m)$ and satisfies all hypotheses (S1), (S2), (S3), (S4) in Karafyllis and Jiang (2011) for the existence and uniqueness of solutions, for the robustness of the equilibrium point

and for the "Boundedness-Implies-Continuation" property. If we define the subspace

$$S := \begin{cases} (x, u) \in \Re^n \times C^0([-r - \varepsilon, 0]; \Re^m) : \\ u(0) = k\exp(Ar)x + k\int_{-r}^0 \exp(-As)Bu(s)ds \end{cases} \qquad (2.2)$$

then we are in a position to guarantee that S is a positively invariant set for system (1.1) with (2.1). Moreover, every solution of (1.1) with (2.1) and initial condition $(x_0, u_0) \in S$ is a solution of (1.1), (1.2) and every solution of (1.1), (1.2) with initial condition $(x_0, u_0) \in S$ is a solution of (1.1) with (2.1). Finally, we notice that, there exist constants $M, L > 0$ such that for every $\varepsilon > 0$, $x_0 \in \Re^n, u_0 \in C^0([-r - \varepsilon, 0]; \Re^m), d \in L^\infty(\Re_+; [-1, 1])$ with $u_0(0) = k\exp(Ar)x_0 + k\int_{-r}^0 \exp(-As)Bu_0(s)ds$ the unique solution $x \in C^0(\Re_+; \Re^n), u \in C^0([-r - \varepsilon, +\infty); \Re^m)$ of system (1.1), (1.2) with initial conditions $x(0) = x_0, u(t) = u_0(t)$ for $t \in [-r - \varepsilon, 0]$ satisfies the exponential growth estimate:

$$|x(t)| + |u(t)| \leq M\exp(Lt)\left(|x_0| + \max_{-r-\varepsilon \leq s \leq 0}|u_0(s)|\right),$$

$$\forall t \geq 0. \qquad (2.3)$$

The existence of constants $M, L > 0$ satisfying estimate (2.3) follows directly from the integral representation of the solution of (1.1) with (2.1) and the Gronwall–Bellman Lemma.

It should be noticed that, discontinuities of $u(t)$ cannot be handled in this framework: the initial condition $u_0 \in C^0([-r - \varepsilon, 0]; \Re^m)$ must be continuous and must satisfy (1.2) for $t = 0$. The reason for this regularity requirement is that, the right hand side of (1.1) and (2.1) must be measurable in $t \geq 0$. Since the disturbance $d \in L^\infty(\Re_+; [-1, 1])$ is measurable, the only way to guarantee this regularity requirement is to demand continuity of $u(t)$ (the composition of a continuous function with a measurable one gives a measurable function).

Our main result is the following theorem, which provides an explicit inequality for the magnitude $\varepsilon > 0$ of the delay perturbation under which robust global exponential stability for the closed-loop system (1.1), (1.2) is guaranteed.

Theorem 2.1. *Consider system* (1.1), (1.2), *where* $0 < \varepsilon \leq r$ *are constants,* $A \in \Re^{n \times n}, B \in \Re^{n \times m}, k \in \Re^{m \times n}$ *and* $(A + Bk)$ *is Hurwitz. There exist constants* $Q, \sigma > 0$ *such that for all* $d \in L^\infty(\Re_+; [-1, 1]), x_0 \in \Re^n, u_0 \in C^0([-r - \varepsilon, 0]; \Re^m)$ *with* $u_0(0) = k\exp(Ar)x_0 + k\int_{-r}^0 \exp(-As)Bu_0(s)ds$ *the solution* $(x(t), u(t)) \in \Re^n \times \Re^m$ *of* (1.1), (1.2) *with initial condition* $x(0) = x_0, u(t) = u_0(t)$ *for* $t \in [-r - \varepsilon, 0]$ *satisfies estimate* (1.3), *provided that the following inequality holds:*

$$\Theta|\exp(Ar)Bk|(\exp(|A + Bk|\varepsilon) - \exp(-\lambda\varepsilon)) < \lambda \qquad (2.4)$$

where $\Theta, \lambda > 0$ *are the constants satisfying* $|\exp((A + Bk)t)| \leq \Theta\exp(-\lambda t)$ *for all* $t \geq 0$. *Moreover, if* $n = 1$ *then the inequality* (2.4) *can be replaced by the inequality*

$$2|Bk|\exp(Ar)(1 - \exp(-|A + Bk|\varepsilon)) < |A + Bk|. \qquad (2.5)$$

Remark 2.2. Since the left hand-side of inequality (2.4) becomes zero for $\varepsilon = 0$, by continuity, there exists $\varepsilon > 0$ (sufficiently small) such that inequality (2.4) holds. The least upper bound value for $\varepsilon > 0$ can be determined numerically.

For the case of constant perturbations of the delay, we obtain the following result.

Corollary 2.3. *Consider the system*

$$\dot{x}(t) = Ax(t) + Bu(t - \tau)$$
$$x(t) \in \Re^n, \qquad u(t) \in \Re^m \tag{2.6}$$

with (1.2), *where* $\tau, r \geq 0$ *are constants,* $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $k \in \Re^{m \times n}$ *and* $(A + Bk)$ *is Hurwitz. The zero solution of the closed-loop system is Globally Exponentially Stable if and only if all roots of the following equation:*

$$\det (sI - (A + Bk) + \exp(Ar)Bk$$
$$\times (\exp(-rs) - \exp(-\tau s))) = 0 \tag{2.7}$$

have negative real parts.

Let $F_r \subseteq \Re_+$ denote the set of all $\tau \geq 0$, for which the roots of Eq. (2.7) have negative real parts, for fixed $r \geq 0, A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$ and $k \in \Re^{m \times n}$. The reader should notice that Theorem 2.1 guarantees that there exists $\varepsilon \in (0, r]$ such that for all $(r - \varepsilon, r + \varepsilon) \subseteq F_r$. However, Corollary 2.3 does not guarantee that $F_r \subseteq \Re_+$ is a convex set of $\Re_+$, i.e., an interval. Indeed, one cannot exclude the possibility of having two delays $\tau_2 > \tau_1$ in F_r and a delay value $\tau \in (\tau_1, \tau_2)$ with $\tau \notin F_r$. The structure and the topological properties of the set $F_r \subseteq \Re_+$ are open problems.

The following example illustrates the use of inequality (2.5) and Corollary 2.3.

Example 2.4. Consider the scalar system

$$\dot{x}(t) = x(t) + u(t - 1 - \varepsilon d(t))$$
$$\text{with } x(t) \in \Re, \ u(t) \in \Re, \ d(t) \in [-1, 1] \tag{2.8}$$

where $\varepsilon > 0$. For this example $A = 1 = B = r$ and we may choose $k = -p$, where $p > 1$. Theorem 2.1 guarantees that the closed-loop system (2.8) with

$$u(t) = -pe\, x(t) - p \int_0^1 \exp(s)u(t - s)ds \tag{2.9}$$

and $d \in L^\infty (\Re_+; [-1, 1])$ is robustly globally exponentially stable provided that $\varepsilon > 0$ satisfies

$$\varepsilon < \frac{1}{p-1} \ln \left(\frac{2pe}{2pe - p + 1} \right). \tag{2.10}$$

In other words, system (2.8) with (2.9) is robustly globally exponentially stable provided that $\tau(t) \in (\tau_{\min}, \tau_{\max})$, where $\tau(t) = 1 + \varepsilon d(t)$, $\tau_{\min} = 1 - \varepsilon$, $\tau_{\max} = 1 + \varepsilon$ and $\varepsilon = \frac{1}{p-1} \ln \left(\frac{2pe}{2pe-p+1} \right)$.

On the other hand, if constant delay perturbations are considered, then the roots of the equation $s + (p - 1) + p \exp(1 - \tau s) - p \exp(1 - s) = 0$ must have negative real parts. For every value of $p > 1$ there exist delay values $0 < \tau_{\min} < 1 < \tau_{\max}$ such that if $\tau \in (\tau_{\min}, \tau_{\max})$ then all roots of the equation $s + (p - 1) + p \exp(1 - \tau s) - p \exp(1 - s) = 0$ have negative real parts. In order to determine the range of values of τ for which the roots of the equation $s + (p-1) + p \exp(1 - \tau s) - p \exp(1 - s) = 0$ have negative real parts, we determine the curves in the parameter plane (the (p, τ) plane) composed of points for which there exists $\omega \in \Re$ such that $\omega j + (p-1) + p \exp(1 - \tau \omega j) - p \exp(1 - \omega j) = 0$, where j is the imaginary unit. The procedure that we follow for every $p > 1$, is:

(i) first we find numerically all solutions $\omega \in (0, 2pe)$ of the equation $(p - 1) \cos(\omega) - \omega \sin(\omega) = \frac{(p-1)^2 + \omega^2}{2pe}$ (which is obtained from the equations $\cos(\omega \tau) - \cos(\omega) = -\frac{p-1}{pe}$ and $\sin(\omega \tau) - \sin(\omega) = \frac{\omega}{pe}$),

(ii) for every $\omega \in (0, 2pe)$ found from the previous step, we determine the unique solution $\phi \in \Re$ of the equations $\cos(\phi) = \cos(\omega) - \frac{p-1}{pe}$ and $\sin(\phi) = \sin(\omega) + \frac{\omega}{pe}$,

(iii) we find the positive solutions of $\tau = \frac{\phi + 2k\pi}{\omega}$, where k is an arbitrary integer, and

(iv) finally, we collect all positive values of $\tau = \frac{\phi + 2k\pi}{\omega}$ from the previous step and we find the highest value of τ that is less than 1 (this is $\tau_{\min}$) and the lowest value of τ that is higher than 1 (this is $\tau_{\max}$).

The results are shown in Fig. 1 both for time-varying delay perturbations which are measurable (where $\tau_{\min} = 1 - \varepsilon$, $\tau_{\max} = 1 + \varepsilon$ and $\varepsilon = \frac{1}{p-1} \ln \left(\frac{2pe}{2pe-p+1} \right)$) and for constant delay perturbations.

The bounds for the magnitude of the delay perturbation obtained from (2.10) are about 50% of the bounds obtained for constant perturbations. However, this is expected since (2.10) applies for time-varying delay perturbations which are measurable. Moreover, notice that, the curves of $\tau_{\min}$ and $\tau_{\max}$ obtained for constant perturbations are not perfectly symmetric around 1. $\square$

The proof of Theorem 2.1 relies on the following theorem.

Theorem 2.5. *Consider the system*

$$\dot{x}(t) = Ax(t) + q(t)C (x(t - r - \varepsilon d(t)) - x(t - r))$$
$$x(t) \in \Re^n, \qquad d(t) \in [-1, 1], \qquad q(t) \in [-1, 1],$$
$$\text{for } t \geq 0, \text{ a.e.} \tag{2.11}$$

where $d \in L^\infty (\Re_+; [-1, 1])$, $q \in L^\infty (\Re_+; [-1, 1])$, $A, C \in \Re^{n \times n}$ *are constant matrices,* $r \geq \varepsilon \geq 0$ *are constants and* $A \in \Re^{n \times n}$ *is Hurwitz. Suppose that*

$$\Theta |C| (\exp(|A| \varepsilon) - \exp(-\lambda \varepsilon)) < \lambda \tag{2.12}$$

where $\Theta, \lambda > 0$ *are constants satisfying* $|\exp(At)| \leq \Theta \exp(-\lambda t)$ *for all* $t \geq 0$. *Then there exist constants* $Q, \sigma > 0$ *such that for all* $d \in L^\infty (\Re_+; [-1, 1])$, $q \in L^\infty (\Re_+; [-1, 1])$, $x_0 \in C^0 ([-r - \varepsilon, 0]; \Re^n)$ *the solution* $x(t) \in \Re^n$ *of* (2.11) *with initial condition* $x(t) = x_0(t)$ *for* $t \in [-r - \varepsilon, 0]$ *that corresponds to inputs* $d \in L^\infty (\Re_+; [-1, 1])$, $q \in L^\infty (\Re_+; [-1, 1])$, *satisfies the following estimate*

$$|x(t)| \leq Q \exp(-\sigma t) \|x_0\|, \quad \forall t \geq 0. \tag{2.13}$$

Moreover, if $n = 1$ *then inequality* (2.12) *can be replaced by the inequality*

$$2 |C| (1 - \exp(-|A| \varepsilon)) < |A|. \tag{2.14}$$

The proof of Theorem 2.5 is based on a small-gain argument and is provided in the following section. The small-gain argument for the proof of Theorem 2.5 was inspired by the results contained in Teel (1998), but the methodology of the proof is essentially different from that followed in Teel (1998).

Finally, the proofs of Theorem 2.1 and Corollary 2.3 are based on the following result, which has its own interest.

Proposition 2.6. *Consider system* (1.1), (1.2), *where* $0 < \varepsilon \leq r$ *are constants,* $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $k \in \Re^{m \times n}$ *and* $(A + Bk)$ *is Hurwitz. Let* $\Omega \subseteq L^\infty (\Re_+; [-1, 1])$ *be a set of time-varying inputs which is invariant under time translation, i.e., if* $d \in \Omega$ *then for every* $s > 0$ *the input* $\tilde{d} : \Re_+ \to [-1, 1]$ *defined by* $\tilde{d}(t) = d(t + s)$ *for all* $t \geq 0$ *is in* $\Omega \subseteq L^\infty (\Re_+; [-1, 1])$. *There exist constants* $Q, \sigma > 0$ *such that for all* $d \in \Omega, x_0 \in \Re^n, u_0 \in C^0 ([-r - \varepsilon, 0]; \Re^m)$ *with* $u_0(0) = k \exp(Ar)x_0 + k \int_{-r}^0 \exp(-As)Bu_0(s)ds$ *the solution* $(x(t), u(t)) \in \Re^n \times \Re^m$ *of* (1.1), (1.2) *with initial condition* $x(0) =$
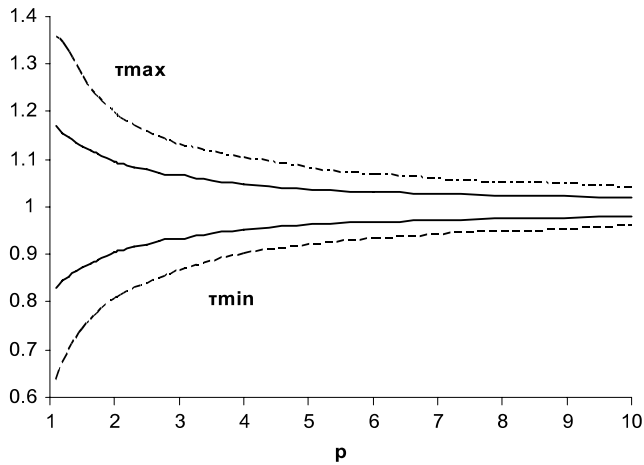
Fig. 1. $\tau_{\min}$ and $\tau_{\max}$ for the closed-loop system (2.8) with (2.9). The solid line is for measurable delay perturbations as calculated by (2.10) and the dashed line is for constant delay perturbations.

x_0, $u(t) = u_0(t)$ for $t \in [-r - \varepsilon, 0]$ satisfies estimate (1.3), if and only if there exist constants $\tilde{Q}$, $\tilde{\sigma} > 0$ such that for all $d \in \Omega$, $p_0 \in C^0 ([-r - \varepsilon, 0]; \Re^n)$, the solution $p(t) \in \Re^n$ of

$$\dot{p}(t) = (A + Bk)p(t) + \exp(Ar)$$
$$\times Bk (p(t - r - \varepsilon d(t)) - p(t - r)) \quad (2.15)$$

with initial condition $p(t) = p_0(t)$ for $t \in [-r - \varepsilon, 0]$ corresponding to input $d \in \Omega$ satisfies the following estimate

$$|p(t)| \leq \tilde{Q} \exp(-\tilde{\sigma} t) \max_{-\varepsilon - r \leq s \leq 0} (|p_0(s)|), \quad \forall t \geq 0. \quad (2.16)$$

Remark 2.7. The proof of Theorem 2.1 relies on showing the exponential stability properties of the system (2.15), where $p(t) = \exp(Ar)x(t) + \int_t^{t+r} \exp(A(t + r - s))Bu(s - r)ds$ is the "predictor state". The exponential stability properties of system (2.15) are guaranteed by means of Theorem 2.5. On the other hand Example 2.4 showed that the allowable magnitude for time-varying delay perturbations which are measurable is less than the magnitude obtained for constant perturbations from Corollary 2.3. We do not know if the conservatism is due to the small-gain approach (which is used for the proof of Theorem 2.5) or if the conservatism is due to the possibility that the stability analysis for delay perturbations depends not only on the magnitude of the perturbation but also on the rate of change of the perturbation. The latter implies that the rate of change of the perturbation may be important in stability analysis. Indeed, the recent work (Bekiaris-Liberis & Krstic, 2013) has provided the construction of a Lyapunov functional for delay perturbations with constrained rate and recent results in Cloosterman, van de Wouw, Heemels, and Nijmeijer (2009) have showed that time-varying delays are more demanding than constant (uncertain) delays. Moreover, it should be noted that, for the time-varying delay perturbations with sufficiently small rate of change there exists a function $\phi : \Re_+ \rightarrow [0, r + \varepsilon]$, which satisfies $\phi(t) = r + \varepsilon d (t + \phi(t))$ for all $t \geq 0$: these are exactly the class of delays considered in Krstic (2010b) for which the following linear time-varying predictor feedback can be applied for the stabilization of (1.1):

$$u(t) = k \exp(A\phi(t))x(t) + k \int_t^{t+\phi(t)} \exp(A(t + \phi(t) - s))$$
$$\times Bu(s - r - \varepsilon d(s))ds, \quad \text{for } t \geq 0 \quad (2.17)$$

provided that the function $d : \Re_+ \rightarrow [-1, 1]$ is known.

3. Proofs of main results

We start with the proof of Theorem 2.5.

Proof of Theorem 2.5. If (2.12) holds, then (by continuity) there exists $\sigma \in (0, \lambda)$ such that:

$$\exp(\sigma(r + \varepsilon)) \frac{\Theta |C|}{\lambda - \sigma} \Big(1 - \exp(-(\lambda - \sigma)\varepsilon)$$
$$+ (\exp(|A| \varepsilon) - 1) \Big) < 1. \quad (3.1)$$

Let $d \in L^\infty (\Re_+; [-1, 1])$, $q \in L^\infty (\Re_+; [-1, 1])$, $x_0 \in C^0 ([-r - \varepsilon, 0]; \Re^n)$ be arbitrary and consider the solution $x(t) \in \Re^n$ of (2.11) with initial condition $x(t) = x_0(t)$ for $t \in [-r - \varepsilon, 0]$ that corresponds to inputs $d \in L^\infty (\Re_+; [-1, 1])$, $q \in L^\infty (\Re_+; [-1, 1])$. We define:

$$v(t) = x(t - r) - x(t - r - \varepsilon d(t)) \quad (3.2)$$
$$\|x\|_{[t_1, t_2]} := \max_{t_1 \leq s \leq t_2} (\exp(\sigma s) |x(s)|),$$
$$\|v\|_{[t_1, t_2]} := \sup_{t_1 \leq s \leq t_2} (\exp(\sigma s) |v(s)|) \quad (3.3)$$

for every $t_1 \leq t_2$ and we distinguish the following cases:

Case 1: $d(t) \leq 0$. In this case, the following formula holds for the solution of system (2.11) for almost all $t \geq r$:

$$-v(t) = (\exp(A\varepsilon |d(t)|) - I) x(t - r)$$
$$- \int_{t-r}^{t-r-\varepsilon d(t)} \exp(A(t - r - \varepsilon d(t) - s)) q(s)Cv(s)ds. \quad (3.4)$$

Using the fact that $|\exp(At)| \leq \Theta \exp(-\lambda t)$ for all $t \geq 0$ and the fact that $|\exp(At) - I| \leq \exp(|A| |t|) - 1$, for all $t \in \Re$, we obtain from (3.4) for almost all $t \geq r$:

$$|v(t)| \exp(\sigma t) \leq \exp(\sigma r) (\exp(|A| \varepsilon) - 1) |x(t - r)|$$
$$\times \exp(\sigma(t - r)) + \Theta \exp(\sigma r) \frac{1 - \exp(-(\lambda - \sigma)\varepsilon)}{\lambda - \sigma}$$
$$\times |C| \sup_{t-r \leq s \leq t-r+\varepsilon} (\exp(\sigma s) |v(s)|). \quad (3.5)$$

Indeed, using the fact that $|\exp(At)| \leq \Theta \exp(-\lambda t)$ for all $t \geq 0$, we get:

$$\left| \int_{t-r}^{t-r-\varepsilon d(t)} \exp(A(t - r - \varepsilon d(t) - s)) q(s)Cv(s)ds \right|$$
$$\leq \int_{t-r}^{t-r-\varepsilon d(t)} |\exp(A(t - r - \varepsilon d(t) - s))| |q(s)| |C| |v(s)| ds$$
$$\leq |C| \int_{t-r}^{t-r-\varepsilon d(t)} |\exp(A(t - r - \varepsilon d(t) - s))| |v(s)| ds$$
$$\leq |C| \Theta \int_{t-r}^{t-r-\varepsilon d(t)} \exp(-\lambda(t - r - \varepsilon d(t) - s)) |v(s)| ds$$
$$\leq |C| \Theta \exp(-\lambda(t - r - \varepsilon d(t)))$$
$$\times \int_{t-r}^{t-r-\varepsilon d(t)} \exp((\lambda - \sigma)s) \exp(\sigma s) |v(s)| ds$$
$$\leq |C| \Theta \exp(-\lambda(t - r - \varepsilon d(t)))$$
$$\times \int_{t-r}^{t-r-\varepsilon d(t)} \exp((\lambda - \sigma)s) ds \sup_{t-r \leq s \leq t-r+\varepsilon} (\exp(\sigma s) |v(s)|)$$
$$\leq |C| \Theta \exp(-\sigma(t - r)) \frac{1 - \exp(-(\lambda - \sigma)\varepsilon)}{\lambda - \sigma}$$
$$\times \sup_{t-r \leq s \leq t-r+\varepsilon} (\exp(\sigma s) |v(s)|).$$

The above inequality in conjunction with (3.4) and the fact that $|\exp(At) - I| \leq \exp(|A||t|) - 1$, for all $t \in \Re$, implies (3.5).

A direct consequence of definition (3.3) and inequality (3.5) is the following inequality which holds for all $t \geq r$:

$$\|v\|_{[r,t]} \leq \exp(\sigma r)(\exp(|A|\,\varepsilon) - 1)\|x\|_{[0,t-r]}$$
$$+ \Theta \exp(\sigma r) \frac{1 - \exp(-(\lambda - \sigma)\varepsilon)}{\lambda - \sigma} |C|\,\|v\|_{[0,t-r+\varepsilon]}. \quad (3.6)$$

Case 2: $d(t) \geq 0$. In this case, the following formula holds for the solution of system (3.13) for almost all $t \geq r + \varepsilon$:

$$v(t) = (\exp(A\varepsilon\,|d(t)|) - I)\,x(t - r - \varepsilon d(t))$$
$$- \int_{t-r-\varepsilon d(t)}^{t-r} \exp(A(t - r - s))\,q(s)Cv(s)ds. \quad (3.7)$$

Similarly as in the previous case, using (3.7), we show that the following inequality holds for all $t \geq r + \varepsilon$:

$$\|v\|_{[r+\varepsilon,t]} \leq \exp(\sigma(r+\varepsilon))(\exp(|A|\,\varepsilon) - 1)\|x\|_{[0,t-r]}$$
$$+ \Theta \exp(\sigma(r+\varepsilon)) \frac{1 - \exp(-(\lambda - \sigma)\varepsilon)}{\lambda - \sigma} |C|\,\|v\|_{[0,t-r]}. \quad (3.8)$$

Consequently, we conclude from (3.6) and (3.8) that the following inequality holds for all $t \geq r + \varepsilon$:

$$\|v\|_{[r+\varepsilon,t]} \leq \exp(\sigma(r+\varepsilon))(\exp(|A|\,\varepsilon) - 1)\|x\|_{[0,t-r]}$$
$$+ \Theta \exp(\sigma(r+\varepsilon)) \frac{1 - \exp(-(\lambda - \sigma)\varepsilon)}{\lambda - \sigma}$$
$$\times |C|\,\|v\|_{[0,t-r+\varepsilon]}. \quad (3.9)$$

Using the fact that $|\exp(At)| \leq \Theta \exp(-\lambda t)$ for all $t \geq 0$ and the variations of constants formula $x(t) = \exp(At)x(0) - \int_0^t \exp(A(t-s))\,q(s)Cv(s)ds$ for all $t \geq 0$, we obtain the estimate:

$$|x(t)|\exp(\sigma t) \leq \Theta \exp(-(\lambda - \sigma)t)|x(0)|$$
$$+ \Theta \frac{1 - \exp(-(\lambda - \sigma)t)}{\lambda - \sigma} |C| \sup_{0 \leq s \leq t}(\exp(\sigma s)|v(s)|),$$
$$\text{for all } t \geq 0. \quad (3.10)$$

Definition (3.3) and inequality (3.10) in conjunction with the fact that $\sigma \in (0, \lambda)$ imply the following inequality:

$$\|x\|_{[0,t]} \leq \Theta |x(0)| + \frac{\Theta |C|}{\lambda - \sigma} \|v\|_{[0,t]}, \quad \text{for all } t \geq 0. \quad (3.11)$$

Combining (3.9) and (3.11), we obtain for all $t \geq r + \varepsilon$:

$$\|v\|_{[r+\varepsilon,t]} \leq \exp(\sigma(r+\varepsilon))(\exp(|A|\,\varepsilon) - 1)\Theta |x(0)|$$
$$+ \exp(\sigma(r+\varepsilon)) \frac{\Theta |C|}{\lambda - \sigma}\left(1 - \exp(-(\lambda - \sigma)\varepsilon)\right.$$
$$+ (\exp(|A|\,\varepsilon) - 1)\left.\right)\|v\|_{[0,t]}. \quad (3.12)$$

Inequality (3.1) in conjunction with (3.12), implies the following inequality for all $t \geq 0$:

$$\|v\|_{[0,t]} \leq \exp(\sigma(r+\varepsilon))$$
$$\times \frac{\exp(|A|\,\varepsilon) - 1}{1 - \delta}\Theta |x(0)| + \|v\|_{[0,r+\varepsilon]} \quad (3.13)$$

where

$$\delta := \exp(\sigma(r+\varepsilon)) \frac{\Theta |C|}{\lambda - \sigma}\left(1 - \exp(-(\lambda - \sigma)\varepsilon)\right.$$
$$+ (\exp(|A|\,\varepsilon) - 1)\left.\right) < 1.$$

Indeed, the equality $\|v\|_{[0,t]} = \max\left(\|v\|_{[0,r+\varepsilon]}, \|v\|_{[r+\varepsilon,t]}\right)$ allows us to consider two cases:

- Case 1: $\|v\|_{[0,t]} = \|v\|_{[0,r+\varepsilon]}$. In this case (3.12), in conjunction with the fact that $\delta := \exp(\sigma(r+\varepsilon))\frac{\Theta|C|}{\lambda-\sigma}$ $(1 - \exp(-(\lambda - \sigma)\varepsilon)) + (\exp(|A|\,\varepsilon) - 1)) < 1$, implies (3.13).
- Case 2: $\|v\|_{[0,t]} = \|v\|_{[r+\varepsilon,t]}$. In this case (3.12) implies $\|v\|_{[r+\varepsilon,t]} \leq \exp(\sigma(r+\varepsilon))\frac{\exp(|A|\varepsilon)-1}{1-\delta}\Theta |x(0)|$ and consequently (3.13) holds.

Inequality (3.13) in conjunction with (3.11) and the fact that there exist constants $L, M > 0$ such that all solutions of (2.11) satisfy the estimate $|x(t)| \leq M \exp(Lt)\max_{-r-\varepsilon \leq s \leq 0}|x(s)|$ and in conjunction with the fact that $\|v\|_{[0,r+\varepsilon]} \leq 2\exp(\sigma(r+\varepsilon))$ $\left(\|x\|_{[0,r+\varepsilon]} + \max_{-r-\varepsilon \leq s \leq 0}|x(s)|\right)$ (a direct consequence of definition (3.2)) imply that there exists a constant $Q > 0$ such that estimate (2.13) holds.

If $n = 1$ then $\Theta = 1$ and $\lambda = |A|$. If (2.14) holds then (by continuity) there exists $\sigma \in (0, |A|)$ such that $\delta := \frac{|C|\exp(\sigma(r+\varepsilon))}{|A|-\sigma}$ $(2 - \exp(-(|A| - \sigma)\varepsilon) - \exp(-|A|\,\varepsilon)) < 1$. Moreover, inequalities (3.6) and (3.8) are replaced by the following inequalities:

$$\|v\|_{[r,t]} \leq \exp(\sigma r)(1 - \exp(-|A|\,\varepsilon))\|x\|_{[0,t-r]}$$
$$+ \exp(\sigma r)\frac{1 - \exp(-(|A| - \sigma)\varepsilon)}{|A| - \sigma}|C|\,\|v\|_{[0,t-r+\varepsilon]}$$

$$\|v\|_{[r+\varepsilon,t]} \leq \exp(\sigma(r+\varepsilon))(1 - \exp(-|A|\,\varepsilon))\|x\|_{[0,t-r]}$$
$$+ \exp(\sigma(r+\varepsilon))$$
$$\times \frac{1 - \exp(-(|A| - \sigma)\varepsilon)}{|A| - \sigma}|C|\,\|v\|_{[0,t-r]}.$$

It follows that, the inequality (3.9) is replaced by

$$\|v\|_{[r+\varepsilon,t]} \leq \exp(\sigma(r+\varepsilon))(1 - \exp(-|A|\,\varepsilon))\|x\|_{[0,t-r]}$$
$$+ \exp(\sigma(r+\varepsilon))$$
$$\times \frac{1 - \exp(-(|A| - \sigma)\varepsilon)}{|A| - \sigma}|C|\,\|v\|_{[0,t-r+\varepsilon]}. \quad (3.14)$$

Combining (3.14) with (3.11) and $\Theta = 1$, $\lambda = |A|$, we obtain the estimate:

$$\|v\|_{[r+\varepsilon,t]} \leq \exp(\sigma(r+\varepsilon))(1 - \exp(-|A|\,\varepsilon))|x(0)|$$
$$+ \frac{|C|\exp(\sigma(r+\varepsilon))}{|A| - \sigma}\left(2 - \exp(-(|A| - \sigma)\varepsilon)\right.$$
$$\left. - \exp(-|A|\,\varepsilon)\right)\|v\|_{[0,t]}.$$

Since

$$\delta := \frac{|C|\exp(\sigma(r+\varepsilon))}{|A| - \sigma}\left(2 - \exp(-(|A| - \sigma)\varepsilon)\right.$$
$$\left. - \exp(-|A|\,\varepsilon)\right) < 1,$$

the above inequality implies the inequality $\|v\|_{[0,t]} \leq \exp$ $(\sigma(r+\varepsilon))\frac{1-\exp(-|A|\varepsilon)}{1-\delta}|x(0)| + \|v\|_{[0,r+\varepsilon]}$. The previous inequality in conjunction with (3.11) and the fact that there exist constants $L, M > 0$ such that all solutions of (3.13) satisfy the estimate $|x(t)| \leq M \exp(Lt)\max_{-r-\varepsilon \leq s \leq 0}|x(s)|$ and in conjunction with the fact that $\|v\|_{[0,r+\varepsilon]} \leq 2\exp(\sigma(r+\varepsilon))$ $\left(\|x\|_{[0,r+\varepsilon]} + \max_{-r-\varepsilon \leq s \leq 0}|x(s)|\right)$ (a direct consequence of definition (3.2)) imply that there exists a constant $Q > 0$ such that estimate (2.13) holds. The proof is complete. □

We are now ready to provide the proof of Theorem 2.1.

Proof of Theorem 2.1. Proposition 2.6 with $\Omega = L^{\infty}(\Re_+; [-1, 1])$ guarantees the conclusion of the theorem provided that system (2.15) is robustly globally exponentially stable. Theorem 2.5 with $A \in \Re^{n \times n}$ replaced by $(A + Bk)$ and $C = \exp(Ar)Bk$ guarantees the robust global exponential stability of system (2.15) provided that (2.4) or (2.5) hold. The proof is complete. □

Next, we provide the proof of Corollary 2.3.

Proof of Corollary 2.3. Corollary 6.1 on p. 215 in Hale and Lunel (1993) implies that: all roots of Eq. (2.7) have negative real parts if and only if the zero solution is Globally Exponentially Stable for the system:

$$\dot{p}(t) = (A + Bk)p(t) + \exp(Ar)Bk\left(p(t - \tau) - p(t - r)\right). \quad (3.15)$$

The rest of the proof is a direct consequence of Proposition 2.6 with $\Omega \subset L^\infty(\Re_+; [-1, 1])$ being the set of constant functions which are identically equal to 1 or -1 and $\tau = r \pm \varepsilon$. The proof is complete. □

Proof of Proposition 2.6. Let arbitrary $(x_0, u_0) \in S$ (where S is defined by (2.2)), $d \in \Omega$ and consider the solution $(x(t), u(t)) \in \Re^n \times \Re^m$ of (1.1), (1.2) with initial conditions $x(0) = x_0, u(t) = u_0(t)$ for $t \in [-r - \varepsilon, 0]$ corresponding to $d \in \Omega$. Define for all $t \geq 0$:

$$p(t) = \exp(Ar)x(t) + \int_t^{t+r} \exp(A(t + r - s))Bu(s - r)ds. \quad (3.16)$$

Notice that, (1.2) and definition (3.16) imply that the following equality holds for all $t \geq 0$:

$$u(t) = kp(t), \quad \text{for all } t \geq 0. \quad (3.17)$$

By using (1.1) and definition (3.16), it follows that the following differential equation holds for almost all $t \geq 0$:

$$\begin{aligned}
\dot{p}(t) = {} & \exp(Ar)Ax(t) + \exp(Ar)Bu(t - r - \varepsilon d(t)) \\
& + A\left(p(t) - \exp(Ar)x(t)\right) \\
& + Bu(t) - \exp(Ar)Bu(t - r). \quad (3.18)
\end{aligned}$$

Using the identity $A\exp(Ar) = \exp(Ar)A$ and (3.17) it follows that, the following differential equation holds for almost all $t \geq r + \varepsilon$:

$$\begin{aligned}
\dot{p}(t) = {} & (A + Bk)p(t) + \exp(Ar) \\
& \times Bk\left(p(t - r - \varepsilon d(t)) - p(t - r)\right). \quad (3.19)
\end{aligned}$$

Since $\Omega \subseteq L^\infty(\Re_+; [-1, 1])$ is a set of time-varying inputs which is invariant under time translation (which implies that the input defined by $\tilde{d}(t) = d(t + r + \varepsilon)$ is in $\Omega \subseteq L^\infty(\Re_+; [-1, 1])$) and since (2.16) holds for certain constants $\tilde{Q}, \tilde{\sigma} > 0$, it follows that the following inequality holds:

$$|p(t)| \leq \tilde{Q} \exp\left(-\tilde{\sigma}(t - r - \varepsilon)\right) \max_{0 \leq s \leq r+\varepsilon} |p(s)|,$$

$$\forall t \geq r + \varepsilon. \quad (3.20)$$

Using (3.20) in conjunction with (3.16), (3.17), (2.3) and the following equality:

$$x(t) = \exp(-Ar)p(t) - \int_t^{t+r} \exp(A(t - s))Bkp(s - r)ds \quad (3.21)$$

which holds for all $t \geq r$ and is a direct consequence of (3.16) and (3.17), we obtain (1.3) with $\sigma := \tilde{\sigma}$ and $Q := M \exp(2(L + \tilde{\sigma} + |A|)(r + \varepsilon))$

$\left((1 + r|Bk| + |k|)\tilde{Q}(1 + r|B|) + 1\right)$.

Conversely, let arbitrary $p_0 \in C^0([-r - \varepsilon, 0]; \Re^n), d \in \Omega$ and consider the solution $p(t) \in \Re^n$ of (2.15), (1.2) with initial condition $p(t) = p_0(t)$ for $t \in [-r - \varepsilon, 0]$ corresponding to $d \in \Omega$. Define $u_0(t) = kp_0(t)$ for $t \in [-r - \varepsilon, 0]$ and $x_0 = \exp(-Ar)\left(p(0) - \int_{-r}^0 \exp(-As)Bu_0(s)ds\right)$. Notice that, $(x_0, u_0) \in S$ (where S is defined by (2.2)). Therefore, the solution $(x(t), u(t)) \in \Re^n \times \Re^m$ of (1.1), (1.2) with initial condition $x(0) = x_0, u(t) = u_0(t)$ for $t \in [-r - \varepsilon, 0]$ satisfies the estimate (1.3) for certain constants

$Q, \sigma > 0$. Notice that, the solution $(x(t), u(t)) \in \Re^n \times \Re^m$ of (1.1), (1.2) with initial condition $x(0) = x_0, u(t) = u_0(t)$ for $t \in [-r - \varepsilon, 0]$ satisfies (3.17) and (3.21) for all $t \geq 0$. Consequently, (3.16) holds for all $t \geq 0$. Estimate (2.16) with $\bar{\tilde{Q}} := Q\exp(2|A|r)(1 + r|B|)(1 + r|Bk|\exp(|A|r) + |k|)$ and $\tilde{\sigma} := \sigma$ is a direct consequence of (1.3), (3.16) and the definitions $x_0 = \exp(-Ar)\left(p(0) - \int_{-r}^0 \exp(-As)Bu_0(s)ds\right)$ and $u_0(t) = kp_0(t)$ for $t \in [-r - \varepsilon, 0]$. The proof is complete. □

4. Concluding remarks

We have provided formulas that allow us to compute estimates of the least upper bound of the magnitude of the delay perturbation that does not destroy the exponential stability properties of the closed-loop system (1.1) with (1.2). Two cases have been considered: the case of measurable perturbations and the case of constant perturbations. As in Krstic (2008), where a Lyapunov analysis in L^2 is pursued, our stability analysis in C^0 separately considers positive and negative perturbations of the delay, whereas the Lyapunov analyses in H^1 in Section 5.3 in Krstic (2009), and in Bekiaris-Liberis and Krstic (2013) simultaneously tackle positive and negative perturbations on the delay.

The obtained formulas can be used easily by the control practitioner in order to estimate the delay error that can be tolerated. For the case of measurable time-varying perturbations, the magnitude of the delay perturbation $\varepsilon > 0$ must satisfy the inequality (2.4) (or (2.5) if $n = 1$). All quantities involved in the inequality (2.4) can be computed easily using software packages to compute the norms of matrices $\exp(Ar)Bk \in \Re^{n \times n}, (A + Bk) \in \Re^{n \times n}$ and to determine the constants $\Theta, \lambda > 0$ by finding a symmetric positive definite matrix $P \in \Re^{n \times n}$ and a constant $\mu > 0$ that satisfies $P(A + Bk) + (A + Bk)'P + 2\mu P \leq 0$ and $P \geq I$ (select $\lambda = \mu$ and $\Theta = \sqrt{|P|}$).

An example showed that the allowable magnitude of measurable delay perturbation is less than the magnitude obtained for constant perturbations from Corollary 2.3. We do not know if the conservatism is due to the small-gain approach (which is used for the proof of Theorem 2.5) or if the conservatism is due to the possibility that the stability analysis for delay perturbations depends not only on the magnitude of the perturbation but also on the rate of change of the perturbation. The latter implies that, the rate of change of the perturbation may be important in stability analysis. There are recent results in Cloosterman et al. (2009) which show that time-varying delays are more demanding than constant (uncertain) delays and recent Lyapunov-based stability studies in Bekiaris-Liberis and Krstic (2013) which utilize the rate of change of the delay. It remains an open problem to construct more accurate expressions for the tolerance of the delay error which may involve the rate of change of the delay perturbation.

Another class of open problems is the extension of the robustness analysis to different types of linear predictor feedback. For example, the linear predictor feedback proposed in Zhou et al. (2012) can be, in principle, studied in the same way.

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