On the geometry of the envelope of a matrix

Panayiotis J. Psarrakos* and Michael J. Tsatsomeros†

June 23, 2014

Abstract

The envelope, $\mathcal{E}(A)$, of a complex square matrix A is a region in the complex plane that contains the spectrum of A and is contained in the numerical range of A. The envelope is compact but not necessarily convex or connected. The connected components of $\mathcal{E}(A)$ have the potential of isolating the eigenvalues of A, leading us to study its geometry, boundary, and number of components. We also examine the envelope of normal matrices and similarities. In the process, we observe that $\mathcal{E}(A)$ contains the 2-rank numerical range of A.

Keywords: eigenvalue bounds, numerical range, cubic curve, envelope.

AMS Subject Classifications: 15A18, 15A60, 65F15.

1 Introduction

The *envelope* of a complex square matrix A, denoted by $\mathcal{E}(A)$, is an eigenvalue containment region that was introduced in [14]. Evidently, the envelope represents a theoretically, computationally and visually attractive way to localize the spectrum of A by isolating the eigenvalues in its connected components.

The concept and definition of the envelope are based on an inequality proven in [1] that the (real and imaginary parts of the) eigenvalues of A must satisfy. This inequality allows one to replace the half-plane to the left of the largest eigenvalue of the hermitian part of A by a smaller region that contains the spectrum of A. Thus, upon rotating a matrix A through all angles in $[0, 2\pi)$, the envelope arises as a region that contains the eigenvalues and is contained in the numerical range, $F(A)$. The precise definition and illustrations of $\mathcal{E}(A)$ can be found in Section 3.

The rendering of $\mathcal{E}(A)$ is akin to the process for $F(A)$, essentially requiring knowledge of the first but also the second largest eigenvalues of the hermitian part of $e^{\mathrm{i}\theta}A$ for a range of angles in $[0, 2\pi)$.

*Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece (ppsarr@math.ntua.gr). Corresponding author.

†Mathematics Department, Washington State University, Pullman, WA 99164-3113, USA (tsat@math.wsu.edu).

The envelope has properties similar to $F(A)$, e.g., it is compact, invariant under unitary similarities and homogeneous; it is not, however, necessarily convex or connected. The aim of this paper is to further understand the properties and features of $\mathcal{E}(A)$ as they pertain to its geometry, boundary, number of components, and containment of eigenvalues. In particular, we study the case of normal matrices and eigenvalues, and make comparisons to the numerical range. In the process, we discover that the envelope contains the 2-rank numerical range of A introduced in [2].

This paper is organized as follows. In Section 2, we describe the notions relevant to the definition and study of the envelope. In Section 3, the envelope is defined formally, its basic properties are reviewed, and its relation to the 2-rank numerical range is established. Section 4 contains results on extremal eigenvalues, normal matrices (Subsection 4.1) and similarities (Subsection 4.2), and the effects of such assumptions on the geometry of the envelope are examined. Finally, a result on the eigenvectors of the right-most eigenvalues is given in Section 5, and some conclusions are presented in Section 6.

2 Definitions and preliminaries

Let $A \in \mathbb{C}^{n\times n}$ ($n \geq 2$) be an $n\times n$ complex matrix with spectrum $\sigma(A)$. Consider the *hermitian* and *skew-hermitian parts* of A, $H(A) = (A+A^*)/2$ and $S(A) = (A-A^*)/2$, respectively, and let $\delta_1(A) \geq \delta_2(A) \geq \cdots \geq \delta_n(A)$ denote the eigenvalues of $H(A)$ in a nonincreasing order. Let also $y_1 \in \mathbb{C}^n$ be a unit (with respect to the Euclidean vector norm) eigenvector of $H(A)$ corresponding to $\delta_1(A)$.

2.1 The standard numerical range

The *numerical range* (also known as the *field of values*) of A is defined as

$$F(A) = \{v^*Av \in \mathbb{C} : v \in \mathbb{C}^n \text{ with } v^*v = 1\}.$$

It is a *compact* and *convex* subset of $\mathbb{C}$ that contains $\sigma(A)$ and is a useful concept in understanding matrices and operators; see [6, Chapter 1] and the references therein.

For an angle $\theta \in [0, 2\pi)$, we consider the largest eigenvalue $\delta_1(e^{\mathrm{i}\,\theta}A)$ and an associated unit eigenvector $y_1(\theta)$ of the hermitian matrix $H(e^{\mathrm{i}\,\theta}A)$. Then, the point $z_\theta = y_1(\theta)^*Ay_1(\theta)$ lies on the boundary of $F(A)$, denoted by $\partial F(A)$, and the line $\mathcal{L}_\theta = \{e^{-\mathrm{i}\,\theta}(\delta_1(e^{\mathrm{i}\,\theta}A)+\mathrm{i}\,t) : t \in \mathbb{R}\}$ is tangential to $\partial F(A)$ at z_θ [6, 7]. Furthermore, $\mathcal{L}_\theta$ defines the closed half-plane $\mathcal{H}_{in}(A,\theta) = \left\{e^{-\mathrm{i}\,\theta}(s+\mathrm{i}\,t) : s,t \in \mathbb{R} \text{ with } s \leq \delta_1(e^{\mathrm{i}\,\theta}A)\right\}$, which contains $F(A)$. Hence, $F(A)$ can be written as an infinite intersection of closed half-planes [6, Theorem 1.5.12], namely,

$$F(A) = \bigcap_{\theta\in[0,2\pi)} \left\{e^{-\mathrm{i}\,\theta}(s+\mathrm{i}\,t) : s,t \in \mathbb{R} \text{ with } s \leq \delta_1(e^{\mathrm{i}\,\theta}A)\right\} = \bigcap_{\theta\in[0,2\pi)} \mathcal{H}_{in}(A,\theta). \quad (1)$$

2.2 The k-rank numerical range

For $1 \le k \le n-1$, the *k-rank numerical range* of matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\begin{aligned} \Lambda_k(A) &= \{\mu \in \mathbb{C} : \ PAP = \mu P \ \text{ for some rank-}k \text{ orthogonal projection } P \in \mathbb{C}^{n\times n}\} \\ &= \left\{\mu \in \mathbb{C} : \ X^*AX = \mu I_k \ \text{ for some } X \in \mathbb{C}^{n\times k} \text{ such that } X^*X = I_k\right\}, \end{aligned}$$

and is a natural generalization of the standard numerical range, in the sense that $\Lambda_1(A)$ coincides with $F(A)$. This set was introduced in [2] and has attracted attention because of its role in quantum information theory; specifically, it is closely connected to the construction of quantum error correction codes for noisy quantum channels (see [2, 3, 8] and the references therein). The range $\Lambda_k(A)$ is a *compact* and *convex* subset of the complex plane [16] and is given by the explicit formula [11, Theorem 2.2]

$$\Lambda_k(A) = \bigcap_{\theta\in[0,2\pi)} \left\{e^{-\mathrm{i}\,\theta}(s+\mathrm{i}\,t) : \ s,t \in \mathbb{R} \ \text{ with } \ s \le \delta_k(e^{\mathrm{i}\,\theta}A)\right\}. \tag{2}$$

Moreover, $\Lambda_k(A)$ is invariant under unitary similarity and satisfies $\Lambda_{n-1}(A) \subseteq \Lambda_{n-2}(A) \subseteq \cdots \subseteq \Lambda_2(A) \subseteq \Lambda_1(A) = F(A)$. For $k \ge 2$, $\Lambda_k(A)$ does not necessarily contain all of the eigenvalues of A and, in fact, may be empty [10].

If the matrix $A \in \mathbb{C}^{n\times n}$ is normal with (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then (2) implies that (see Corollary 2.4 of [11])

$$\Lambda_k(A) = \bigcap_{1\le j_1<j_2<\cdots<j_{n-k+1}\le n} \operatorname{conv}\left\{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_{n-k+1}}\right\}, \tag{3}$$

where $\operatorname{conv}\{\cdot\}$ denotes the convex hull. Efficient techniques to generate $\Lambda_k(A)$ for normal A, using half-planes determined by the eigenvalues instead of formula (3), are proposed in [4].

2.3 The cubic curve $\Gamma(A)$

For matrix $A \in \mathbb{C}^{n\times n}$, define the nonnegative quantities $\mathrm{v}(A) = \|S(A)y_1\|_2^2$ and $\mathrm{u}(A) = \mathrm{Im}(y_1^* S(A) y_1) \le \|S(A)y_1\|_2 = \sqrt{\mathrm{v}(A)}$, where $\|\cdot\|_2$ denotes the *spectral matrix norm* (i.e., the norm subordinate to the Euclidean vector norm). Adam and Tsatsomeros [1], extending a methodology of [12], derived the following theorem.

Theorem 2.1. [1, Theorem 3.1] *Let $A \in \mathbb{C}^{n\times n}$. Then, for every eigenvalue $\lambda \in \sigma(A)$,*

$$(\mathrm{Re}\lambda - \delta_2(A))(\mathrm{Im}\lambda - \mathrm{u}(A))^2 \le (\delta_1(A) - \mathrm{Re}\lambda)[\mathrm{v}(A) - \mathrm{u}(A)^2 + (\mathrm{Re}\lambda - \delta_2(A))(\mathrm{Re}\lambda - \delta_1(A))].$$

Motivated by the above result, the authors of [1] introduced and studied the algebraic curve

$$\begin{aligned} \Gamma(A) = \big\{ & s+\mathrm{i}\,t : \ s,t \in \mathbb{R}, \ (\delta_2(A) - s)[(\delta_1(A) - s)^2 + (\mathrm{u}(A) - t)^2] \\ & + (\delta_1(A) - s)(\mathrm{v}(A) - \mathrm{u}(A)^2) = 0\big\}. \end{aligned}$$

This is a cubic algebraic curve in $s, t \in \mathbb{R}$ (a suggested general reference on this type of curves is [13]), which defines the region

$$\begin{aligned}\Gamma_{in}(A) &= \big\{s + \mathrm{i}\, t : s, t \in \mathbb{R},\ (\delta_2(A) - s)[(\delta_1(A) - s)^2 + (\mathrm{u}(A) - t)^2] \\ &\quad + (\delta_1(A) - s)(\mathrm{v}(A) - \mathrm{u}(A)^2) \geq 0\big\}\,.\end{aligned}$$

By Theorem 2.1, it follows that $\sigma(A) \subset \Gamma_{in}(A)$. If $s > \delta_1(A)$ or $s < \delta_2(A)$, then $s + \mathrm{i}\, t$ cannot satisfy the defining equation of $\Gamma(A)$ (always for $s, t \in \mathbb{R}$), and thus, the curve $\Gamma(A)$ lies in the vertical zone $\{z \in \mathbb{C} : \delta_2(A) \leq \operatorname{Re} z \leq \delta_1(A)\}$. As a consequence, (2) yields $\Lambda_2(A) \subseteq \Gamma_{in}(A)$. It is also straightforward to verify that $\Gamma(A)$ is symmetric with respect to the horizontal line $\mathcal{L} = \{z \in \mathbb{C} : \operatorname{Im} z = \mathrm{u}(A)\}$ which it intercepts at the point $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$, and is asymptotic to the vertical line $\{z \in \mathbb{C} : \operatorname{Re} z = \delta_2(A)\}$. Apparently, the point $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is a right most point of $\Gamma(A)$ and $F(A)$. Furthermore, if $\partial F(A)$ has a flat portion (i.e., a non-degenerate line segment) on the vertical line $\mathcal{L}_0 = \{z \in \mathbb{C} : \operatorname{Re} z = \delta_1(A)\}$, then Lemma 1.5.7 of [6] implies that $\delta_1(A) = \delta_2(A)$, in which case the curve $\Gamma(A)$ reduces to the line $\mathcal{L}_0$ and the region $\Gamma_{in}(A)$ coincides with the half-plane $\mathcal{H}_{in}(A, 0)$.

When $\delta_1(A) > \delta_2(A)$, $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is the unique right most point of $\Gamma(A)$ (i.e., the only point of the curve with real part equal to $\delta_1(A)$). Moreover, the vertical line $\mathcal{L}_0$ is tangential to $\Gamma(A)$ at $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$. This means that $\mathcal{L}_0$ is a common tangent to the curve $\Gamma(A)$ and the numerical range $F(A)$ at $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$.

For $t = \mathrm{u}(A)$ and $\delta_2(A) < s < \delta_1(A)$, the defining equation of $\Gamma(A)$ becomes

$$(\delta_1(A) - s)[s^2 - (\delta_1(A) + \delta_2(A))s + \delta_1(A)\delta_2(A) + \mathrm{v}(A) - \mathrm{u}(A)^2] = 0,$$

and the discriminant of its quadratic factor is $\Delta = (\delta_1(A) - \delta_2(A))^2 - 4(\mathrm{v}(A) - \mathrm{u}(A)^2)$. Hence, we have the following cases [1, 14], which are illustrated in Figure 1 for three appropriately chosen 9×9 matrices (the eigenvalues are marked as +'s)[1].

(a) If $\Delta < 0$, then $\Gamma(A)$ intercepts the horizontal line $\mathcal{L}$ only once, at $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$, and is an unbounded simple open curve which has all the eigenvalues of A lying to its left.

(b) If $\Delta = 0$, then $\Gamma(A)$ intercepts $\mathcal{L}$ at $\frac{\delta_1(A)+\delta_2(A)}{2} + \mathrm{i}\,\mathrm{u}(A)$ and $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$, where the first point (double root) is the node point (cusp) of $\Gamma(A)$.

(c) If $\Delta > 0$, then $\Gamma(A)$ comprises two branches, a closed bounded branch that lies in the vertical zone $\left\{z \in \mathbb{C} : \frac{\delta_1(A)+\delta_2(A)+\sqrt{\Delta}}{2} \leq \operatorname{Re} z \leq \delta_1(A)\right\}$, intercepts $\mathcal{L}$ at $\frac{\delta_1(A)+\delta_2(A)+\sqrt{\Delta}}{2} + \mathrm{i}\,\mathrm{u}(A)$ and $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ and encompasses exactly one eigenvalue of matrix A which is simple [14, Theorem 3.2], and an open unbounded branch which lies in $\left\{z \in \mathbb{C} : \delta_2(A) \leq \operatorname{Re} z \leq \frac{\delta_1(A)+\delta_2(A)-\sqrt{\Delta}}{2}\right\}$, intercepts $\mathcal{L}$ at $\frac{\delta_1(A)+\delta_2(A)-\sqrt{\Delta}}{2} + \mathrm{i}\,\mathrm{u}(A)$ and has the remaining eigenvalues of A to its left.

[1] These three possible cases and configurations of $\Gamma(A)$ are paramount to the geometric features of the envelope discussed subsequently, so we choose to repeat here part of the illustrations and analysis found in [1, 14].

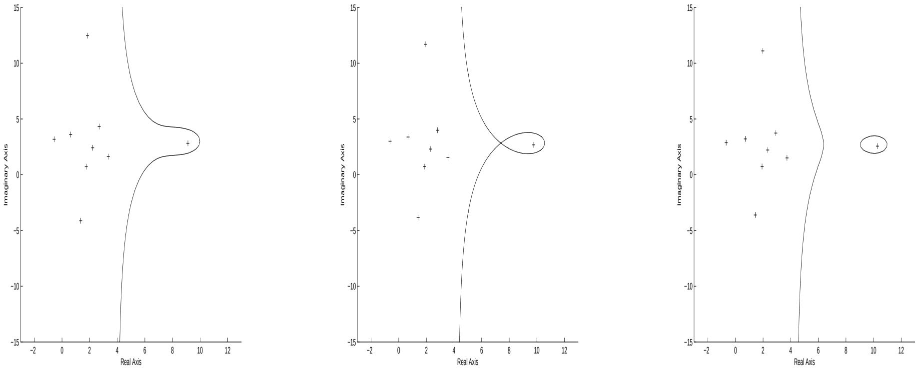

Figure 1: The cases of $\Gamma(A)$ with $\Delta < 0$ (left), $\Delta = 0$ (middle), and $\Delta > 0$ (right).

3 The envelope $\mathcal{E}(A)$

Motivated by (1) and the fact that for every $\theta \in [0, 2\pi)$,

$$\sigma(A) = e^{-\mathrm{i}\theta}\sigma(e^{\mathrm{i}\theta}A) \subseteq e^{-\mathrm{i}\theta}\Gamma_{in}(e^{\mathrm{i}\theta}A) \subseteq \mathcal{H}_{in}(A,\theta), \tag{4}$$

the *cubic envelope* (or simply, the *envelope*) of A was defined in [14] as the set

$$\mathcal{E}(A) = \bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\theta}\Gamma_{in}(e^{\mathrm{i}\theta}A). \tag{5}$$

Next, it is formally shown that $\mathcal{E}(A)$ lies in the (convex) numerical range $F(A) = \Lambda_1(A)$ and that it contains the (convex) 2-rank numerical range $\Lambda_2(A)$, as well as the spectrum $\sigma(A)$. (Recall that $\Lambda_2(A)$ does not necessarily contain $\sigma(A)$.)

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$. Then the following hold:*

$$\sigma(A) \subseteq \mathcal{E}(A) = \bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\theta}\Gamma_{in}(e^{\mathrm{i}\theta}A) \subseteq \bigcap_{\theta\in[0,2\pi)} \mathcal{H}_{in}(A,\theta) = F(A) \tag{6}$$

and

$$\begin{aligned}\Lambda_2(A) &= \bigcap_{\theta\in[0,2\pi)} \left\{e^{-\mathrm{i}\theta}(s+\mathrm{i}\,t) : s,t\in\mathbb{R} \text{ with } s \le \delta_2(e^{\mathrm{i}\theta}A)\right\} \\ &\subseteq \bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\theta}\Gamma_{in}(e^{\mathrm{i}\theta}A) = \mathcal{E}(A).\end{aligned}$$

Proof. The relations in (6) follow by (1), (4) and (5). The rest of the relations in this theorem follow by (2) and (5). $\square$

We continue by establishing that the envelope, being an infinite intersection of complex regions, can be approximated to arbitrary precision by finite intersections.

This indeed provides the foundation for our method to render the envelope. Recall that for two compact subsets Ω_1 and Ω_2 of a metric space $(\mathcal{X},\rho)$, the *Hausdorff distance* between Ω_1 and Ω_2 is defined by

$$d_H(\Omega_1,\Omega_2) = \max\left\{\max_{x_1\in\Omega_1}\min_{x_2\in\Omega_2}\rho(x_1,x_2),\ \max_{x_2\in\Omega_2}\min_{x_1\in\Omega_1}\rho(x_1,x_2)\right\}.$$

For any $x_0 \in \mathcal{X}$ and $\delta > 0$, we define the open ball $\mathcal{B}(x_0,\delta) = \{x \in \mathcal{X} :\ \rho(x_0,x) < \delta\}$. Adopting arguments from the proof of [9, Lemma 2.5], we obtain the following general result.

Lemma 3.2. *Let $\{\mathcal{G}_a : a \in \mathcal{A}\}$ be an infinite family of closed subsets of $\mathbb{C}^n$, such that the set $\mathcal{F} = \bigcap\limits_{a\in\mathcal{A}} \mathcal{G}_a$ is non-empty and compact. Then, for every $\varepsilon > 0$, there exist $a_1, a_2, \ldots, a_k \in \mathcal{A}$ such that*

$$d_H\left(\mathcal{F}, \bigcap_{j=1}^{k} \mathcal{G}_{a_j}\right) \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since $\mathcal{F}$ is compact, there is a compact set $\Omega \subset \mathbb{C}^n$ such that $\mathcal{F} + \mathcal{B}(0,\varepsilon)$ lies in the interior of Ω. Then the set $\Omega\backslash(\mathcal{F} + \mathcal{B}(0,\varepsilon))$ is compact and lies in the union $\bigcup\limits_{a\in\mathcal{A}} (\mathbb{C}^n\backslash\mathcal{G}_a)$. As a consequence, compactness implies that there exist $a_1, a_2, \ldots, a_k \in \mathcal{A}$ such that

$$\Omega\backslash(\mathcal{F} + \mathcal{B}(0,\varepsilon)) \subseteq \bigcup_{j=1}^{k} \left(\mathbb{C}^n\backslash\mathcal{G}_{a_j}\right).$$

Thus,

$$\mathcal{F} \subseteq \bigcap_{j=1}^{k} \mathcal{G}_{a_j} \subseteq \mathcal{F} + \mathcal{B}(0,\varepsilon),$$

and the proof is complete. □

The above lemma yields readily the following desired approximation result, which can be modified to also hold for the numerical range and the k-rank numerical range.

Corollary 3.3. *Let $A \in \mathbb{C}^{n\times n}$. Then, for every $\varepsilon > 0$, there exist $\theta_1, \theta_2, \ldots, \theta_k \in [0, 2\pi)$ such that*

$$d_H\left(\mathcal{E}(A), \bigcap_{j=1}^{k} e^{-\mathrm{i}\,\theta_j}\Gamma_{in}(e^{\mathrm{i}\,\theta_j}A)\right) \leq \varepsilon.$$

Example 3.4. Consider the 4×4 complex matrix

$$A = \begin{bmatrix} 14+\mathrm{i}\,19 & -4-\mathrm{i} & -55-\mathrm{i}\,13 & -32+\mathrm{i}\,13 \\ 27+\mathrm{i}\,2 & 14-\mathrm{i}\,25 & 64 & 72 \\ 54+\mathrm{i} & 47-\mathrm{i}\,3 & 14+\mathrm{i}\,44 & -32-\mathrm{i}\,42 \\ 76 & 73 & 4-\mathrm{i}\,2 & -11+\mathrm{i}\,24 \end{bmatrix}.$$

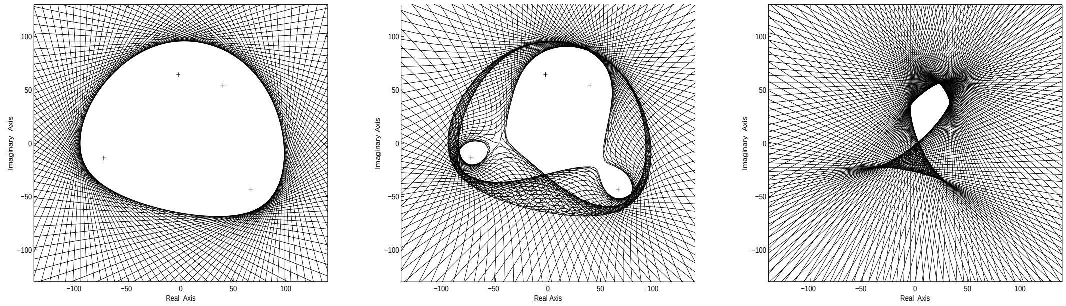

Figure 2: The sets $F(A)$ (left), $\mathcal{E}(A)$ (middle), and $\Lambda_2(A)$ (right).

The numerical range of A is drawn as the intersection of 120 closed half-planes on the left of Figure 2. In the middle part of the figure, $\mathcal{E}(A)$ is the unshaded region[2] resulting from having drawn 120 curves $e^{-\mathrm{i}\,\theta}\Gamma(e^{\mathrm{i}\,\theta}A)$. In both of these parts, the eigenvalues are marked as $+$'s. In the right part of Figure 2, $\Lambda_2(A)$ is the unshaded region resulting from having sketched 120 lines (applying (2)), and does not contain any eigenvalue of A. Notice that the cubic envelope $\mathcal{E}(A)$ consists of two connected components, is a significantly improved localization of the spectrum $\sigma(A)$ as compared to $F(A)$, and clearly contains $\Lambda_2(A)$. Finally, notice that the numerical range of A appears in our plot of the envelope (middle part) as a by-product; specifically, $F(A)$ is depicted as the outer outlined region.

The envelope $\mathcal{E}(A)$ is *compact*, since it is a closed subset of the compact numerical range $F(A)$. It is not necessarily convex or connected, as illustrated by Example 3.4. It satisfies, however, some of the basic properties of $F(A)$, $\Lambda_k(A)$ and, more importantly, of $\sigma(A)$ listed next (see [14]).

(P$_1$) $\Gamma(A^T) = \Gamma(A)$, $\Gamma(A^*) = \Gamma(\overline{A}) = \overline{\Gamma(A)}$, $\mathcal{E}(A^T) = \mathcal{E}(A)$ and $\mathcal{E}(A^*) = \mathcal{E}(\overline{A}) = \overline{\mathcal{E}(A)}$. In particular, if $A \in \mathbb{R}^{n\times n}$, then the curve $\Gamma(A)$ and the envelope $\mathcal{E}(A)$ are symmetric with respect to the real axis.

(P$_2$) For any unitary matrix $U \in \mathbb{C}^{n\times n}$, $\Gamma(U^*AU) = \Gamma(A)$ and $\mathcal{E}(U^*AU) = \mathcal{E}(A)$.

(P$_3$) For any $b \in \mathbb{C}$, $\Gamma(A + bI_n) = \Gamma(A) + b$ and $\mathcal{E}(A + bI_n) = \mathcal{E}(A) + b$ (where I_n denotes the $n \times n$ identity matrix).

(P$_4$) For any real $r > 0$ and any $a \in \mathbb{C}$, $\Gamma(rA) = r\,\Gamma(A)$ and $\mathcal{E}(aA) = a\,\mathcal{E}(A)$.

By Properties (P$_1$) and (P$_2$), it is clear that for any unitary matrix $U \in \mathbb{C}^{n\times n}$, the linear mappings $A \mapsto U^*AU$ and $A \mapsto U^*A^TU$ preserve the envelope.

---

[2] A Matlab function for rendering the envelope $\mathcal{E}(A)$, which is based on the defining relation (5) of the envelope and has been used in our numerical experiments, can be found in `http://www.math.ntua.gr/~ppsarr/envelope.m` and `http://www.math.wsu.edu/faculty/tsat/files/matlab/envelope.m`.

Recall that an eigenvalue $\lambda_0 \in \sigma(A)$ is called *normal* if its algebraic and geometric multiplicities are equal and the eigenvectors of A corresponding to λ_0 are orthogonal to the eigenvectors corresponding to any other eigenvalue of A. By Theorem 1.6.6 of [6], every eigenvalue of A that lies on the boundary of $F(A)$ is a normal eigenvalue of A. Moreover, the non-differentiable points (corners) of $\partial F(A)$ are necessarily eigenvalues of A [6, Theorem 1.6.3].

Suppose now that $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is an eigenvalue of A. Then, $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is a normal eigenvalue of A that lies on $\partial F(A)$. Furthermore, $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$, $\delta_1(A)$ and $\mathrm{i}\,\mathrm{u}(A)$ are eigenvalues of A, $H(A)$ and $S(A)$, respectively, and they share the same eigenspace. If, in addition, $\delta_1(A)$ is a simple eigenvalue of $H(A)$, then $\mathrm{v}(A) - \mathrm{u}(A)^2 = 0$, and the cubic curve $\Gamma(A)$ reduces to the union of the point $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ and the vertical line $\{z \in \mathbb{C} : \mathrm{Re}\, z = \delta_2(A)\}$. Otherwise, i.e., when $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is a normal eigenvalue of A on $\partial F(A)$ and $\delta_1(A)$ is a multiple eigenvalue of $H(A)$, the curve $\Gamma(A)$ reduces to the vertical line $\{z \in \mathbb{C} : \mathrm{Re}\, z = \delta_1(A)\}$ and $\Gamma_{in}(A)$ coincides with $\mathcal{H}_{in}(A, 0)$. As a consequence, we have the following.

Proposition 3.5. [14, Proposition 5.1] *Let λ_0 be a simple eigenvalue of A on the boundary of $F(A)$. If λ_0 does not lie on a flat portion of $\partial F(A)$, or it is a non-differentiable point of $\partial F(A)$, then λ_0 is an isolated point of the envelope $\mathcal{E}(A)$.*

4 Normal matrices and similarity

Let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the spectrum of matrix $A \in \mathbb{C}^{n\times n}$ $(n \geq 2)$, where multiple eigenvalues (if any) are listed in successive positions. Consider the diagonal matrix $D(A) = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We call an eigenvalue λ_j of A *extremal* if λ_j is a vertex of the convex hull of $\sigma(A)$, denoted by $\mathrm{conv}\{\sigma(A)\}$.

4.1 The envelope of normal matrices

Suppose that A is normal. Then the numerical range $F(A)$ coincides with the convex hull of $\sigma(A)$ [6, Property 1.2.9]. If $\hat\lambda_1, \hat\lambda_2, \ldots, \hat\lambda_k$ are the simple extremal eigenvalues of A, then Proposition 3.5 implies that $\hat\lambda_1, \hat\lambda_2, \ldots, \hat\lambda_k$ are isolated points of the cubic envelope $\mathcal{E}(A)$. Since A is normal, by Property (P_2), and without loss of generality, we may assume that A is diagonal. Then for every $\theta \in [0, 2\pi)$, $e^{\mathrm{i}\,\theta}A$ is also diagonal, and $\delta_1(e^{\mathrm{i}\,\theta}A) + \mathrm{i}\,\mathrm{u}(e^{\mathrm{i}\,\theta}A)$, $\delta_1(e^{\mathrm{i}\,\theta}A)$ and $\mathrm{i}\,\mathrm{u}(e^{\mathrm{i}\,\theta}A)$ are eigenvalues of $e^{\mathrm{i}\,\theta}A$, $H(e^{\mathrm{i}\,\theta}A)$ and $S(e^{\mathrm{i}\,\theta}A)$, respectively, having the same eigenspace. If, in addition, $\delta_1(e^{\mathrm{i}\,\theta}A)$ is a simple eigenvalue of $H(e^{\mathrm{i}\,\theta}A)$, then $\mathrm{v}(e^{\mathrm{i}\,\theta}A) - \mathrm{u}(e^{\mathrm{i}\,\theta}A)^2 = 0$, and the cubic curve $\Gamma(e^{\mathrm{i}\,\theta}A)$ reduces to the union of the point $\delta_1(e^{\mathrm{i}\,\theta}A) + \mathrm{i}\,\mathrm{u}(e^{\mathrm{i}\,\theta}A)$ and the vertical line $\{z \in \mathbb{C} : \mathrm{Re}\, z = \delta_2(e^{\mathrm{i}\,\theta}A)\}$. Otherwise, i.e., when A is normal and $\delta_1(e^{\mathrm{i}\,\theta}A)$ is a multiple eigenvalue of $H(e^{\mathrm{i}\,\theta}A)$, $\Gamma(e^{\mathrm{i}\,\theta}A)$ reduces to the vertical line $\{z \in \mathbb{C} : \mathrm{Re}\, z = \delta_1(e^{\mathrm{i}\,\theta}A)\}$ and $\Gamma_{in}(e^{\mathrm{i}\,\theta}A)$ coincides with the closed half-plane $\mathcal{H}_{in}(e^{\mathrm{i}\,\theta}A, 0)$ (i.e., $e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A)$

coincides with $\mathcal{H}_{in}(A,\theta)$). Hence, recalling (2), we have

$$\begin{aligned}\mathcal{E}(A)\setminus\{\hat\lambda_1,\hat\lambda_2,\dots,\hat\lambda_k\} &= \left(\bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A)\right)\setminus\{\hat\lambda_1,\hat\lambda_2,\dots,\hat\lambda_k\}\\ &= \bigcap_{\theta\in[0,2\pi)}\left\{e^{-\mathrm{i}\,\theta}(s+\mathrm{i}\,t):\, s,t\in\mathbb{R}\ \text{with}\ s\le\delta_2(e^{\mathrm{i}\,\theta}A)\right\}\\ &= \Lambda_2(A),\end{aligned}$$

where $\Lambda_2(A)$ is explicitly described by (3). In [4], it is obtained that if the normal matrix A has m distinct eigenvalues, then $\Lambda_2(A)$ is either an empty set, a singleton, a line segment, or a nondegenerate convex polygon with at most m vertices (which vertices are not necessarily eigenvalues of A), and efficient ways to generate it are proposed.

The above discussion yields directly the following result (see also Corollary 2.3 and Theorem 2.4 of [2]).

Theorem 4.1. *Suppose $A\in\mathbb{C}^{n\times n}$ is a normal matrix, and $\hat\lambda_1,\hat\lambda_2,\dots,\hat\lambda_k$ are (exactly) the simple extremal eigenvalues of A. Then $\Lambda_2(A)\cap\{\hat\lambda_1,\hat\lambda_2,\dots,\hat\lambda_k\}=\emptyset$ and*

$$\mathcal{E}(A) = \Lambda_2(A)\cup\{\hat\lambda_1,\hat\lambda_2,\dots,\hat\lambda_k\}.$$

Corollary 4.2. *Let $A\in\mathbb{C}^{n\times n}$ be a normal matrix.*

(i) *If all the eigenvalues of A are simple and extremal, then $\Lambda_2(A)\cap\sigma(A)=\emptyset$ and $\mathcal{E}(A)=\Lambda_2(A)\cup\sigma(A)$.*

(ii) *If all the extremal eigenvalues of A are multiple, then $\mathcal{E}(A)=\Lambda_2(A)=\mathrm{conv}\{\sigma(A)\}$ $=F(A)$. In particular, for any $a\in\mathbb{C}$, $\mathcal{E}(aI_n)=\Lambda_2(aI_n)=F(aI_n)=\{a\}$.*

(iii) *If $n=2$ or 3, then $\mathcal{E}(A)=\sigma(A)$.*

(iv) *Let $n=4$, and suppose that all the eigenvalues of A are extremal. If all the eigenvalues are simple, then $\mathcal{E}(A)\backslash\Lambda_2(A)=\sigma(A)$ (see (i)) and $\Lambda_2(A)$ is a singleton. If exactly one of the eigenvalues is double, then $\Lambda_2(A)$ coincides with this double eigenvalue and $\mathcal{E}(A)=\sigma(A)$.*

Corollary 4.3. *Let $A\in\mathbb{C}^{n\times n}$ be a hermitian matrix (i.e., $A=H(A)$), with eigenvalues $\delta_1(A)\ge\delta_2(A)\ge\cdots\ge\delta_n(A)$. Then,*

$$\mathcal{E}(A) = \{\delta_n(A)\}\cup[\delta_{n-1}(A),\delta_2(A)]\cup\{\delta_1(A)\}\subseteq[\delta_n(A),\delta_1(A)] = F(A).$$

Example 4.4. Consider the diagonal matrices

$$D_1 = \mathrm{diag}\{1,2,3,4\},\quad D_2 = \mathrm{diag}\{1,1,2,3,4\},$$

$$D_3 = \mathrm{diag}\{\mathrm{i}\,3,5,2+\mathrm{i}\,3,1-\mathrm{i}\,2,-3\}\ \text{ and }\ D_4 = \mathrm{diag}\{\mathrm{i}\,3,\mathrm{i}\,3,5,2+\mathrm{i}\,3,1-\mathrm{i}\,2,3\}.$$

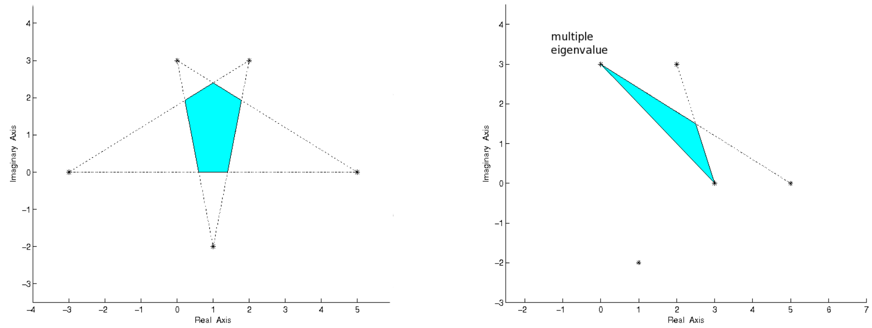

Figure 3: The envelopes of the diagonal matrices D_3 (left) and D_4 (right).

The envelopes of D_1 and D_2 are

$$\mathcal{E}(D_1) = \Lambda_2(D_1) \cup \{1, 4\} = [2, 3] \cup \{1, 4\} \text{ and } \mathcal{E}(D_2) = \Lambda_2(D_2) \cup \{4\} = [1, 3] \cup \{4\},$$

and clearly verify Corollary 4.3. The envelopes of D_3 and D_4 are depicted in the left and right parts of Figure 3, respectively, where the eigenvalues are marked as $*$'s and the dotted lines are auxiliary. The polygons $\Lambda_2(D_3)$ and $\Lambda_2(D_4)$ are shaded, and the envelopes $\mathcal{E}(D_3)$ and $\mathcal{E}(D_4)$ confirm Theorem 4.1; in particular, $\mathcal{E}(D_3)$ illustrates case (i) of Corollary 4.2. Note also that the scalar $\mathrm{i}\,3$ is a multiple eigenvalue of D_4, and as a consequence, the eigenvalue $1 - \mathrm{i}\,2 \in \sigma(D_4)$ does not give rise to an edge of $\Lambda_2(D_4)$.

4.2 Similarity classes

Every Jordan matrix is (diagonally) similar to a bidiagonal matrix with the modulii of its nonzero entries on the super-diagonal arbitrarily small [15, p. 21]. For example, for a $k \times k$ Jordan block associated to a scalar $\lambda \in \mathbb{C}$ and any nonzero $\alpha \in \mathbb{C}$, we have the similarity

$$\begin{bmatrix} \alpha^{-1} & 0 & \dots & 0 \\ 0 & \alpha^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha^{-k} \end{bmatrix} \begin{bmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda \end{bmatrix} \begin{bmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha^{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha^{k} \end{bmatrix} = \begin{bmatrix} \lambda & \alpha & \dots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha \\ 0 & \dots & 0 & \lambda \end{bmatrix}.$$

As a consequence, the continuity of the numerical range of a general matrix $A \in \mathbb{C}^{n\times n}$ with respect to the entries of A yields [5]

$$\bigcap \left\{ F(R^{-1}AR) :\ R \in \mathbb{C}^{n\times n},\ \det(R) \neq 0 \right\} = \operatorname{conv}\{\sigma(A)\}. \tag{7}$$

An analogous result holds for the envelope.

Theorem 4.5. *For any matrix $A \in \mathbb{C}^{n\times n}$, we have*

$$\bigcap \left\{ \Gamma_{in}(R^{-1}AR) :\ R \in \mathbb{C}^{n\times n},\ \det(R) \neq 0 \right\} \subseteq \Gamma_{in}(D(A)) \tag{8}$$

and

$$\bigcap \left\{\mathcal{E}(R^{-1}AR) :\, R\in\mathbb{C}^{n\times n},\, \det(R)\neq 0\right\} \subseteq \mathcal{E}(D(A)). \tag{9}$$

Proof. If A is diagonalizable, then there exists a nonsingular matrix $R\in\mathbb{C}^{n\times n}$ such that $R^{-1}AR = D(A)$, and the result is apparent.

Suppose that A is not diagonalizable. By the definition of the cubic envelope, we have

$$\begin{aligned}
&\bigcap \left\{\mathcal{E}(R^{-1}AR) :\, R\in\mathbb{C}^{n\times n},\, \det(R)\neq 0\right\}\\
&= \bigcap \left\{e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}R^{-1}AR) :\, R\in\mathbb{C}^{n\times n},\, \det(R)\neq 0,\, \theta\in[0,2\pi)\right\}\\
&= \bigcap_{\theta\in[0,2\pi)} \bigcap \left\{e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}R^{-1}AR) :\, R\in\mathbb{C}^{n\times n},\, \det(R)\neq 0\right\}
\end{aligned}$$

and

$$\mathcal{E}(D(A)) = \bigcap \left\{e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}D(A)) :\, \theta\in[0,2\pi)\right\}.$$

Thus, it is enough to prove the first inclusion relation of the theorem. In particular, we consider a scalar $\mu\in\mathbb{C}\setminus\Gamma_{in}(D(A))$, and we will verify that $\mu\notin\Gamma_{in}(R^{-1}AR)$ for some nonsingular $R\in\mathbb{C}^{n\times n}$.

If $\delta_1(D(A))$ is a simple eigenvalue of $H(D(A))$, then μ lies to the right of the vertical line $\{z\in\mathbb{C} :\, \mathrm{Re}\,z = \delta_2(D(A))\}$, where $\delta_2(D(A))$ coincides with the real part of the second right most eigenvalue of $D(A)$ and A. Moreover, μ is different than $\delta_1(D(A)) + \mathrm{i}\,\mathrm{u}(D(A))$ (that is, the right most eigenvalue of $D(A)$ and A). As mentioned above (see also the discussion in [15, p. 21]), for any $\varepsilon>0$, there is a nonsingular $R_\varepsilon\in\mathbb{C}^{n\times n}$ such that $R_\varepsilon^{-1}AR_\varepsilon$ is a bidiagonal matrix with the modulii of its nonzero entries on the super-diagonal less than or equal to ε. The continuity of eigenvalues, eigenvectors and norms as functions of the matrix entries implies that for sufficiently small $\varepsilon>0$, $\mathrm{v}(R_\varepsilon^{-1}AR_\varepsilon) - \mathrm{u}(R_\varepsilon^{-1}AR_\varepsilon)^2$ and $|\delta_2(R_\varepsilon^{-1}AR_\varepsilon) - \delta_2(D(A))|$ can be arbitrarily small. As a consequence, the curve $\Gamma(R_\varepsilon^{-1}AR_\varepsilon)$ can be assumed to be disconnected, with its unbounded open branch arbitrarily close to the vertical line $\{z\in\mathbb{C} :\, \mathrm{Re}\,z = \delta_2(D(A))\}$ and its bounded closed branch arbitrarily close to the singleton $\{\delta_1(D(A)) + \mathrm{i}\,\mathrm{u}(D(A))\}$. Hence, there is a nonsingular $R\in\mathbb{C}^{n\times n}$ such that $\mu\notin\Gamma_{in}(R^{-1}AR)$.

If $\delta_1(D(A))$ is a multiple eigenvalue of $H(D(A))$, then μ lies to the right of the vertical line $\{z\in\mathbb{C} :\, \mathrm{Re}\,z = \delta_1(D(A))\}$. We can now apply the above continuity arguments to obtain that for appropriate nonsingular $R\in\mathbb{C}^{n\times n}$, the real numbers $\delta_1(D(A))$, $\delta_1(H(R^{-1}AR))$ and $\delta_2(H(R^{-1}AR))$ can be arbitrarily close. As a consequence, $\Gamma(R^{-1}AR)$ can be arbitrarily close to $\Gamma(D(A)) = \{z\in\mathbb{C} :\, \mathrm{Re}\,z = \delta_1(D(A))\}$, and the proof is complete. $\square$

The inclusion relation (9) in the above theorem and the equality (7) yield the following corollary.

Corollary 4.6. *If $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_k$ are the simple extremal eigenvalues of $A \in \mathbb{C}^{n\times n}$, then*

$$\begin{aligned}\sigma(A) &\subseteq \bigcap \big\{\mathcal{E}(R^{-1}AR) : R \in \mathbb{C}^{n\times n}, \det(R) \neq 0\big\} \\ &\subseteq \mathcal{E}(D(A)) \;=\; \Lambda_2(D(A)) \cup \{\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_k\} \\ &\subseteq \operatorname{conv}\{\sigma(A)\} \;=\; \bigcap \big\{F(R^{-1}AR) : R \in \mathbb{C}^{n\times n}, \det(R) \neq 0\big\} \\ &\subseteq F(A).\end{aligned}$$

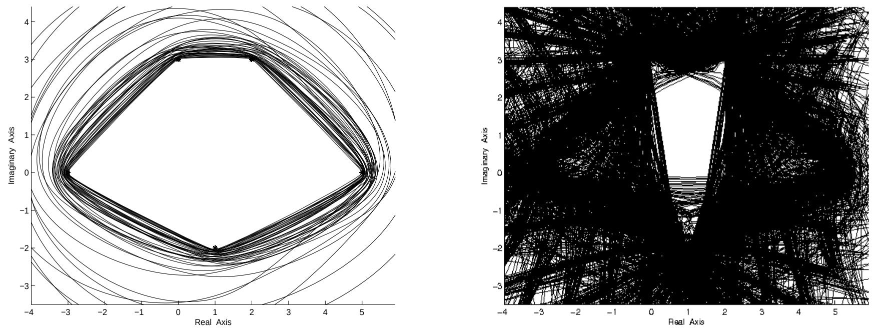

Figure 4: Numerical ranges (left) and envelopes (right) of similar matrices.

Example 4.7. Recall the diagonal matrix $D_3 = \operatorname{diag}\{\mathrm{i}\,3, 5, 2+\mathrm{i}\,3, 1-\mathrm{i}\,2, -3\}$ in Example 4.4 and the envelope $\mathcal{E}(D_3)$ in the right part of Figure 3. In Figure 4, the numerical ranges (left part) and the envelopes (right part) of 60 randomly chosen matrices similar to D_3 are depicted. The unshaded region in the left part of the figure is an estimation of $\operatorname{conv}\{\sigma(D_3)\}$ (the eigenvalues are marked as $*$'s), confirming (7), and the unshaded region in the left part is an estimation of the polygon $\Lambda_2(D_3)$, verifying Theorem 4.5. As expected, since all the eigenvalues are extremal, they are not visible in the right part of Figure 4.

Remark 4.8. One can easily construct a non-normal matrix $A \in \mathbb{C}^{n\times n}$ such that the curve $\Gamma(A)$ comprises two branches and $\delta_2(A)$ is an eigenvalue of $H(A)$ of algebraic multiplicity $n-1$ (see [12, Example 3.4]). Then $\delta_2(A) < \delta_1(A)$, and at least two eigenvalues of A lie in the interior of the numerical range $F(A)$ [6, Theorem 1.6.6] and have their real parts lying in the open (real) interval $(\delta_2(A), \delta_1(A))$. As a consequence, the real part of at least one eigenvalue of A lies in the interval $\Big(\delta_2(A), (\delta_1(A)+\delta_2(A)-\sqrt{\Delta}\,)/2\Big]$, and thus, $\delta_2(D(A)) > \delta_2(A)$. Hence, $\Gamma_{in}(D(A)) \nsubseteq \Gamma_{in}(A)$, and we conclude that the inclusion relation (8) cannot be replaced by equality. On the other hand, it is not known whether the inclusion relation (9) for the envelope always holds as an equality or not.

5 On the eigenvectors of the right most eigenvalues

Perhaps the most interesting configuration of the curve $\Gamma(A)$ is when it consists of two branches. The closed branch must then contain a simple eigenvalue λ_1 of matrix A, and thus, it forces the envelope $\mathcal{E}(A)$ to have a connected component that contains λ_1. In this section, we examine the relation (angle) among the eigenvectors corresponding to λ_1 and the right most eigenvalue of the hermitian part.

For any two vectors $x, y \in \mathbb{C}^n$, consider the (real) cosine of their angle given by

$$\cos(\widehat{x,y}) \;=\; \frac{|y^*x|}{\|x\|_2\|y\|_2}.$$

Note that this definition ignores the direction of the vectors and describes the (acute) angle between the one-dimensional subspaces $\mathrm{span}\{x\}$ and $\mathrm{span}\{y\}$.

Consider now a matrix $A \in \mathbb{C}^{n\times n}$ with the discriminant $\Delta = (\delta_1(A) - \delta_2(A))^2 - 4(\mathrm{v}(A) - \mathrm{u}(A)^2)$ being positive. By Theorem 3.2 of [14], the cubic curve $\Gamma(A)$ has a closed branch, and exactly one eigenvalue of A which (is simple and) lies inside or on this closed branch of $\Gamma(A)$.

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$ be such that $\Delta > 0$, and let $\delta_1(A)$ be a simple eigenvalue of $H(A)$ with an associated unit eigenvector $y_1 \in \mathbb{C}^n$. Let also λ_1 be the simple eigenvalue of A that lies inside or on the closed branch of $\Gamma(A)$ (i.e., λ_1 is the right most eigenvalue of A), and assume that $\mathrm{Re}\,\lambda_1 \neq \delta_1(A)$. Then, for any unit eigenvector $x_1 \in \mathbb{C}^n$ of A corresponding to the eigenvalue λ_1,*

$$\cos(\widehat{x_1,y_1}) \;\geq\; \sqrt{\frac{1}{2} + \frac{\sqrt{(\delta_1(A)-\delta_2(A))^2 - 4(\mathrm{v}(A)-\mathrm{u}(A)^2)}}{2(\delta_1(A)-\delta_2(A))}} \;\geq\; \frac{\sqrt{2}}{2} \;=\; \cos\left(\frac{\pi}{4}\right).$$

Proof. Let $x_1 \in \mathbb{C}^n$ be a unit eigenvector of A corresponding to $\lambda_1 \in \sigma(A)$. This vector is written in the form $x_1 = \hat{y}_1 + v_1$, where $\hat{y}_1 \in \mathrm{span}\{y_1\}$ (i.e., $\hat{y}_1$ is an eigenvector of $H(A)$ corresponding to $\delta_1(A)$) and v_1 lies in the orthogonal complement of $\mathrm{span}\{y_1\}$, $\mathrm{span}\{y_1\}^{\perp}$. Since x_1 is unit, it follows

$$\begin{aligned}
\lambda_1 &= x_1^*Ax_1 \\
&= (\hat{y}_1+v_1)^*H(A)(\hat{y}_1+v_1) + x_1^*K(A)x_1 \\
&= v_1^*H(A)v_1 + \delta_1(A)\|\hat{y}_1\|_2^2 + x_1^*K(A)x_1,
\end{aligned}$$

where $\delta_n(A)(1-\|\hat{y}_1\|_2^2) \leq v_1^*H(A)v_1 \leq \delta_2(A)(1-\|\hat{y}_1\|_2^2)$.

It is clear that $\mathrm{Re}\,\lambda_1 = v_1^*H(A)v_1 + \delta_1(A)\|\hat{y}_1\|_2^2$ (for example, see [6]), and hence,

$$\delta_n(A)(1-\|\hat{y}_1\|_2^2) + \delta_1(A)\|\hat{y}_1\|_2^2 \;\leq\; \mathrm{Re}\,\lambda_1 \;\leq\; \delta_2(A)(1-\|\hat{y}_1\|_2^2) + \delta_1(A)\|\hat{y}_1\|_2^2.$$

Recall also that since $\Delta > 0$, we have

$$\frac{\delta_1(A)+\delta_2(A)+\sqrt{\Delta}}{2} \;\leq\; \mathrm{Re}\,\lambda_1 \;\leq\; \delta_1(A).$$

As a consequence,

$$\frac{\delta_1(A) + \delta_2(A) + \sqrt{\Delta}}{2} \leq \delta_2(A)(1 - \|\hat{y}_1\|_2^2) + \delta_1(A)\|\hat{y}_1\|_2^2,$$

or equivalently,

$$\sqrt{(\delta_1(A) - \delta_2(A))^2 - 4(\mathrm{v}(A) - \mathrm{u}(A)^2)} \leq (\delta_1(A) - \delta_2(A))(2\|\hat{y}_1\|_2^2 - 1).$$

Hence, it follows that

$$\|\hat{y}_1\|_2 \geq \sqrt{\frac{1}{2} + \frac{\sqrt{(\delta_1(A) - \delta_2(A))^2 - 4(\mathrm{v}(A) - \mathrm{u}(A)^2)}}{2(\delta_1(A) - \delta_2(A))}} \geq \frac{\sqrt{2}}{2}.$$

The proof is completed by observing that $\|\hat{y}_1\|_2 = |y_1^* x_1| = \cos(\widehat{x_1, y_1})$. $\square$

6 Conclusions

The envelope of a matrix A is an infinite intersection of regions defined by cubic curves and it contains the eigenvalues of A. In this paper, we proved that the envelope can indeed be approximated to arbitrary precision by a finite number of intersections, thus justifying our methodology to visually render the envelope. Since the envelope is typically neither convex nor connected, it is important to understand the properties of eigenvalues that are either isolated points of the envelope or are contained in its connected components. In this respect, we studied the geometry of the envelope in the fundamental case of normal matrices and under similarities. We also examined the angle among the eigenvector of the rightmost eigenvalue of A contained in a connected component of the envelope and the eigenvector of the largest eigenvalue of the hermitian part. Finally, we established that the envelope contains the 2-rank numerical range of A. Illustrative examples were provided, along with access to a Matlab function for rendering the envelope of a matrix.

References

[1] M. Adam and M.J. Tsatsomeros, An eigenvalue inequality and spectrum localization for complex matrices, *Electron. J. Linear Algebra*, 15 (2006) 239–250.

[2] M.D. Choi, D.W. Kribs and K. Życzkowski, Higher-rank numerical ranges and compression problems, *Linear Algebra Appl.*, 418 (2006) 828–839.

[3] M.D. Choi, D.W. Kribs and K. Życzkowski, Quantum error correcting codes from the compression formalism, *Rep. Math. Phys.*, 58 (2006) 77–91.

[4] H.-L. Gau, C.-K. Li, Y.-T. Poon and N.-S. Sze, Higher rank numerical range of normal matrices, *SIAM J. Matrix Anal. Appl.*, 32 (2011) 23–43.

[5] W. Givens, Field of values of a matrix, *Proc. Amer. Math. Soc.*, 3 (1952) 206–209.

[6] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[7] C.R. Johnson, Numerical determination of the field of values of a general complex matrix, *SIAM J. Numer. Anal.*, 15 (1978) 595–602.

[8] D.W. Kribs, R. Laflamme, D. Poulin and M. Lesosky, Operator quantum error correction, *Quantum Inf. Comput.*, 6 (2006) 383–399.

[9] H.W.J. Lenferink and M.N. Spijker, A generalization of the numerical ranges of a matrix, *Linear Algebra Appl.*, 140 (1990) 251–266.

[10] C.K. Li, Y.T. Poon and N.S. Sze, Condition for the higher rank numerical range to be non-empty, *Linear Multilinear Algebra*, 57 (2009) 365–368.

[11] C.K. Li and N.S. Sze, Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations, *Proc. Amer. Math. Soc.*, 136 (2008) 3013–3023.

[12] J.J. McDonald, P.J. Psarrakos and M.J. Tsatsomeros, Almost skew-symmetric matrices, *Rocky Mountain J. Math.*, 34 (2004) 269–288.

[13] J.S. Milne, *Elliptic Curves*, BookSurge Publishers, Charleston, 2006.

[14] P.J. Psarrakos and M.J. Tsatsomeros, An envelope for the spectrum of a matrix, *Cent. Eur. J. Math.*, 10 (2012) 292–302.

[15] G.W. Stewart and J.-G. Sun, *Matrix Perturbation Theory*, Academic Press, New York, 1990.

[16] H. Woerdeman, The higher rank numerical range is convex, *Linear Multilinear Algebra*, 56 (2008) 65–67.