Finite Dimensionality of a Klein - Gordon - Schrödinger Type System *

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Abstract

In this paper we study the finite dimensionality of the global attractor for the following system of Klein-Gordon-Schrödinger type

$$\begin{aligned} i\psi_t + \kappa\psi_{xx} + i\alpha\psi &= \phi\psi + f, \\ \phi_{tt} - \phi_{xx} + \phi + \lambda\phi_t &= -Re\psi_x + g, \\ \psi(x,0) = \psi_0(x),\ \phi(x,0) &= \phi_0(x),\ \phi_t(x,0) = \phi_1(x), \\ \psi(x,t) = \phi(x,t) = 0, &\qquad x \in \partial\Omega,\ \ t > 0, \end{aligned}$$

where $x \in \Omega,\ t > 0,\ \kappa > 0,\ \alpha > 0,\ \lambda > 0,\ f$ and g are driving terms and Ω is a bounded interval of $\mathbb{R}$. With the help of the Lyapunov exponents we give an estimate of the upper bound of its Hausdorff and Fractal dimension.

1 Introduction

The aim of this paper is to prove the finite dimensionality of the global attractor for the following Klein-Gordon-Schrödinger type system

$$i\psi_t + \kappa\psi_{xx} + i\alpha\psi = \phi\psi + f, \tag{1.1}$$

$$\phi_{tt} - \phi_{xx} + \phi + \lambda\phi_t = -Re\psi_x + g, \tag{1.2}$$

$$\psi(x,0) = \psi_0(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), \tag{1.3}$$

$$\psi(x,t) = \phi(x,t) = 0, \qquad x \in \partial\Omega,\ \ t > 0, \tag{1.4}$$

*Key Phrases: Klein-Gordon-Schrödinger System; Global Attractor; Absorbing Set; Lyapunov Exponents; Hausdorff and Fractal Dimension.

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where $x \in \Omega,\ t > 0,\ \kappa > 0,\ \alpha > 0,\ \lambda > 0,\ \Omega$ is a bounded interval of $\mathbb{R}$ and f and g are the driving terms. The variable ψ stands for the dimensionless low frequency electron field, whereas the (real) variable ϕ denotes the dimensionless low frequency density. System (1.1)-(1.4) describes the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field, adapted to model the UHH plasma heating scheme. This modeling process appeared for first time in the work [12] (see also [15]), where for the undriven case $(f \equiv 0,\ g \equiv 0)$ the global existence and uniqueness of the solutions were proved and necessary conditions were established for the system to manifest exponential energy decay. These results were extended by the authors to the more realistic driven case system (1.1) - (1.4) (see, [13]), where the driving terms $f, g \in L^2(\Omega)$. Here the existence of a global attractor is derived in the space $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$, which attracts all bounded sets of $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$ in the norm topology. To this end, some useful estimates on the solutions of the system (1.1) - (1.4) are derived in $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$. Then, based on a method first introduced by John Ball in [1], the continuous dependence of the solutions on the initial data in the space $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$ is proved and the asymptotic compactness of the dynamical system is shown. Finally, the existence of a global attractor is established.

Xanthopoulos and Zouraris in a recently published paper (see [16]) propose a linearly implicit finite difference method to approximate the solution of the system (1.1) - (1.4), the convergence of which is ensured by deriving a second order error estimate in a discrete energy norm that is stronger than the discrete maximum norm. The numerical implementation of the method gives a computational confirmation of its order of convergence and recovers known theoretical results for the behavior of the solution, while revealing additional nonlinear features.

The finite dimensionality of an attractor has been extensively studied. The dimension of an attractor is one of the few mathematical information one may have on the geometry of such (invariant) sets. While on the physical and numerical side, the dimension gives an idea of the necessary number of degrees of freedom of a system and therefore the size of the computations needed for its numerical simulation.

To estimate upper bounds for the Hausdorff and Fractal dimensions various properties of local and global Lyapunov exponents are used. Both of these notions where implemented in the study of the Hausdorff dimension of a global attractor for the 2D Navier-Stokes equations (see [2], [3]). Later on, global Lyapunov exponents became the standard tools to study attractors for the dissipative partial differential equations, see [14]. The global Lyapunov exponents measure, uniformly over the attractor, the exponential rate of change of solutions with respect to time. In [6] the authors successively replaced the global Lyapunov exponents with the Local ones which measure the exponential rate of the growth along a single trajectory which simplifies the task of estimating the dimension of an attractor.

General results on the dimension of attractors were proved in [8] for a weakly damped driven Schrödinger one-dimensional equation defined in Ω (bounded) $\subset \mathbb{R}$. Later the authors of [9] followed similar arguments with the ones introduced in [3] proving the finite dimensionality of a global attractor for a nonlinear wave equation. Based on the results of [3] the authors of [10] derive an upper bound for the dimension of an attractor for the Navier-Stokes equation in space dimension three as well as improve previous results on the lower and upper bounds for the two-dimensional case. In [7] the long time behaviour of solutions for a Zakharov system in a bounded domain is studied and by using the linearized flow and Lyapunov exponents, the existence of a weak global attractor with a finite fractal and Hausdorff dimension is proved. A later work by [11] improves the previous results by showing the existence of a strong global attractor equivalent to the weak one.

The rest of the paper is divided into three parts. In Section 2, we study the linearization of the system (1.1)-(1.4) and obtain the energy estimates necessary to find bounds for the dimension of the attractor in the space $\mathcal{E}_1 = (H_0^1(\Omega) \cap H^2(\Omega))^2(\Omega) \times H_0^1(\Omega)$. In Section 3, we apply the general method based on the uniform Lyapunov exponents and find upper bounds for the Hausdorff and Fractal dimension of the attractor X_1 in $\mathcal{E}_1$. Finally, taking into consideration the results of [11] and [13] we prove that $X_1^w = X_1$ Two main questions are raised a) is there any relation between the dimensions of the two sets X_1^w and X_1? and b) is it possible to estimate a positive lower bound for the dimensions of the attractors X_1^w and/or X_1?

***Notation*:** Denote by H^s both the standard real and complex Sobolev spaces. For *simplicity reasons* sometimes we use H^s, L^s for $H^s(\Omega)$, $L^s(\Omega)$ and $||.||$, $(.,.)$ for the norm and the inner product of $L^2(\Omega)$, respectively. $\int dx$ denotes the integration over the domain Ω. Finally, C is a general symbol for any positive constant.

2 Energy Estimates

Let A be the unbounded linear operator defined by

$$Au = -u_{xx}, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Since the embedding of $D(A)$ into $L^2(\Omega)$ is compact, A^{-1} is a compact, self adjoint operator on $L^2(\Omega)$. Therefore there exists a Hilbert basis on $L^2(\Omega)$ made of eigenvectors of A. Let $\{\lambda_i\}_{i=0}^{\infty}$ denote the nondecreasing sequence of eigenvalues counting multiplicities

$$0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_j \leqq \ldots \rightarrow +\infty, \quad as\ j \rightarrow \infty,$$

with corresponding orthonormalized eigenvectors $\{w_j\}_{j=0}^{\infty}$. The following product spaces will be proved useful

$$\mathcal{E}_0 = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega),$$

$$\mathcal{E}_1 = (H_0^1(\Omega) \cap H^2(\Omega))^2(\Omega) \times H_0^1(\Omega).$$

It is well known that the embedding $\mathcal{E}_1 \hookrightarrow \mathcal{E}_0$ is compact.

Let us introduce the transformation $\theta = \phi_t + \delta\phi$, where θ is real and δ a small positive constant to be specified later. Then, the system (1.1)-(1.2) takes the form

$$i\psi_t + \kappa\psi_{xx} + i\alpha\psi = \phi\psi + f, \tag{2.1}$$

$$\phi_t + \delta\phi = \theta, \tag{2.2}$$

$$\theta_t + (\lambda - \delta)\theta - \phi_{xx} + (1 - \delta(\lambda - \delta))\phi = -Re\psi_x + g. \tag{2.3}$$

Also the initial and boundary conditions (1.3)-(1.4) become

$$\psi(x,0) = \psi_0(x),\ \phi(x,0) = \phi_0(x),\ \theta(x,0) = \theta_0(x), \quad x \in \Omega, \tag{2.4}$$

$$\psi(x,t) = \phi(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{2.5}$$

The linearization of (2.1)-(2.3) is the following system

$$iv_t + \kappa v_{xx} + i\alpha v = u\psi + v\phi, \tag{2.6}$$

$$u_t + \delta u = F, \tag{2.7}$$

$$F_t + (\lambda - \delta)F - u_{xx} + (1 - \delta(\lambda - \delta))u = -Re\ v_x, \tag{2.8}$$

where $(\psi, \phi, \theta) = S(t)(\psi_0, \phi_0, \theta_0)$, with $(\psi_0, \phi_0, \theta_0) \in \mathcal{E}_1$, $(v_0, u_0, F_0) \in \mathcal{E}_1$ and $S(t)$ is defined as in [13]. Since $(\psi, \phi, \theta) \in L^\infty(\mathbb{R}_+; \mathcal{E}_1)$, one may prove that (v, u, F) admits a unique solution in $L^\infty(\mathbb{R}_+; \mathcal{E}_1)$ (see [13]).

Let $(v(t), u(t), F(t)) = DS(t)(\psi_0, \phi_0, \theta_0)(v_0, u_0, F_0)$, where $DS(t)(\psi_0, \phi_0, \theta_0)$ is the differential of $S(t)$ at the point $(\psi_0, \phi_0, \theta_0)$. The following lemma shows that $S(t)$ is uniformly differentiable on bounded sets of $\mathcal{E}_1$. This fact is important for the proof of the finite dimensionality of the global attractor. The proof of the lemma may be omitted as it follows the same techniques as in [[13], Lemma 3.3]

Lemma 2.1 *Let* R, T *be two positive constants. Then there exists a constant* $C(R,T)$ *such that for every* $(\psi_0^i, \phi_0^i, \theta_0^i)$ *with*

$$||(\psi_0^i, \phi_0^i, \theta_0^i)||_{\mathcal{E}_1} \leq R, \quad i = 1,2 \text{ for every } |t| \leq T,$$

we have

$$\begin{aligned} &||S(t)(\psi_0^2, \phi_0^2, \theta_0^2) - S(t)(\psi_0^1, \phi_0^1, \theta_0^1) - DS(t)(\psi_0^1, \phi_0^1, \theta_0^1)(\psi_0, \phi_0, \theta_0)||_{\mathcal{E}_1} \\ &\qquad \leq C||(\psi_0, \phi_0, \theta_0)||^2_{\mathcal{E}_1}, \end{aligned} \tag{2.9}$$

where

$$\psi_0 = \psi_0^2 - \psi_0^1, \ \phi_0 \ = \phi_0^2 - \phi_0^1, \ \theta_0 = \theta_0^2 - \theta_0^1.$$

It will be convenient to rewrite the system (2.6)-(2.8) by using the following change of variables

$$p = e^{\sigma t} v, \quad q = e^{\sigma t} u, \quad G = e^{\sigma t} F.$$

Then the linearized system (2.6)-(2.8) can be rewritten in the form

$$ip_t + \kappa p_{xx} + i(\alpha - \sigma)p = q\psi + p\phi, \tag{2.10}$$

$$q_t + (\delta - \sigma)q = G, \tag{2.11}$$

$$G_t + (\lambda - \sigma - \delta)G - q_{xx} + (1 - \delta(\lambda - \delta))q = -Rep_x. \tag{2.12}$$

Let μ, ν be positive constants large enough to be fixed later in the proof (see relation 2.29). Then the following *energy equivalence* is of basic importance for the continuation of the discussion

Proposition 2.2 *Let* α, δ, λ *satisfy the following relations*

$$\delta - 2\kappa\alpha < 0, \ \ 3\delta - 2\lambda < 0, \ \ 1 - \delta(\lambda - \delta) > 0, \tag{2.13}$$

and introduce a new parameter σ *such that* $\sigma < min(\alpha/2, \delta/4)$. *Then the following energy estimates are valid*

$$min(\frac{\kappa^2}{2}, \frac{\nu}{2})(||p_{xx}||^2 + ||G_x||^2 + ||q_{xx}||^2) \leq Q(t) \leq max(\mu, \nu)(||p_{xx}||^2 + ||G_x||^2 + ||q_{xx}||^2),$$

where

$$\begin{aligned} Q(t) = \kappa^2 ||p_{xx}||^2 - 2\kappa Re \int q\psi \bar{p}_{xx} - 2\kappa Re \int p\phi \bar{p}_{xx} + \frac{\nu}{2}||G_x||^2 + \frac{\nu}{2}||q_{xx}||^2 \\ + \nu\frac{\delta}{8}||q_x||^2 + \frac{\mu}{2}||p||^2. \end{aligned}$$

Proof Take in L^2 the scalar product of equation (2.10) by $\bar{p}$. The imaginary part of the resulting relation forms the *first energy equation*

$$\frac{1}{2}\frac{d}{dt}||p||^2 + (\alpha - \sigma)||p||^2 = Im \int q\psi\bar{p}. \tag{2.14}$$

Next, multiplying equation (2.10) by $\bar{p}_{xx,t} + \alpha\bar{p}_{xx}$ in L^2 and taking the real part we have the *second energy equation*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\kappa||p_{xx}||^2 + \kappa\alpha||p_{xx}||^2 &= Re\int q\psi\bar{p}_{xx,t} + \alpha Re\int q\psi\bar{p}_{xx} + Re\int p\phi\bar{p}_{xx,t} \\ &\quad + \alpha Re\int p\phi\bar{p}_{xx}.\end{aligned} \tag{2.15}$$

But

$$\frac{d}{dt}\int q\psi\bar{p}_{xx} = \int q_t\psi\bar{p}_{xx} + \int q\psi_t\bar{p}_{xx} + \int q\psi\bar{p}_{xx,t}. \tag{2.16}$$

Substitution of the above relation (2.16) into equation (2.15) produces

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\kappa||p_{xx}||^2 + \kappa\alpha||p_{xx}||^2 = \frac{d}{dt}\int q\psi\bar{p}_{xx} - \int q_t\psi\bar{p}_{xx} - \int q\psi_t\bar{p}_{xx} \\ + \alpha Re\int q\psi\bar{p}_{xx} + Re\int p\phi\bar{p}_{xx,t} + \alpha Re\int p\phi\bar{p}_{xx}.\end{aligned} \tag{2.17}$$

Also, we have

$$\kappa\int G\psi\bar{p}_{xx} = \int qG|\psi|^2 + \int G\psi p\phi$$

and

$$\frac{d}{dt}\int p\phi\bar{p}_{xx} = \int p_t\phi\bar{p}_{xx} + \int p\phi_t\bar{p}_{xx} + \int p\phi\bar{p}_{xx,t}. \tag{2.18}$$

Now,

$$\int p_t\phi\bar{p}_{xx} = (\alpha - \sigma)Re\int p\phi\bar{p}_{xx} - Im\int q\psi\phi\bar{p}_{xx} - Im\int p|\phi|^2\bar{p}_{xx}. \tag{2.19}$$

Hence relation (2.15) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\kappa^2||p_{xx}||^2 - 2\kappa Re\int q\psi\bar{p}_{xx} - 2\kappa Re\int p\phi\bar{p}_{xx}\right) + \kappa^2\alpha||p_{xx}||^2 \\ &\quad = \kappa(2\alpha - \sigma)Re\int p\phi\bar{p}_{xx} + \kappa(\alpha + \delta)Re\int q\psi\bar{p}_{xx} + \kappa Re\int q\psi_t\bar{p}_{xx} \\ &\quad - \kappa Re\int |\psi|^2Gq - \kappa Re\int G\psi p\phi - \kappa Im\int q\psi\phi\bar{p}_{xx} \\ &\quad - \kappa Im\int p|\phi|^2\bar{p}_{xx} + \kappa Re\int p\phi_t\bar{p}_{xx}.\end{aligned} \tag{2.20}$$

To proceed, we multiply relation (2.12) by $-G_{xx}$ and integrate to obtain the *third energy equation*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(||G_x||^2+||q_{xx}||^2+\frac{\delta}{4}||q_x||^2\right)&+(\lambda-\sigma-\delta)||G_x||^2\\ &+(\delta-\sigma)||q_{xx}||^2+\frac{\delta}{4}(\delta-\sigma)||q_x||^2\le -Re\int p_{xx}G_x.\end{aligned}\tag{2.21}$$

Combination of the energy estimates in the following way

$$\mu\times(2.14)+2\times(2.20)+\nu\times(2.21),$$

leads to the definition of the energy functional

$$\begin{aligned}Q(t)=&\kappa^2||p_{xx}||^2-2\kappa Re\int q\psi\bar{p}_{xx}-2\kappa Re\int p\phi\bar{p}_{xx}+\frac{\nu}{2}||G_x||^2+\frac{\nu}{2}||q_{xx}||^2\\ &+\nu\frac{\delta}{8}||q_x||^2+\frac{\mu}{2}||p||^2,\end{aligned}$$

which satisfies

$$\begin{aligned}\frac{d}{dt}\,Q(t)+2\kappa^2\alpha||p_{xx}||^2&+\nu(\lambda-\sigma-\delta)||G_x||^2+\nu(\delta-\sigma)||q_{xx}||^2\\ &+\nu\frac{\delta}{4}(\delta-\sigma)||q_x||^2+\mu(\alpha-\sigma)||p||^2\le A,\end{aligned}\tag{2.22}$$

where

$$\begin{aligned}A=:&\ 2\kappa(2\alpha-\sigma)Re\int p\phi\bar{p}_{xx}+2\kappa(\alpha+\delta)Re\int q\psi\bar{p}_{xx}+2\kappa Re\int q\psi_t\bar{p}_{xx}\\ &-2\kappa Re\int|\psi|^2Gq-2\kappa Re\int G\psi p\phi-2\kappa Im\int q\psi\phi\bar{p}_{xx}\\ &-2\kappa Im\int p|\phi|^2\bar{p}_{xx}+2\kappa Re\int p\phi_t\bar{p}_{xx}-\nu\ Re\int p_{xx}G_x+\mu\ Im\int q\psi\bar{p}.\end{aligned}\tag{2.23}$$

Majorization of A produces the following inequality

$$\begin{aligned}A\le&2|2\alpha-\sigma|\left|\int p\phi\bar{p}_{xx}\right|+2|\alpha+\delta|\left|\int q\psi\bar{p}_{xx}\right|+2\left|\int q\psi_t\bar{p}_{xx}\right|\\ &-2\left|\int|\psi|^2Gq\right|-2\left|\int G\psi p\phi\right|-2\left|\int q\psi\phi\bar{p}_{xx}\right|-2\left|\int p|\phi|^2\bar{p}_{xx}\right|\\ &+2\left|\int p\phi_t\bar{p}_{xx}\right|-|\nu|\left|\int p_{xx}G_x\right|+|\mu|\left|\int q\psi\bar{p}\right|.\end{aligned}\tag{2.24}$$

A further mazorization of A is obtained by applying Hölder inequality in the right hand side of the relation (2.24)

$$\begin{aligned} A \leq & 2\kappa|2\alpha-\sigma|\ ||p||\ ||\phi||_\infty\ ||p_{xx}|| + 2\kappa|\alpha+\delta|\ ||q||\ ||\psi||_\infty\ ||p_{xx}|| \\ & + 2\kappa||q||\ ||\psi_t||\ ||p_{xx}|| - 2\kappa||\psi||_\infty^2\ ||G||\ ||q|| - 2\kappa||\psi||_\infty\ ||\phi||_\infty\ ||G||\ ||p|| \\ & - 2\kappa||\psi||_\infty\ ||\phi||_\infty\ ||q||\ ||p_{xx}|| - 2\kappa||\phi||_\infty^2\ ||p||\ ||p_{xx}|| \\ & + 2\kappa||\phi_t||\ ||p||\ ||p_{xx}|| - |\nu|\ ||p_{xx}||\ ||G_x|| + |\mu|\ ||q||\ ||\psi||_\infty\ ||p||, \end{aligned} \tag{2.25}$$

where the integrals in (2.24) have been evaluated as follows

$$\begin{aligned} \left|\int p\phi\bar{p}_{xx}\right| &\leq ||p||\ ||\phi||_\infty\ ||p_{xx}|| \leq \frac{c^2}{2}||p_x||^2||\phi||_\infty^2 + \frac{1}{2}||p_{xx}||^2, \\ \left|\int q\psi\bar{p}_{xx}\right| &\leq ||q||\ ||\psi||_\infty\ ||p_{xx}|| \leq c||q_x||\ ||\psi||_\infty\ ||p_{xx}|| \leq \frac{1}{2}||p_{xx}||^2 + \frac{c^2}{2}||q_x||^2||\psi||_\infty^2, \\ \left|\int |\psi|^2 Gq\right| &\leq ||q||\ ||\psi||_\infty^2\ ||G|| \leq c||q_x||\ ||\psi||_\infty^2\ ||G_x|| \leq \frac{c^2}{2}||q_x||^2\ ||\psi||_\infty^4 + \frac{1}{2}||G_x||^2, \\ \left|\int G\psi p\phi\right| &\leq c||\phi||_\infty||\psi||_\infty\ ||G_x||\ ||p|| \leq \frac{1}{2}||G_x||^2 + \frac{c^2}{2}||\phi||_\infty^2||\psi||_\infty^2||p_x||^2, \\ \left|\int q\psi_t\bar{p}_{xx}\right| &\leq c||q_x||\ ||\psi_t||_\infty\ ||p_{xx}|| \leq \frac{c^2}{2}||q_x||^2\ ||\psi_t||_\infty^2 + \frac{1}{2}||p_{xx}||^2, \\ \left|\int q\psi\phi\bar{p}_{xx}\right| &\leq ||\phi||_\infty\ ||\psi||_\infty\ ||q_x||\ ||p_{xx}|| \leq \frac{1}{2}||p_{xx}||^2 + \frac{c^2}{2}||\phi||_\infty^2\ ||\psi||_\infty^2\ ||q_x||^2, \\ \left|\int p|\phi|^2\bar{p}_{xx}\right| &\leq c||\phi||_\infty^2\ ||p_x||\ ||p_{xx}|| \leq \frac{1}{2}||p_{xx}||^2 + \frac{c^2}{2}||\phi||_\infty^4\ ||p_x||^2, \\ \left|\int p\phi_t\bar{p}_{xx}\right| &\leq c||p_x||\ ||\phi_t||_\infty\ ||p_{xx}|| \leq \frac{1}{2}||p_{xx}||^2 + \frac{c^2}{2}||\phi_t||^2\ ||p_x||^2, \\ \left|\int p_{xx}G_x\right| &\leq ||G_x||\ ||p_{xx}|| \leq \frac{1}{2}||p_{xx}||^2 + \frac{1}{2}||G_x||^2, \\ \left|\int q\psi\bar{p}\right| &\leq c||q_x||\ ||\psi||_\infty\ ||p|| \leq \frac{c^2}{2}||\psi||_\infty^2||q_x||^2 + \frac{1}{2}||p||^2. \end{aligned}$$

Here we have used the compact embeddings $H^1 \hookrightarrow L^2$, $\mathcal{E}_1 \hookrightarrow \mathcal{E}_0$ and the fact that the assumption $(\psi_0, \phi_0, \theta_0) \in \mathcal{E}_1$ implies that the solution $(\psi, \phi, \theta) \in \mathcal{E}_1$ (see, [13]). Therefore $||\psi||$, $||\phi||$, $||\psi_t||$ $||\phi_t||$ are time-uniform bounded. Hence there exist some constants c_0, $c_1 > 0$, where $c_0 = c_0(\kappa, \delta, \alpha, \sigma, ||\psi||, ||\psi_t||, ||\phi||)$ and $c_1 = c_1(\alpha, \sigma, \nu, ||\psi||,\ ||\phi_t||,\ ||\phi||)$ such that relation (2.25) obtains the form

$$A \leq c_0||q_x||^2 + c_1||p_x||^2 + \nu\frac{\delta}{4}||G_x||^2 + \delta\kappa||p_{xx}||^2 + \mu\frac{\alpha}{2}||p||^2. \tag{2.26}$$

From the inequalities (2.21) and (2.24) we obtain

$$\frac{d}{dt}\,Q(t)+\kappa(2\kappa\alpha-\delta)||p_{xx}||^2+\nu(\lambda-\sigma-\delta-\frac{\delta}{4})||G_x||^2+\nu(\delta-\sigma)||q_{xx}||^2 \\ +\nu\frac{\delta}{4}(\delta-\sigma)||q_x||^2+\mu(\alpha-\sigma-\frac{\alpha}{2})||p||^2\leq c_0||q_x||^2+c_1||p_x||^2. \tag{2.27}$$

Until now there has been no condition imposed on σ. Considering the estimates made for δ in (2.13) and taking δ small enough, one may choose $\sigma<\min\{\alpha/2,\delta/4\}$. Thereby, we have $\lambda-\sigma-\delta-\frac{\delta}{4}=\lambda-\sigma-\frac{5\delta}{4}>0$, $\alpha-\sigma-\frac{\alpha}{2}=\frac{\alpha}{2}-\sigma>0$, $\delta-\sigma>0$ and $2\kappa\alpha-\delta>0$. Therefore it is clear that

$$\frac{d}{dt}\,Q(t)\leq c_0||q_x||^2+c_1||p_x||^2\leq c_2(||q_x||^2+||p_x||^2), \tag{2.28}$$

where $c_2=max\{c_0,c_1\}$. For well chosen μ, ν it can be proven that the norm introduced by the functional $Q^{1/2}$ is equivalent to the norm of $\mathcal{E}_1$. That is taking into consideration the definition of $Q^{1/2}$ the following integrals need to be evaluated

$$2\kappa\left|\int q\psi\bar{p}_{xx}\right|\leq 2\kappa c||q_x||\;||\psi||_\infty\;||p_{xx}||\leq\frac{\nu}{8}||q_x||^2+\frac{8\kappa^2c^2||\psi||_\infty^2||p_{xx}||^2}{\nu}$$

and

$$2\kappa\left|\int p\phi\bar{p}_{xx}\right|\leq 2\kappa||\phi||_\infty||p||\;||p_{xx}||\leq\frac{\mu}{2}||p||^2+\frac{2\kappa^2||\phi||_\infty^2||p_{xx}||^2}{\mu}.$$

Consequently we obtain the following minorization

$$Q(t)\geq\left(\kappa^2-\frac{8\kappa^2c^2||\psi||_\infty^2}{\nu}-\frac{2\kappa^2||\phi||_\infty^2}{\mu}\right)||p_{xx}||^2+\frac{\nu}{2}||G_x||^2+\frac{\nu}{2}||q_{xx}||^2.$$

Since $||\psi||_\infty^2$, $||\phi||_\infty^2$ lie in a uniform in time bounded set then for large enough μ,ν we may have

$$\kappa^2-\frac{8\kappa^2c^2||\psi||_\infty^2}{\nu}-\frac{2\kappa^2||\phi||_\infty^2}{\mu}\geq\frac{\kappa^2}{2},\qquad\text{for all }\ t\geq 0. \tag{2.29}$$

This infers that $Q(t)\geq c(||p_{xx}||^2+||G_x||^2+||q_{xx}||^2)$ with $c=\min\{\nu/2,\kappa^2/2\}$. On the other hand the following is also true

$$Q(t)\leq|Q(t)|\leq max\{\mu,\nu\}(||p_{xx}||^2+||G_x||^2+||q_{xx}||^2).$$

Hence

$$c(||p_{xx}||^2+||G_x||^2+||q_{xx}||^2)\leq Q(t)\leq c^{'}(||p_{xx}||^2+||G_x||^2+||q_{xx}||^2), \tag{2.30}$$

where $c^{'}=\max\{\mu,\nu\}$. Thus the functional $Q(t)^{1/2}$ defines a norm equivalent to the norm of $\mathcal{E}_1$. $\lhd$

3 Upper Bounds on the Dimension of Attractor

Let us define the following bilinear form associated to the energy functional $Q(t)$:

$$\begin{aligned} q(t;\zeta_1,\zeta_2) =&\kappa^2\int p_{x,1}\bar{p}_{x,2} - 2\kappa\ Re\int q_1\psi\bar{p}_{xx,2} - 2\kappa\ Re\int q_2\psi\bar{p}_{xx,1} \\ &- 2\kappa Re\int p_1\phi\bar{p}_{xx,2} - 2\kappa Re\int p_2\phi\bar{p}_{xx,1} + \frac{\nu}{2}\int G_{x,1}\bar{G}_{x,2} \\ &+ \frac{\nu}{2}\int q_{xx,1}\bar{q}_{xx,2} + \nu\frac{\delta}{8}\int q_{x,1}\bar{q}_{x,2} + \frac{\mu}{2}\int p_1\bar{p}_2, \end{aligned} \tag{3.1}$$

where $\zeta_i = (p_i, q_i, G_i), i = 1, 2$ are the solutions of the linear system (2.10)-(2.12). Consider an invariant set X_1 which is bounded in $\mathcal{E}_1$. Our aim is to study how the operator $DS(t)(\psi_0,\phi_0,\theta_0)(\psi_0,\phi_0,\theta_0)$ transforms m-dimensional volumes in $\mathcal{E}_1$.

Introduce the operator $L \in \mathcal{L}(\mathcal{E}_1)$, where the exterior product $\wedge^m L$ is m-linear and continuous from the space $\mathcal{E}_1^m$ to $\wedge^m\mathcal{E}_1$. Let $\omega_m^2(L)$ denote the norm of the mth exterior product of L in $\wedge^m\mathcal{E}_1$:

$$\omega_m^2(L) = ||\wedge^m L||^2_{\mathcal{L}(\wedge^m\mathcal{E}_1)}. \tag{3.2}$$

The norm $\omega_m^2(L)$ is defined by

$$||\wedge^m L||^2_{\mathcal{L}(\wedge^m\mathcal{E}_1)} = ||L\xi^1\wedge ...\wedge L\xi^m||^2_{\mathcal{E}_1}, \tag{3.3}$$

where

$$||L\xi^1\wedge ...\wedge L\xi^m||^2_{\mathcal{E}_1} = \sup_{Gram(\xi^1,...,\xi^m)=1} Gram(L\xi^1, ..., L\xi^m). \tag{3.4}$$

The $Gram$ denotes the Gram determinant, i.e.,

$$Gram(L\xi^1, ..., L\xi^m) = \det_{1\le i,\, j\le m}(L\xi^i, L\xi^j), \tag{3.5}$$

where the supremum is taken over all $\{\xi^i\}_{i=1}^m$ with $\det_{1\le i,\, j\le m}(\xi^i,\xi^j)_{\mathcal{E}_0} \le 1$. For a detailed presentation of ideas related to this subject we refer to [8].

Consider next, m linearly independent elements $V_0^1, ..., V_0^m \in \mathcal{E}_1$, and denote by $V(t)$ the solution of the system (2.10)-(2.12). Now according to the above arguments we set $V^i(t) = L\xi^i = DS(t)(\eta)V_0^i$, where $\eta = (\psi_0,\phi_0,\theta_0)$. The $Gram$ determinant represents the square of m! times the volume of the m-dimensional polyhedron defined by the vectors $V^1(t), ..., V^m(t)$. The next result states that, for sufficiently large m, the volume of the m-dimensional polyhedron, decays exponentially, as $t \to +\infty$.

Theorem 3.1 *Let* X_1 *be an invariant set which is bounded in* $\mathcal{E}_1$. *There exist two constants* $C_1 > 0$ *and* $C_2 > 0$, *such that for every* $V_0 \in X_1$, $m \geq 1$ *and* $t \geq 0$,

$$||V^1(t) \wedge ... \wedge V^m(t)||^2_{\mathcal{E}_1} \\ \leq ||V^1_0(t) \wedge ... \wedge V^m_0(t)||^2_{\mathcal{E}_1} C_2^m \; exp\Big(C_1 S_\lambda - 2\sigma m\Big)t, \;\; for \; all \;\; V^i_0(t) \in \mathcal{E}_1, \tag{3.6}$$

where $V^i(t) = DS(t)(\eta)V^i_0$ *and* $\eta = (\psi_0, \phi_0, \theta_0)$.

Proof As mentioned earlier $(p, q, G) = e^{\sigma t}(u, v, F)$. We set $w(t) = e^{\sigma t}(u, v, F) = e^{\sigma t}V(t)$. Then $(V^i, V^j) = e^{-2\sigma t}(w^i(t), w^j(t))$ and therefore the following equality holds

$$\det_{1\leq i,\, j\leq m}(V^i, V^j) = e^{-2m\sigma t} \det_{1\leq i,\, j\leq m}(w^i(t), w^j(t)).$$

The next step is to introduce the Gram determinants

$$G_m(t) = \det_{1\leq i,\, j\leq m}(w^i(t), w^j(t))$$

and

$$H_m(t) = \det_{1\leq i,\, j\leq m} q(t; w^i(t), w^j(t)),$$

where $q(t; w^i(t), w^j(t))$ is the bilinear form which is associated to the quadratic form $Q(t)$ (see equation (3.1)). Hence taking into consideration the definition of $G_m(t)$ above and equation (3.4) concludes that

$$||V^1(t) \wedge ... \wedge V^m(t)||^2_{\mathcal{E}_1} = e^{-2m\sigma t} G_m(t). \tag{3.7}$$

But also from relation (2.30) we get

$$c^m G_m(t) \leq H_m(t) \leq c'^m G_m(t). \tag{3.8}$$

Therefore it is equivalent to estimate the two Gram determinants $H_m(t)$ and $G_m(t)$. The Gram determinant of m vectors in a Hilbert space with scalar product $(.,.)$ is also the determinant of the quadratic form on $\mathbb{R}^m$

$$(x_1, ..., x_m) \to \Bigg(\Bigg(\sum_{j=1}^m x_j w^j(t), \sum_{j=1}^m x_j w^j(t)\Bigg)\Bigg).$$

Taking into consideration the quantities $H_m(t)$, $G_m(t)$ and the fact that the above determinant is equal to the product of the m eigenvalues of the quadratic form one obtains

$$\det_{1\leq i,\, j\leq m}(w^i(t), w^j(t)) = \prod_{i=1}^m \lambda_i(t). \tag{3.9}$$

Hence for any subset G of $\mathbb{R}^m$, with $\dim G = l$ and any $x \in G \setminus \{0\}$ the relation above becomes

$$\det_{1\le i,\, j\le m}(w^i(t), w^j(t)) \le \prod_{i=1}^{m} \max_{\substack{G\subset \mathbb{R}^m\\ \dim G = l}} \min_{\substack{x\in G\\ \sum_{i=1}^m x_i^2 = 1}} q\Big(t; \sum_{j=1}^m x_j w^j(t), \sum_{j=1}^m x_j w^j(t)\Big), \quad (3.10)$$

where the mini-max principle is applied. For the continuation of the proof procedure the following estimation of the Gram determinant $H_m(t)$ is necessary

Lemma 3.2 *Assume that* Ψ *and* Ψ_1 *are two bilinear symmetric forms on* $\mathbb{R}^m$ *and assume that* Ψ *is definite and positive. Then denoting by* $\{\kappa_l\}_{l=1}^m$ *the ordered eigenvalues of* Ψ_1 *with respect to* Ψ, *i.e.,*

$$\kappa_l = \max_{\substack{F\subset \mathbb{R}^m\\ \dim F = l}} \min_{\substack{r\in F\\ r\neq 0}} \frac{\Psi_1(r,r)}{\Psi(r,r)}, \quad (3.11)$$

for every family $\{\psi^1, \ldots \psi^m\}$, *we have*

$$\sum_{l=1}^m \det_{1\le i,\, j\le m}\{\Psi(\xi^i,\xi^j)_l\} = \Big(\sum_{l=1}^m \kappa_l\Big) \det_{1\le i,\, j\le m}\{\Psi(\xi^i,\xi^j)\}, \quad (3.12)$$

where

$$\Psi(\xi^i,\xi^j)_l = (1-\delta_{kj})\Psi(\xi^i,\xi^j) + \delta_{kj}\frac{d}{dt}\Psi_1(\xi^i,\xi^j), \quad (3.13)$$

in which case the symbol of Kronecker is defined by

$$\delta_{kj} = \begin{cases} 1, & j = l; \\ 0, & j \neq l. \end{cases}$$

Proof See [8, Lemma 3.2]. $\lhd$

In order to estimate the time derivative of $H_m(t)$ one may use the classical rule of differentiation to obtain

$$\frac{d}{dt}H_m(t) = \sum_{k=1}^m \det_{1\le i,\, j\le m}\Big((1-\delta_{kj})q(t; w^i(t), w^j(t)) + \delta_{kj}\frac{d}{dt}q(t; w^i(t), w^j(t))\Big). \quad (3.14)$$

Due to equation (3.1) the last quantity on the right hand side of relation (3.14) becomes

$$\frac{d}{dt}q(t; w^i(t), w^j(t)) = \frac{1}{4}\frac{d}{dt}Q(t; w^i(t)+w^j(t)) - \frac{1}{4}\frac{d}{dt}Q(t; w^i(t)-w^j(t)). \quad (3.15)$$

Introduce the following quantity:

$$\frac{d}{dt}q(t; w^i(t), w^j(t)) = \frac{R(t; w^i(t)+w^j(t)) - R(t; w^i(t)-w^j(t))}{4} = \rho(t; w^i(t), w^j(t)),$$

where $R(t) = \dfrac{dQ(t)}{dt}$ and ρ is the bilinear form associated to R. Therefore relations (3.12), (3.14) and (3.15) give

$$\frac{d}{dt}H_m(t) = \left(\sum_{l=1}^{m} \kappa_l\right) H_m(t), \tag{3.16}$$

where

$$\kappa_l = \max_{\substack{G \subset \mathbb{R}^m \\ \dim G = l}} \; \min_{\substack{x \in G \\ x \neq 0}} \frac{R(t; \sum_{j=1}^m x_j w^j(t))}{Q(t; \sum_{j=1}^m x_j w^j(t))} \quad (1 \leq l \leq m) \tag{3.17}$$

and $Q(t; \sum_{j=1}^m x_j w^j(t)) \neq 0$. Now according to relation (2.27) we have that

$$R(t; (p, q, G)) \leq c_2(||p_x||^2 + ||q_x||^2).$$

But the norms $||p_x||^2 + ||q_x||^2$ can be written as $(K(p, q, G), (p, q, G))_{\mathcal{E}_1}$, where K is defined by

$$K(\xi_1, \xi_2, \xi_3) = ((-\Delta)^{-1}\xi_1, (-\Delta)^{-1}\xi_2, 0), \;\; \text{with} \;\; \xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{E}_1.$$

Since Ω is bounded one may prove that K is a compact symmetric operator. Consequently, using this last remark and relation (2.30) we obtain

$$\frac{d}{dt}H_m(t) \leq H_m(t) \sum_{l=1}^{m} \max_{\substack{G \subset \mathbb{R}^m \\ \dim G = l}} \; \min_{\substack{x \in G \\ x \neq 0}} \frac{c_2(K(\sum_{j=1}^m x_j w^j), \sum_{j=1}^m x_j w^j)_{\mathcal{E}_1}}{c|| \sum_{j=1}^m x_j w^j ||^2_{\mathcal{E}_1}}. \tag{3.18}$$

Notice that $F \subset \mathbb{R}^m$, $dim\ F = l$ is given. Then for $x \in F$, the $\sum_{j=1}^m x_j w^j(t)$ span a space $F(t) \subset \mathcal{E}_1$ with dimension l so that

$$\min_{\substack{x \in F \\ x \neq 0}} \frac{c_2(K(\sum_{j=1}^m x_j w^j), \sum_{j=1}^m x_j w^j)_{\mathcal{E}_1}}{c|| \sum_{j=1}^m x_j w^j ||^2_{\mathcal{E}_1}} \leq \max_{\substack{\mathcal{F} \subset \mathcal{E}_1 \\ \dim \mathcal{F} = l}} \; \min_{\substack{\xi \in \mathcal{F} \\ \xi \neq 0}} \frac{(K(\xi), \xi)_{\mathcal{E}_1}}{||\xi||^2_{\mathcal{E}_1}}. \tag{3.19}$$

Now, the right hand side of inequality (3.19) is the l-th eigenvalue of K, denoted by κ_l. But, according to the definition of K, it can be shown that $\kappa_l = (\lambda_l)^{-1}$, where λ_l is the l-th eigenvalue of the Laplacian operator. Hence relation (3.16) becomes

$$\frac{d}{dt}H_m(t) \leq C_1 \left(\sum_{l=1}^{m} \lambda_l^{-1}\right) H_m(t), \tag{3.20}$$

where $C_1 = \dfrac{c_2}{c}$ and $\lambda_l = C_0 l^2 L^{-2}$. Then Gronwall's Lemma implies

$$H_m(t) \leq H_m(0) exp\Big(C_1 S_\lambda t\Big),$$

where $S_\lambda = \sum_{l=1}^{m} \lambda_l^{-1}$. Therefore inequality (3.8) gives

$$G_m(t) \leq \frac{1}{c^m} H_m(0) exp\Big(C_1 S_\lambda t \Big)$$

and

$$G_m(t) \leq \Big(\frac{c^{'}}{c} \Big)^m G_m(0) exp\Big(C_1 S_\lambda t \Big).$$

But according to the definition of $G_m(t)$

$$\det_{1 \leq i,\, j \leq m} (V^i, V^j)_{\mathcal{E}_1} \leq C_2 G_m(0) exp\Big(C_1 S_\lambda - 2\sigma m \Big) t, \tag{3.21}$$

where $C_2 = \Big(\frac{c^{'}}{c} \Big)^m$. Finally using relations (3.7) and (3.21) we have

$$||V^1(t) \wedge ... \wedge V^m(t)||^2_{\mathcal{E}_1} \leq ||V^1_0(t) \wedge ... \wedge V^m_0(t)||^2_{\mathcal{E}_1} C_2 \; exp\Big(C_1 S_\lambda - 2\sigma m \Big) t, \tag{3.22}$$

for all $V_0^m \in \mathcal{E}_1$. Hence the proof of Theorem 3.1 is completed. $\triangleleft$

Let

$$\bar{\omega}_m(L) = \sup_{x \in X_1} \omega_m(L), \tag{3.23}$$

where the quantity $\omega_m(L)$ is introduced in relation (3.2). Define the uniform Lyapunov exponents on the invariant set X_1 by

$$\mu_1 = Log \bar{\omega}_1, \quad \mu_j = Log \bar{\omega}_j - Log \bar{\omega}_{j-1}, \quad j \geq 2. \tag{3.24}$$

Hence we can state the following.

Theorem 3.3 *If for some* $m \geq 0$

$$\mu_1 + \mu_2 + ... + \mu_{m+1} < 0, \tag{3.25}$$

then the global attractor X_1 *of the Klein-Gordon-Schrödinger type system has finite fractal and Hausdorff dimension in* $\mathcal{E}_1$

$$d_H(X) \leq m + 1, \quad with \;\; m\sigma > \frac{C_1 S_\lambda}{2}, \quad \sigma < min(\frac{\alpha}{2}, \frac{\delta}{4})$$

and

$$d_F \leq (m+1) \max\Big(1 + \frac{|\mu_1 + \mu_2 + ... + \mu_l|}{|\mu_1 + \mu_2 + ... + \mu_{m+1}|} \Big).$$

Proof From relations (3.2)-(3.7) we know that

$$\omega_m^2(L) = \sup_{Gram(\xi^1,...,\xi^m)=1} Gram(L\xi^1, ..., L\xi^m) = \sup_{Gram(\xi^1,...,\xi^m)=1} \det_{1\le i,\, j\le m}(L\xi^i, L\xi^j).$$

For fixed $t_0 > 0$, consider the mapping $S = S(nt_0)$, where $n \geq 1$ will be chosen later. Let

$$L = DS(nt_0)(m_0, n_0, E_0), \ \ U_0 \ = (m_0, n_0, E_0), \ \ V^i(t) = DS(nt_0)U_0 \times V_0^i.$$

Taking into consideration relation (3.21) and (3.23) gives

$$\bar{\omega}_m(L((m_0, n_0, E_0))) \leq C_2^{1/2} exp\bigg(\Big(C_1 S_\lambda - 2\sigma m\Big)\frac{nt_0}{2}\bigg).$$

But the right hand side is independent of $(m_0, n_0, E_0) \in X_1$ hence one may deduce that

$$\bar{\omega}_m(L) \leq \sup_{x\in X_1} C_2^{1/2} exp\bigg(\Big(C_1 S_\lambda - 2\sigma m\Big)\frac{nt_0}{2}\bigg).$$

For fix $m \in \mathbb{N}$ such that $m\sigma > \dfrac{C_1 S_\lambda}{2}$ and sufficiently large n, there exists n_0 such that for $S = S(n_0, t_0)$

$$\bar{\omega}_m < 1.$$

Therefore hypothesis (3.25) is satisfied and the proof of Theorem 3.3 is completed. $\lhd$

To complete the present work we point out two open problems, which are of independent interest.

Remark 3.4 *(Open Problem I) Flahaut [7] studied the Zakharov system in one dimension, where she succeeded to prove the existence of a Global Attractor in the weak topology. Her results were later improved by Goubet and Moise [11], where they proved the existence of a Global Attractor for the same system in the strong topology. Furthermore, they proved that the Strong and the Weak Attractors are equivalent, namely*

$$\mathcal{P}_1^w = \mathcal{P}_1 = \mathcal{P}_2^w = \mathcal{P}_2,$$

where $\mathcal{P}_1^w, \mathcal{P}_1 \ \in D(A^{1/2}) \times D(A) \times D(A^{3/2})$ *and* $\mathcal{P}_2^w, \mathcal{P}_2 \ \in D(A^{3/2}) \times D(A^2) \times D(A^{5/2})$, *where* A *is the Laplacian operator. We conjecture that following the reasoning of the above mentioned authors we may prove that* $X_1^w = X_1$, *for the system (2.1)-(2.5). It seems that the investigation of the relationship between Hausdorff and Fractal Dimensions of the two attractors* X_1^w, X_1 *is an interesting open problem.*

Remark 3.5 *(Open Problem II) So far we have proved the existence of an upper bound for the Hausdorff and Fractal dimensions of the global (strong) attractor* $X_1 \in \mathcal{E}_1$. *The study of the existence of a positive lower bound for Hausdorff and Fractal Dimensions of the system (2.1)-(2.5) seems to be an other difficult but interesting open question.*

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