HIGH DIMENSIONAL RANDOM SECTIONS OF ISOTROPIC CONVEX BODIES

DAVID ALONSO, JESÚS BASTERO, JULIO BERNUÉS, AND GRIGORIS PAOURIS

Abstract. Let $K \subset \mathbb{R}^n$ be a centrally symmetric isotropic convex body. We provide sharp estimates for the section function $|F^\perp \cap K|_{n-k}^{1/k}$ for random $F \in G_{n,k}$ answering a question raised by V. Milman and A. Pajor (see [MP]). We also show that every symmetric convex body has a high dimensional section F with isotropy constant bounded by $c(\log(dimF))^{1/2} \log\log(dimF)$.

1. Introduction and notation

Throughout the paper $K \subset \mathbb{R}^n$ will denote a symmetric convex body. K is called isotropic if it is of volume 1 and its inertia matrix is a multiple of the identity. Equivalently, there exists a constant $L_K > 0$ called isotropy constant of K such that $L_K^2 = \int_K \langle x, \theta\rangle^2\, dx, \forall \theta \in S^{n-1}$.

The relation between the isotropy constant and the size of the central sections of an isotropic convex appears [H], [B] or [MP] where it is proved that for every $1 \le k \le n$ there exist $c_1(k), c_2(k) > 0$ such that for every subspace $F \in G_{n,k}$ (the Grassmann space) and $K \subset \mathbb{R}^n$ isotropic

$$\frac{c_1(k)}{L_K} \le |F^\perp \cap K|_{n-k}^{1/k} \le \frac{c_2(k)}{L_K}$$

where $|\cdot|_m$ is the Lebesgue measure in the appropiate m dimensional space.

More precisely, it is proved in [MP] that $|F^\perp \cap K|_{n-k}^{1/k} \sim L_{B_{k+1}(K,F)}/L_K$, see Lemma 2.2 below ($a \sim b$ means $a \cdot c_1 \le b \le a \cdot c_2$ for some numerical constans $c_1, c_2 > 0$). From now on the letters c, C, c_1... will denote absolute numerical constants, whose value may change from line to line. Well known estimates imply $c_1(k) \ge c_1$, [H], and $c_2(k) \le c_2 k^{1/4}$, [Kl]. We remark that these bounds are valid for *every* subspace $F \in G_{n,k}$.

Our first result in the second section of the paper is an improvement of this general estimate that holds for "most" subspaces. We make use of the tools developed in [Pa1].

Date: December 17, 2007.

2000 *Mathematics Subject Classification.* 52A40, 46B20, 52A39.

Key words and phrases. isotropic convex bodies, section function, concentration of measure phenomenum.

The three first named authors partially supported by MCYT Grants(Spain) MTM2004-03036, MTM2007-61446; Marie Curie RTN CT-2004-511953 and by DGA E-64.

Theorem 2.1 *There exist absolute constants* $c_1, c_2, c_3 > 0$ *with the following property: If* K *is an isotropic convex body in* $\mathbb{R}^n$ *and* $1 \leqslant k \leqslant \sqrt{n}$ *then*

$$\mu\{F \in G_{n,k} : \frac{c_1}{L_K} \leqslant |K \cap F^\perp|_{n-k}^{1/k} \leqslant \frac{c_2}{L_K}\} \geq 1 - e^{-c_3 \frac{n}{k}}$$

In the third section we reach optimal bounds for $c_1(k)$ and $c_2(k)$ but for a worse dependence on k. For that matter, we need to estimate the Lipschitz constant of the central section function $|F^\perp \cap K|_{n-k}$. For $k = 1$ this was proved in [ABP]. Then, we exploit the concentration of measure on $G_{n,k}$ together with some results from [BB] and [Kl2]. Precisely, we show
Theorem 3.8. *Let* $K \subset \mathbb{R}^n$ *isotropic. For all* $\varepsilon > 0$, $1 \le k \le \frac{c\varepsilon \log n}{(\log \log n)^2}$, *the set* A *of subspaces* $F \in G_{n,k}$ *such that*

$$\frac{1-\varepsilon}{\sqrt{2\pi} L_K} \le |K \cap F^\perp|_{n-k}^{1/k} \le \frac{1+\varepsilon}{\sqrt{2\pi} L_K} \tag{1.1}$$

holds, has probability $\mu(A) \ge 1 - c_1 e^{-c_2 n^{0.9}}$.

Independently, in a recent paper [EK], the authors proved a multidimensional central limit theorem for convex bodies from which one can deduce (1.1) for $k \le c(\varepsilon) n^{c_1}$ and $\mu(A) \ge 1 - c_2 e^{-n^{c_3}}$.

In the last section we give upper bounds for the isotropic constant of high dimensional central sections $L_{K \cap F}$. In particular,

Proposition 1.1. *For every symmetric convex body* K *in* $\mathbb{R}^n$ *and* $k \leqslant (1 - \frac{1}{\log n}) n$, *there exists an* $F \in G_{n,k}$ *such that*

$$L_{K \cap F} \leqslant c (\log k)^{1/2} \log \log k$$

where $c > 0$ *is a universal constant.*

We denote by $|\cdot|$ the Euclidean norm in the appropiate space, D_n the Euclidean ball in $\mathbb{R}^n$ and by ω_n its Lebesgue measure. The surface area of the unit sphere is $|S^{n-1}| = n\ \omega_n$. For any k-dimensional subspace $F \subset \mathbb{R}^n$ we denote $S_F = S^{n-1} \cap F$, the Haar probability on S_F by σ_F, $D_F = D_n \cap F$ and by P_F the orthogonal projection onto F. The Haar probability on the grassmaniann $G_{n,k}$ is denoted by μ. For $T \in GL(n)$, $\|T\|$ denotes the operator norm and $\|T\|_{HS} := \left(\sum_{j=1}^n |T(e_j)|^2 \right)^{1/2}$, for (any) orthonormal basis (e_j) of $\mathbb{R}^n$, its Hilbert-Schmidt norm. K° denotes the polar body of K. For any convex body $L \subset \mathbb{R}^n$ we will write $\widetilde{L} = L/|L|_n^{1/n}$. We will denote $W(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta)$, the mean width of the convex body K.

2. First improvement via random sections

Theorem 2.1. *There exist absolute constants* $c_1, c_2, c_3 > 0$ *with the following property: If* K *is an isotropic convex body in* $\mathbb{R}^n$ *and* $1 \leqslant k \leqslant \sqrt{n}$

then

$$\mu\{F \in G_{n,k} : \frac{c_1}{L_K} \leqslant |K \cap F^{\perp}|_{n-k}^{1/k} \leqslant \frac{c_2}{L_K}\} \geq 1 - e^{-c_3 \frac{n}{k}} \tag{2.2}$$

Through the section K will be (a symmetric convex body) of volume 1.
Let F a k-dimensional subspace of $\mathbb{R}^n$ and denote by E the orthogonal subspace of F. For every $\phi \in S_F$ we define $E(\phi) = \mathrm{span}\{E, \phi\}$.
K. Ball (see [B]) proved the following theorem: For every $q \geqslant 0$ and $\phi \in F$, the function

$$\phi \mapsto |\phi|^{1+\frac{q}{q+1}} \left(\int_{K \cap E(\phi)} |\langle x, \phi \rangle|^q dx \right)^{-\frac{1}{q+1}}$$

is a norm on F. We denote by $B_q(K, F)$ the unit ball of this norm.
Under this notation it was proved in [MP] the following

Lemma 2.2. *If K is isotropic then $B_{k+1}(K, F)$ is also isotropic for every $F \in G_{n,k}$, and*

$$|K \cap F^{\perp}|_{n-k}^{1/k} \sim \frac{L_{B_{k+1}(K,F)}}{L_K} \qquad \forall\ F \in G_{n,k} \tag{2.3}$$

A generalization for L_q centroid bodies of this approach appeared in [Pa1]. For any $q \geq 1$ we define the L_q centroid body of K, the symmetric convex body that has support function

$$h_{Z_q(K)}(z) := \left(\int_K |\langle x, z \rangle|^q dx \right)^{1/q}, \forall z \in S^{n-1}$$

The following equality was proved in [Pa1]: For every $1 \leqslant k \leqslant n-1$, $F \in G_{n,k}$ and $q \geq 1$,

$$P_F(Z_q(K)) = \left(\frac{k+q}{2} \right)^{1/q} |B_{k+q-1}(K, F)|_k^{1/k+1/q} Z_q(\widetilde{B}_{k+q-1}(K, F)) \tag{2.4}$$

Proposition 2.3. *Let $K \subset \mathbb{R}^n$, $1 \leqslant k \leqslant n-1$, $F \in G_{n,k}$ and $E = F^{\perp}$. Then*

$$c_1 \leqslant |P_F Z_k(K)|_k^{1/k} |K \cap E|_{n-k}^{1/k} \leqslant c_2$$

where $c_1, c_2 > 0$ are universal constants.

Proof. We choose $q = k$ in (2.4). Then by taking volumes we have that

$$|P_F(Z_k(K))|_k^{1/k} = k^{1/k} |B_{2k-1}(K, F)|_k^{2/k} |Z_k(\widetilde{B}_{2k-1}(K, F))|_k^{1/k}$$

It is known that there exists a universal constant $c > 0$ such that for any symmetric convex body K of volume 1 in $\mathbb{R}^k$ and $q \geq k$, $cK \subseteq Z_q(K) \subseteq K$. (see [Pa2] for a proof). So,

$$c \leqslant |Z_k(\widetilde{B}_{2k-1}(K, F))|_k^{1/k} \leqslant 1.$$

So, it is enough to prove that there exists $c > 0$ such that

$$\frac{1}{|K \cap E|_{n-k}^{1/k}} \leqslant k^{1/k} |B_{2k-1}(K, F)|_k^{\frac{2}{k}} \leqslant c \frac{1}{|K \cap E|_{n-k}^{1/k}} \tag{2.5}$$

The right hand side inequality was proved in [Pa1]. The left hand side inequality follows the same line. We will need the following fact (see [MP] for a proof):

Let C be a symmetric convex body in $\mathbb{R}^m$. If $s \leqslant r$ are non-negative integers and $\theta \in S^{m-1}$, we have that

$$\left(\frac{r+1}{2}\frac{\int_C |\langle x,\theta\rangle|^r dx}{|C\cap\theta^\perp|_{m-1}}\right)^{1/(r+1)} \geqslant \left(\frac{s+1}{2}\frac{\int_C |\langle x,\theta\rangle|^s dx}{|C\cap\theta^\perp|_{m-1}}\right)^{1/(s+1)} \tag{2.6}$$

Writing in polar coordinates we get

$$|B_{2k-1}(K,F)|_k = \omega_k \int_{S_F}\left(\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{2k-1} dx\right)^{\frac{1}{2}} d\sigma_F(\phi) \tag{2.7}$$

Applying (2.6) with $C = K\cap E(\phi)$, $m = n-k+1$, $r = 2k-1$ and $s = k-1$, we get

$$\left(k\frac{\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{2k-1} dx}{|K\cap E|_{n-k}}\right)^{1/(2k)} \geqslant \left(\frac{k}{2}\frac{\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{k-1} dx}{|K\cap E|_{n-k}}\right)^{1/k}$$

It follows that

$$\left(\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{2k-1} dx\right)^{\frac{1}{2}} \geqslant (k|K\cap E|_{n-k})^{-1/2}\frac{k}{2}\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{k-1} dx$$

Then formula (2.7) becomes

$$|B_{2k-1}(K,F)|_k \geqslant (k|K\cap E|_{n-k})^{-1/2}\frac{k}{2}\omega_k\int_{S_F}\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{k-1} dx d\sigma_F(\phi)$$

Observe that (see also [Pa1])

$$|K|_n = \frac{k\omega_k}{2}\int_{S_F}\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{k-1} dx\, d\sigma_F(\phi).$$

So we get $|B_{2k-1}(K,F)|_k \geqslant \dfrac{1}{k^{1/2}}|K\cap E|_{n-k}^{-1/2}$, that is

$$k^{1/k}|B_{2k-1}(K,F)|_k^{2/k} \geqslant \frac{1}{|K\cap E|_{n-k}^{1/k}}$$

That proves formula (2.5) and the Proposition. □

We will use the isomorphic version of Dvoretzky theorem proved by V. Milman (see [M] , [MS2]):

Proposition 2.4. *Let C a symmetric convex body in $\mathbb{R}^n$. If $k \leqslant c_1 n\left(\frac{W(C)}{R(C)}\right)^2$,*

$$\mu\{F\in G_{n,k} : \frac{W(C)}{2}D_F \subseteq P_F(C) \subseteq 2W(C)D_F\} \geq 1-\exp\left(-c_2 n\left(\frac{W(C)}{R(C)}\right)^2\right) \tag{2.8}$$

where $c_1, c_2 > 0$ *are universal constants,* $R(C) = \max\{|x| \; ; \; x \in C\}$ *and* $W(C) = \int_{S^{n-1}} h_C(\theta)\, d\sigma(\theta)$

We will denote $k_*(C^\circ) := n \left(\frac{W(C)}{R(C)}\right)^2$. Furthermore, (see [LMS]) we have that for $p \leqslant k_*(C^\circ)$,

$$W(C) \sim W_p(C) := \left(\int_{S^{n-1}} h_C^p(\theta) d\sigma(\theta)\right)^{1/p} \tag{2.9}$$

Definition 2.5. Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ and let $\alpha \in [1, 2]$. We say that K is a ψ_α-body with constant b_α if

$$\left(\int_K |\langle x, \theta\rangle|^q dx\right)^{1/q} \leqslant b_\alpha q^{1/\alpha} \left(\int_K |\langle x, \theta\rangle|^2 dx\right)^{1/2}$$

for all $q \geqslant \alpha$ and all $\theta \in S^{n-1}$. Equivalently, if

$$Z_q(K) \subseteq b_\alpha q^{1/\alpha} Z_2(K) \qquad \text{for all} \quad q \geqslant \alpha$$

The following definition appeared in [Pa3]:

Definition 2.6. Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$. We define

$$q_*(K) = \max\{q \in \mathbb{N} : k_*(Z_q^\circ(K)) \geqslant q\}$$

where $Z_q^\circ(K) := (Z_q(K))^\circ$.

We will need the following lower bounds for the quantities $k_*(Z_q^\circ(K))$ and $q_*(K)$, (see [Pa1]):

Proposition 2.7. *Let* K *be an isotropic* ψ_α*-body with constant* b_α *and* $1 \leqslant q \leqslant n$ *then*

$$k_*(Z_q^\circ(K)) \geq c_1 \frac{n}{q^{\frac{2-\alpha}{\alpha}} b_\alpha^2} \qquad \textit{and} \qquad q_*(K) \geqslant c_2 \left(\frac{\sqrt{n}}{b_\alpha}\right)^\alpha$$

Proposition 2.8. *Let* $K \subset \mathbb{R}^n$ *isotropic. Then for* $q \leqslant q_*(K)$ *we have*

$$W(Z_q(K)) \sim \sqrt{q} L_K \tag{2.10}$$

Proof. A direct computation shows that for $q \leqslant n$,

$$\left(\int_K |x|^q dx\right)^{1/q} \sim \sqrt{\frac{n}{q}} W_q(Z_q(K))$$

It was proved in [Pa1] that for every $q \leqslant q_*(K)$ we have

$$\left(\int_K |x|^q dx\right)^{1/q} \sim \sqrt{n} L_K$$

Also by (2.9) and the definition of $q_*(K)$ we have that

$$W(Z_q(K)) \sim W_q(Z_q(K)) \qquad \forall \; q \leqslant q_*(K)$$

By putting these results together we conclude the proof.

□

A well known application of Brunn-Minkowski inequality implies that every convex body is ψ_1 body with a constant c, where c is universal. So, Theorem 2.1 is a direct consequence of the following:

Theorem 2.9. *Let K be an isotropic ψ_α-body with constant b_α and $1 \leqslant k \leqslant c\left(\frac{\sqrt{n}}{b_\alpha}\right)^\alpha$. Then*

$$\mu\{F \in G_{n,k} : \frac{c_1}{L_K} \leqslant |K \cap F^\perp|_{n-k}^{1/k} \leqslant \frac{c_2}{L_K}\} \geq 1 - \exp\{-c_3 \frac{n}{k^{\frac{2-\alpha}{\alpha}} b_\alpha^2}\}$$

Proof. Let $1 \leqslant k \leqslant q_*(K)$. We will apply Proposition 2.4 for the symmetric convex body $Z_k(K)$. So, we have that there exists an $A \subseteq G_{n,k}$ of measure greater that $1 - e^{c_2 k_*(Z_k^\circ(K))}$ such that for every $F \in A$

$$\frac{W(Z_k(K))}{2} D_F \subseteq P_F(Z_k(K)) \subseteq 2W(Z_k(K)) D_F$$

By taking volumes we get

$$|P_F(Z_k(K))|_k^{1/k} \sim \frac{W(Z_k(K))}{\sqrt{k}} \sim L_K$$

where we used Proposition 2.8 and the fact that $|D_k|_k^{1/k} \sim \frac{1}{\sqrt{k}}$.
By Proposition 2.3 we get that for every $F \in A$,

$$\frac{c_1}{L_K} \leqslant |K \cap E|_{n-k}^{1/k} \leqslant \frac{c_2}{L_K}$$

The result follows by Proposition 2.7.

□

3. Second improvement via random sections

In the first part we estimate the Lipschitz constant of the function $F \to |F^\perp \cap K|_{n-k}$ and also review concentration inequalities with respect to several natural distances on $G_{n,k}$. We start with the latter.

The following lemma constructs a suitable orthonormal basis for two subspaces E and F and will be very useful for our purposes

Lemma 3.1 ([GM], Lemma 4.1)**.** *Let $E, F \in G_{n,k}$ such that $F^\perp \cap E = 0$. Then there exists $u_1, \dots u_k$ orthonormal basis of E such that the family $v_1, \dots v_k$ given by $v_j = \frac{P_F(u_j)}{|P_F(u_j)|}$ is an orthonormal basis of F. In particular, $\langle u_j, v_i \rangle = |P_F(u_j)| \; \delta_i^j$.*

The space $G_{n,k}$ appears in the literature equipped with a number of different distances. In the following Proposition, we estimate the equivalence constants between them. It is probably folklore but we include for the reader's convenience. The fact that one can move from one distance to another will be useful while computing the Lipschitz constant and also when considering the concentration phenomena on $G_{n,k}$.

The elements of the orthogonal group $O(n)$ will be denoted by $U = (u_1 \dots u_n)$ so the columns (u_i) form an orthonormal basis in $\mathbb{R}^n$.

Proposition 3.2. *For $E, F \in G_{n,k}$ we consider the following distances*

$d_0(E,F) = \max\{d(x,S_F) \mid x \in S_E\}$, *$d$ is the euclidean distance.*

$d_1(E,F) = \inf\{\varepsilon > 0 \mid S_E \subset S_F + \varepsilon D_n, S_F \subset S_E + \varepsilon D_n\}$

$$d_2(E,F) = \inf\{\Big(\sum_{j=1}^{k} |u_j - v_j|^2\Big)^{1/2} E = \langle u_j\rangle_1^k,\ F = \langle v_j\rangle_1^k \text{ orthon. basis}\}$$

$$d_3(E,F) = \inf\{\Big(\sum_{j=1}^{n} |u_j - v_j|^2\Big)^{1/2} E = \langle u_j\rangle_1^k,\ F = \langle v_j\rangle_1^k \text{ orthon. basis}\}$$

$d_4(E,F) = \|P_E - P_F\|_{HS}$

$d_5(E,F) = \inf\{\|U - V\|_{HS} \mid U, V \in O(n), E = \langle u_1 \dots u_k\rangle, F = \langle v_1 \dots v_k\rangle\}$

$d_6(E,F) = \|P_E - P_F\|$

Then, d_2, d_3, d_4, d_5 are equivalent with numerical equivalence constants, $d_0 = d_1$, $d_1 \le d_2 \le \sqrt{2k}\ d_1$ and $d_6 \le d_4 \le \sqrt{2k} d_6$.

Proof. $d_0 = d_1$: d_1 is the Hausdorff distance between S_E and S_F which also reads $d_1(E,F) = \max\big\{\max_{x \in S_E} d(x,S_F), \max_{y \in S_F} d(y,S_E)\big\}$, so $d_0 \le d_1 \le \sqrt{2}$ and it is enough to check that the two inner maxima are equal.

If $E \cap F^\perp \neq 0$ then $d_0(E,F) = \sqrt{2}$. Suppose $E \cap F^\perp = 0$.

For any $x \in S_E, y \in S_F$, $|x-y|^2 = 2 - 2\langle x,y\rangle = 2 - 2\langle P_F(x), y\rangle$. So, $d^2(x,S_F) = 2 - 2\sup_{y \in S_F} \langle P_F(x), y\rangle = 2 - 2|P_F(x)| = \big|x - \dfrac{P_F(x)}{|P_F(x)|}\big|^2$. Let $x_0 \in S_E$ that maximizes $d(x,S_F)$ on S_E or equivalently that minimizes $|P_F(x)|$. Denote $y_0 = \frac{P_F(x_0)}{|P_F(x_0)|}$ (observe $P_F(x_0) \neq 0$). By the arguments in [GM] Lemma 4.1, $P_F(x_0)$ is orthogonal to $E \cap x_0^\perp$ and so $P_E P_F(x_0)$ is parallel to x_0. Write $P_E(y_0) = \lambda x_0$. Then $\lambda = \langle P_E(y_0), x_0\rangle = \langle y_0, P_E(x_0)\rangle = |P_F(x_0)|$ and $\frac{P_E P_F(x_0)}{|P_E P_F(x_0)|} = x_0$. Therefore, $d(y_0, S_E) = d(x_0, S_F)$ and so $\max\{d(y,S_E) \mid y \in S_F\} \ge \max\{d(x,S_F) \mid x \in S_E\}$. Exchange E, F and equality follows.

$d_1 \le d_2 \le \sqrt{2k}\ d_1$: It is proved in [GM], Lemma 4.1.

$\frac{1}{\sqrt{2}} d_2 \le d_4 \le \sqrt{2}\ d_2$: Let $F^\perp \cap E := E_0$ and write the orthogonal decomposition $E = E_0 \oplus E_1$ with $E_1 \cap F^\perp = 0$. By Lemma 3.1, there exists an orthonormal basis in E_1, (u_j), such that $v_j = \frac{P_F(u_j)}{|P_F(u_j)|}$ is an orthonormal system in F. Now add vectors to complete an orthonormal basis in E (by adding vectors in E_0) and in F that we also denote as u_j and v_j. Trivially,

$$\|P_E - P_F\|_{HS}^2 \ge \sum_{j=1}^{k} |(P_E - P_F)(u_j)|^2$$

If $u_j \in E_1$ then, since $\langle u_j, v_j\rangle = |P_F(u_j)|$ (Lemma 3.1),

$$|(P_E - P_F)(u_j)|^2 = 1 - |P_F(u_j)|^2 \geq 1 - |P_F(u_j)| = \frac{1}{2}|u_j - v_j|^2$$

If $u_j \in E_0$ and $v_j \in F$ then $|(P_E - P_F)(u_j)|^2 = 1$. Also, since $\langle u_j, v_j\rangle = 0$ and so $|u_j - v_j|^2 = 2$.

For the second inequality, let $(u_j), (v_j)$ be orthonormal basis of $E, F \in G_{n,k}$ we write $P_E = \sum_{j=1}^k u_j \otimes u_j$ and $P_F = \sum_{i=1}^k v_i \otimes v_i$ and by definition

$$\|P_E - P_F\|_{HS}^2 = 2k - 2\sum_{i,j=1}^k \langle u_j, v_i\rangle^2 \leq 2\sum_{j=1}^k (1 - \langle u_j, v_j\rangle^2) \leq 2\sum_{j=1}^k |u_j - v_j|^2$$

since $1 - \langle u_j, v_j\rangle^2 \leq 2(1 - \langle u_j, v_j\rangle) = |u_j - v_j|^2$.

$d_2 \leq d_3 \leq \sqrt{5}d_2$: By definition $d_3^2(E,F) = d_2^2(E,F) + d_2^2(E^\perp, F^\perp)$. Now, $d_2^2(E^\perp, F^\perp) \leq 2d_4^2(E^\perp, F^\perp) = 2d_4^2(E,F) \leq 4d_2^2(E,F)$. With similar arguments one proves $d_2 \leq d_5 \leq 3d_2$.

$d_6 \leq d_4 \leq \sqrt{2k}d_6$: For $T \in GL(n)$ $\|T\| \leq \|T\|_{HS} \leq \sqrt{\dim(T(\mathbb{R}^n))}\|T\|$. □

Proposition 3.3. *Let $K \subset \mathbb{R}^n$ isotropic. The function given by $G_{n,k} \ni E \to |E^\perp \cap K|_{n-k}$ is Lipschitz and for all $E, F \in G_{n,k}$ we have the estimate*

$$\big|\, |E^\perp \cap K|_{n-k} - |F^\perp \cap K|_{n-k}\big| \leq \frac{(c\mathcal{L}_k)^{2k}}{L_K^k}\|P_E - P_F\|_{HS}$$

where $\mathcal{L}_k := \sup\{L_M \mid M \subset \mathbb{R}^k,$ convex body isotropic$\}$.

In order to prove it, one more lemma will be used. An equivalent version of it for $k = 1$ is due to Busemann.

Lemma 3.4 ([MP]). *If K is a convex body and $E \in G_{n,k}$ then the function given by*

$$E^\perp \ni \theta \to \|\theta\| := \frac{|\theta|}{|K \cap E(\theta)|}$$

is a norm on $E^\perp$.

Proof of Proposition 3.3. Suppose $F^\perp \cap E = 0$ and let $E = \langle u_1 \dots u_k\rangle, F = \langle v_1 \dots v_k\rangle$ the orthonormal basis in Lemma 3.1. Denote $E_0^\perp = E^\perp$, $E_j^\perp = v_1^\perp \cap \dots \cap v_j^\perp \cap u_{j+1}^\perp \cap \dots \cap u_k^\perp$ and $E_k^\perp = F^\perp$. Then

$$\big|\, |E^\perp \cap K|_{n-k} - |F^\perp \cap K|_{n-k}\big| \leq \sum_{j=1}^k \big|\, |E_j^\perp \cap K|_{n-k} - |E_{j-1}^\perp \cap K|_{n-k}\big|$$

Let us estimate (say) the first summand. Set $\bar{E} = E^\perp \cap v_1^\perp = E_1^\perp \cap u_1^\perp$. Then, by Lemma 3.1, $E^\perp = \bar{E} \oplus P_{E^\perp}(v_1)$ and $E_1^\perp = \bar{E} \oplus P_{E_1^\perp}(u_1)$ so we can apply Lemma 3.4 to $\bar{E}$

$$\big|\, |E^\perp \cap K|_{n-k} - |E_1^\perp \cap K|_{n-k}\big| = \left| \frac{|P_{E^\perp}(v_1)|}{\|P_{E^\perp}(v_1)\|} - \frac{|P_{E_1^\perp}(u_1)|}{\|P_{E_1^\perp}(u_1)\|} \right|$$

and since $|P_{E_1}(u_1)| = |\langle u_1, v_1\rangle| = |P_E(v_1)|$ and the triangle inequality,

$$\left| \frac{|P_{E^\perp}(v_1)|}{\|P_{E^\perp}(v_1)\|} - \frac{|P_{E_1^\perp}(u_1)|}{\|P_{E_1^\perp}(u_1)\|} \right| \leq \frac{|P_{E_1^\perp}(u_1)|}{\|P_{E_1^\perp}(u_1)\|\, \|P_{E^\perp}(v_1)\|} \|P_{E_1^\perp}(u_1) - P_{E^\perp}(v_1)\|$$

Finally, observe that $|P_{E_1^\perp}(u_1) - P_{E^\perp}(v_1)| = (1 - \langle u_1, v_1\rangle)|u_1 - v_1|$ and apply Lemma 2.2 to conclude with

$$\big|\, |E^\perp \cap K|_{n-k} - |E_1^\perp \cap K|_{n-k}\big| \leq \frac{(1 - \langle u_1, v_1\rangle)}{(1 - \langle u_1, v_1\rangle^2)^{1/2}} |u_1 - v_1| \frac{(c\mathcal{L}_k)^{2k}}{L_K^k}$$

Since we can also suppose $\langle u_1, v_1\rangle \geq 0$, the first quotient above is bounded by 1. So,

$$\big|\, |E^\perp \cap K|_{n-k} - |F^\perp \cap K|_{n-k}\big| \leq \sqrt{k}\big(\sum_{j=1}^{k} |u_j - v_j|^2\big)^{1/2} \frac{(c\mathcal{L}_k)^{2k}}{L_K^k}$$

By the proof of Proposition 3.2, $\big(\sum_{j=1}^{k} |u_j - v_j|^2\big)^{1/2} \leq \sqrt{2}\|P_E - P_F\|_{HS}$. In the general case, if $F^\perp \cap E := E_0$ then we can write $E = E_0 \oplus E_1$ with $E_1 \cap F^\perp = 0$. Choose an orthonormal basis of E_0 and proceed as in the previous case. □

Theorem 3.5 (Concentration of measure, [MS])**.** *There exist absolute constants $c_1, c_2 > 0$ such that*

i) *For every $A \subset G_{n,k}$ and every $\delta > 0$*

$$\mu(A_\delta) \geq 1 - \frac{c_1}{\mu(A)} \exp\left(-c_2\delta^2 n\right)$$

where $A_\delta = \{E \in G_{n,k};\ \exists\ F \in A, d_5(E,F) \leq \delta\}$

ii) *For $f\colon G_{n,k} \to \mathbb{R}$ a Lipschitz function with Lipschitz constant σ, that is $|f(E) - f(F)| \leq \sigma d_5(E,F)$,*

$$\mu\left\{E \in G_{n,k}; |f(E) - \mathbb{E}\,(f)| \leq a\right\} \geq 1 - c_1 \exp\left(-\frac{c_2 a^2 n}{\sigma^2}\right) \qquad \forall\ a > 0$$

Remark 3.6. If $d, \tilde{d}$ are two distances on $G_{n,k}$ such that $d \leq M\tilde{d}$ for some $M > 0$ then a concentration inequality for $\tilde{d}$ with bound $c_1 \exp\left(-c_2\delta^2 n\right)$ implies one for d with bound $c_1 \exp\left(\frac{-c_2\delta^2 n}{M^2}\right)$. Similarly for Lipschitz functions. It is then possible to state concentration inequalities for the different distances (Proposition 3.2) on $G_{n,k}$.

The last main ingredient of the section is

Theorem 3.7. [Kl2]. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body. Then,*

$$|\{x \in K : \big||x| - \sqrt{n}L_K\big| > t\sqrt{n}L_K\}|_n \leq c\exp(-Cn^\alpha t^\beta) \tag{3.11}$$

for all $0 \leq t \leq 1$ and $\alpha = 0.33, \beta = 3.33$.

It was proved by [So] (with sharp exponents α and β) for normalized unit balls of $\ell_p^n, 1 \leq p$ and in full generality in [Kl2].

As an application of the results in this section we show the announced

Theorem 3.8. *Let $K \subset \mathbb{R}^n$ isotropic. For all $\varepsilon > 0$, $1 \leq k \leq \frac{\varepsilon \log n}{(\log\log n)^2}$, the set A of subspaces $E \in G_{n,k}$ such that*

$$\frac{1-\varepsilon}{\sqrt{2\pi}L_K} \leq |E^\perp \cap K|_{n-k}^{1/k} \leq \frac{1+\varepsilon}{\sqrt{2\pi}L_K}$$

holds, has probability $\mu(A) \geq 1 - c_1 \exp -c_2 n^{0.9}$

Proof. Consider the function $f\colon G_{n,k} \to \mathbb{R}$, $f(E) = |E^\perp \cap K|_{n-k}$. By Proposition 3.3 and Theorem 3.5 we have

$$\mu\left\{E \in G_{n,k}; |f(E) - \mathbb{E}\,(f)| \leq \varepsilon\mathbb{E}\,(f)\right\} \geq 1 - c_1 \exp\left(-\frac{c_2^k L_K^{2k}(\mathbb{E}\,(f))^2\varepsilon^2 n}{(\mathcal{L}_k)^{2k}}\right)$$

On the other hand, Theorem 3.5 in [BB] and Theorem 3.7 readily imply

$$\left|\frac{\int_{G_{n,k}} F_K(t,E)\,d\mu(E)}{\Gamma_K^k(t)} - 1\right| \leq \frac{c_1}{n^{0.09}} \qquad \forall\, t \geq 0$$

Taking limits as $t \to 0$ (see Corollary 3.6 in [BB]) yields

$$\left|\frac{\mathbb{E}\,(f)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1\right| \leq \frac{c_1}{n^{0.09}}\Big(\leq \frac{\varepsilon}{3}\Big)$$

By the triangle inequality

$$\left|\frac{f(E)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1\right| \leq \frac{\mathbb{E}\,(f)}{\frac{1}{(\sqrt{2\pi}L_K)^k}}\left|\frac{f(E)}{\mathbb{E}(f)} - 1\right| + \left|\frac{\mathbb{E}\,(f)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1\right|$$

So, if $\left|\frac{f(E)}{\mathbb{E}(f)} - 1\right| \leq \frac{\varepsilon}{3}$, then $\left|\frac{f(E)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1\right| \leq (1+\frac{\varepsilon}{3})\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \leq \varepsilon$ and conclude, using also $\mathcal{L}_k \leq ck^{1/4}$

$$\mu\big\{E \in G_{n,k}; \big|f(E) - \frac{1}{(\sqrt{2\pi}L_K)^k}\big| \leq \frac{\varepsilon}{(\sqrt{2\pi}L_K)^k}\big\} \geq$$
$$\geq \mu\left\{E \in G_{n,k}; |f(E) - \mathbb{E}\,(f)| \leq \frac{\varepsilon}{3}\mathbb{E}\,(f)\right\} \geq 1 - c_1 \exp\left(-\frac{c_2^k \varepsilon^2 n}{k^{k/2}}\right)$$

The hypothesis on k implies $\varepsilon \geq \frac{(\log\log n)^2}{\log n}$ and $k^{k/2} \ll n^{0,1}$, so

$$\mu\{E \in G_{n,k}; \big|f(E) - \frac{1}{(\sqrt{2\pi}L_K)^k}\big| \leq \frac{\varepsilon}{(\sqrt{2\pi}L_K)^k}\} \geq 1 - c_1\exp(-c_2 n^{0.9})$$

□

4. The isotropy constant of high dimensional sections

The purpose of this section is to show that for any convex body K in $\mathbb{R}^n$ there exists a high dimensional section that its isotropic constant is bounded(up to a logarithmic to the dimension factor). In particular:

Proposition 4.1. *For every symmetric convex body K in $\mathbb{R}^n$ and $k \leqslant (1 - \frac{1}{\log n})n$, there exists an $F \in G_{n,k}$ such that*

$$L_{K\cap F} \leqslant c(\log k)^{1/2}\log\log k$$

where $c > 0$ is a universal constant.

Proof. We will make use of the following fact that is a particular version of a result by Pisier, see [P], Chapter 7:

Let $0 < p < 2$. There exists a linear map $u\colon \ell_2^n \to (\mathbb{R}^n, \|\cdot\|_K)$ and universal constants $c, c' > 0$ such that for all $1 \leq k \leq n$,

- $c_k(u^{-1}) \leq \frac{c}{\sqrt{2-p}}\left(\frac{n}{k}\right)^{1/p}$ *and*
- $\log N(u(B_2^n), tK) \leq \left(\frac{c'}{\sqrt{2-p}}\right)^p \frac{n}{t^p}, \quad \forall\, t \geq \frac{c'}{\sqrt{2-p}}$

where $c_k(u^{-1}) = \inf\{\|u^{-1}|_S\|\,; codim(S) < k\}$ are the Gelfand numbers of u^{-1} and $N(K,L) = \inf\{N \in \mathbb{N} \mid \exists x_1, \dots x_N, K \subset \cup_{i=1}^N (x_i + L)\}$ is the covering number of K by L.

Let K be of volume 1. Define $K_1 := |det(u)|^{1/n}u^{-1}(K)$, also of volume 1, that is, we take K_1 into the so called Milman's position.
By definition of covering number and $t = \frac{c'}{\sqrt{2-p}}$ we have $N(u(B_2^n), tK) \leq e^n$ and so

$$|u(B_2^n)|_n = |det(u)|\omega_n \leq \Big(\frac{c'}{\sqrt{2-p}}\Big)^n e^n$$

Thus, by the estimate $\omega_n^{-1/n} \sim \sqrt{n}$,

$$|det(u)|^{1/n} \leq \frac{c'\sqrt{n}}{\sqrt{2-p}}$$

On the other hand, since in particular $c_1(u^{-1}) \leq \frac{c}{\sqrt{2-p}}n^{1/p}$, we have

$$|u^{-1}(x)| \leq \frac{c}{\sqrt{2-p}}n^{1/p} \qquad \forall\, x \in K$$

or equivalently, $u^{-1}(K) \subset \frac{c}{\sqrt{2-p}}n^{1/p}B_2^n$.
Therefore the two estimates above yield

$$K_1 \subset \frac{c_1}{\sqrt{2-p}}n^{\frac{1}{p}+\frac{1}{2}}B_2^n \quad \text{or equivalently} \quad R(K_1) \leq \frac{c_1}{\sqrt{2-p}}n^{\frac{1}{p}+\frac{1}{2}} \tag{4.12}$$

Now, by definition of Gelfand numbers, there exists a subspace S of codimension $< k$ such that

$$|u^{-1}(x)| \leq \frac{c}{\sqrt{2-p}}\Big(\frac{n}{k}\Big)^{1/p} \quad \forall\ x \in K \cap S$$

and so it follows (be the definition of K_1) that for every $1 \leq k \leq n$, there exists a subspace $F \in G_{n,k}$ such that

$$R(K_1 \cap F) \leq \frac{c_2\sqrt{n}}{2-p}\Big(\frac{n}{n-k}\Big)^{1/p} \tag{4.13}$$

Also, for any $F \in G_{n,k}$ we have that (see [Pa1])

$$\frac{1}{|K_1 \cap F|_k^{\frac{1}{n-k}}} \leqslant \hat{c}\frac{L(K_1, F^\perp)}{L(B_{n-k+1}(K_1, F^\perp))}$$

where $L^2(C) := \frac{1}{n}\int_{\widetilde{C}} |x|^2 dx$ and $L^2(K,F) := \frac{1}{k}\int_{\widetilde{K}} |P_F(x)|^2 dx$. We have that $L(C) \geq L_C \geq L_{D_n}$ (see [MP]) . So,

$$\frac{1}{|K_1 \cap F|_k^{\frac{1}{n-k}}} \leqslant c_3\frac{R(P_{F^\perp}(K_1))}{\sqrt{n-k}} \leqslant c_3\frac{R(K_1)}{\sqrt{n-k}}$$

Let $k = n\frac{\log{(n+2)}}{1+\log{(n+2)}}$ or $\frac{n-k}{k} = \frac{1}{\log{(n+2)}}$. Then if F is as before we have

$$kL^2_{K_1\cap F} \leqslant kL(K_1 \cap F)^2 = \frac{1}{|K_1 \cap F|_k^{1+\frac{2}{k}}}\int_{K_1\cap F} |x|^2 dx \leqslant$$

$$\frac{\int_{K_1\cap F}|x|^2dx}{|K_1 \cap F|_k}\left(\frac{1}{|K_1 \cap F|_k^{\frac{1}{n-k}}}\right)^{2\frac{n-k}{k}} \leqslant R^2(K_1 \cap F)\left(c_3\frac{R(K_1)}{\sqrt{n-k}}\right)^{\frac{2}{\log{(n+2)}}}$$

or, using the inequalities (4.12) and (4.13)

$$L^2_{K_1\cap F} \leqslant \frac{c_2 n}{k(2-p)^2}\Big(\frac{n}{n-k}\Big)^{2/p}\Big(\frac{c_1 n}{n-k}\frac{n^{\frac{2}{p}}}{2-p}\Big)^{\frac{1}{\log{(n+2)}}}$$

By taking $p = 2 - \frac{1}{\log\log n}$ and the choice of k we have $\frac{n}{k} \sim 1$, $\frac{n}{n-k} \sim \log n$ and $\frac{2}{p} = 1 + O(\frac{1}{\log\log n})$, so

$$L^2_{K_1\cap F} \leq c_4 \log n (\log\log n)^2$$

For any $T \in GL(n)$ we have $TK \cap F = T(K \cap T^{-1}F)$. So we have proved that for every K in $\mathbb{R}^n$ there exists a subspace of dimension $(1 - \frac{1}{\log{(n+2)}})n$ such that

$$L_{K\cap F} \leqslant c_5(\log n)^{1/2}\log\log n$$

Let $1 \leq k \leq (1 - \frac{1}{\log(n+2)})n$. Let $\lambda \in (0,1]$ so that $k = (1 - \frac{1}{\log(\lambda n+2)})\lambda n$ (note that $k \sim \lambda n$). If $E \in G_{n,\lambda n}$ and $K_0 = K \cap E$ then there exists $F \in G_{n,k}$ such that

$$L_{K\cap F} = L_{K_0\cap F} \leqslant c_5(\log \lambda n)^{1/2} \log\log \lambda n \sim (\log k)^{1/2} \log\log k$$

$\square$

References

[ABP] M. Anttila, K. Ball, I. Perissinaki, *The central limit theorem for convex bodies*, Trans. Amer. Math. Soc. **355** (2003), pp. 4723-4735.

[B] K. Ball, *Logarithmic concave functions and sections of convex sets in* $\mathbb{R}^n$, Studia Math. **88** (1988), pp. 69-84.

[BB] J. Bastero and J. Bernués, *Asymptotic behaviour of averages of k-dimensional marginals of measures on* $\mathbb{R}^n$, preprint.

[EK] R. Eldan and B. Klartag, *Pointwise Estimates for Marginals of Convex Bodies*, preprint.

[G] A. Giannopoulos, *Notes on isotropic convex bodies*, Warsaw University Notes (2003).

[GM] A. Giannopoulos and V. Milman, *Mean width and diameter of proportional sections of a symmetric convex body*, J. Reine Angew. Math. **497** (1998), pp. 113-139.

[H] D. Hensley, *Slicing convex bodies, bounds of slice area in terms of the body's covariance*, Proc. Amer. Math. Soc. **79** (1980), pp. 619-625.

[Kl] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. and Funct. Anal. (GAFA) **16** (2006) 1274-1290.

[Kl2] B. Klartag, *Power-law estimates for the central limit theorem for convex sets*, Journal Functional Analysis **245** (2007), pp. 284-310.

[LMS] A. Litvak, V.D. Milman and G. Schechtman, *Averages of norms and quasi-norms*, Math. Ann. **312** (1998), pp. 95124.

[M] V.D. Milman , *A new proof of A. Dvoretzky's theorem in cross-sections of convex bodies*, (Russian), Funkcional. Anal. i Prilozen. **5** (1971), no.4, 28-37.

[MP] V. Milman and A. Pajor, *Isotropic positions and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, GAFA Seminar 87-89, Springer Lecture Notes in Math. **1376** (1989), pp. 64-104.

[MS] V. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. **1200**, Springer, (1986).

[MS2] V.D. Milman and G. Schechtman, *Global versus Local asymptotic theories of finite-dimensional normed spaces*, Duke Math. Journal **90** (1997), 73-93.

[Pa1] G. Paouris, *Concentration of mass on convex bodies*, Geometric and Functional Analysis **16** (2006), pp. 1021-1049.

[Pa2] G. Paouris, *Ψ_2- estimates for linear functionals on zonoids*, Geom. Aspects of Funct. Analysis, Lecture Notes in Math. **1807** (2003), 211-222.

[Pa3] G. Paouris, *On the Ψ_2-behavior of linear functionals on isotropic convex bodies*, Studia Math. **168** (2005), no. 3, 285-299.

[P] G. Pisier, *The volume of convex bodies and Banach space geometry* Cambridge Univ. Press, Cambridge 1989.

[So] S. Sodin, *Tail-sensitive Gaussian asymptotics for marginals of concentrated measures in high dimension*, preprint.

Dpto. de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address, (David Alonso): 498220@celes.unizar.es

E-mail address, (Jesús Bastero): bastero@unizar.es

E-mail address, (Julio Bernués): bernues@unizar.es

E-mail address, (Grigoris Paouris): paouris@courant.nyu.edu

Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, U.S.A.