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REVERSE LOOMIS-WHITNEY INEQUALITIES VIA ISOTROPICITY

DAVID ALONSO-GUTIÉRREZ AND SILOUANOS BRAZITIKOS

ABSTRACT. Given a centered convex body $K \subseteq \mathbb{R}^n$, we study the optimal value of the constant $\tilde{\Lambda}(K)$ such that there exists an orthonormal basis $\{w_i\}_{i=1}^n$ for which the following reverse dual Loomis-Whitney inequality holds:

$$|K|^{n-1} \leqslant \tilde{\Lambda}(K) \prod_{i=1}^n |K \cap w_i^\perp|.$$

We prove that $\tilde{\Lambda}(K) \leqslant (CL_K)^n$ for some absolute $C > 1$ and that this estimate in terms of L_K, the isotropic constant of K, is asymptotically sharp in the sense that there exists another absolute constant $c > 1$ and a convex body K such that $(cL_K)^n \leqslant \tilde{\Lambda}(K) \leqslant (CL_K)^n$. We also prove more general reverse dual Loomis-Whitney inequalities as well as reverse restricted versions of Loomis-Whitney and dual Loomis-Whitney inequalities.

1. Introduction and notation

The classical Loomis-Whitney inequality [11] states that given a fixed orthonormal basis $\{e_i\}_{i=1}^n$, for any convex body $K \subseteq \mathbb{R}^n$ we have that

$$|K| \leqslant \prod_{i=1}^n |P_{e_i^\perp} K|^{\frac{1}{n-1}}, \tag{1.1}$$

where $|\cdot|$ denotes the volume (i.e., the Lebesgue measure) in the corresponding subspace and, for any k-dimensional linear subspace $H \in G_{n,k}$, P_H denotes the orthogonal projection onto H. Convex body is a compact convex set with non-empty interior and the set of all convex bodies $K \subseteq \mathbb{R}^n$ will be denoted by $\mathcal{K}^n$. The barycentre of a convex body $K \in \mathbb{R}^n$ is the vector

$$\text{bar}(K) = \frac{1}{|K|} \int_K x\, dx.$$

We call K centered if $\text{bar}(K) = 0$ and the set of all centered convex bodies will be denoted by $\mathcal{K}_c^n$. Finally, the set of all centrally symmetric convex bodies will be denoted by $\mathcal{K}_0^n$.

In [12], Meyer proved the following dual inequality: For any convex body $K \subseteq \mathbb{R}^n$

$$|K| \geqslant \frac{(n!)^{\frac{1}{n-1}}}{n^{\frac{n}{n-1}}} \prod_{i=1}^n |K \cap e_i^\perp|^{\frac{1}{n-1}}. \tag{1.2}$$

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In [5], Campi, Gritzmann and Gronchi considered the following problem. Given any convex body $K \subseteq \mathbb{R}^n$, find the largest constant $\Lambda(K)$ such that there exists an orthonormal basis $\{w_i\}_{i=1}^n$ for which the following inequality, reverse to the classical Loomis-Whitney inequality (1.1), holds:

$$|K|^{n-1} \geqslant \Lambda(K) \prod_{i=1}^{n} |P_{w_i^{\perp}} K|. \tag{1.3}$$

In the aforementioned paper the authors were interested in finding the value of $\Lambda(n) := \inf_{K\in\mathcal{K}^n} \Lambda(K)$. They found the exact value of this constant in the planar case and gave a lower bound for its value in any dimension. Subsequently, in [10], Koldobsky, Saroglou and Zvavitch gave the right asymptotic estimate for the value of the constant, of the order $\Lambda(n)^{\frac{1}{n}} \simeq n^{-\frac{1}{2}}$. Here, and through the whole text, the notation $a \simeq b$ is used to denote the existence of two absolute constants $c_1, c_2 > 0$ such that $c_1 b \leqslant a \leqslant c_2 b$.

In [6], Feng, Huang and Li considered the dual problem. Given any centered convex body $K \subseteq \mathbb{R}^n$, find the best constant $\tilde{\Lambda}(K)$ such that there exists an orthonormal basis $\{w_i\}_{i=1}^n$ for which the following inequality, reverse to the dual Loomis-Whitney inequality (1.2) holds:

$$|K|^{n-1} \leqslant \tilde{\Lambda}(K) \prod_{i=1}^{n} |K \cap w_i^{\perp}|. \tag{1.4}$$

They proved that if K is a centrally symmetric convex body in $\mathbb{R}^n$ then $\tilde{\Lambda}(K) \leqslant ((n-1)!)^n$. In other words, given a centered convex body $K \subseteq \mathbb{R}^n$, we are interested in the value of

$$\tilde{\Lambda}(K) = \min \frac{|K|^{n-1}}{\prod_{i=1}^{n} |K \cap w_i^{\perp}|}, \tag{1.5}$$

where the minimum is taken over all the orthogonal bases $\{w_i\}_{i=1}^n$ of $\mathbb{R}^n$. Moreover, we define

$$\tilde{\Lambda}(n) = \sup_{K\in\mathcal{K}_c^n} \tilde{\Lambda}(K) \quad \text{and} \quad \tilde{\Lambda}_0(n) = \sup_{K\in\mathcal{K}_0^n} \tilde{\Lambda}(K),$$

where the supremum is taken over all centered convex bodies K in $\mathbb{R}^n$ and over all centrally symmetric convex bodies respectively.

In this note, we describe the exact asymptotic behavior of $\tilde{\Lambda}(n)$ given by the following theorem. The precise definition of L_K, the isotropic constant of K, will be given in Section 2.

Theorem 1.1. *For every centered convex body $K \in \mathcal{K}_c^n$, we have that*

$$\tilde{\Lambda}(K) \leqslant \left(2\sqrt{3}L_K\right)^n.$$

Furthermore,

$$\left(\sqrt{2}L_n\right)^n \leqslant \tilde{\Lambda}(n) \leqslant \left(2\sqrt{3}L_n\right)^n,$$

where $L_n = \max_{K\in\mathcal{K}^n} L_K$, is the maximal isotropic constant.

Remark. Notice that the best known general upper bound for the isotropic constant (see section 2) gives an estimate $\tilde{\Lambda}(n) \leqslant (Cn^{\frac{1}{4}})^n$, improving the estimate $\tilde{\Lambda}(n) \leqslant ((n-1)!)^n$. Moreover, if we assume that the hyperplane conjecture is true, we have that $\tilde{\Lambda}(n)^{\frac{1}{n}} \simeq 1$.

As a consequence we obtain that for every centrally symmetric planar convex body $K \in \mathcal{K}_0^2$, we have that $\tilde{\Lambda}(K) \leq 1$. This inequality was proved in [6], where the equality cases were claimed to be characterized. Unfortunately, such characterization is not correct and, while it is true that $\tilde{\Lambda}_0(2) = 1$, the equality cannot be attained for any convex body (see Section 4).

Moreover, we prove the following general reverse inequality for sections of arbitrary dimension. Before stating the theorem, we need a more general definition for $\tilde{\Lambda}(K)$ and $\tilde{\Lambda}(n)$. Let $m \geqslant 1$ and let $\mathcal{S} = (S_1, \dots, S_m)$ be a uniform cover of $[n] := \{1, \dots, n\}$ with weights $(p_1, \dots, p_m)$, that is $S_j \subseteq [n]$ for every $1 \leqslant j \leqslant m$ and for every $1 \leqslant i \leqslant n$

$$\sum_{j=1}^{m} p_j \chi_{S_j}(i) = 1.$$

For any basis $\{w_i\}_{i=1}^n$ of $\mathbb{R}^n$, let $H_j = \mathrm{span}\{w_k : k \in S_j\}$, $d_j = \mathrm{dim} H_j = |S_j|$, and $p = \sum_{j=1}^m p_j$.

For every $\mathcal{S}$, we are interested in the value of

$$\tilde{\Lambda}_{\mathcal{S}}(K) = \min \frac{|K|^{p-1}}{\prod_{i=1}^n |K \cap H_j^{\perp}|^{p_j}},$$

where the minimum is taken over all the orthogonal bases $\{w_i\}_{i=1}^n$ of $\mathbb{R}^n$. Moreover, let

$$\tilde{\Lambda}_{\mathcal{S}}(n) = \sup_{K \in \mathcal{K}_c^n} \tilde{\Lambda}_{\mathcal{S}}(K),$$

where the supremum is taken over all centered convex bodies K in $\mathbb{R}^n$. Then, we have the following

Theorem 1.2. *There exists an absolute constant $C > 0$, such that for every centered convex body $K \in \mathcal{K}_c^n$ for any uniform cover $\mathcal{S} = (S_1, \dots, S_m)$ of $[n]$ with weights $(p_1, \dots, p_m)$, we have that*

$$\tilde{\Lambda}_{\mathcal{S}}(K) \leqslant (CL_K)^n.$$

Furthermore, there exist absolute constants c, C such that

$$\frac{(cL_n)^n}{\prod_{j=1}^m L_{d_j}^{p_j d_j}} \leqslant \tilde{\Lambda}_{\mathcal{S}}(n) \leqslant (CL_n)^n,$$

where $L_d = \max_{K \in \mathcal{K}^d} L_K$ is the maximal isotropic constant in $\mathbb{R}^d$.

Remark. Again, if we assume that the hyperplane conjecture is true we have $\tilde{\Lambda}_{\mathcal{S}}(n)^{\frac{1}{n}} \simeq 1$.

In [3], the following restricted Loomis-Whitney inequality was obtained; if $S \subseteq [n]$ has cardinality $|S| = d$ and $(S_1, \dots, S_m)$ form a uniform cover of S with the same weights $(\frac{1}{k}, \dots, \frac{1}{k})$, where $m > k$, then for every convex body $K \subseteq \mathbb{R}^n$ and any orthogonal basis $\{e_i\}_{i=1}^n$

$$|P_{H^{\perp}} K| |K|^{\frac{m}{k}-1} \leqslant \frac{\binom{n-\frac{kd}{m}}{n-d}^{\frac{m}{k}}}{\binom{n}{d}^{\frac{m}{k}-1}} \prod_{j=1}^{m} |P_{H_j^{\perp}} K|^{\frac{1}{k}}.$$

where $H_j = \mathrm{span}\{e_k : k \in S_j\}$ and $H = \mathrm{span}\{e_k : k \in S\}$. In particular, for every convex body $K \subseteq \mathbb{R}^n$ and any d-dimensional subspace $H \in G_{n,d}$, we have that for any orthogonal basis $\{e_i\}_{i=1}^d$ of H

$$|P_{H^\perp}K||K|^{d-1} \leqslant \frac{\binom{n-1}{n-d}^d}{\binom{n}{d}^{d-1}} \prod_{j=1}^{d} |P_{e_j^\perp}K|. \tag{1.6}$$

Dual restricted inequalities were also proved in [3]. We will also consider the problem of finding reverse restricted Loomis-Whitney inequalities and restricted dual Loomis-Whitney inequalities. We will prove the following two results:

Theorem 1.3. *Let $K \in \mathcal{K}^n$ be a convex body and let $2 \leqslant d \leqslant n-1$. For any $H \in G_{n,d}$ there exists an orthonormal basis $\{w_j\}_{j=1}^d$ of H such that if we denote $H = \mathrm{span}\{w_1, \dots, w_d\}$ then we have that*

$$|P_{H^\perp}K||K|^{d-1} \geqslant \frac{\binom{n+d}{n}}{(2n)^d} \prod_{i=1}^{d} |P_{w_i^\perp}K|.$$

Remark. Notice that if $d = 2$ then the constant in Theorem 1.3 and the constant in equation (1.6) are of the same order.

Theorem 1.4. *There exists an absolute constant C such that for every centered convex body $K \in \mathcal{K}_c^n$ and every $H \in G_{n,d}$ there exists an orthonormal basis $\{w_j\}_{j=1}^d$ of H such that*

$$|K||K \cap H^\perp|^{d-1} \leq C^{d(d-1)} d^{\frac{d}{2}} \prod_{j=1}^{d} |K \cap (H^\perp \oplus \langle w_j \rangle)|.$$

The paper is organized as follows: In Section 2 we provide the preliminary definitions and results that we use in order to prove our results. In Section 3 we prove the reverse dual Loomis-Whitney inequalities given by Theorem 1.1 and Theorem 1.2. In Section 4 we study the situation in the centrally symmetric planar case. Finally, in Section 5 we prove the restricted versions provided in Theorems 1.3 and 1.4.

2. Preliminaries

A convex body $K \in \mathcal{K}^n$ is called isotropic if $|K| = 1$, K is centered, and for every $\theta \in S^{n-1}$

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2,$$

where L_K is a constant depending on K, but not on θ, which is called the isotropic constant of K. Given any convex body $K \subseteq \mathbb{R}^n$ there exists an affine map $a + T$, with $a \in \mathbb{R}^n$ and $T \in GL(n)$ (unique up to orthogonal transformations), such that $a + TK$ is isotropic. The isotropic constant of K is then defined as the isotropic constant of any of its isotropic images. Such an affine map is the solution of a minimization problem, which allows to alternatively define L_K in the following way

$$nL_K^2 = \min\left\{ \frac{1}{|K|^{1+\frac{2}{n}}} \int_{a+TK} |x|^2 \,:\, a \in \mathbb{R}^n, T \in GL(n) \right\}.$$

It is well known that the Euclidean ball B_2^n is the n-dimensional convex body with the smallest isotropic constant and, as a consequence, there exists an absolute constant $c > 0$ such that $L_K \geqslant c$ for every convex body $K \subseteq \mathbb{R}^n$ and any $n \in \mathbb{N}$ (see, for instance, [4, Proposition 3.3.1]. However, it is still a major open problem (known as the slicing problem) whether there exists an absolute constant $C > 0$ such that $L_n := \max_{K\in\mathcal{K}^n} L_K \leqslant C$. This question was posed by Bourgain, who proved the upper bound $L_n \leqslant Cn^{\frac{1}{4}} \log n$ in [2]. This was improved to $L_n \leqslant Cn^{\frac{1}{4}}$ by Klartag in [9] and it is the currently best known bound. In the planar case, it is known (see [4, Theorem 3.5.7] and the results in [15]) that $L_2 = L_{\Delta^2} = \frac{1}{\sqrt{6}\sqrt[4]{3}}$. If we restrict ourselves to centrally symmetric convex bodies and denote $L_{n,0} := \max_{K\in\mathcal{K}^n_0} L_K$, then $L_{2,0} = L_{B^2_\infty} = \frac{1}{\sqrt{12}}$. Here Δ^n denotes the n-dimensional regular simplex and B^n_∞ denotes the n-dimensional cube. These (and their affine images) are the only convex bodies on which the maximums in $\mathcal{K}^n$ (and in $\mathcal{K}^n_0$) are attained.

Given a centered convex body $K \in \mathcal{K}^n_c$ with $|K| = 1$ and $p > 1$, its L_p-centroid body $Z_p(K)$ is defined by

$$h_{Z_p(K)}(y) = \left(\int_K |\langle x, y\rangle|^p dx\right)^{\frac{1}{p}}, \quad y \in \mathbb{R}^n,$$

where for any convex body $L \in \mathcal{K}^n$, $h_L(y) = \max\{\langle x, y\rangle : x \in L\}$ is the support function of L. Notice that, by Hölder's inequality, if $1 \leqslant p \leqslant q$ then $Z_p(K) \subseteq Z_q(K)$. Moreover, for any linear map $T \in SL(n)$, with $|\det T| = 1$, $Z_p(TK) = TZ_p(K)$, and that K is isotropic if and only if $Z_2(K) = L_K B^n_2$. If K is not isotropic and $|K| = 1$ then $Z_2(K)$ is an ellipsoid whose volume is $|Z_2(K)| = L^n_K |B^n_2|$ (see, for instance [4, Proposition 3.1.7]). In [8], Hensley proved that there exist two absolute constants c_1, c_2 such that for every centered convex body $K \in \mathcal{K}^n_c$ with $|K| = 1$ and every $\theta \in S^{n-1}$

$$\frac{c_1}{|K \cap \theta^\perp|} \leqslant h_{Z_2(K)}(\theta) \leqslant \frac{c_2}{|K \cap \theta^\perp|}. \tag{2.1}$$

The value of these two constants are known to be (see [13, Corollaries 2.5 and 2.7] and [7, Theorem 3]) $c_1 = \frac{1}{2\sqrt{3}}$ and $c_2(n) = \frac{n}{\sqrt{2(n+1)(n+2)}} \le \frac{1}{\sqrt{2}}$. Furthermore, there is equality in the left-hand side inequality if and only if K is cylindrical in the direction θ (i.e., $K = K \cap \theta^\perp + [-x, x]$ for some $x \in \mathbb{R}^n$) and there is equality in the right hand-side inequality if and only if K is a double cone in the direction θ.

The latter equation shows that for any isotropic convex body and any $\theta \in S^{n-1}$

$$|K \cap \theta^\perp| \simeq \frac{1}{L_K}.$$

More generally, in [13, Proposition 3.11] (see also [4, Proposition 5.1.15]) it was proved that for any isotropic convex body K and any d-dimensional linear subspace $H \in G_{n,d}$, there exists a d-dimensional convex body $B(K, H)$ such that

$$|K \cap H^\perp|^{\frac{1}{d}} \simeq \frac{L_{B(K,H)}}{L_K}. \tag{2.2}$$

It was proved by Paouris (see [4, Theorem 5.1.14]) that there exist two absolute constants c_1, c_2 such that for every centered convex body $K \in \mathcal{K}^n$ with $|K| = 1$ and every d-dimensional linear subspace $H \in G_{n,d}$

$$c_1 \leqslant |K \cap H^\perp|^{\frac{1}{d}} |P_H Z_d(K)|^{\frac{1}{d}} \leqslant c_2. \tag{2.3}$$

Given a convex body $K \in \mathcal{K}^n$, its polar projection body $\Pi^* K$ is the closed unit ball of the norm given by

$$\|x\|_{\Pi^* K} = |x||P_{x^\perp} K|,$$

which is a centrally symmetric convex body. Equivalently, its radial function is given by $\rho_{\Pi^* K}(\theta) = \frac{1}{|P_{\theta^\perp} K|}$, where for every convex body $L \in \mathcal{K}^n$ containing the origin in its interior, its radial function is defined for every $\theta \in S^{n-1}$ by $\rho_L(\theta) = \max\{\lambda > 0 \,:\, \lambda\theta \in L\}$. It is well known that for any convex body $K \in \mathcal{K}^n$, the affinely invariant quantity $|K|^{n-1}|\Pi^* K|$ is maximized when K is an ellipsoid and minimized when K is a simplex (see [14] and [16]). Thus, for every convex body $K \subseteq \mathbb{R}^n$

$$\frac{\binom{2n}{n}}{n^n} \leq |K|^{n-1}|\Pi^* K| \leq \left(\frac{|B_2^n|}{|B_2^{n-1}|}\right)^n.$$

In [1, Proposition 5.2], it was proved that for any convex body $K \in \mathcal{K}^n$ and any d-dimensional linear subspace $H \in G_{n,d}$

$$|K|^{d-1}|\Pi^* K \cap H| \geqslant \frac{\binom{n+d}{n}}{n^d |P_{H^\perp} K|}. \tag{2.4}$$

3. Proof of the reverse dual Loomis Whitney inequality

We begin this section by proving Theorem 1.1

Proof of Theorem 1.1. Let K be a centered convex body. We can assume without loss of generality that $|K| = 1$. Let $Z_2(K) \subseteq \mathbb{R}^n$ be the ellipsoid whose support function is given by

$$h_{Z_2(K)}(w) = \left(\int_K \langle x, w\rangle^2 dx\right)^{\frac{1}{2}}$$

for every $w \in S^{n-1}$. We have that $|Z_2(K)| = L_K^n |B_2^n|$. By (2.1) there exist two absolute constants $c_1 = \frac{1}{2\sqrt{3}}, c_2 = \frac{1}{\sqrt{2}}$ such that for every centered convex body $K \subseteq \mathbb{R}^n$ with volume 1 and every $w \in S^{n-1}$

$$\frac{c_1}{|K \cap w^\perp|} \leqslant h_{Z_2(K)}(w) \leqslant \frac{c_2}{|K \cap w^\perp|}.$$

Therefore, taking $\{w_i\}_{i=1}^n$ the orthonormal basis given by the principal axes of the ellipsoid $Z_2(K)$ we have

$$\prod_{i=1}^n |K \cap w_i^\perp| \geqslant \frac{c_1^n}{\prod_{i=1}^n h_{Z_2(K)}(w_i)} = \frac{c_1^n |B_2^n|}{|Z_2(K)|} = \frac{c_1^n}{L_K^n},$$

which proves that

$$\tilde{\Lambda}(K) \leq (CL_K)^n$$

with $C = \frac{1}{c_1} = 2\sqrt{3}$. To conclude the proof of Theorem 1.1 we first notice that from the above

$$\tilde{\Lambda}(n) \leqslant (2\sqrt{3}L_n)^n.$$

On the other hand, if we consider an isotropic convex body with isotropic constant $L_K = L_n$ we have that for every orthonormal basis $\{w_i\}$ of $\mathbb{R}^n$

$$\frac{c_1^n}{L_K^n} \leqslant \prod_{i=1}^n |K \cap w_i^\perp| \leqslant \frac{c_2^n}{L_K^n},$$

and, since $L_K = L_n$,

$$\tilde{\Lambda}(n) \geqslant (cL_n)^n,$$

with $c = \frac{1}{c_2} = \sqrt{2}$. This concludes the proof. $\square$

Remark. The latter proof shows that for every isotropic convex body, $\tilde{\Lambda}(K)^{\frac{1}{n}} \simeq L_K$.

We now move to the general case.

Proof of Theorem 1.2. Let $m \geqslant 1$ and let $\mathcal{S} = (S_1, \dots, S_m)$ be a uniform cover of $[n]$ with weights $(p_1, \dots, p_m)$. Let K be a centered convex body. We can assume without loss of generality that $|K| = 1$. Let $\{w_i\}_{i=1}^n$ be the orthonormal basis given by the principal axes of the ellipsoid $Z_2(K)$, whose support function is given by

$$h_{Z_2(K)}(w) = \left(\int_K \langle x, w \rangle^2 dx \right)^{\frac{1}{2}}.$$

Let $T \in GL(n)$ be the diagonal map with respect to the orthonormal basis $\{w_i\}_{i=1}^n$ given by $T(w_i) = \lambda_i w_i$ such that TK is isotropic. By (2.2), there exists an absolute constant c_1 such that for any $1 \leqslant j \leqslant m$ there exists a d_j-dimensional convex body $B(K, H_j)$, depending on K and $H_j = \operatorname{span}\{w_k : k \in S_j\}$, verifying

$$\begin{aligned} |K \cap H_j^{\perp}| &= |T^{-1}T(K \cap H_j^{\perp})| = \prod_{k \notin S_j} \frac{1}{\lambda_k} |TK \cap H_j^{\perp}| \\ &\geqslant \left(\frac{c_1 L_{B(K,H_j)}}{L_K} \right)^{d_j} \prod_{k \notin S_j} \frac{1}{\lambda_k}. \end{aligned}$$

Note that $\displaystyle\sum_{j=1}^m p_j d_j = n$, the m-tuple $(S_1^c, \dots, S_m^c)$ forms a uniform cover of $[n]$ with weights $(p_1', \dots, p_m')$, where $p_i' = \frac{p_i}{p-1}$, and $\prod_{i=1}^n \lambda_i = |T| = 1$ since $|K| = |TK| = 1$. Combining the above and calling $p = \displaystyle\sum_{i=1}^n p_i$ we get

$$\begin{aligned} \prod_{j=1}^m |K \cap H_j^{\perp}|^{p_j} &\geqslant \left(\frac{c_1}{L_K} \right)^n \prod_{j=1}^m \left(L_{B(K,H_j)} \right)^{p_j d_j} \frac{1}{\prod_{k \notin S_j} \lambda_k^{p_j}} \\ &= \left(\frac{c_1}{L_K} \right)^n \frac{\prod_{j=1}^m \left(L_{B(K,H_j)} \right)^{p_j d_j}}{\prod_{i=1}^n \lambda_i^{\sum_{j=1}^m p_j \chi_{S_j^c(i)}}} \\ &= \left(\frac{c_1}{L_K} \right)^n \frac{\prod_{j=1}^m \left(L_{B(K,H_j)} \right)^{p_j d_j}}{\prod_{i=1}^n \lambda_i^{p-1}} \\ &= \frac{c_1^n \prod_{j=1}^m \left(L_{B(K,H_j)} \right)^{p_j d_j}}{L_K^n}. \end{aligned}$$

This means that

$$\tilde{\Lambda}_{\mathcal{S}}(K) \leqslant \frac{(L_K)^n}{c_1^n \prod_{j=1}^m (L_{B(K,H_j)})^{p_j d_j}}.$$

Taking now the supremum over all orthonormal bases, and taking into account that there exists a universal constant $\tilde{c} > 0$ bounding from below the isotropic constant of any convex body in any dimension, we get that

$$\tilde{\Lambda}_{\mathcal{S}}(K) \leqslant \max \frac{(L_K)^n}{c_1^n \prod_{j=1}^m (L_{B(K,H_j)})^{p_j d_j}} \leqslant (C L_K)^n,$$

with $C = \frac{1}{\tilde{c}c_1}$.

If K is isotropic then, by (2.2), there exists a universal constant c_2 such that for any orthonormal basis $\{w_i\}_{i=1}^n$ and any uniform cover $\mathcal{S} = (S_1, \dots, S_m)$ of $[n]$ with weights $(p_1, \dots, p_m)$, we have that for every $1 \leqslant j \leqslant m$ the d_j-dimensional convex bodies $B(K, H_j)$ associated to K and $H_j = \mathrm{span}\{w_k : k \in S_j\}$ verifies

$$|K \cap H_j^\perp| \leqslant \left(\frac{c_2 L_{B(K,H_j)}}{L_K}\right)^{d_j},$$

and then for any orthonormal basis $\{w_i\}_{i=1}^n$ and any uniform cover $\mathcal{S} = (S_1, \dots, S_m)$ of $[n]$ with weights $(p_1, \dots, p_m)$

$$|K|^{p-1} \geqslant \frac{(L_K)^n}{c_2^n \prod_{j=1}^m (L_{B(K,H_j)})^{p_j d_j}} \prod_{j=1}^m |K \cap H_j^\perp|^{p_j}.$$

Therefore, taking $c = \frac{1}{c_2}$

$$\tilde{\Lambda}_{\mathcal{S}}(K) \geqslant \min \frac{(cL_K)^n}{\prod_{j=1}^m (L_{B(K,H_j)})^{p_j d_j}} \geqslant \frac{(cL_K)^n}{\prod_{j=1}^m L_{d_j}^{p_j d_j}},$$

where the minimum is taken over all the orthogonal basis $\{w_i\}_{i=1}^n$ in $\mathbb{R}^n$. Taking the convex body with maximal isotropic constant in $\mathbb{R}^n$, we get the reverse bound for $\tilde{\Lambda}_{\mathcal{S}}(n)$. □

Remark. Notice that if K is isotropic then one has that for any orthonormal basis $\{w_i\}_{i=1}^n$

$$\frac{(cL_K)^n}{\prod_{j=1}^m (L_{B(K,H_j)})^{p_j d_j}} \leqslant \frac{|K|^{p-1}}{\prod_{j=1}^m |K \cap H_j^\perp|^{p_j}} \leqslant \frac{(CL_K)^n}{\prod_{j=1}^m (L_{B(K,H_j)})^{p_j d_j}},$$

where c, C are absolute constants and so

$$\tilde{\Lambda}_{\mathcal{S}}(K)^{\frac{1}{n}} \simeq \min \frac{L_K}{\prod_{j=1}^m (L_{B(K,H_j)})^{\frac{p_j d_j}{n}}},$$

where the minimum is taken over all orthonormal bases $\{w_i\}_{i=1}^n$ in $\mathbb{R}^n$.

4. The centrally symmetric planar case

In this section we will study the centrally symmetric planar case and prove the following:

Proposition 4.1. *The value of* $\tilde{\Lambda}_0(2)$ *is*

$$\tilde{\Lambda}_0(2) = 1.$$

However, there exists no centrally symmetric planar convex body $K \in \mathcal{K}_0^2$ *such that* $\tilde{\Lambda}(K) = 1$.

In order to prove the proposition we will make use of the following lemma, which shows that when K is a centrally symmetric planar box, one of the two orthogonal vectors for which we obtain the minimum defining $\tilde{\Lambda}(K)$ has to be the direction of one of the diagonals.

Lemma 4.1. *Let $K \in \mathcal{K}_0^2$ be a centrally symmetric box (i.e. a centrally symmetric orthogonal parallelepiped) with $|K| = 1$. Then*

$$\tilde{\Lambda}(K) = \frac{|K|}{|K \cap w_1^\perp||K \cap w_2^\perp|} = \frac{l^4}{l^4+1} < 1,$$

where w_1 is the direction of a diagonal of K and w_2 is orthogonal to w_1 and $l \geq 1$ is the length of the largest side of K.

Remark. If we do not assume $|K| = 1$, then $l^2 > 1$ is the quotient of the largest side and the shortest side of the box.

Proof. We can assume without loss of generality that the sides of K are parallel to the coordinate axes. Let l denote the length of the vertical side of the box, which we can assume to be the longest one. Then $l \geqslant 1$ and

$$K = \operatorname{conv}\left\{ \left(\frac{1}{2l}, \frac{l}{2}\right), \left(-\frac{1}{2l}, \frac{l}{2}\right), \left(\frac{1}{2l}, -\frac{l}{2}\right), \left(-\frac{1}{2l}, -\frac{l}{2}\right) \right\}.$$

Let us take $w_2^\perp = \{(x,y) \in \mathbb{R}^2 : y = ax, a \in \mathbb{R}\}$ a generic linear hyperplane and w_1 an orthogonal vector to w_2. Thus, $w_1^\perp = \left\{(x,y) \in \mathbb{R}^2 : y = -\frac{1}{a}x\right\}$. Notice that if $a \in \left[l^2, \infty\right)$ then $w_2^\perp$ intersects with the boundary of K, ∂K, in the horizontal sides at the points $P_1 = \left(\frac{l}{2a}, \frac{l}{2}\right)$ and $-P_1$ and $w_1^\perp$ in the vertical sides at the points $P_2 = \left(\frac{1}{2l}, -\frac{1}{2al}\right)$ and $-P_2$, while if $a \in \left[\frac{1}{l^2}, l^2\right]$ both $w_1^\perp, w_2^\perp$ intersect ∂K in the vertical sides, being $w_2^\perp \cap \partial K$ the points $P_1' = \left(\frac{1}{2l}, \frac{a}{2l}\right)$ and $-P_1'$, and $w_1^\perp \cap \partial K$ the points $P_2' = \left(\frac{1}{2l}, -\frac{1}{2al}\right)$ and $-P_2'$.

Therefore, if $a \in \left[l^2, \infty\right)$, we have that

$$|K \cap w_1^\perp||K \cap w_2^\perp| = 1 + \frac{1}{a^2}$$

and if $a \in \left[\frac{1}{l^2}, l^2\right]$

$$|K \cap w_1^\perp||K \cap w_2^\perp| = \frac{1}{l^2}\left(a + \frac{1}{a}\right)$$

we have that $|K \cap w_1^\perp||K \cap w_2^\perp|$ is maximized in $a \in \left[\frac{1}{l^2}, \infty\right)$ for the values $a = l^2$ and $a = \frac{1}{l^2}$, which correspond to the cases in which either $w_2^\perp$ or $w_1^\perp$ passes through one of the vertices of the box. If this is the case,

$$|K \cap w_1^\perp||K \cap w_2^\perp| = \frac{l^4+1}{l^4}.$$

Since K is symmetric with respect to the coordinate axes, we have that for any $a \in \left(-\infty, \frac{1}{l^2}\right)$ there exists another pair of orthogonal lines $\tilde{w}_1^\perp, \tilde{w}_2^\perp$ described as before by a parameter $a_1 \in \left[\frac{1}{l^2}, \infty\right)$ for which

$$|K \cap w_1^\perp||K \cap w_2^\perp| = |K \cap \tilde{w}_1^\perp||K \cap \tilde{w}_2^\perp|.$$

Since in the case where w_1, w_2 are the coordinate vectors we have $|K \cap w_1^\perp||K \cap w_2^\perp| = 1$, it follows that

$$\max |K \cap w_1^\perp||K \cap w_2^\perp| = \frac{l^4+1}{l^4},$$

where the maximum is taken over all the pairs of orthogonal vectors in $\mathbb{R}^2$, and it is attained when one of the two vectors is the direction of the diagonal of K. □

Let us now prove Proposition 4.1:

Proof of Proposition 4.1. We argue like in the proof of Theorem 1.1. For any $K \in \mathcal{K}_0^2$ with $|K| = 1$, if w_1, w_2 are the principal axes of the inertia ellipsoid $Z_2(K)$ of K, and taking into account that $L_{2,0} = \frac{1}{\sqrt{12}}$, we have

$$\tilde{\Lambda}(K) \leqslant \frac{|K|}{|K \cap w_1^\perp||K \cap w_2^\perp|} \leqslant 12L_K^2 \leqslant 1. \tag{4.1}$$

Besides, by Lemma 4.1, we have that

$$\tilde{\Lambda}_0(2) \geqslant \lim_{l\to\infty} \frac{l^4}{l^4+1} = 1.$$

Therefore, $\tilde{\Lambda}_0(2) = 1$. If there exists a convex body K (we can assume that $|K| = 1$) such that $\tilde{\Lambda}(K) = 1$ then for such K all the inequalities in (4.1) are equalities. In particular, if we have equality in the second inequality, K is cylindrical both with respect to w_1 and w_2, which implies that K is a box. But in this case, the Lemma 4.1 gives $\tilde{\Lambda}(K) < 1$. □

Remark. In [6], the authors claimed that if $K \in \mathcal{K}_0^2$, then $\tilde{\Lambda}(K) = 1$ if and only if K is a parallelogram with one of its diagonals perpendicular to the edges. The following example shows that such characterization was not correct. Let

$$K = \text{conv}\left\{\left(0, \frac{1}{2}\right), \left(1, \frac{1}{2}\right), \left(0, -\frac{1}{2}\right), \left(-1, -\frac{1}{2}\right)\right\},$$

which is a symmetric parallelogram with the diagonal from $\left(0, \frac{1}{2}\right)$ to $\left(0, -\frac{1}{2}\right)$ perpendicular to the edge from $\left(0, \frac{1}{2}\right)$ to $\left(1, \frac{1}{2}\right)$. Notice that $|K| = 1$ and if we take w_1 in the direction of the diagonal from $\left(1, \frac{1}{2}\right)$ to $\left(-1, -\frac{1}{2}\right)$, we have that $w_1^\perp$ intersects the boundary of K at the points $P = \left(-\frac{1}{6}, \frac{1}{3}\right)$ and $-P$ and then, taking w_2 orthogonal to w_1 we have that

$$\frac{|K|}{|K \cap w_1^\perp||K \cap w_2^\perp|} = \frac{3}{5} < 1.$$

Thus, it is not true that $\tilde{\Lambda}(K) = 1$.

5. Restricted Versions

In this section we will prove reverse versions of restricted Loomis-Whitney and restricted dual Loomis-Whitney inequalities. We start proving Theorem 1.3.

Proof of Theorem 1.3. Let $K \in \mathcal{K}^n$, $H \in G_{n,d}$ and let Π^*K be the polar projection body of K. Since Π^*K is a centrally symmetric convex body, $\Pi^*K \cap H$ is a centrally symmetric convex body in H, using [5, Lemma 5.5], there exists a rectangular cross-polytope C contained in $\Pi^*K \cap H$ such that

$$|\Pi^*K \cap H| \leqslant d!|C|.$$

That is, there exist d orthogonal vectors $\{w_i\}_{i=1}^d \in S^{n-1} \cap H$ such that

$$C = \text{conv}\{\pm\rho_{\Pi^*K}(w_i)w_i\}_{i=1}^d \subseteq \Pi^*K \cap H$$

and

$$|\Pi^*K \cap H| \leqslant d!|C| = \prod_{i=1}^d 2\rho_{\Pi^*K}(w_i) = \frac{2^d}{\prod_{i=1}^d |P_{w_i^\perp}K|}.$$

Since, by (2.4), we have

$$|\Pi^* K \cap H| \geqslant \frac{\binom{n+d}{n}}{n^d |K|^{d-1} |P_{H^\perp} K|},$$

we obtain

$$|P_{H^\perp} K||K|^{d-1} \geqslant \frac{\binom{n+d}{n}}{(2n)^d} \prod_{i=1}^{d} |P_{w_i^\perp} K|.$$

□

Let us now prove the restricted dual Loomis-Whitney inequality given in Theorem 1.4.

Proof of Theorem 1.4. Let $K \in \mathcal{K}_c^n$ be a centered convex body. We can assume, without loss of generality, that $|K| = 1$. If $H \in G_{n,d}$, by the reverse Loomis-Whitney inequality (1.3) applied to the convex body $P_H(Z_d(K))$, with the value of the constant estimated in [10], there exists an absolute constant c and an orthonormal basis $\{w_j\}_{j=1}^d$ of H such that

$$|P_H(Z_d(K))|^{d-1} \geqslant \frac{1}{(cd)^{\frac{d}{2}}} \prod_{j=1}^{d} |P_{H\cap w_j^\perp} Z_d(K)|.$$

Using (2.3), we get that there exist two absolute constants c_1, c_2 such that

$$c_1^d \leqslant |K \cap H^\perp||P_H Z_d(K)| \leqslant c_2^d.$$

Therefore, for every $1 \leqslant j \leqslant d$

$$c_1^{d-1} \leqslant |K \cap (H^\perp \oplus \langle w_j \rangle)||P_{H\cap w_j^\perp} Z_{d-1}(K)| \leqslant c_2^{d-1}.$$

Combining the above with the fact that $Z_{d-1}(K) \subseteq Z_d(K)$, it follows that

$$\frac{c_2^{d(d-1)}}{|K \cap H^\perp|^{d-1}} \geqslant \frac{c_1^{d(d-1)}}{(cd)^{\frac{d}{2}}} \frac{1}{\prod_{j=1}^d |K \cap (H^\perp \oplus \langle w_j \rangle)|},$$

which gives the result. □

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Área de análisis matemático, Departamento de matemáticas, Facultad de Ciencias, Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza (Spain), IUMA
E-mail address, (David Alonso): alonsod@unizar.es

School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, JCMB, Peter Guthrie Tait Road King's Buildings, Mayfield Road, Edinburgh, EH9 3FD, Scotland.
E-mail address, (Silouanos Brazitikos): silouanos.brazitikos@ed.ac.uk