Brascamp-Lieb inequality and quantitative versions of Helly's theorem

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Abstract

We provide a number of new quantitative versions of Helly's theorem. For example, we show that for every family $\{P_i : i \in I\}$ of closed half-spaces

$$P_i = \{x \in \mathbb{R}^n : \langle x, w_i \rangle \leqslant 1\}$$

in $\mathbb{R}^n$ such that $P = \bigcap_{i \in I} P_i$ has positive volume, there exist $s \leqslant \alpha n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (Cn)^n \, |P|,$$

where $\alpha, C > 0$ are absolute constants. These results complement and improve previous work of Bárány-Katchalski-Pach and Naszódi. Our method combines the work of Srivastava on approximate John's decompositions with few vectors, a new estimate on the corresponding constant in the Brascamp-Lieb inequality and an appropriate variant of Ball's proof of the reverse isoperimetric inequality.

1 Introduction

Our starting point is a quantitative version of Helly's theorem on convex sets in Euclidean space. Helly's theorem states that if $\mathcal{P} = \{P_i : i \in I\}$ is a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$ and if any $n+1$ members of $\mathcal{P}$ have non-empty intersection then $\bigcap_{i \in I} P_i \neq \emptyset$. Bárány, Katchalski and Pach proved in [5] (see also [6]) the following quantitative "volume version":

Let $\mathcal{P} = \{P_i : i \in I\}$ be a finite family of convex sets in $\mathbb{R}^n$. If the intersection of any $2n$ or fewer members of $\mathcal{P}$ has volume greater than or equal to 1, then $\left|\bigcap_{i \in I} P_i\right| \geqslant c_n$, where $c_n > 0$ is a constant depending only on n.

Using the fact that every (closed) convex set is the intersection of a family of closed half-spaces and a simple compactness argument (see [5]) one can remove the restriction that $\mathcal{P}$ is finite and also assume that each P_i is a closed half-space. Therefore, the theorem of Bárány, Katchalski and Pach is equivalently stated as follows:

Let $\mathcal{P} = \{P_i : i \in I\}$ be a family of closed half-spaces in $\mathbb{R}^n$ such that $\left|\bigcap_{i \in I} P_i\right| > 0$. There exist $s \leqslant 2n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant C_n \left| \bigcap_{i \in I} P_i \right|, \tag{1.1}$$

where $C_n > 0$ is a constant depending only on n.

Note that the cube $[-1,1]^n$ in $\mathbb{R}^n$ can be written as the intersection of the $2n$ closed half-spaces $H_j^{\pm} := \{x : \langle x, \pm e_j \rangle \leqslant 1\}$ and that the intersection of any $2n-1$ of these half-spaces has infinite volume; this shows that one cannot replace $2n$ by $2n-1$ in the statement above. In [5] the authors offered a bound $C_n \leqslant n^{2n^2}$ for the constant C_n and they conjectured that one might actually have $C_n \leqslant n^{cn}$ for an absolute constant $c > 0$. Naszódi [15] has recently verified this conjecture; namely, he proved a volume version of Helly's theorem with $C_n \leqslant (Cn)^{2n}$, where $C > 0$ is an absolute constant. In Section 3 we present a slight modification of Naszódi's argument which leads to the exponent $\frac{3n}{2}$ instead of $2n$:

Theorem 1.1. *Let $\mathcal{P} = \{P_i : i \in I\}$ be a family of closed half-spaces such that $\left|\bigcap_{i\in I} P_i\right| > 0$. We may find $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that*

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (Cn)^{\frac{3n}{2}} \left|\bigcap_{i\in I} P_i\right|, \tag{1.2}$$

where $C > 0$ is an absolute constant.

The aim of this work is to study a natural question that arises from Theorem 1.1. Given $N > 2n$ we would like to estimate the quantity

$$C_{n,N} = \sup \frac{|P_{i_1} \cap \cdots \cap P_{i_N}|}{\left|\bigcap_{i\in I} P_i\right|} \tag{1.3}$$

where the supremum is over all families $\mathcal{P} = \{P_i : i \in I\}$ of closed half-spaces with $\left|\bigcap_{i\in I} P_i\right| > 0$. We would also like to study the same question in the case of families of symmetric strips in $\mathbb{R}^n$.

Starting with the symmetric case, our main result is the next theorem.

Theorem 1.2. *Let $\{P_i : i \in I\}$ be a family of symmetric strips*

$$P_i = \{x \in \mathbb{R}^n : |\langle x, w_i\rangle| \leqslant 1\} \tag{1.4}$$

in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume. For every $d > 1$ there exist $s \leqslant dn$ and $i_1, \dots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant \left(\frac{4\gamma_d}{\pi}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right) |P|, \tag{1.5}$$

where $\gamma_d := \left(\frac{\sqrt{d}+1}{\sqrt{d}-1}\right)^2$.

Note that if $d \gg 1$ then the constant $C_{n,\lfloor dn\rfloor}$ is bounded by $(Cn)^{\frac{n}{2}}$. In the non-symmetric case we first use a similar strategy (whose details are of course more delicate) to obtain an estimate comparable to the one in Theorem 1.1.

Theorem 1.3. *Let $\{P_i : i \in I\}$ be a family of closed half-spaces*

$$P_i = \{x \in \mathbb{R}^n : \langle x, v_i\rangle \leqslant 1\} \tag{1.6}$$

in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume. For every $d > 1$ there exist $s \leqslant (d+1)(n+1)$ and $i_1, \dots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant \gamma_d^{\frac{n+1}{2}} \frac{n^{n/2}(n+1)^{3(n+1)/2}}{\pi^{\frac{n}{2}} n!} \Gamma\left(\frac{n}{2} + 1\right) |P| \leqslant \gamma_d^{\frac{n+1}{2}} (Cn)^{\frac{3n}{2}} |P|, \tag{1.7}$$

where $C > 0$ is an absolute constant.

Note that Theorem 1.3 gives the same dependence on n as Theorem 1.1. In fact, Theorem 1.1 is stronger if what matters is to use (the smallest possible number of) $2n$ of the half-spaces P_i. On the other hand, there is a (small) difference in the value of the constant C involved in the two statements: the proof of Theorem 1.1 works with $C = 2\sqrt[3]{\pi}$, while the proof of Theorem 1.3 works with $C_d = \left(\frac{e\gamma_d}{2\pi}\right)^{\frac{1}{3}}$ (which is smaller than C if d is large enough).

However, if we relax the condition on the number s of half-spaces that we use (but still require that it is proportional to the dimension) we are able to (significantly) improve the exponent in the constant $C_{n,N}$ from $\frac{3n}{2}$ to n:

Theorem 1.4. *There exists an absolute constant $\alpha > 1$ with the following property: for every family $\{P_i : i \in I\}$ of closed half-spaces*

$$P_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leqslant 1\} \tag{1.8}$$

in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume, there exist $s \leqslant \alpha n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (Cn)^n \, |P|, \tag{1.9}$$

where $C > 0$ is an absolute constant.

Let us note that, in the recent paper [14], De Loera, La Haye, Rolnick and Soberón have presented many interesting results, both continuous and discrete, that may be viewed as quantitative versions of Carathéodory's, Helly's and Tverberg's theorems. For example, they prove that for every $n \geqslant 1$ and $\varepsilon > 0$ there exists a positive integer $N(n, \varepsilon)$ with the following property: if $\mathcal{F} = \{F_i : i \in I\}$ is a finite family of convex sets in $\mathbb{R}^n$ such that $|F_{i_1} \cap \cdots \cap F_{i_s}| \geqslant 1$ for all $s \leqslant Nn$ and all $i_1, \ldots, i_s \in I$, then

$$\left| \bigcap_{i\in I} F_i \right| \geqslant \frac{1}{1+\varepsilon}. \tag{1.10}$$

They also obtain a variant of this statement in which volume is replaced by diameter, as well as a "colorful" volume version of Helly's theorem. We would like to emphasize that the "philosophy" of all these results is completely different from the one in our work. The parameter $N(n, \varepsilon)$ is defined as the smallest integer such that, for every convex set $K \subset \mathbb{R}^n$ of positive volume there exists a polytope $P \supseteq K$ with at most $N(n, \varepsilon)$ facets such that $|P| \leqslant (1 + \varepsilon)|K|$, and it is known that $N(n, \varepsilon)$ is exponential in n and ε: one has

$$\left(\frac{c_1 n}{\varepsilon}\right)^{\frac{n-1}{2}} \leqslant N(n, \varepsilon) \leqslant \left(\frac{c_2 n}{\varepsilon}\right)^{\frac{n-1}{2}}. \tag{1.11}$$

We are interested in the best lower bound that one can obtain for $\left|\bigcap_{i\in I} F_i\right|$ in terms of a lower bound for the volume of the intersection of any $N \simeq n$ of the sets F_i (the main point is that N is assumed proportional to the dimension).

We close this introductory section by briefly explaining the main ideas behind the proof of our results in the non-symmetric case. We may assume that $P = \bigcap_{i\in I}\{x \in \mathbb{R}^n : \langle x, v_i\rangle \leqslant 1\}$ has finite volume and, since the statements are affinely invariant, that P is in John's position, i.e. the ellipsoid of maximal volume inscribed in P is the Euclidean unit ball B_2^n. Then, we have John's decomposition of the identity (see Section 2 for background information): there exists $J \subseteq I$ such that v_j, $j \in J$ are contact points of P and B_2^n and there are positive scalars a_j, $j \in J$ such that

$$I_n = \sum_{j\in J} a_j v_j \otimes v_j \quad \text{and} \quad \sum_{j\in J} a_j v_j = 0. \tag{1.12}$$

Given $d > 1$ we would like to extract a subset σ of J, of cardinality dn, which still forms an approximate John's decomposition of the identity with suitable weigths. To this end, for the proof of Theorem 1.3 we use a result of Batson, Spielman and Srivastava from [7]: there exists a subset $\sigma \subseteq J$ with $|\sigma| \leqslant dn$ and $b_j > 0$, $j \in \sigma$, such that

$$I_n \preceq \sum_{j\in\sigma} b_j a_j v_j \otimes v_j \preceq \gamma_d I_n, \tag{1.13}$$

where $\gamma_d := \left(\frac{\sqrt{d}+1}{\sqrt{d}-1}\right)^2$. For the proof of Theorem 1.4 we use a second, more delicate, theorem of Srivastava from [19] (see Section 4 for the precise statement).

Then, we would like to exploit an appropriate variant of Ball's proof of the reverse isoperimetric inequality in [3] in order to estimate the volume of $Q := \bigcap_{j\in\sigma} P_j$ using the Brascamp-Lieb inequality (see Section 5).

The main problem now is to obtain an estimate for the constant in the Brascamp-Lieb inequality that corresponds to an approximate John's decomposition. To the best of our knowledge this question had not been studied. Our main technical result is the next theorem; we feel that it is a useful tool of independent interest.

Theorem 1.5. *Let $\gamma > 1$. Let $u_1, \ldots, u_s \in S^{n-1}$ and $c_1, \ldots, c_s > 0$ satisfy*

$$I_n \preceq A := \sum_{j=1}^{s} c_j u_j \otimes u_j \preceq \gamma I_n \tag{1.14}$$

and set $\kappa_j = c_j \langle A^{-1} u_j, u_j \rangle > 0$, $1 \leqslant j \leqslant s$. If $f_1, \ldots, f_s : \mathbb{R} \to \mathbb{R}^+$ are measurable functions then

$$\int_{\mathbb{R}^n} \prod_{j=1}^{s} f_j^{\kappa_j}(\langle x, u_j \rangle) dx \leqslant \gamma^{\frac{n}{2}} \prod_{j=1}^{s} \left(\int_{\mathbb{R}} f_j(t) dt \right)^{\kappa_j}. \tag{1.15}$$

In Section 6 we present the proofs of the main results.

2 Notation and background

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $| \cdot |$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto F. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

We refer to the book of Schneider [18] for basic facts from the Brunn-Minkowski theory and to the book of Artstein-Avidan, Giannopoulos and V. Milman [1] for basic facts from asymptotic convex geometry.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$, and that K is centered if its barycenter

$$\mathrm{bar}(K) = \frac{1}{|K|} \int_K x \, dx \tag{2.1}$$

is at the origin. The polar body K° of K is defined by

$$K^\circ := \{ y \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } x \in K \}. \tag{2.2}$$

The Blaschke-Santaló inequality states that for every centered convex body K in $\mathbb{R}^n$ one has $|K||K^\circ| \leqslant \omega_n^2$, with equality if and only if K is an ellipsoid. The reverse Santaló inequality of Bourgain and V. Milman [8] states that there exists an absolute constant $c > 0$ such that

$$\left(|K||K^\circ| \right)^{1/n} \geqslant c/n, \tag{2.3}$$

where $c > 0$ is an absolute constant, for every convex body K in $\mathbb{R}^n$ which contains 0 in its interior.

We say that a convex body K is in John's position if the ellipsoid of maximal volume inscribed in K is the Euclidean unit ball B_2^n. John's theorem [13] states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist $u_1, \ldots, u_m \in \mathrm{bd}(K) \cap S^{n-1}$ (contact points of K and B_2^n) and positive real numbers $c_1, \ldots, c_m$ such that

$$\sum_{j=1}^{m} c_j u_j = 0 \tag{2.4}$$

and the identity operator I_n is decomposed in the form

$$I_n = \sum_{j=1}^{m} c_j u_j \otimes u_j, \tag{2.5}$$

where $(u_j \otimes u_j)(y) = \langle u_j, y\rangle u_j$. In the case where K is symmetric, the second condition (2.5) is enough (for any contact point u we have that $-u$ is also a contact point, and hence, having (2.5) we may easily produce a decomposition for which (2.4) is also satisfied). In analogy to John's position, we say that a convex body K is in Löwner's position if the ellipsoid of minimal volume containing K is the Euclidean unit ball B_2^n. One can check that this holds true if and only if K° is in John's position; in particular, we have a decomposition of the identity similar to (2.5).

Assume that $u_1, \ldots, u_m$ are unit vectors that satisfy John's decomposition (2.5) with some positive weights c_j. Then, one has the useful identities

$$\sum_{j=1}^{m} c_j = \mathrm{tr}(I_n) = n \quad \text{and} \quad \sum_{j=1}^{m} c_j \langle u_j, z\rangle^2 = 1 \tag{2.6}$$

for all $z \in S^{n-1}$. Moreover,

$$\mathrm{conv}\{u_1, \ldots, u_m\} \supseteq \frac{1}{n} B_2^n. \tag{2.7}$$

In the symmetric case we actually have

$$\mathrm{conv}\{\pm u_1, \ldots, \pm u_m\} \supseteq \frac{1}{\sqrt{n}} B_2^n. \tag{2.8}$$

Another useful fact, which goes back to the classical article of Dvoretzky and Rogers [11], is that we may choose $v_1, \ldots, v_n$, among the u_i's, which satisfy

$$\mathrm{dist}(v_k, \mathrm{span}(v_1, v_2, \ldots, v_{k-1})) \geqslant \sqrt{\frac{n-k+1}{n}} \tag{2.9}$$

for all $k = 2, \ldots, n$.

Finally, we state as a lemma a useful fact from linear algebra that will be used in Section 5.

Lemma 2.1. *Let A be an $n \times n$ invertible matrix. For any $u, v \in \mathbb{R}^n$ we have*

$$\det(A + u \otimes v) = \det(A)(1 + \langle A^{-1}u, v\rangle). \tag{2.10}$$

Proof. Let $u, v \in \mathbb{R}^n$. Starting with the identity

$$\begin{pmatrix} I_n & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} I_n + u \otimes v & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -v & 1 \end{pmatrix} = \begin{pmatrix} I_n & u \\ 0 & 1 + \langle u, v\rangle \end{pmatrix} \tag{2.11}$$

and taking determinants we see that $\det(I + u \otimes v) = 1 + \langle u, v\rangle$, which is the assertion of the lemma in the case $A = I_n$. Given any $n \times n$ invertible matrix A we write

$$A + u \otimes v = A(I_n + A^{-1}(u \otimes v)) = A(I_n + (A^{-1}u \otimes v)), \tag{2.12}$$

and applying the previous special case we obtain

$$\det(A + u \otimes v) = \det(A)\det(I_n + (A^{-1}u \otimes v)) = \det(A)(1 + \langle A^{-1}u, v\rangle) \tag{2.13}$$

as claimed. □

3 A refinement of Naszódi's argument

We start with a refinement of Naszódi's argument from [15]; our only new ingredient is the fact that every convex body K contains a centrally symmetric convex body K_1 of volume $|K_1| \geqslant 2^{-n}|K|$. Incorporating this in the original proof we obtain a better estimate.

Theorem 3.1. *Let $\mathcal{P} = \{P_i : i \in I\}$ be a family of closed half-spaces such that $\left|\bigcap_{i\in I} P_i\right| > 0$. We may find $s \leqslant 2n$ and $i_1, \ldots, i_s \in I$ such that*

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (Cn)^{\frac{3n}{2}} \left|\bigcap_{i\in I} P_i\right|, \tag{3.1}$$

where $C > 0$ is an absolute constant.

Proof. We start with a family $\mathcal{P} = \{P_i : i \in I\}$ of closed half-spaces $P_i = \{x : \langle x, u_i\rangle \leqslant 1\}$ such that $\left|\bigcap_{i\in I} P_i\right| < \infty$. We may assume that $\mathcal{P}$ is a finite family, therefore $P = \bigcap_{i\in I} P_i$ is a polytope. By affine invariance, we may also assume that P is in John's position. From John's theorem there exists $J \subseteq I$ such that u_j, $j \in J$ are contact points of P and B_2^n, and $a_j > 0$, $j \in J$ such that

$$I_n = \sum_{j\in J} a_j u_j \otimes u_j \quad \text{and} \quad \sum_{j\in J} a_j u_j = 0. \tag{3.2}$$

By the Dvoretzky-Rogers lemma, we may choose n of these contact points, which we denote by $v_1, \ldots, v_n$, so that

$$\mathrm{dist}(v_k, \mathrm{span}(v_1, v_2, \ldots, v_{k-1})) \geqslant \sqrt{\frac{n-k+1}{n}} \tag{3.3}$$

for all $k = 2, \ldots, n$. It follows that the simplex $S = \mathrm{conv}\{v_0 = 0, v_1, \ldots, v_n\} \subseteq P$ has volume

$$|S| = \frac{1}{n!} \prod_{k=1}^{n} \mathrm{dist}(v_k, \mathrm{span}(v_1, v_2, \ldots, v_{k-1})) \geqslant \frac{1}{n^{\frac{n}{2}} \sqrt{n!}}. \tag{3.4}$$

Now we use the fact (see [1, Theorem 4.1.20]) that if w is the center of mass of S then $S - w$ contains an origin symmetric convex body T_1 of volume $|T_1| \geqslant 2^{-n}|S - w| = 2^{-n}|S|$, and hence the convex body $T = T_1 + w \subseteq S$ has a center of symmetry at w and satisfies

$$|T| \geqslant 2^{-n}|S|. \tag{3.5}$$

Consider the ray ℓ from the origin in the direction of $-w$. Then, ℓ intersects the boundary of $\mathrm{conv}\{u_j, j \in J\}$ at a point $z \in \mathrm{conv}\{v_{n+1}, \ldots, v_{n+k}\}$ for some $v_{n+i} \in \{u_j, j \in J\}$ and $k \leqslant n$ (this follows from Carathéodory's theorem). Also, note that $\mathrm{conv}\{u_j, j \in J\} \supseteq \frac{1}{n} B_2^n$, and hence $\|z\|_2 \geqslant \frac{1}{n}$. Applying a contraction with center z and ratio

$$\lambda = \frac{\|z\|_2}{\|z - w\|_2} = \frac{\|z\|_2}{\|z\|_2 + \|w\|_2} \geqslant \frac{\|z\|_2}{1 + \|z\|_2} \geqslant \frac{1}{n+1}$$

to T, we obtain an origin symmetric convex body

$$Q \subseteq \mathrm{conv}\{z, v_1, \ldots, v_n\} \subseteq \mathrm{conv}\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+k}\} \tag{3.6}$$

with volume

$$|Q| \geqslant \frac{1}{(n+1)^n}|T| \geqslant \frac{1}{2^n(n+1)^n}|S| \geqslant \frac{1}{2^n(n+1)^n n^{\frac{n}{2}} \sqrt{n!}}. \tag{3.7}$$

Consider the intersection of $n+k \leqslant 2n$ half-spaces

$$R = \bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leqslant 1\}. \tag{3.8}$$

Using the Blaschke-Santaló inequality for Q and the fact that $B_2^n \subseteq P$ and $R \subseteq Q^\circ$ we get

$$\frac{|R|}{|P|} \leqslant \frac{|Q^\circ|}{|B_2^n|} \leqslant \frac{|B_2^n|}{|Q|}. \tag{3.9}$$

Finally, from (3.7) we see that

$$|R| \leqslant \frac{\pi^{\frac{n}{2}} 2^n (n+1)^n n^{\frac{n}{2}} \sqrt{n!}}{\Gamma\left(\frac{n}{2}+1\right)} |P| \tag{3.10}$$

and the result follows (with constant $C = 2\sqrt[3]{\pi}$ as one can check using Stirling's formula) . □

4 Approximate John's decompositions

Our first main tool is the work of Batson, Spielman and Srivastava [7] on spectral sparcification of graphs, in which they introduced a deterministic method extracting an approximate John's decomposition starting from a John's decomposition of the form (2.5). Their result is the following:

Theorem 4.1 (Batson-Spielman-Srivastava)**.** *Let $v_1, \ldots, v_m \in S^{n-1}$ and $a_1, \ldots, a_m > 0$ such that*

$$I_n = \sum_{j=1}^{m} a_j v_j \otimes v_j. \tag{4.1}$$

Then, for every $d > 1$ there exists a subset $\sigma \subseteq \{1, \ldots, m\}$ with $|\sigma| \leqslant dn$ and $b_j > 0$, $j \in \sigma$, such that

$$I_n \preceq \sum_{j \in \sigma} b_j a_j v_j \otimes v_j \preceq \gamma_d I_n, \tag{4.2}$$

where

$$\gamma_d := \left(\frac{\sqrt{d}+1}{\sqrt{d}-1} \right)^2. \tag{4.3}$$

Here, given two symmetric positive definite matrices A and B we write $A \preceq B$ if $\langle Ax, x \rangle \leqslant \langle Bx, x \rangle$ for all $x \in \mathbb{R}^n$. Using this fact, Srivastava [19] obtained an improved version of Rudelson's theorem [17] on the approximation of a symmetric convex body K by a symmetric convex body T which has few contact points with its maximal volume ellipsoid: for any symmetric convex body K in $\mathbb{R}^n$ and any $\varepsilon > 0$ there exists a symmetric convex body T such that $T \subseteq K \subseteq (1+\varepsilon)T$ and T has at most $32n/\varepsilon^2$ contact points with its John ellipsoid.

In order to deal with the not-necessarily symmetric case of this question, Srivastava proved in [19] the next variant of Theorem 4.1:

Theorem 4.2 (Srivastava)**.** *Let $v_1, \ldots, v_m \in S^{n-1}$ and $a_1, \ldots, a_m > 0$ such that*

$$I_n = \sum_{j=1}^{m} a_j v_j \otimes v_j \quad \text{and} \quad \sum_{j=1}^{m} a_j v_j = 0. \tag{4.4}$$

Given $\varepsilon>0$ we can find a subset σ of $\{1,\dots,m\}$ of cardinality $|\sigma|=O_\varepsilon(n)$, positive scalars c_i, $i\in\sigma$ and a vector v with

$$\|v\|_2^2\leqslant\frac{\varepsilon}{\sum_{i\in\sigma}c_i}, \tag{4.5}$$

such that

$$I_n\preceq\sum_{i\in\sigma}c_i(v_i+v)\otimes(v_i+v)\preceq(4+\varepsilon)I_n \tag{4.6}$$

and

$$\sum_{i\in\sigma}c_i(v_i+v)=0. \tag{4.7}$$

Using Theorem 4.2, Srivastava showed that for any convex body K in $\mathbb{R}^n$ and any $\varepsilon>0$ there exists a convex body T such that $T\subseteq K\subseteq(\sqrt{5}+\varepsilon)T$ and T has at most $O_\varepsilon(n)$ contact points with its John ellipsoid. We will use Theorem 4.2 in order to deal with the not-necessarily symmetric case of our problem, which is clearly much more interesting than the symmetric one.

5 Brascamp-Lieb inequality and approximate John decompositions

The Brascamp-Lieb inequality [9] estimates the norm of the multilinear operator $G:L^{p_1}(\mathbb{R})\times\cdots\times L^{p_m}(\mathbb{R})\to\mathbb{R}$ defined by

$$G(f_1,\dots,f_m)=\int_{\mathbb{R}^n}\prod_{j=1}^m f_j(\langle x,u_j\rangle)\,dx, \tag{5.1}$$

where $m\geqslant n$, $p_1,\dots,p_m\geqslant 1$ with $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=n$, and $u_1,\dots,u_m\in\mathbb{R}^n$. Brascamp and Lieb proved that the norm of G is the supremum D of

$$\frac{G(g_1,\dots,g_m)}{\prod_{j=1}^m\|g_j\|_{p_j}} \tag{5.2}$$

over all centered Gaussian functions $g_1,\dots,g_m$, i.e. over all functions of the form $g_j(t)=e^{-\lambda_j t^2}$, $\lambda_j>0$.

If we set $c_j=1/p_j$ and replace f_j by $f_j^{c_j}$ then we can state the Brascamp-Lieb inequality in the following form.

Theorem 5.1 (Bracamp-Lieb). *Let $m\geqslant n$, and let $u_1,\dots,u_m\in\mathbb{R}^n$ and $c_1,\dots,c_m>0$ with $c_1+\cdots+c_m=n$. Then,*

$$\int_{\mathbb{R}^n}\prod_{j=1}^m f_j^{c_j}(\langle x,u_j\rangle)dx\leqslant D\prod_{j=1}^m\left(\int_{\mathbb{R}}f_j\right)^{c_j} \tag{5.3}$$

for all integrable functions $f_j:\mathbb{R}\to[0,\infty)$, where $D=1/\sqrt{F}$ and

$$F=\inf\left\{\frac{\det\left(\sum_{j=1}^m c_j\lambda_j u_j\otimes u_j\right)}{\prod_{j=1}^m\lambda_j^{c_j}}:\lambda_j>0\right\}. \tag{5.4}$$

Calculating the constant $F = F(\{u_j\},\{c_j\})$ in Theorem 5.1 seems difficult. An important observation of Ball (see e.g. [2]) is that if $u_1,\ldots,u_m \in S^{n-1}$ and $c_1,\ldots,c_m > 0$ satisfy John's decomposition of the identity (2.5) then the constant $F = F(\{u_j\},\{c_j\})$ in Theorem 5.1 is equal to 1.

The next proposition shows that we still have a Brascamp-Lieb inequality with a reasonable constant when an approximate John's decomposition is available.

Proposition 5.2. *Let $\gamma > 1$. If $u_1,\ldots,u_s \in S^{n-1}$ and $c_1,\ldots,c_s > 0$ satisfy*

$$I_n \preceq A := \sum_{j=1}^{s} c_j u_j \otimes u_j \preceq \gamma I_n \tag{5.5}$$

then

$$\gamma^n \det\left(\sum_{j=1}^{s} \kappa_j \lambda_j u_j \otimes u_j\right) \geqslant \prod_{j=1}^{s} \lambda_j^{\kappa_j} \tag{5.6}$$

for all $\lambda_1,\ldots,\lambda_s > 0$, where $\kappa_j = c_j\langle A^{-1}u_j, u_j\rangle > 0$, $1 \leqslant j \leqslant s$.

Proof. For every $M \subset \{1,\ldots,s\}$ with cardinality $|M| = n$ we define

$$\lambda_M = \prod_{j\in M} \lambda_j \quad \text{and} \quad U_M = \det\left(\sum_{j\in M} c_j u_j \otimes u_j\right). \tag{5.7}$$

By the Cauchy-Binet formula we have

$$\det\left(\sum_{j=1}^{s} c_j \lambda_j u_j \otimes u_j\right) = \sum_{|M|=n} \lambda_M U_M. \tag{5.8}$$

Choosing $\lambda_j = 1$ in (5.8) we get

$$\sum_{|M|=n} U_M = \det(A). \tag{5.9}$$

By the arithmetic-geometric means inequality,

$$\sum_{|M|=n} \lambda_M \frac{U_M}{\sum_{|M|=n} U_M} \geqslant \prod_{|M|=n} \lambda_M^{\frac{U_M}{\sum_{|M|=n} U_M}} = \prod_{j=1}^{s} \lambda_j^{\frac{\sum_{\{M:j\in M\}} U_M}{\sum_{|M|=n} U_M}}. \tag{5.10}$$

Applying the Cauchy-Binet formula again, we get

$$\begin{aligned}\frac{\sum_{\{M:j\in M\}} U_M}{\sum_{|M|=n} U_M} &= \frac{\sum_{|M|=n} U_M - \sum_{\{M:j\notin M\}} U_M}{\sum_{|M|=n} U_M} = 1 - \frac{\det\left(A - c_j u_j \otimes u_j\right)}{\det(A)} \\ &= 1 - (1 - c_j\langle A^{-1}u_j, u_j\rangle) = c_j\langle A^{-1}u_j, u_j\rangle\end{aligned}$$

for every $j = 1,\ldots,s$, where in the last equality we used Lemma 2.1. Going back to (5.8) and (5.10) we see that

$$\frac{\det\left(\sum_{j=1}^{s} c_j \lambda_j u_j \otimes u_j\right)}{\det(A)} \geqslant \prod_{j=1}^{s} \lambda_j^{c_j\langle A^{-1}u_j, u_j\rangle} \tag{5.11}$$

We set

$$\kappa_j = c_j\langle A^{-1}u_j, u_j\rangle, \qquad j = 1, \dots, s. \tag{5.12}$$

Since $I_n \preceq A \preceq \gamma I_n$ we have that $\det(A) \geqslant 1$ and $\gamma\kappa_j = c_j\gamma\langle A^{-1}u_j, u_j\rangle \geqslant c_j$ for all $1 \leqslant j \leqslant s$. This implies that, for all $\lambda_1, \dots, \lambda_s > 0$,

$$\sum_{j=1}^{s} c_j\lambda_j u_j \otimes u_j \preceq \gamma\left(\sum_{j=1}^{s}\kappa_j\lambda_j u_j \otimes u_j\right). \tag{5.13}$$

Combining (5.11) and (5.13) we get

$$\gamma^n \det\left(\sum_{j=1}^{s}\kappa_j\lambda_j u_j \otimes u_j\right) \geqslant \prod_{j=1}^{s}\lambda_j^{\kappa_j} \tag{5.14}$$

as claimed. □

Remark 5.3. Setting $\lambda_1 = \cdots = \lambda_s = \lambda > 0$ in the conclusion of Proposition 5.2, we get

$$\gamma^n\lambda^n \det\left(\sum_{j=1}^{s}\kappa_j u_j \otimes u_j\right) \geqslant \lambda^{\sum_{j=1}^{s}\kappa_j}. \tag{5.15}$$

Since this holds true for any $\lambda > 0$, we must have

$$\sum_{j=1}^{s}\kappa_j = n. \tag{5.16}$$

We can also check this directly: note that

$$\begin{aligned}\sum_{j=1}^{s}\kappa_j &= \sum_{j=1}^{s}c_j\langle A^{-1}u_j, u_j\rangle = \sum_{j=1}^{s}c_j\,\mathrm{tr}(u_j\otimes A^{-1}u_j) = \mathrm{tr}\left(\sum_{j=1}^{s}c_j(u_j\otimes A^{-1}u_j)\right)\\ &= \mathrm{tr}\left(\sum_{j=1}^{s}c_jA^{-1}(u_j\otimes u_j)\right) = \mathrm{tr}\left(A^{-1}\Big(\sum_{j=1}^{s}c_j(u_j\otimes u_j)\Big)\right) = \mathrm{tr}(A^{-1}A) = \mathrm{tr}(I_n) = n.\end{aligned} \tag{5.17}$$

Having verified condition (5.16), we conclude from Proposition 5.2 that the constant in the Brascamp-Lieb inequality that corresponds to $\{u_j\}_{j=1}^s$ and $\{\kappa_j\}_{j=1}^s$ is bounded by $\gamma^{n/2}$. We will use this observation in the following form:

Theorem 5.4. *Let $\gamma > 1$. Let $u_1, \dots, u_s \in S^{n-1}$ and $c_1, \dots, c_s > 0$ satisfy*

$$I_n \preceq A := \sum_{j=1}^{s}c_ju_j\otimes u_j \preceq \gamma I_n \tag{5.18}$$

and set $\kappa_j = c_j\langle A^{-1}u_j, u_j\rangle > 0$, $1 \leqslant j \leqslant s$. If $f_1, \dots, f_s : \mathbb{R} \to \mathbb{R}^+$ are integrable functions then

$$\int_{\mathbb{R}^n}\prod_{j=1}^{s}f_j^{\kappa_j}(\langle x, u_j\rangle)dx \leqslant \gamma^{\frac{n}{2}}\prod_{j=1}^{s}\left(\int_{\mathbb{R}}f_j(t)dt\right)^{\kappa_j}. \tag{5.19}$$

6 Volume approximation by convex bodies with few facets

In this section we prove the main theorems of this article. We show that the intersection of any family of closed half-spaces is contained in an intersection of $N \simeq n$ of these half-spaces whose volume is reasonably small. This implies our quantitative versions of Helly's theorem as explained in the introduction.

We start with the symmetric case.

Theorem 6.1. *Let $\{P_i : i \in I\}$ be a family of symmetric strips*

$$P_i = \{x \in \mathbb{R}^n : |\langle x, v_i\rangle| \leqslant 1\} \tag{6.1}$$

in $\mathbb{R}^n$, and let $P = \bigcap_{i\in I} P_i$. For every $d > 1$ there exist $s \leqslant dn$ and $i_1, \dots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant \left(\frac{2}{\sqrt{\pi}}\frac{\sqrt{d}+1}{\sqrt{d}-1}\right)^n \Gamma\left(\frac{n}{2}+1\right)|P|. \tag{6.2}$$

Proof. We may assume that P is in John's position. From John's theorem there exists $J \subseteq I$ so that the vectors v_j, $j \in J$ are contact points of P and S^{n-1} and there exist $a_j > 0$, $j \in J$, such that

$$I_n = \sum_{j\in J} a_j v_j \otimes v_j. \tag{6.3}$$

Theorem 4.1 shows that there exists a subset $\sigma \subseteq J$ with $|\sigma| = s \leqslant dn$ and $b_j > 0$, $j \in \sigma$, such that

$$I_n \preceq \sum_{j\in\sigma} b_j a_j v_j \otimes v_j \preceq \gamma_d I_n, \tag{6.4}$$

where $\gamma_d = \left(\frac{\sqrt{d}+1}{\sqrt{d}-1}\right)^2$. We rewrite the vectors v_j, $j \in \sigma$, as $w_1, \dots, w_s$ and we set $c_j = a_j b_j$. Now, we apply Theorem 5.4 to find $\kappa_j > 0$, $1 \leqslant j \leqslant s$ such that $\sum_{j=1}^s \kappa_j = n$ and

$$\int_{\mathbb{R}^n} \prod_{j=1}^s f_j^{\kappa_j}(\langle x, w_j\rangle)dx \leqslant \gamma_d^{\frac{n}{2}} \prod_{j=1}^s \left(\int_{\mathbb{R}} f_j(t)dt\right)^{\kappa_j} \tag{6.5}$$

for any choice of non-negative integrable functions $f_1, \dots, f_s$ on $\mathbb{R}^n$. Note that

$$|P_1 \cap \cdots \cap P_s| = \int_{\mathbb{R}^n} \prod_{j=1}^s \mathbf{1}_{[-1,1]}(\langle x, w_j\rangle)^{\kappa_j} dx. \tag{6.6}$$

Since $\int_{\mathbb{R}} \mathbf{1}_{[-1,1]}(t)dt = 2$, from Theorem 5.4 we get

$$|P_1 \cap \cdots \cap P_s| \leqslant 2^n \gamma_d^{\frac{n}{2}}. \tag{6.7}$$

Since $B_2^n \subseteq P$, we also have

$$|P| \geqslant |B_2^n| = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \tag{6.8}$$

and the result follows. □

Remark 6.2. The proof of Theorem 6.1 shows that if K is a symmetric convex body in John's position then for every $d > 1$ there exist $s \leqslant dn$ and $w_1, \dots, w_s \in S^{n-1}$ such that

$$K \subseteq P := \bigcap_{j=1}^s \{x \in \mathbb{R}^n : |\langle x, w_j\rangle| \leqslant 1\} \tag{6.9}$$

and

$$|P|^{\frac{1}{n}} \leqslant 2\frac{\sqrt{d}+1}{\sqrt{d}-1}. \tag{6.10}$$

This estimate should be compared to well-known lower bounds for the volume of intersections of strips, due to Carl-Pajor [10], Gluskin [12] and Ball-Pajor [4]. If we fix $d > 1$ and set $N = \lfloor dn \rfloor$ then for any choice of vectors $w_1, \ldots, w_N$ spanning $\mathbb{R}^n$, with $\|w_i\|_2 \leqslant 1$ for all $1 \leqslant i \leqslant N$, we know that the body $P = \bigcap_{j=1}^N \{x \in \mathbb{R}^n : |\langle x, w_j\rangle| \leqslant 1\}$ satisfies

$$|P|^{\frac{1}{n}} \geqslant \frac{2}{\sqrt{e}\sqrt{\log(1+d)}}. \tag{6.11}$$

which is of the same order (up to the dependence on d).

On the other hand, even if we ask that $N = n$ (which corresponds to $d = 1$), one may find upper estimates of the form (6.10) in the literature: for example, if K is a symmetric convex body in John's position and if $v_1, \ldots, v_n$ are the vectors in (2.9) then the parallelepiped

$$P = \{x \in \mathbb{R}^n : |\langle x, v_j\rangle| \leqslant 1,\ j = 1, \ldots, n\}, \tag{6.12}$$

satisfies $K \subseteq P$ and

$$|P|^{\frac{1}{n}} = 2|\det(v_1, v_2, \ldots, v_n)|^{-\frac{1}{n}} \leqslant \frac{2\sqrt{n}}{(n!)^{\frac{1}{2n}}} \sim 2\sqrt{e}. \tag{6.13}$$

This result is due to Dvoretzky and Rogers, and an estimate of the same order (but improving in a sense the constants involved) was obtained by Pelczynski and Szarek in [16]. Comparing $2\sqrt{e}$ with $2\frac{\sqrt{d}+1}{\sqrt{d}-1}$ we see that our estimate provides a better bound if we allow a larger, but still proportional to the dimension, number of strips.

Next, we pass to the not-necessarily symmetric case; we consider a family $\{P_i : i \in I\}$ of closed half-spaces and ask for a collection of s half-spaces P_j such that $|P_1 \cap \cdots \cap P_s| \leqslant c_{n,s} \left|\bigcap_{i\in I} P_i\right|$. We give two arguments. The first one is based on the ideas of Theorem 6.1 and establishes (for any $d > 1$) a choice of $s \leqslant (d+1)(n+1)$ half-spaces and a bound of the order of $n^{3n/2}$ for the constant $c_{n,s}$.

Theorem 6.3. *Let $\{P_i : i \in I\}$ be a family of closed half-spaces*

$$P_i = \{x \in \mathbb{R}^n : \langle x, u_i\rangle \leqslant 1\} \tag{6.14}$$

in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume. For every $d > 1$ there exist $s \leqslant (d+1)(n+1)$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant \gamma_d^{\frac{n+1}{2}} \frac{n^{n/2}(n+1)^{3(n+1)/2}}{\pi^{\frac{n}{2}} n!} \Gamma\left(\frac{n}{2}+1\right) |P| \leqslant \gamma_d^{\frac{n+1}{2}} (Cn)^{\frac{3n}{2}} |P|, \tag{6.15}$$

where $C > 0$ is an absolute constant.

Proof. We may assume that P is in John's position. From John's theorem there exists $J \subseteq I$ so that the vectors u_j, $j \in J$ are contact points of P and S^{n-1} and there exist $a_j > 0$, $j \in J$, such that

$$I_n = \sum_{j\in J} a_j u_j \otimes u_j \quad \text{and} \quad \sum_{j\in J} a_j u_j = 0. \tag{6.16}$$

Set

$$v_j = \sqrt{\frac{n}{n+1}}\left(-u_j, \frac{1}{\sqrt{n}}\right) \quad \text{and} \quad b_j = \frac{n+1}{n} a_j. \tag{6.17}$$

Then

$$I_{n+1} = \sum_{j\in J} b_j v_j \otimes v_j. \tag{6.18}$$

Theorem 4.1 shows that there exists a subset $\sigma \subseteq J$ with $|\sigma| = s \leqslant d(n+1)$ and $\delta_j > 0$, $j \in \sigma$, such that

$$I_{n+1} \preceq A := \sum_{j\in\sigma} \delta_j b_j v_j \otimes v_j \preceq \gamma_d I_{n+1}, \tag{6.19}$$

where $\gamma_d = \left(\frac{\sqrt{d}+1}{\sqrt{d}-1}\right)^2$. We fix the vectors v_j, $j \in \sigma$, and set $c_j = \delta_j b_j$. We also consider the vector

$$w := -\frac{1}{n(n+1)} \sum_{j\in\sigma} \kappa_j u_j, \tag{6.20}$$

where $\kappa_j = c_j\langle A^{-1}u_j, u_j\rangle > 0$, $j \in \sigma$ are the scalars provided by Proposition 5.2. Recall that, by John's theorem, $\mathrm{conv}\{u_j, j \in J\} \supseteq \frac{1}{n}B_2^n$, and $\|w\|_2 \leqslant \frac{1}{n}$ by the triangle inequality and the fact that $\sum_{j\in\sigma} \kappa_j = n+1$. From Carathéodory's theorem we get that there exists $\tau \subseteq J$ with $|\tau| \leqslant n+1$ and $\rho_i > 0$ with $\sum_{i\in\tau} \rho_i = 1$ so that

$$w = \sum_{i\in\tau} \rho_i u_i. \tag{6.21}$$

We define

$$Q = \{x \in \mathbb{R}^n : \langle x, u_j\rangle < 1 \ \text{ for all } j \in \sigma\} \tag{6.22}$$

and

$$Q' = Q \cap \{x \in \mathbb{R}^n : \langle x, u_i\rangle \leqslant 1 \ \text{ for all } i \in \tau\}. \tag{6.23}$$

From Theorem 5.4 we know that if $f_j : \mathbb{R} \to \mathbb{R}^+$, $j \in \sigma$ are integrable functions, then

$$\int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) dy \leqslant \gamma_d^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\int_{\mathbb{R}} f_j(t)dt\right)^{\kappa_j}. \tag{6.24}$$

For $j \in \sigma$ we define $f_j(t) = e^{-t}\mathbf{1}_{[0,\infty)}(t)$. Let $y = (x, r) \in \mathbb{R}^{n+1}$. We easily check that if $r > 0$ and $x \in \frac{r}{\sqrt{n}}Q$ then $\langle x, u_j\rangle < \frac{r}{\sqrt{n}}$ for all $j \in \sigma$. This implies that $\langle y, v_j\rangle > 0$ for all $j \in \sigma$, and hence $\prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) > 0$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) dy &\geqslant \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) dy \\
&= \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q} \exp\left(-\sum_{j\in\sigma} \kappa_j\langle (x,r), v_j\rangle\right) dx\, dr \\
&= \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q} \exp\left(\sqrt{\frac{n}{n+1}} \sum_{j\in\sigma} \kappa_j\langle x, u_j\rangle - \frac{r}{\sqrt{n+1}} \sum_{j\in\sigma} \kappa_j\right) dx\, dr \\
&= \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q} e^{-r\sqrt{n+1}} \exp\left(-n^{3/2}\sqrt{n+1}\langle x, w\rangle\right) dx\, dr \\
&\geqslant \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q'} e^{-r\sqrt{n+1}} \exp\left(-n^{3/2}\sqrt{n+1}\,\langle x, w\rangle\right) dx\, dr,
\end{aligned} \tag{6.25}$$

where, in the last step, we use the fact that $Q' \subseteq Q$. Now, observe that if $x \in \frac{r}{\sqrt{n}}Q'$ then

$$\langle x, w\rangle = \sum_{i\in\tau} \rho_i \langle x, u_i\rangle \leqslant \frac{r}{\sqrt{n}}. \tag{6.26}$$

So, we get

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle)dy &\geqslant \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q'} e^{-r\sqrt{n+1}-rn\sqrt{n+1}}dx\,dr \\ &= \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q'} e^{-r(n+1)^{3/2}}dx\,dr \\ &= |Q'|\cdot\frac{1}{n^{n/2}}\int_0^\infty r^n e^{-r(n+1)^{3/2}}\,dr \\ &= |Q'|\cdot\frac{1}{n^{n/2}}\frac{n!}{(n+1)^{3(n+1)/2}}.\end{aligned} \tag{6.27}$$

Note that

$$\prod_{j\in\sigma}\left(\int_{\mathbb{R}} f_j(t)dt\right)^{\kappa_j} = 1, \tag{6.28}$$

and hence (6.24) gives us

$$|Q'| \leqslant \gamma_d^{\frac{n+1}{2}}\frac{n^{n/2}(n+1)^{3(n+1)/2}}{n!}. \tag{6.29}$$

Since Q' is an intersection of at most $(d+1)(n+1)$ half-spaces and $B_2^n \subseteq P \subseteq Q'$, the result follows as in the symmetric case. Using Stirling's formula one can check that the statement holds true with $C_d = \left(\frac{e\gamma_d}{2\pi}\right)^{\frac{1}{3}}$. $\square$

Our next argument provides (for an absolute constant $\alpha \gg 1$) a choice of $s \leqslant \alpha n$ half-spaces and a much better bound of the order of n^n for the constant $c_{n,s}$.

Theorem 6.4. *There exists an absolute constant $\alpha > 1$ with the following property: for every family $\{P_i : i \in I\}$ of closed half-spaces*

$$P_i = \{x \in \mathbb{R}^n : \langle x, u_i\rangle \leqslant 1\} \tag{6.30}$$

in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume, there exist $s \leqslant \alpha n$ and $i_1, \dots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (Cn)^n\,|P|, \tag{6.31}$$

where $C > 0$ is an absolute constant.

Proof. As in the proof of Theorem 6.3 we assume that P is in John's position, and we find $J \subseteq I$ so that the vectors u_j, $j \in J$ are contact points of P and S^{n-1} and there exist $a_j > 0$, $j \in J$, such that

$$I_n = \sum_{j\in J} a_j u_j \otimes u_j \quad \text{and} \quad \sum_{j\in J} a_j u_j = 0. \tag{6.32}$$

We apply Theorem 4.2 to find a subset $\sigma \subseteq J$ with $|\sigma| \leqslant \alpha_1(\varepsilon)n$, positive scalars c_j, $j \in \sigma$ and a vector u such that

$$I_n \preceq \sum_{j\in\sigma} c_j(u_j + u) \otimes (u_j + u) \preceq (4+\varepsilon)I_n \tag{6.33}$$

and

$$\sum_{j\in\sigma} c_j(u_j+u)=0 \quad\text{and}\quad \|u\|_2^2 \leqslant \frac{\varepsilon}{\sum_{j\in\sigma} c_j}. \tag{6.34}$$

Note that

$$\begin{aligned}\operatorname{tr}\left(\sum_{j\in\sigma} c_j(u_j+u)\otimes(u_j+u)\right) &= \sum_{j\in\sigma} c_j\|u_j+u\|_2^2 \\ &= \sum_{j\in\sigma} c_j\|u_j\|_2^2 + 2\sum_{j\in\sigma}\langle u, c_ju_j\rangle + \left(\sum_{j\in\sigma} c_j\right)\|u\|_2^2 \\ &= \sum_{j\in\sigma} c_j + 2\Big\langle u, -\Big(\sum_{j\in\sigma} c_j\Big)u\Big\rangle + \left(\sum_{j\in\sigma} c_j\right)\|u\|_2^2 \\ &= \sum_{j\in\sigma} c_j - \left(\sum_{j\in\sigma} c_j\right)\|u\|_2^2\end{aligned} \tag{6.35}$$

and hence from (6.33) we get that

$$n \leqslant \sum_{j\in\sigma} c_j - \left(\sum_{j\in\sigma} c_j\right)\|u\|_2^2 \leqslant (4+\varepsilon)n.$$

Now, using (6.34) we get

$$n \leqslant \sum_{j\in\sigma} c_j \leqslant (4+2\varepsilon)n. \tag{6.36}$$

In particular,

$$\|u\|_2^2 \leqslant \frac{\varepsilon}{\sum_{j\in\sigma} c_j} \leqslant \frac{\varepsilon}{n}. \tag{6.37}$$

Recall that $\mathrm{conv}\{u_j, j\in J\} \supseteq \frac{1}{n}B_2^n$. Then, for the vector $w=\frac{u}{\sqrt{\varepsilon n}}$ we have $\|w\|_2 \leqslant \frac{1}{n}$ and hence $w\in \mathrm{conv}\{u_j, j\in J\}$. Carathéodory's theorem shows that there exist $\tau\subseteq J$ with $|\tau|\leqslant n+1$ and $\rho_i>0$, $i\in\tau$ such that

$$w=\sum_{i\in\tau}\rho_i u_i \quad\text{and}\quad \sum_{i\in\tau}\rho_i = 1. \tag{6.38}$$

Note that

$$\left(\sum_{j\in\sigma} c_j\right)(-u) = \sum_{j\in\sigma} c_ju_j, \tag{6.39}$$

and this shows that $-u\in \mathrm{conv}\{u_j : j\in\sigma\}$. It follows that the segment

$$\left[-u, \frac{u}{\sqrt{\varepsilon n}}\right] \subset \mathrm{conv}\{u_j : j\in\sigma\cup\tau\}. \tag{6.40}$$

For $j \in \sigma$ we set

$$v_j = \sqrt{\frac{n}{n+1}}\left(-u_j, \frac{1}{\sqrt{n}}\right) \quad \text{and} \quad b_j = \frac{n+1}{n}c_j. \tag{6.41}$$

We also set $-v = \sqrt{\frac{n}{n+1}}(u,0)$. Then, using (6.34) we get

$$\sum_{j\in\sigma} b_j(v_j+v)\otimes(v_j+v) = \begin{pmatrix} \sum_{j\in\sigma} c_j(u_j+u)\otimes(u_j+u) & 0 \\ 0 & \frac{\sum_{j\in\sigma} c_j}{n} \end{pmatrix}, \tag{6.42}$$

which implies, with the help of (6.36), that

$$I_{n+1} \preceq \sum_{j\in\sigma} b_j(v_j+v)\otimes(v_j+v) \preceq (4+2\varepsilon)I_{n+1}. \tag{6.43}$$

We rewrite the last one as follows:

$$\begin{aligned} I_{n+1} - \sum_{j\in\sigma} b_j v_j\otimes v - \sum_{j\in\sigma} v\otimes b_j v_j - \left(\sum_{j\in\sigma} b_j\right) v\otimes v \\ \preceq \sum_{j\in\sigma} b_j v_j\otimes v_j \preceq 5I_{n+1} - \sum_{j\in\sigma} b_j v_j\otimes v - \sum_{j\in\sigma} v\otimes b_j v_j - \left(\sum_{j\in\sigma} b_j\right) v\otimes v. \end{aligned} \tag{6.44}$$

Note that

$$\sum_{j\in\sigma} b_j v_j = \sqrt{\frac{n+1}{n}}\left(-\sum_{j\in\sigma} c_j u_j, \frac{\sum_{j\in\sigma} c_j}{\sqrt{n}}\right) = \sqrt{\frac{n+1}{n}}\left(\left(\sum_{j\in\sigma} c_j\right)u, \frac{\sum_{j\in\sigma} c_j}{\sqrt{n}}\right), \tag{6.45}$$

so

$$\begin{aligned} \left(\sum_{j\in\sigma} b_j v_j\right)\otimes v &= \left(\left(\sum_{j\in\sigma} c_j\right)u, \frac{\sum_{j\in\sigma} c_j}{\sqrt{n}}\right)\otimes(-u,0) \\ &= \begin{pmatrix} -\left(\sum_{j\in\sigma} c_j\right)u\otimes u & 0 \\ -\frac{\left(\sum_{j\in\sigma} c_j\right)u}{\sqrt{n}} & 0 \end{pmatrix}. \end{aligned} \tag{6.46}$$

Computing in a similar way we finally have that

$$T := \sum_{j\in\sigma} b_j v_j\otimes v + \sum_{j\in\sigma} v\otimes b_j v_j + \left(\sum_{j\in\sigma} b_j\right) v\otimes v = \begin{pmatrix} V & z \\ z & 0 \end{pmatrix}. \tag{6.47}$$

where $V = -\left(\sum_{j\in\sigma} c_j\right)u\otimes u$ and $z = -\frac{(\sum_{j\in\sigma} c_j)u}{\sqrt{n}}$. Now, for every $(x,t)\in S^n$ we have

$$\begin{aligned} \langle T(x,t),(x,t)\rangle &= \langle Vx,x\rangle + 2\langle z,t\rangle \leqslant \|V\| + 2\|z\|_2 \\ &= \left(\sum_{j\in\sigma} c_j\right)\|u\|_2^2 + \left(\sum_{j\in\sigma} c_j\right)\frac{2\|u\|_2}{\sqrt{n}} \\ &\leqslant \varepsilon + (4+2\varepsilon)n\frac{2\sqrt{\varepsilon}}{n} = \varepsilon + 2\sqrt{\varepsilon}(4+2\varepsilon). \end{aligned} \tag{6.48}$$

Choosing $\varepsilon = 10^{-3}$ we get

$$\left\| \sum_{j\in\sigma} b_j v_j \otimes v + \sum_{j\in\sigma} v \otimes b_j v_j + \left(\sum_{j\in\sigma} b_j\right) v \otimes v \right\| \leqslant \frac{1}{2}, \tag{6.49}$$

and going back to (6.44) we get

$$\frac{1}{2} I_{n+1} \preceq \sum_{j\in\sigma} b_j v_j \otimes v_j \preceq 5 I_{n+1}. \tag{6.50}$$

Now, we apply Proposition 5.2 to find $\kappa_j > 0$, $j \in \sigma$ such that if $f_j : \mathbb{R} \to \mathbb{R}^+$ are measurable functions, then

$$\int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) dy \leqslant 10^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\int_{\mathbb{R}} f_j(t) dt\right)^{\kappa_j}. \tag{6.51}$$

For $j \in \sigma$ we define $f_j(t) = e^{-\frac{b_j}{\kappa_j} t} \mathbf{1}_{[0,\infty)}(t)$. Then,

$$\int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) dy \leqslant 10^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\int_{\mathbb{R}} f_j(t) dt\right)^{\kappa_j} = 10^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\frac{\kappa_j}{c_j}\right)^{\kappa_j} \leqslant 40^{\frac{n+1}{2}}, \tag{6.52}$$

recalling from the proof of Proposition 5.2 that $\frac{\kappa_j}{b_j} = \langle A^{-1} u_j, u_j\rangle \leqslant 2$ (the last inequality is a consequence of $\frac{1}{2} I_{n+1} \preceq A = \sum_{j\in\sigma} b_j v_j \otimes v_j$).

Let

$$Q = \{x \in \mathbb{R}^n : \langle x, u_j\rangle < 1, \quad j \in \sigma \cup \tau\}. \tag{6.53}$$

We write $y = (x, r) \in \mathbb{R}^{n+1}$ and assume that $r > 0$ and $x \in \frac{r}{\sqrt{n}} Q$. Then, we have $\langle x, u_j\rangle < \frac{r}{\sqrt{n}}$ for all $j \in \sigma$. This implies that $\langle y, v_j\rangle > 0$ for all $j \in \sigma$, and hence $\prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) > 0$. We also have

$$\begin{aligned} \frac{1}{\left(\sum_{j\in\sigma} c_j\right)} \left\langle \sum_{j\in\sigma} c_j u_j, x \right\rangle &= \langle -u, x\rangle = \sqrt{\varepsilon n} \langle -w, x\rangle = \sqrt{\varepsilon n} \left\langle -\sum_{i\in\tau} \rho_i u_i, x \right\rangle \\ &\geqslant -\sqrt{\varepsilon} r, \end{aligned} \tag{6.54}$$

where the last inequality holds since $x \in \frac{r}{\sqrt{n}} Q$. It follows that

$$\left\langle \sum_{j\in\sigma} c_j u_j, x \right\rangle \geqslant -5\sqrt{\varepsilon} r n. \tag{6.55}$$

Using the above (and recalling our choice of $\varepsilon = 10^{-3} < 1$) we see that if $y = (x, r) \in \frac{r}{\sqrt{n}} Q \times (0, \infty)$ then

$$\begin{aligned} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle) &= \exp\left(-\sum_{j\in\sigma} b_j \left(\frac{r}{\sqrt{n}} - \sqrt{\frac{n}{n+1}} \langle x, u_j\rangle\right)\right) \\ &= \exp\left(-\frac{r}{\sqrt{n}} \sum_{j\in\sigma} b_j\right) \exp\left(\left\langle x, \sum_{j\in\sigma} b_j u_j \right\rangle\right) \\ &\geqslant \exp\left(-5r \frac{n+1}{\sqrt{n}} - 5\sqrt{\varepsilon} r(n+1)\right) \geqslant \exp\left(-10 r(n+1)\right). \end{aligned} \tag{6.56}$$

Now, (6.52) gives us

$$\frac{|Q|}{n^{\frac{n}{2}}}\int_0^\infty r^n e^{-10r(n+1)}\,dr = \int_0^\infty \int_{\frac{r}{\sqrt{n}}Q} e^{-10r(n+1)}dx\,dr \leqslant \int_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{\kappa_j}(\langle y, v_j\rangle)dy \leqslant 40^{\frac{n+1}{2}}. \tag{6.57}$$

Direct computation and then Stirling's approximation show that

$$|Q| \leqslant C_1^n \frac{n^{\frac{3n}{2}}}{n!} \leqslant C_2^n n^{\frac{n}{2}} \tag{6.58}$$

and Q is the intersection of at most $|\sigma|+|\tau| \leqslant \alpha_1(10^{-3})n+n+1 \leqslant \alpha n$ half-spaces, where $\alpha = \alpha_1(10^{-3})+2$. Since $B_2^n \subseteq P \subseteq Q$, the result follows as in the symmetric case. $\square$

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