Continuous version of the approximate geometric Brascamp-Lieb inequalities

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Abstract

Given $\gamma > 1$ we say that a Borel measure ν on S^{n-1} is a γ-approximation of an isotropic measure if

$$I_n \preceq T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u) \preceq \gamma I_n,$$

where I_n is the identity matrix. We provide a generalization of Barthe's continuous version of the Brascamp-Lieb inequalities to the context of these approximate isotropic measures, and we apply these inequalities to obtain stability results for some classical positions of convex bodies.

1 Introduction

This article is a continuation of the work [11] of the first named author. Our purpose is to extend Barthe's continuous version of the Brascamp-Lieb inequalities to the setting of approximately isotropic Borel measures on the sphere. Recall that a Borel measure ν on S^{n-1} is called isotropic if

$$I_n = \int_{S^{n-1}} u \otimes u \, d\nu(u), \tag{1.1}$$

where I_n is the identity matrix and $(u \otimes v)(y) = \langle v, y\rangle u$. Barthe's theorem is a pair of inequalities for a family of functions (f_u), $u \in S^{n-1}$, that satisfy some mild continuity conditions; namely,

- There exist a continuous function $F : S^{n-1} \times \mathbb{R} \longrightarrow (0, +\infty)$ and two functions a, b on S^{n-1} with $a < b$ (a, b are either real-valued continuous or constant with value $\pm\infty$) such that for all $(u, t) \in S^{n-1} \times \mathbb{R}$

$$f_u(t) = \mathbf{1}_{a(u) \leqslant t \leqslant b(u)} F(u, t).$$

- There exists a function $U \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ such that $0 \leqslant f_u \leqslant U$ for all $u \in S^{n-1}$.

Then, we say that (f_u) *satisfies condition* (H).

Theorem 1.1 (Barthe)**.** *Let ν be an isotropic Borel measure on S^{n-1} and let (f_u), $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \longrightarrow [0, +\infty)$ that satisfies condition (H). Then,*

$$\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle) d\nu(u)\right) dx \leqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) d\nu(u)\right). \tag{1.2}$$

Also, if h is a measurable function such that

$$h\left(\int_{S^{n-1}} \theta(u) u \, d\nu(u)\right) \geqslant \exp\left(\int_{S^{n-1}} \log f_u(\theta(u)) \, d\nu(u)\right) \tag{1.3}$$

for every integrable function θ, then

$$\int_{\mathbb{R}^n} h(x) \, dx \geqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) d\nu(u)\right). \tag{1.4}$$

The discrete analogues of the two statements in Theorem 1.1, Ball's version of the Brascamp-Lieb inequality and Barthe's reverse Brascamp-Lieb inequality, have been the key to a number of sharp geometric inequalities and play an important role in convex geometric analysis. Recall that if $m \geqslant n$, $u_1, \dots, u_m \in \mathbb{R}^n$ and $c_1, \dots, c_m > 0$ with $c_1 + \cdots + c_m = n$, then the Brascamp-Lieb inequality [10], as reformulated by Ball (see e.g. [2]), states that

$$G(f_1, \dots, f_m) := \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{c_j}(\langle x, u_j \rangle) dx \leqslant \frac{1}{\sqrt{D}} \prod_{j=1}^{m} \left(\int_{\mathbb{R}} f_j \right)^{c_j} \tag{1.5}$$

for all integrable functions $f_j : \mathbb{R} \longrightarrow [0, \infty)$, where

$$D = \inf \left\{ \frac{\det \left(\sum_{j=1}^{m} c_j \lambda_j u_j \otimes u_j \right)}{\prod_{j=1}^{m} \lambda_j^{c_j}} : \lambda_j > 0 \right\}. \tag{1.6}$$

A reverse form of (1.5) was proved by Barthe in [6] (see also [7] for a multidimensional extension). Under the same assumptions on the data $\{u_j, c_j\}_{j \leqslant m}$, we have

$$K(h_1, \dots, h_m) := \int_{\mathbb{R}^n}^{*} \sup \left\{ \prod_{j=1}^{m} h_j^{c_j}(\theta_j) : \theta_j \in \mathbb{R} \, , \; x = \sum_{j=1}^{m} \theta_j c_j u_j \right\} dx \geqslant \sqrt{D} \prod_{j=1}^{m} \left(\int_{\mathbb{R}} h_j \right)^{c_j} \tag{1.7}$$

for all integrable functions $h_1, \dots, h_m : \mathbb{R} \longrightarrow [0, \infty)$, where $\int^*$ denotes outer integral.

In order to apply (1.5) and (1.7) for a given set of vectors u_j and weights c_j, one has to compute the constant $D = D(\{u_j, c_j\}_{1 \leqslant j \leqslant m})$; this is not a simple problem. Ball's crucial observation is that if $u_1, \dots, u_m \in S^{n-1}$ and $c_1, \dots, c_m > 0$ satisfy the condition

$$I_n = \sum_{j=1}^{m} c_j u_j \otimes u_j, \tag{1.8}$$

then

$$D = D(\{u_j, c_j\}_{1 \leqslant j \leqslant m}) = 1. \tag{1.9}$$

Ball first used this fact in [2] to obtain estimates on the volume of sections and projections of the unit cube. A well-known decomposition of the form (1.8) appears in John's theorem regarding a convex body whose maximal volume ellipsoid is the Euclidean unit ball (see Section 2). The first and well-known application of (1.5) in this context is Ball's sharp reverse isoperimetric inequality in [3]. Subsequently, both (1.5) and (1.7) have been systematically used for the proof of several other sharp inequalities in convex geometric analysis (see [4] for references and the history of these ideas).

Note that $u_j \in S^{n-1}$ and $c_j > 0$ satisfy (1.8) if and only if the measure ν with $\mathrm{supp}(\nu) = \{u_1, \dots, u_m\}$ and $\nu(\{u_j\}) = c_j$, $j = 1, \dots, m$ is an isotropic measure on S^{n-1}. It was later understood (and this point of view was put forward in [12]) that various classical positions of convex bodies are also characterized by the fact that an appropriate Borel measure on the sphere is isotropic (see Section 2 for background information). For example:

- A convex body K has minimal surface area among all its affine images of the same volume if and only if the surface area measure σ_K of K is an isotropic measure on S^{n-1}.
- A convex body K has minimal mean width among all its affine images of the same volume if and only if the measure ν_K with density h_K with respect to σ is an isotropic measure on S^{n-1}.

Barthe's theorem (Theorem 1.1) provides a continuous version (and extension) of (1.5) and (1.7) which potentially allows one to obtain geometric applications in situations where a non-discrete isotropic measure appears. This is certainly the case in the two examples above (see also [18] and [19] for a sample of applications).

We present a variant of Theorem 1.1 for any non-negative finite Borel measure ν on S^{n-1}, and in particular we are interested in the case where ν is *approximately isotropic*. For any non-negative finite Borel measure ν on S^{n-1} we define the symmetric positive semi-definite $n \times n$ matrix

$$T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u). \tag{1.10}$$

We say that ν is a γ-*approximation of an isotropic measure* (for some $\gamma > 1$) if

$$I_n \preceq T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u) \preceq \gamma I_n. \tag{1.11}$$

First, we obtain a generalization of the next result of Lutwak, Yang and Zhang [17] which is the continuous version of Ball's observation (1.9): if ν is an isotropic measure on S^{n-1} and $t : \mathrm{supp}(\nu) \longrightarrow (0, \infty)$ is continuous, then

$$\det\left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u)\right) \geqslant \exp\left[\int_{S^{n-1}} \log t(u) \, d\nu(u)\right]. \tag{1.12}$$

This fact is a basic ingredient in Barthe's proof of Theorem 1.1. In the general case it takes the following form:

Theorem 1.2. *Let ν be a non-negative finite Borel measure on S^{n-1}. For every continuous function $t : \mathrm{supp}(\nu) \longrightarrow (0, \infty)$ one has*

$$\det\left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u)\right) \geqslant \det(T_\nu) \exp\left[\int_{S^{n-1}} \log t(u) \, \langle T_\nu^{-1} u, u\rangle \, d\nu(u)\right]. \tag{1.13}$$

Then, with a modification of Barthe's argument from [8], we obtain a general continuous version of the Brascamp-Lieb inequality.

Theorem 1.3. *Let ν be a non-negative finite Borel measure on S^{n-1} and let (f_u), $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \longrightarrow [0, +\infty)$ that satisfies condition (H). Let*

$$A_\nu = \sqrt{\det(T_\nu)} \exp\left(\int_{S^{n-1}} \log \sqrt{\langle T_\nu^{-1} u, u\rangle} d\mu(u)\right),$$

where μ is the measure on S^{n-1} with $d\mu(u) = \langle T_\nu^{-1} u, u\rangle d\nu(u)$. Then,

$$A_\nu \int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle) \langle T_\nu^{-1} u, u\rangle d\nu(u)\right) dx \leqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) \langle T_\nu^{-1} u, u\rangle \, d\nu(u)\right). \tag{1.14}$$

Also, if h is a measurable function such that

$$h\left(\int_{S^{n-1}} \theta(u) u \, d\mu(u)\right) \geqslant \exp\left(\int_{S^{n-1}} \log f_u(\theta(u)) d\mu(u)\right) \tag{1.15}$$

for every integrable function θ, then

$$\int_{\mathbb{R}^n} h(y) \, dy \geqslant A_\nu \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) \langle T_\nu^{-1} u, u\rangle \, d\nu(u)\right). \tag{1.16}$$

In particular, if ν a γ-approximation of an isotropic measure, we obtain a generalization of the discrete analogues of Theorem 1.2 and Theorem 1.3 from [11] (see the precise statements in Section 2) where it was shown that if an approximate John's decomposition is available by a set of $u_j \in S^{n-1}$ and $c_j > 0$ then we still have a Brascamp-Lieb inequality with a reasonable constant $D(\{u_j, c_j\}_{1 \leqslant j \leqslant m})$.

We also observe in Proposition 3.4 that the constant A_ν in Theorem 1.3 is at most 1, with equality if the measure ν is isotropic. This agrees with a result of Valdimarsson from [22] for the discrete Brascamp-Lieb inequality.

In the last section of this article we apply Theorem 1.3 to obtain stability results for some classical positions of convex bodies. The main idea is to use the next fact.

Theorem 1.4. *Let ν be a non-negative finite Borel measure on S^{n-1}. Let $C(\nu)$ be the symmetric convex body whose support function is the cosine transform of ν*

$$C_\nu(x) = \int_{S^{n-1}} |\langle x, u\rangle| \, d\nu(u), \qquad x \in \mathbb{R}^n, \tag{1.17}$$

and write $C^*(\nu)$ for the polar body of ν. Then,

$$\nu(S^{n-1})|C^*(\nu)|^{1/n} \geqslant \frac{n\omega_n^{\frac{n+1}{n}}}{2\omega_{n-1}} \geqslant c, \tag{1.18}$$

where $c > 0$ is an absolute constant. Conversely, if ν is a γ-approximation of an isotropic measure on S^{n-1} for some $\gamma > 1$, then

$$\nu(S^{n-1})|C^*(\nu)|^{1/n} \leqslant 2e\gamma. \tag{1.19}$$

The isotropic analogue of Theorem 1.4 has first appeared in [14], in the particular case where $\nu = \sigma_K$ is the area measure of a convex body K. Theorem 1.4 allows us to use it for approximate isotropic measures; as an example of application we show that if the area measure of a convex body K is almost isotropic then K has almost minimal surface area (this fact appears in [14] but we feel that the present argument is more transparent and may be the model for new applications of this type).

2 Notation and background

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $| \cdot |$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto F. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

We refer to the book of Schneider [21] for basic facts from the Brunn-Minkowski theory and to the book of Artstein-Avidan, Giannopoulos and V. Milman [1] for basic facts from asymptotic convex geometry.

2.1. Convex bodies

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$, and that K is centered if its barycenter

$$\text{bar}(K) = \frac{1}{|K|} \int_K x \, dx \tag{2.1}$$

is at the origin. If $0 \in \text{int}(K)$ then the polar body K° of K is defined by

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\}. \tag{2.2}$$

The Blaschke-Santaló inequality states that for every centered convex body K in $\mathbb{R}^n$ one has $|K||K^\circ| \leqslant \omega_n^2$, with equality if and only if K is an ellipsoid. The reverse Santaló inequality of Bourgain and V. Milman [9] states that there exists an absolute constant $c > 0$ such that

$$(|K||K^\circ|)^{1/n} \geqslant c/n, \tag{2.3}$$

where $c > 0$ is an absolute constant, for every convex body K in $\mathbb{R}^n$ which contains 0 in its interior.

The support function of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\}. \tag{2.4}$$

The mean width of K is the quantity

$$w(K) = \int_{S^{n-1}} h_K(x)\, d\sigma(x). \tag{2.5}$$

The area measure σ_K of a convex body K in $\mathbb{R}^n$ is defined on S^{n-1} and corresponds to the usual surface measure on K via the Gauss map: for every Borel $A \subseteq S^{n-1}$, we set

$$\sigma_K(A) = \lambda\left(\{x \in \mathrm{bd}(K) : \text{ the outer normal to } K \text{ at } x \text{ is in } A\}\right), \tag{2.6}$$

where λ is the $(n-1)$-dimensional surface measure on K. The surface area $\partial(K)$ of K is obviously equal to $\sigma_K(S^{n-1})$.

The projection body ΠK of K is the symmetric convex body whose support function is defined by $h_{\Pi K}(\theta) = |P_{\theta^\perp}(K)|$, $\theta \in S^{n-1}$. We write $\Pi^* K$ for its polar body (the polar projection body of K).

The cosine transform of a non-negative finite Borel measure ν on S^{n-1} is defined by

$$C_\nu(x) = \int_{S^{n-1}} |\langle x, u\rangle|\, d\nu(u), \qquad x \in \mathbb{R}^n. \tag{2.7}$$

If ν is not concentrated on a great subsphere then C_ν is the support function of a symmetric convex body in $\mathbb{R}^n$, which we denote by $C(\nu)$. For example, if K is a convex body in $\mathbb{R}^n$ then

$$C_{\sigma_K}(\theta) = \int_{S^{n-1}} |\langle \theta, u\rangle|\, d\sigma_K(u) = 2|P_{\theta^\perp}(K)| \tag{2.8}$$

for every $\theta \in S^{n-1}$, and hence $C(\sigma_K) = 2\Pi K$. We also write $C^*(\nu)$ for the polar body of $C(\nu)$.

2.2. Discrete approximate decompositions of the identity

It was mentioned in the introduction that if $u_1, \ldots, u_m \in S^{n-1}$ and $c_1, \ldots, c_m > 0$ satisfy (1.8) then $D(\{u_j, c_j\}_{1\leqslant j\leqslant m}) = 1$. The next result from [11] provides a substitute of this fact in the case of an approximate John's decomposition.

Theorem 2.1 (Brazitikos). *Let $\gamma > 1$. If $u_1, \ldots, u_m \in S^{n-1}$ and $c_1, \ldots, c_m > 0$ satisfy*

$$I_n \preceq A := \sum_{j=1}^m c_j u_j \otimes u_j \preceq \gamma I_n \tag{2.9}$$

then

$$\gamma^n \det\left(\sum_{j=1}^m \kappa_j \lambda_j u_j \otimes u_j\right) \geqslant \det\left(\sum_{j=1}^m c_j \lambda_j u_j \otimes u_j\right) \geqslant \prod_{j=1}^m \lambda_j^{\kappa_j} \tag{2.10}$$

for all $\lambda_1, \ldots, \lambda_m > 0$, where $\kappa_j = c_j\langle A^{-1}u_j, u_j\rangle > 0$, $1 \leqslant j \leqslant m$.

As a direct consequence one has the next approximate geometric Brascamp-Lieb inequality and its reverse counterpart.

Theorem 2.2 (Brazitikos). *Let $\gamma > 1$. Assume that $u_1, \ldots, u_m \in S^{n-1}$ and $c_1, \ldots, c_m > 0$ satisfy* (2.9) *and set $\kappa_j = c_j\langle A^{-1}u_j, u_j\rangle > 0$, $1 \leqslant j \leqslant m$. If $f_1, \ldots, f_m : \mathbb{R} \longrightarrow [0, +\infty)$ are integrable functions then*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{\kappa_j}(\langle x, u_j\rangle)dx \leqslant \gamma^{\frac{n}{2}} \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j(t)dt\right)^{\kappa_j}. \tag{2.11}$$

Also, if $w, h_1, \dots, h_m : \mathbb{R} \longrightarrow [0,\infty)$ *are integrable functions and* $w(x) \geqslant \sup\left\{ \prod_{j=1}^m h_j^{\kappa_j}(\theta_j) : \theta_j \in \mathbb{R}\ ,\ x = \sum_{j=1}^m \theta_j c_j u_j \right\}$*, then*

$$\int_{\mathbb{R}^n} w(x)dx \geqslant \gamma^{-\frac{n}{2}} \prod_{j=1}^m \left(\int_{\mathbb{R}} h_j(t)\, dt \right)^{\kappa_j} . \tag{2.12}$$

2.3. Isotropic measures and classical positions of convex bodies

A Borel measure ν on S^{n-1} is called isotropic if

$$I_n = \int_{S^{n-1}} u \otimes u \, d\nu(u). \tag{2.13}$$

Note that $u_j \in S^{n-1}$ and $c_j > 0$ satisfy (1.8) if and only if the measure ν with $\mathrm{supp}(\nu) = \{u_1, \dots, u_m\}$ and $\nu(\{u_j\}) = c_j$, $j = 1, \dots, m$ is an isotropic measure on S^{n-1}.

For any non-negative finite Borel measure ν on S^{n-1} we define the symmetric positive semi-definite $n \times n$ matrix

$$T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u). \tag{2.14}$$

Equivalently, we have

$$\langle T_\nu x, x \rangle = \int_{S^{n-1}} \langle x, u \rangle^2 \, d\nu(u) \tag{2.15}$$

for all $x \in \mathbb{R}^n$. Note that

$$\mathrm{tr}(T_\nu) = \nu(S^{n-1}). \tag{2.16}$$

In particular, whenever (1.8) is satisfied we have that

$$\sum_{j=1}^m c_j = \mathrm{tr}(I_n) = n \quad \text{and} \quad \sum_{j=1}^m c_j \langle u_j, z \rangle^2 = 1 \tag{2.17}$$

for any $z \in S^{n-1}$.

We say that K has *minimal mean width* if $w(K) \leqslant w(TK)$ for every $T \in SL(n)$. It was proved in [12] that a smooth enough convex body K in $\mathbb{R}^n$ has minimal mean width if and only if

$$\int_{S^{n-1}} h_K(x) \langle x, \theta \rangle^2 d\sigma(x) = \frac{w(K)}{n} \tag{2.18}$$

for every $\theta \in S^{n-1}$. Equivalently, if the measure ν_K on S^{n-1} with density h_K with respect to σ is a scalar multiple of an isotropic measure. Moreover, this minimal mean width position is unique up to orthogonal transformations.

We say that K has minimal surface area if $\partial(K) \leqslant \partial(TK)$ for every $T \in SL(n)$. A characterization of the minimal surface area position through the area measure was given by Petty in [20] (see also [14]): a convex body K has minimal surface area if and only if σ_K is a scalar multiple of an isotropic measure. Moreover, this minimal surface area position is unique up to orthogonal transformations.

A convex body K of volume 1 in $\mathbb{R}^n$ is called isotropic if it is centered and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2 \tag{2.19}$$

for all $\theta \in S^{n-1}$. Moreover, the isotropic position is unique up to orthogonal transformations. Writing (2.19) in polar coordinates we get

$$\frac{n\omega_n}{n+2}\int_{S^{n-1}} \langle u,\theta\rangle^2 \, \rho_K^{n+2}(u)d\sigma(u) = L_K^2 \tag{2.20}$$

for all $\theta \in S^{n-1}$, where $\rho_K(u) = \max\{t > 0 : tu \in K\}$ is the radial function of K. Therefore, K is isotropic if and only if the measure λ_K on S^{n-1} with density ρ_K^{n+2} with respect to σ is a scalar multiple of an isotropic measure.

We say that a convex body K is in John's position if the ellipsoid of maximal volume inscribed in K is the Euclidean unit ball B_2^n. John's theorem [16] states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist $u_1, \ldots, u_m \in \mathrm{bd}(K) \cap S^{n-1}$ (contact points of K and B_2^n) and positive real numbers $c_1, \ldots, c_m$ such that

$$\sum_{j=1}^m c_j u_j = 0 \tag{2.21}$$

and the identity operator I_n is decomposed in the form

$$I_n = \sum_{j=1}^m c_j u_j \otimes u_j. \tag{2.22}$$

2.4. Mixed discriminants

For the proof of Theorem 1.2 we shall modify the argument that Lutwak, Yang and Zhang used in [17] for the isotropic case, which employs some basic properties of mixed discriminants. Recall that if $T_1, \ldots, T_m$ are positive semi-definite $n \times n$ matrices then the determinant of $a_1T_1 + \cdots + a_mT_m$ can be expanded as a homogeneous polynomial of degree n in $a_1, \ldots, a_m \geqslant 0$; one has

$$\det(a_1T_1 + \cdots + a_mT_m) = \sum_{1\leqslant i_1,\ldots,i_n\leqslant m} D(T_{i_1}, \ldots, T_{i_n})a_{i_1}\cdots a_{i_n}, \tag{2.23}$$

where the coefficient $D(T_{i_1}, \ldots, T_{i_n})$ depends only on $i_1, \ldots, i_n$ and is invariant under permutations of the i_j's. This coefficient is the mixed discriminant of $T_{i_1}, \ldots, T_{i_n}$.

We shall use a number of properties of mixed discriminants (see [5] for a proof).

Lemma 2.3. *If S, T, T_i, T_i' are positive semi-definite $n \times n$ matrices, then:*

(i) $D(T_1, \ldots, T_n) \geqslant 0$.

(ii) $D(T, T, \ldots, T) = \det(T)$. *In particular,* $D(I_n, \ldots, I_n) = 1$.

(iii) $nD(T, I_n, \ldots, I_n) = \mathrm{tr}(T)$.

(iv) $D(aT_1 + bT_1', T_2, \ldots, T_n) = aD(T_1, T_2, \ldots, T_n) + bD(T_1', T_2, \ldots, T_n)$ *for all* $a, b \geqslant 0$.

(v) $D(T_1S, T_2S, \ldots, T_nS) = |\det(S)|D(T_1, \ldots, T_n)$ *and* $D(ST_1, ST_2, \ldots, ST_n) = |\det(S)|D(T_1, \ldots, T_n)$.

We will be interested in n-tuples $(T_{\mu_1}, \ldots, T_{\mu_n})$ where $\mu_1, \ldots, \mu_n$ are non-negative finite Borel measures on S^{n-1}. We will use the fact that

$$D(T_{\mu_1}, \ldots, T_{\mu_n}) = \frac{1}{n!}\int_{S^{n-1}} \cdots \int_{S^{n-1}} [u_1, \ldots, u_n]^2 d\mu_1(u_1)\cdots d\mu_n(u_n), \tag{2.24}$$

where $[u_1, \ldots, u_n]$ denotes the volume of the parallelotope defined by $u_1, \ldots, u_n$ (see [17] for a proof).

Note also that if $u \in S^{n-1}$ and δ_u is the probability measure supported by $\{u\}$, then

$$\frac{1}{n} = \frac{\mathrm{tr}(T_{\delta_u})}{n} = D(T_{\delta_u}, I_n, \ldots, I_n). \tag{2.25}$$

3 Proof of the main results

We start with the proof of Theorem 1.2. Let ν be a non-negative finite Borel measure on S^{n-1}. Given a continuous function $t : \mathrm{supp}(\nu) \longrightarrow (0,\infty)$ we apply (2.24) for the measures $\mu_1 = \cdots = \mu_n = \mu$ where $d\mu = t\, d\nu$, to get

$$
\begin{aligned}
(3.1)\qquad \det\left(\int_{S^{n-1}} t(u) u\otimes u\, d\nu(u)\right) &= \det(T_\mu) = D(T_\mu,\dots,T_\mu)\\
&= \frac{1}{n!}\int_{S^{n-1}}\cdots\int_{S^{n-1}} t(u_1)\cdots t(u_n)[u_1,\dots,u_n]^2 d\nu(u_1)\cdots d\nu(u_n)\\
&= \det(T_\nu)\frac{1}{n!\det(T_\nu)}\int_{S^{n-1}}\cdots\int_{S^{n-1}} t(u_1)\cdots t(u_n)[u_1,\dots,u_n]^2 d\nu(u_1)\cdots d\nu(u_n).
\end{aligned}
$$

Replacing t by the constant function $\mathbf{1}$ in the equation above, we see that

$$
(3.2)\qquad \frac{1}{n!\det(T_\nu)}\int_{S^{n-1}}\cdots\int_{S^{n-1}} [u_1,\dots,u_n]^2 d\nu(u_1)\cdots d\nu(u_n) = 1.
$$

Then, from Jensen's inequality we see that

$$
\begin{aligned}
(3.3)\qquad &\det\left(\int_{S^{n-1}} t(u) u\otimes u\, d\nu(u)\right)\\
&\geqslant \det(T_\nu)\exp\left[\frac{1}{n!\det(T_\nu)}\int_{S^{n-1}}\cdots\int_{S^{n-1}} \log(t(u_1)\cdots t(u_n))[u_1,\dots,u_n]^2 d\nu(u_1)\cdots d\nu(u_n)\right]\\
&= \det(T_\nu)\exp\left[\frac{1}{n!\det(T_\nu)}\sum_{j=1}^{n}\int_{S^{n-1}}\cdots\int_{S^{n-1}} \log t(u_j)[u_1,\dots,u_n]^2 d\nu(u_1)\cdots d\nu(u_n)\right].
\end{aligned}
$$

Note that for every $u\in S^{n-1}$ we also have that

$$
\begin{aligned}
(3.4)\qquad \int_{S^{n-1}}\cdots\int_{S^{n-1}} [u,u_2,\dots,u_n]^2 d\nu(u_2)\cdots d\nu(u_n) &= n! D(T_{\delta_u}, T_\nu,\dots,T_\nu)\\
&= n!\det(T_\nu) D(T_\nu^{-1}T_{\delta_u}, I_n,\dots,I_n)\\
&= (n-1)!\det(T_\nu)\mathrm{tr}(T_\nu^{-1}T_{\delta_u})\\
&= (n-1)!\det(T_\nu)\left\langle T_\nu^{-1}u,u\right\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(3.5)\qquad &\int_{S^{n-1}}\cdots\int_{S^{n-1}} \log t(u_1)[u_1,\dots,u_n]^2 d\nu(u_1)\cdots d\nu(u_n)\\
&\qquad = (n-1)!\det(T_\nu)\int_{S^{n-1}} \log t(u)\left\langle T_\nu^{-1}u,u\right\rangle\, d\nu(u),
\end{aligned}
$$

and then, (3.3) and (3.5) give:

$$
(3.6)\qquad \det\left(\int_{S^{n-1}} t(u) u\otimes u\, d\nu(u)\right) \geqslant \det(T_\nu)\exp\left[\int_{S^{n-1}} \log t(u)\left\langle T_\nu^{-1}u,u\right\rangle\, d\nu(u)\right],
$$

which completes the proof. □

Applying Theorem 1.2 to a γ-approximation of an isotropic measure we get a generalization of Theorem 2.1.

Theorem 3.1. *Let $u_1,\dots,u_m\in S^{n-1}$ and $c_1,\dots,c_m>0$ satisfy $I_n\preceq A:=\sum_{j=1}^m c_ju_j\otimes u_j\preceq \gamma I_n$ for some $\gamma>1$. Then, for all $\lambda_1,\dots,\lambda_m>0$,*

$$\gamma^n\det\left(\sum_{j=1}^m c_j\lambda_j\langle A^{-1}u_j,u_j\rangle u_j\otimes u_j\right)\geqslant \exp\left[\sum_{j=1}^m \log(\lambda_j)\,c_j\langle A^{-1}u_j,u_j\rangle\right]=\prod_{j=1}^m \lambda_j^{c_j\langle A^{-1}u_j,u_j\rangle}. \tag{3.7}$$

Proof. We apply Theorem 1.2 to the discrete measure ν with $\nu(\{u_j\})=c_j\langle A^{-1}u_j,u_j\rangle$ and to the function $t:\{u_1,\dots,u_m\}\longrightarrow(0,\infty)$ with $t(u_j)=\lambda_j$. Note that $\langle A^{-1}u_j,u_j\rangle\geqslant\gamma^{-1}$ and hence

$$\det(T_\nu)=\det\left(\sum_{j=1}^m c_j\langle A^{-1}u_j,u_j\rangle u_j\otimes u_j\right)\geqslant\det(\gamma^{-1}A)\geqslant\gamma^{-n}$$

by the assumption that $I_n\preceq A$. □

We pass to the proof of Theorem 1.3. In the next lemma, which is essentially the main lemma in [8], $(f_u),(g_u)$, $u\in S^{n-1}$ are two families of functions $f_u,g_u:\mathbb{R}\longrightarrow[0,+\infty)$ that satisfy condition (H):

- There exist two continuous functions $F,G:S^{n-1}\times\mathbb{R}\longrightarrow(0,+\infty)$ and functions a,b,c,d on S^{n-1} with $a<b$ and $c<d$ (a,b,c,d are either real-valued continuous or constant with value $\pm\infty$) such that for all $(u,t)\in S^{n-1}\times\mathbb{R}$
$$f_u(t)=\mathbf{1}_{a(u)\leqslant t\leqslant b(u)}F(u,t)\quad\text{and}\quad g_u(t)=\mathbf{1}_{c(u)\leqslant t\leqslant d(u)}G(u,t). \tag{3.8}$$
- There exist two functions $U,V\in L_1(\mathbb{R})\cap L_\infty(\mathbb{R})$ such that $0\leqslant f_u\leqslant U$ and $0\leqslant g_u\leqslant V$ for all $u\in S^{n-1}$.

Lemma 3.2. *Let ν be a non-negative finite Borel measure on S^{n-1} and let μ be the measure on S^{n-1} with $d\mu(u)=\langle T_\nu^{-1}u,u\rangle d\nu(u)$. If $(f_u),(g_u)$, $u\in S^{n-1}$ are two families of functions that satisfy (H) then*

$$\begin{aligned}&\det(T_\nu)\exp\left(\int_{S^{n-1}}\log\left(\int_{\mathbb{R}}g_u\right)d\mu(u)\right)\int_{\mathbb{R}^n}\exp\left(\int_{S^{n-1}}\log f_u(\langle x,u\rangle)d\mu(u)\right)dx\\ &\leqslant\exp\left(\int_{S^{n-1}}\log\left(\int_{\mathbb{R}}f_u\right)d\mu(u)\right)\int_{\mathbb{R}^n}^{*}\sup_{y=\int\theta(u)ud\nu(u)}\exp\left(\int_{S^{n-1}}\log g_u(\theta(u))d\mu(u)\right)dy.\end{aligned} \tag{3.9}$$

Proof. We sketch the proof, following Barthe's argument, just in order to make the necessary modifications. We assume that the left hand side of (3.9) is positive and note that $\exp\left(\int_{S^{n-1}}\log f_u(\langle x,u\rangle)d\mu(u)\right)$ is equal to zero outside the closed and convex set

$$D=\{x\in\mathbb{R}^n:a(u)\leqslant\langle x,u\rangle\leqslant b(u)\text{ for all }u\in\mathrm{supp}(\nu)\}. \tag{3.10}$$

For every $u\in S^{n-1}$ we define $M_u:(a(u),b(u))\longrightarrow(c(u),d(u))$ by the equation

$$\frac{\int_{a(u)}^t f_u}{\int f_u}=\frac{\int_{c(u)}^{M_u(t)}g_u}{\int g_u}. \tag{3.11}$$

Note that the map $(u,t)\mapsto M_u(t)$ is continuous on the open set $\{(u,t):u\in S^{n-1},a(u)<t<b(u)\}$ and that for every u the function M_u is strictly increasing and differentiable, with

$$f_u(t)\int_{\mathbb{R}}g_u=g_u(M_u(t))M_u'(t)\int_{\mathbb{R}}f_u \tag{3.12}$$

for all $t\in(a(u),b(u))$. It follows that the functions $u\mapsto M_u(\langle x,u\rangle)$ and $u\mapsto M_u'(\langle x,u\rangle)$ are continuous on S^{n-1} for every $x\in\mathrm{int}(D)$.

We define $M : \mathrm{int}(D) \longrightarrow \mathbb{R}^n$ by

$$M(x) = \int_{S^{n-1}} M_u(\langle x, u\rangle) u \, d\nu(u). \tag{3.13}$$

Then,

$$dM(x) = \int_{S^{n-1}} M_u'(\langle x, u\rangle) u \otimes u \, d\nu(u). \tag{3.14}$$

Using Theorem 1.2 we see that

$$\det(dM(x)) \geqslant \det(T_\nu) \exp\left(\int_{S^{n-1}} \log M_u'(\langle x, u\rangle) \, d\mu(u) \right). \tag{3.15}$$

In particular, this implies that M is injective. Then, for any measurable function h that satisfies

$$h\left(\int_{S^{n-1}} \theta(u) u \, d\nu(u) \right) \geqslant \exp\left(\int_{S^{n-1}} \log g_u(\theta(u)) \, d\mu(u) \right) \tag{3.16}$$

for every integrable function θ, we may write

(3.17)

$$\begin{aligned}
&\exp\Big(\int_{S^{n-1}} \log\Big(\int_{\mathbb{R}} f_u \Big) d\mu(u) \Big) \int_{\mathbb{R}^n} h(y) \, dy \\
&\geqslant \exp\Big(\int_{S^{n-1}} \log\Big(\int_{\mathbb{R}} f_u \Big) d\mu(u) \Big) \int_{\mathrm{int}(D)} h(M(x)) \det(dM(x)) \, dx \\
&\geqslant \exp\Big(\int_{S^{n-1}} \log\Big(\int_{\mathbb{R}} f_u \Big) d\mu(u) \Big) \int_{\mathrm{int}(D)} \exp\Big(\int_{S^{n-1}} \log g_u(M_u(\langle x, u\rangle)) \, d\mu(u) \Big) \det(dM(x)) \, dx \\
&\geqslant \det(T_\nu) \int_{\mathrm{int}(D)} \exp\Big(\int_{S^{n-1}} \log\Big(g_u(M_u(\langle x, u\rangle)) \int_{\mathbb{R}} f_u \Big) \, d\mu(u) \Big) \, dx \times \exp\Big(\int_{S^{n-1}} \log M_u'(\langle x, u\rangle) \, d\mu(u) \Big) \, dx \\
&= \det(T_\nu) \int_{\mathrm{int}(D)} \exp\Big(\int_{S^{n-1}} \log\Big(g_u(M_u(\langle x, u\rangle)) M_u'(\langle x, u\rangle) \int_{\mathbb{R}} f_u \Big) \, d\mu(u) \Big) \, dx \\
&= \det(T_\nu) \int_{\mathrm{int}(D)} \exp\Big(\int_{S^{n-1}} \log\Big(f_u(\langle x, u\rangle) \int_{\mathbb{R}} g_u \Big) \, d\mu(u) \Big) \, dx \\
&= \det(T_\nu) \exp\Big(\int_{S^{n-1}} \log\Big(\int_{\mathbb{R}} g_u \Big) d\mu(u) \Big) \int_{\mathbb{R}^n} \exp\Big(\int_{S^{n-1}} \log f_u(\langle x, u\rangle) d\mu(u) \Big) \, dx
\end{aligned}$$

using (3.15) and (3.12). The lemma follows. □

Proof of Theorem 1.3. Let ν be a non-negative finite Borel measure on S^{n-1}. Let μ be the measure on S^{n-1} with $d\mu(u) = \langle T_\nu^{-1} u, u\rangle d\nu(u)$ and let (f_u), $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \longrightarrow [0, +\infty)$ that satisfies (H).

Note that if $y = \int \theta(u) u d\nu(u)$ then

$$\begin{aligned}
\langle T_\nu^{-1} y, y\rangle &= \Big\langle T_\nu^{-1} y, \int_{S^{n-1}} \theta(u) u d\nu(u) \Big\rangle = \int_{S^{n-1}} \theta(u) \langle T_\nu^{-1} y, u\rangle d\nu(u) \\
&\leqslant \Big(\int_{S^{n-1}} \theta^2(u) d\nu(u) \Big)^{1/2} \Big(\int_{S^{n-1}} \langle T_\nu^{-1} y, u\rangle^2 d\nu(u) \Big)^{1/2} \\
&= \Big(\int_{S^{n-1}} \theta^2(u) d\nu(u) \Big)^{1/2} \sqrt{\langle T_\nu^{-1} y, y\rangle}
\end{aligned} \tag{3.18}$$

taking into account (2.15) in the last step. Therefore,

$$\int_{S^{n-1}} \theta^2(u)d\nu(u) \geqslant \langle T_\nu^{-1}y, y\rangle. \tag{3.19}$$

We consider the functions $g_u(t) = \exp\left(\frac{-\pi t^2}{\langle T_\nu^{-1}u,u\rangle}\right)$, $u \in S^{n-1}$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n}^{*} \sup_{y=\int \theta(u)ud\nu(u)} \exp\left(\int_{S^{n-1}} \log g_u(\theta(u))d\mu(u)\right) dy \\ = \int_{\mathbb{R}^n}^{*} \sup_{y=\int \theta(u)ud\nu(u)} \exp\left(-\int_{S^{n-1}} \pi\theta^2(u)d\nu(u)\right) dy \\ \leqslant \int_{\mathbb{R}^n} \exp(-\pi\langle T_\nu^{-1}y, y\rangle)\, dy = \frac{1}{\sqrt{\det(T_\nu^{-1})}} = \sqrt{\det(T_\nu)}.\end{aligned} \tag{3.20}$$

Applying Lemma 3.2 we get

$$\begin{aligned}\sqrt{\det(T_\nu)}\exp\left(\int_{S^{n-1}} \log\sqrt{\langle T_\nu^{-1}y, y\rangle}d\mu(u)\right)\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle)d\mu(u)\right) dx \\ \leqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) d\mu(u)\right).\end{aligned} \tag{3.21}$$

For the second assertion of the theorem, assume that

$$h\left(\int_{S^{n-1}} \theta(u)u\, d\nu(u)\right) \geqslant \exp\left(\int_{S^{n-1}} \log g_u(\theta(u))d\mu(u)\right) \tag{3.22}$$

for every integrable function θ. We apply Lemma 3.2 with $f_u(t) = \exp\left(\frac{-\pi t^2}{\langle T_\nu^{-1}u,u\rangle}\right)$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle)d\mu(u)\right) dx = \int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} -\pi\langle x, u\rangle^2 d\nu(u)\right) dx \\ = \int_{\mathbb{R}^n} e^{-\pi\langle T_\nu x,x\rangle} dx = \frac{1}{\sqrt{\det(T_\nu)}},\end{aligned}$$

and hence

$$\begin{aligned}\sqrt{\det(T_\nu)}\exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} g_u\right) d\mu(u)\right) \\ \leqslant \exp\left(\int_{S^{n-1}} \log\sqrt{\langle T_\nu^{-1}u, u\rangle}d\mu(u)\right)\int_{\mathbb{R}^n}^{*} \sup_{y=\int \theta(u)ud\nu(u)} \exp\left(\int_{S^{n-1}} \log g_u(\theta(u))d\mu(u)\right) dy \\ \leqslant \exp\left(\int_{S^{n-1}} \log\sqrt{\langle T_\nu^{-1}u, u\rangle}d\mu(u)\right)\int_{\mathbb{R}^n} h\left(\int_{S^{n-1}} \theta(u)u\, d\nu(u)\right) dy \\ = \exp\left(\int_{S^{n-1}} \log\sqrt{\langle T_\nu^{-1}u, u\rangle}d\mu(u)\right)\int_{\mathbb{R}^n} h(y)\, dy.\end{aligned} \tag{3.23}$$

If we define $\theta'(u) = \theta(u)\langle T_\nu^{-1}u, u\rangle$ and $g_u'(t) = g_u(t\langle T_\nu^{-1}u, u\rangle)$, then the assumption (3.22) is satisfied by the pair (θ', g_u'), and making a change of variables we have

$$\begin{aligned}\exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} g_u'(t/\langle T_\nu^{-1}u, u\rangle)\, dt\right) d\mu(u)\right) \\ = \exp\left(\int_{S^{n-1}} \log\langle T_\nu^{-1}u, u\rangle d\mu(u)\right)\exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} g_u'(t)\, dt\right) d\mu(u)\right).\end{aligned}$$

We now complete the proof applying (3.23) for the pair (θ', g_u'), making some cancellations and recalling what A_ν is. □

Note that if ν is a γ-approximation of an isotropic measure then we recover the continuous analogue of Theorem 2.2.

Theorem 3.3. *Let ν be a γ-approximation of an isotropic Borel measure in $\mathbb{R}^n$ and let (f_u), $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \longrightarrow [0, +\infty)$ that satisfies (H). Then,*

$$\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle)\langle T_\nu^{-1}u, u\rangle d\nu(u)\right) dx \leqslant \gamma^{\frac{n}{2}} \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) \langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right). \tag{3.24}$$

Also, if h is a measurable function such that

$$h\left(\int_{S^{n-1}} \theta(u) u \langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right) \geqslant \exp\left(\int_{S^{n-1}} \log f_u(\theta(u))\langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right) \tag{3.25}$$

for every integrable function θ, then

$$\gamma^{\frac{n}{2}} \int_{\mathbb{R}^n} h(y)\, dy \geqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) \langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right). \tag{3.26}$$

One can also check that the constant A_ν which appears in Theorem 1.3 is at most 1, therefore the best constant in the continuous Brascamp-Lieb inequality of theorem is attained when the measure is isotropic. This agrees with the result in [22] for the discrete Brascamp-Lieb inequality.

Proposition 3.4. *Let ν be a non-negative finite Borel measure on S^{n-1}. Then*

$$A_\nu \leqslant 1,$$

with equality if ν is isotropic.

Proof. We will use the continuous version of Hadamard's inequality; for any non-negative finite and isotropic Borel measure μ on S^{n-1} and any $n \times n$ matrix T we have

$$\det(T) \leqslant \exp\left(\int_{S^{n-1}} \log(\|Tu\|_2)\, d\nu(u)\right). \tag{3.27}$$

Indeed, since the inequality is invariant under orthogonal transformations we can assume that T is positive definite and can be written as

$$T = \sum_{i=1}^{n} a_i e_i \otimes e_i$$

for some $a_i > 0$. Note that for any $u \in S^{n-1}$ we have $\sum_{i=1}^{n} \langle u, e_i\rangle^2 = \|u\|_2^2 = 1$. Therefore, by the AM-GM inequality we get

$$\|Tu\|_2^2 = \sum_{i=1}^{n} a_i^2 \langle u, e_i\rangle^2 \geq \prod_{i=1}^{n} a_i^{2\langle u, e_i\rangle^2}.$$

Using the fact that ν is isotropic we check that

$$\begin{aligned}
\exp\left(\int_{S^{n-1}} \log(\|Tu\|_2)\, d\nu(u)\right) &\geqslant \exp\left(\int_{S^{n-1}} \sum_{i=1}^{n} \log(a_i) \langle u, e_i\rangle^2\, d\nu(u)\right) \\
&= \exp\left(\sum_{i=1}^{n} \log(a_i) \int_{S^{n-1}} \langle u, e_i\rangle^2\, d\nu(u)\right) \\
&= \exp\left(\sum_{i=1}^{n} \log(a_i)\right) = \prod_{i=1}^{n} a_i = \det(T).
\end{aligned}$$

Since

$$T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u),$$

we have that

$$I_n = \int_{S^{n-1}} \frac{T_\nu^{-1/2}u}{\|T_\nu^{-1/2}u\|_2} \otimes \frac{T_\nu^{-1/2}u}{\|T_\nu^{-1/2}u\|_2} \, \|T_\nu^{-1/2}u\|_2^2 d\nu(u),$$

therefore the measure $\mu(u) := \|T_\nu^{-1/2}u\|_2^2(\nu \circ M^{-1})(u)$, where $M(u) = \frac{T_\nu^{-1/2}u}{\|T_\nu^{-1/2}u\|_2}$, is isotropic. This means that we can apply (3.27) for $T = T_\nu^{1/2}$ and for the isotropic measure μ. We get

$$\det(T_\nu^{1/2}) \leqslant \exp\left(\int_{S^{n-1}} \log\left(\frac{1}{\|T_\nu^{-1/2}u\|_2}\right) \|T_\nu^{-1/2}u\|_2^2 \, d\nu(u)\right).$$

Since $\|T_\nu^{-1/2}u\|_2^2 = \langle T_\nu^{-1}u, u\rangle$, we get the desired result. □

4 Approximate classical positions of convex bodies

Let ν be a non-negative finite Borel measure on S^{n-1}. Consider the cosine transform C_ν of ν and the symmetric convex body $C(\nu)$ defined by

$$h_{C(\nu)}(x) = C_\nu(x) = \int_{S^{n-1}} |\langle x, u\rangle| \, d\nu(u). \tag{4.1}$$

Note that

$$\begin{aligned} w(C(\nu)) &= \int_{S^{n-1}} h_{C(\nu)}(x) d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, u\rangle| d\nu(u) d\sigma(x) \\ &= \int_{S^{n-1}} \left(\int_{S^{n-1}} |\langle x, u\rangle| d\sigma(x)\right) d\nu(u) = c_n \nu(S^{n-1}), \end{aligned} \tag{4.2}$$

where

$$c_n = \int_{S^{n-1}} |\langle x, u\rangle| d\sigma(x) = \frac{2\omega_{n-1}}{n\omega_n} \simeq \frac{1}{\sqrt{n}}. \tag{4.3}$$

Combining this fact with the inequality

$$\frac{1}{w(C(\nu))} \leqslant \mathrm{vrad}(C^*(\nu))^{1/n} := \left(\frac{|C^*(\nu)|}{\omega_n}\right)^{1/n} \tag{4.4}$$

which follows immediately by expressing the volume of $C^*(\nu)$ in polar coordinates, we get the next simple fact.

Proposition 4.1. *Let ν be a non-negative finite Borel measure on S^{n-1}. Then,*

$$\nu(S^{n-1})|C^*(\nu)|^{1/n} \geqslant \frac{n\omega_n^{\frac{n+1}{n}}}{2\omega_{n-1}} \geqslant c, \tag{4.5}$$

where $c > 0$ is an absolute constant.

The next proposition establishes a reverse inequality for approximate isotropic measures.

Proposition 4.2. *Let $\gamma > 1$ and let ν be a γ-approximation of an isotropic measure on S^{n-1}. Then,*

$$\nu(S^{n-1})|C^*(\nu)|^{1/n} \leqslant \gamma n \frac{2}{(n!)^{1/n}} \leqslant 2e\gamma. \tag{4.6}$$

Proof. Using Fubini's theorem we check that

$$n!|C^*(\nu)| = \int_{\mathbb{R}^n} e^{-h_{C(\nu)}(x)} dx = \int_{\mathbb{R}^n} \exp\left(-\int_{S^{n-1}} |\langle x, u\rangle| d\nu(u)\right) dx. \tag{4.7}$$

Then, applying Theorem 1.3 with $f_u(t) = \exp\left(-\frac{|t|}{\langle T_\nu^{-1}u,u\rangle}\right)$ we see that

$$\begin{aligned} A_\nu \int_{\mathbb{R}^n} \exp\left(-\int_{S^{n-1}} |\langle x, u\rangle| d\nu(u)\right) dx &= A_\nu \int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log(f_u(\langle x, u\rangle)) \langle T_\nu^{-1}u, u\rangle \, d\nu(u)\right) dx \\ &\leqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) \langle T_\nu^{-1}u, u\rangle \, d\nu(u)\right) \\ &= \exp\left(\int_{S^{n-1}} \log\left(2\langle T_\nu^{-1}u, u\rangle\right) \langle T_\nu^{-1}u, u\rangle \, d\nu(u)\right) \\ &= 2^n \exp\left(\int_{S^{n-1}} \log\left(\langle T_\nu^{-1}u, u\rangle\right) \langle T_\nu^{-1}u, u\rangle \, d\nu(u)\right), \end{aligned} \tag{4.8}$$

using the fact that

$$\begin{aligned} \int_{S^{n-1}} \langle T_\nu^{-1}u, u\rangle \, d\nu(u) &= \int_{S^{n-1}} \mathrm{tr}(T_\nu^{-1}(u \otimes u)) \, d\nu(u) = \mathrm{tr}\left[T_\nu^{-1}\left(\int_{S^{n-1}} u \otimes u \, d\nu(u)\right)\right] \\ &= \mathrm{tr}(T_\nu^{-1}T_\nu) = \mathrm{tr}(I_n) = n. \end{aligned} \tag{4.9}$$

Simplifying with the common part of A_ν, we get

$$n!\sqrt{\det(T_\nu)}|C^*(\nu)| \leqslant 2^n \exp\left(\int_{S^{n-1}} \log\left(\sqrt{\langle T_\nu^{-1}u, u\rangle}\right) \langle T_\nu^{-1}u, u\rangle \, d\nu(u)\right). \tag{4.10}$$

Since $I_n \preceq T_\nu$ we have that $\det(T_\nu) \geqslant 1$ and $\log\left(\sqrt{\langle T_\nu^{-1}u, u\rangle}\right) \leqslant 0$ for all $u \in S^{n-1}$. Therefore, (4.10) implies that

$$n!|C^*(\nu)| \leqslant 2^n \tag{4.11}$$

On the other hand, since $\langle T_\nu^{-1}u, u\rangle \geqslant \gamma^{-1}$ for all $u \in S^{n-1}$, we also have

$$n = \int_{S^{n-1}} \langle T_\nu^{-1}u, u\rangle \, d\nu(u) \geqslant \gamma^{-1}\nu(S^{n-1}). \tag{4.12}$$

Multiplying (4.11) and (4.12) we see that

$$\nu(S^{n-1})|C^*(\nu)|^{1/n} \leqslant \gamma n \frac{2}{(n!)^{1/n}} \leqslant 2e\gamma, \tag{4.13}$$

as claimed. □

Remark 4.3. Note that the estimate is sharp for $\nu = \sigma_C$, where C is the cube.

From (4.11) we see that if ν is a γ-approximation of an isotropic measure on S^{n-1} then

$$\int_{S^{n-1}} h_{C(\nu)}^{-n} d\sigma(\theta) = \frac{|C^*(\nu)|}{\omega_n} \leqslant \frac{2^n}{n!\omega_n}, \tag{4.14}$$

and an application of Markov's inequality shows that a random $\theta \in S^{n-1}$ statisfies

$$h_{C(\nu)}(\theta) \geqslant \frac{(n!\omega_n)^{\frac{1}{n}}}{4} \geqslant c\sqrt{n} \tag{4.15}$$

with probability greater than $1 - 2^{-n}$, where $c > 0$ is an absolute constant. On the other hand, from the Cauchy-Schwarz inequality we have

$$\begin{aligned} h_{C(\nu)}(\theta) &= \int_{S^{n-1}} |\langle u, \theta\rangle| \, d\nu(u) \leqslant \left(\int_{S^{n-1}} \langle u, \theta\rangle^2 \, d\nu(u)\right)^{\frac{1}{2}} \sqrt{\nu(S^{n-1})} \\ &\leqslant \sqrt{\gamma}\sqrt{\nu(S^{n-1})} \leqslant \gamma\sqrt{n} \end{aligned} \tag{4.16}$$

In other words, with probability greater than $1 - 2^{-n}$ we have

$$c\sqrt{n} \leqslant \int_{S^{n-1}} |\langle u, \theta\rangle| \, d\nu(u) \leqslant \gamma\sqrt{n}. \tag{4.17}$$

This observation applies to all the classical positions of a convex body that we discussed in Section 2:

Fact 4.4. *Let K be a convex body in $\mathbb{R}^n$.*

(i) *If σ_K is a γ-approximation of an isotropic measure then*

$$\frac{c\partial(K)}{\sqrt{n}} \leqslant 2|P_{\theta^\perp}(K)| = \int_{S^{n-1}} |\langle u, \theta\rangle| \, d\sigma_K(u) \leqslant \gamma\frac{\partial(K)}{\sqrt{n}}. \tag{4.18}$$

with probability greater than $1 - 2^{-n}$ on S^{n-1}.

(ii) *If ν_K is a γ-approximation of an isotropic measure then*

$$\frac{cw(K)}{\sqrt{n}} \leqslant \int_{S^{n-1}} |\langle u, \theta\rangle| \, h_K(u) \, d\sigma(u) \leqslant \gamma\frac{w(K)}{\sqrt{n}} \tag{4.19}$$

with probability greater than $1 - 2^{-n}$ on S^{n-1}.

(iii) *If λ_K is a γ-approximation of an isotropic measure then*

$$c\sqrt{n}L_K^2 \leqslant \int_K |\langle x, \theta\rangle| \, \|x\|_2 \, dx \leqslant \gamma\sqrt{n}L_K^2 \tag{4.20}$$

with probability greater than $1 - 2^{-n}$ on S^{n-1}.

Remark 4.5. Note that $(t\nu)(S^{n-1})|C^*(t\nu)|^{1/n} = \nu(S^{n-1})|C^*(\nu)|^{1/n}$ for every Borel measure ν on S^{n-1} and every $t > 0$.

As an application of Proposition 4.1 and Proposition 4.2 we provide an alternative proof of a result from [14] on the stability of the minimal surface area position.

Theorem 4.6. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ such that*

$$I_n \preceq \frac{1}{\alpha}\int_{S^{n-1}} u \otimes u \, d\sigma_K(u) \preceq \gamma I_n \tag{4.21}$$

for some $\gamma > 1$ and $\alpha > 0$. Then,

$$\partial(TK) \leqslant \partial(K) \leqslant c\gamma\partial(TK), \tag{4.22}$$

where $c > 0$ is an absolute constant, $T \in SL(n)$ and TK is in the minimal surface area position.

Proof. Recall that $\Pi^*K = 2C^*(\sigma_K)$ and $\Pi^*(TK) = 2C^*(\sigma_{TK})$. Applying Proposition 4.1 and Proposition 4.2 to suitable multiples of the measure σ_K and σ_{TK} (also, taking into account Remark 4.5) we get

$$\sigma_K(S^{n-1})|\Pi^*K|^{1/n} \leqslant 4e\gamma \leqslant \frac{2e}{c}\gamma\sigma_{TK}(S^{n-1})|\Pi^*(TK)|^{1/n}. \tag{4.23}$$

Now, we use the observation of Petty [20] that

$$\Pi^*(TK) = T(\Pi^*K) \tag{4.24}$$

(this holds true for every convex body K and every $T \in SL(n)$) and hence, $|\Pi^*(TK)| = |\Pi^*K|$. Going back to (4.23) we conclude that

$$\partial(K) = \sigma_K(S^{n-1}) \leqslant \frac{2e}{c}\gamma\sigma_{TK}(S^{n-1}) = \frac{2e}{c}\gamma\partial(TK). \tag{4.25}$$

The inequality $\partial(TK) \leqslant \partial(K)$ is obvious since TK has minimal surface area. □

Remark 4.7. Proposition 4.2 and the Bourgain-Milman inequality (2.3) give upper and lower bounds for $|C(\nu)|$ if ν is a γ-approximation of an isotropic measure on S^{n-1} for some $\gamma > 1$. One has,

$$\frac{c_1\nu(S^{n-1})}{n} \geqslant |C(\nu)|^{\frac{1}{n}} \geqslant \frac{c_2\nu(S^{n-1})}{\gamma n}, \tag{4.26}$$

where $c_1, c_2 > 0$ are absolute constants. In fact the sharp reverse Santaló inequality is known for zonoids (see e.g. [15] for a very elegant proof) and since $C^*(\nu)$ is a zonoid we can specify the constants c_i further.

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