Σιλουανός Μπραζιτίκος: 1/7/2024.

Μετασχηματισμός Cramér, Συνάρτησις βάθους τοῦ Tukey, thresholds γιὰ λογαριθμικῶς κοῖλα μέτρα.

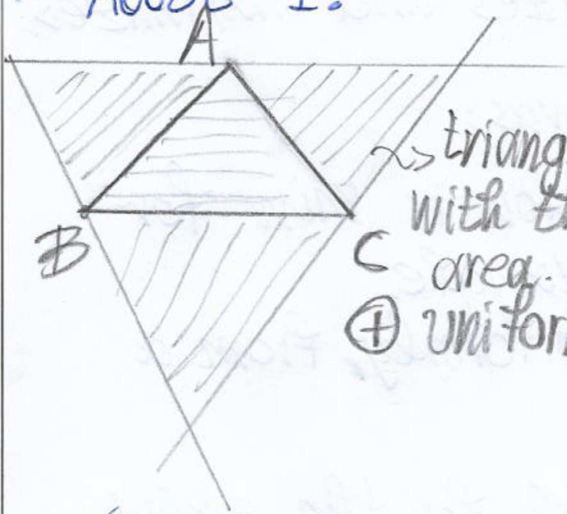
• 1864: Educational Times.

Sylvester: What is the chance of getting a convex quadrilateral after taking 4 points from the infinite plane?

→ 6 different answers to the problem:

1. Sylvester and Cayley $\rightsquigarrow \frac{3}{4}$.
2. Woolhouse $\rightsquigarrow 1-\frac{35}{12\pi^2} \sim 0{,}704$

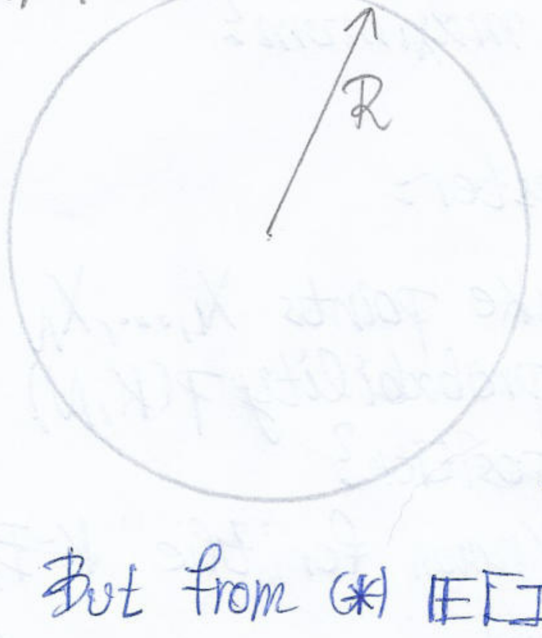
About 1:



$\rightsquigarrow$ triangles with the same area.
⊕ uniform measure
$\Rightarrow \mathbb{P}=\frac{1}{4}$ for ABC.

About 2:

X, Y, Z, W:



• Introduce the random variables I_X, I_Y, I_Z, I_W as:

• $I_X := \begin{cases} 1, & \text{if } X \in \text{inside } YZW \\ 0, & \text{otherwise} \end{cases}$

Let $I = I_X + I_Y + I_Z + I_W$ (*).

• We want the probability of $\{I=0\}$ $(=p)$

• $\mathbb{E}[I] = 0 \cdot p + 1\cdot(1-p) = 1-p$.

But from (*) $\mathbb{E}[I] = \mathbb{E}I_X + \mathbb{E}I_Y + \mathbb{E}I_Z + \mathbb{E}I_W =$

$= 4 \cdot \mathbb{E} I_X$

And $\mathbb{E}[I_X] = \mathbb{E}[\mathbb{E}[I_X | Y, Z, W]] = \mathbb{E}[\mathbb{P}[I_X | Y, Z, W]]$

$= \mathbb{E}\left[\frac{\text{area}(YZW)}{\text{area}(D)}\right]$ and the radius R doesn't play any role so we can take $R \to +\infty$.

$*$: After the 6 answers it was concluded that the problem wasn't well posed. So:

If we take our points from a convex set, what is the maximum and the minimum probability?

- This is equivalent to find the convex set K for which $\mathbb{E}\left[\frac{\text{area}(YZW)}{\text{area}(K)}\right]$ minimizes and maximizes.

The first to answer the question was:

Blaschke (1929): The above expectation is min for the disc and maximum for the triangle.

Indeed, if we take $X_1, X_2, \dots, X_N$ uniformly from a convex set K, with $|K| = 1$, then:

$\mathbb{E}[|\text{conv}\{X_1, X_2, \dots, X_N\}|]$ is minimal for the euclidean ball.

Groemer (1973): What about the maximum?

Open, even for $n = 3$.

The general problem of Sylvester:

Take $K \subseteq \mathbb{R}^n$ convex body and take points $X_1, \dots, X_N$ uniformly from K. What is the probability $P(K, N)$ that the points are in convex position?

$\rightsquigarrow$ Exact computation is only known for the $K = B_2^n$ for N, $P(\Delta^n, n+2)$ is also known.

About the behaviour:

<u>Theorem (Barany, Füredi 1988):</u>

If $K = B_2^n$ and $N < \frac{2^{n/2}}{n}$ then $p(B_2^n, N) \xrightarrow[n\to+\infty]{} 0$

$\gg$ and $N > n \cdot 2^{n/2}$ then $p(B_2^n, N) \xrightarrow[n\to+\infty]{} 0$

▷ <u>Analogous Question:</u> If $X_1, X_2, \ldots, X_N$ uniformly from $K \subseteq \mathbb{R}^n$ and $K_N := \mathrm{conv}\{X_1, \ldots, X_N\}$

$\frac{\mathbb{E}[|K_N|]}{|K|} \in [0,1]$. So we ask for the number of points N for which $\frac{\mathbb{E}[|K_N|]}{|K|} \xrightarrow[n\to+\infty]{} 0$, $\frac{\mathbb{E}[|K_N|]}{|K|} \xrightarrow[n\to+\infty]{} 1$.

The first result (Dyer, Füredi, McDiarmid '92):

If $K = B_\infty^n = [-1,1]^n$ and fix $\varepsilon \in (0,1)$ and $\kappa := \log \frac{2}{\sqrt{e} \cdot e^{\gamma/2}}$. Then if $N > e^{\kappa \cdot n \cdot (1+\varepsilon)}$, then:

$$\frac{\mathbb{E}[|K_N|]}{|[-1,1]^n|} \xrightarrow[n\to+\infty]{} 1$$

and if $N < e^{\kappa n (1-\varepsilon)}$, then:

$$\frac{\mathbb{E}[|K_N|]}{|[-1,1]^n|} \xrightarrow[n\to+\infty]{} 0$$

The second Result [After Break]

If $K = B_2^n$, then fix $\varepsilon \in (0,1)$. Then, if $N > e^{(1+\varepsilon)\cdot \frac{n+1}{2} \cdot \log n}$

then $\frac{\mathbb{E}[|K_N|]}{|K|} \to 1$ and if $N < e^{(1-\varepsilon)\cdot \frac{n+1}{2} \log n}$

we have: $\frac{\mathbb{E}[|K_N|]}{|K|} \xrightarrow[n\to+\infty]{} 0$

Giannopoulos, Pafis, B.: In the case of B_2^n we can take $\varepsilon = \varepsilon(n) = \frac{1}{\log(n)}$

2020: Tkocz, Frieze, Pedgen. The case of the simplex. There are constants c_1 and c_2 s.t. if $N > e^{c_1 \cdot n}$, then:

$$\frac{\mathbb{E}[|K_N|]}{|\Delta^n|} \xrightarrow[n\to+\infty]{} 1$$ and if $N < e^{c_2 \cdot n}$ then

$$\frac{\mathbb{E}[|K_N|]}{|\Delta^n|} \xrightarrow[n\to+\infty]{} 0$$

Generalizations: For a random vector X in $\mathbb{R}^n$, we define $\Lambda(\xi) := \log \mathbb{E}[e^{\langle \xi, X\rangle}]$ (log-Laplace Transformation) for $\xi \in \mathbb{R}^n$.

- Λ: convex function $\Rightarrow \exists$ dual of Λ.
- For $x \in \mathbb{R}^n$, we define: $\Lambda_X^*(x) := \sup_{\xi \in \mathbb{R}^n} (\langle x, \xi\rangle - \Lambda_X(\xi))$

[Cramér Transformation $\rightsquigarrow$ Large Deviation Theory].

- If $\{X_i\}_{i=1}^n$ are r.v then $\forall \delta > 0$
  - $\mathbb{P}\left[\left|\frac{\sum X_i}{n}\right| \geq \delta\right] \xrightarrow[n\to+\infty]{} 0$ [Law of large numbers].

- $\mathbb{P}\left[\left|\frac{\sum X_i}{\sqrt{n}}\right| \in A\right] = \int_A e^{-x^2/2} dx$ [Central Limit Theorem]

$\rightsquigarrow$ Large Deviation Principle (Cramér):

$$\log \mathbb{P}\left(\frac{|\sum X_i|}{n} \geq \delta\right) \xrightarrow[n\to+\infty]{} -\Lambda^*(\sigma)$$

1st Generalization (Giannopoulos, Gantzouras):

Let μ an even measure in $\mathbb{R}$ with compact support $[-x^*, x^*]$ and suppose that $\lim_{x\to x^*} \frac{-\log(\mu([x,+\infty)))}{\Lambda^*(x)} = 1$ (*)

If $\mu^n = \underbrace{\mu \otimes \dots \otimes \mu}_{n\text{-times}}$ (product measure), then $\forall \varepsilon \in (0,1)$ if $N > e^{(1+\varepsilon)\cdot \mathbb{E}[\Lambda^*_{\mu^n}}$, then $\frac{\mathbb{E}[|K_N|]}{|[-x^*, x^*]^n|} \xrightarrow[n\to+\infty]{} 1$.

2nd Generalization. (Pafis, 2023):

The dream of Sylvester:

- Let μ be an even probability measure in $\mathbb{R}$ and suppose (*)
- Let μ be a good measure. Then $\forall \varepsilon \in (0,1)$ if $N > e^{(1+\varepsilon)\cdot \mathbb{E}[\Lambda^*_{\mu^n}]}$ then $\mathbb{E}[\mu^n(K_N)] \xrightarrow[n\to+\infty]{} 1$.

if $N < e^{(1-\varepsilon)\cdot \mathbb{E}[\Lambda^*_{\mu^n}]}$ then $\mathbb{E}[\mu^n(K_N)] \xrightarrow[n\to+\infty]{} 0$.

(for the Gaussian measure: $\mathbb{E}[\Lambda^*_{\mu^n}] = \frac{n}{2}$)

(B., Chasapis, 2024)

*: A non trivial example for before is the case of $f(x) = c_p \cdot e^{-|x|^p}$

- The condition (*) holds iff there's a convex function V: $\lim_{x \to x^*} \frac{-\log(\mu([x,+\infty)))}{V(x)} = 1$.

(the tails of the measure behave log-concavely).

- If μ is any log-concave measure in $\mathbb{R}$ and $\mu^n = \mu \otimes \mu \otimes \dots \otimes \mu$. Then, if $N > e^{(1+\varepsilon) \cdot \mathbb{E}[\Lambda^*_{\mu^n}]}$ then $\mathbb{E}[\mu^n(K_N)] \longrightarrow 1$ and if $N < e^{(1-\varepsilon)\mathbb{E}[\Lambda^*_{\mu^n}]}$, then $\mathbb{E}[\mu^n(K_N)] \longrightarrow 0$. (A measure μ in $\mathbb{R}^n$ is log-concave if $\forall A, B$ Borel and $\lambda \in (0,1)$

$$\mu((1-\lambda) A + \lambda B) \geq \mu(A)^{1-\lambda} \cdot \mu(B)^{\lambda} \,)$$

For all even log-concave measures: (Remark).

$\mathbb{E}[\Lambda^*_{\mu^n}]$ is minimum for the measure with density $f(x) = \frac{1}{2} \cdot e^{-|x|}$

Reduction:

If $\frac{\mathbb{V}[\Lambda_\mu^*]}{(\mathbb{E}[\Lambda^*_\mu])^2} \xrightarrow[n \to +\infty]{} 0$

Σιλουανός Μπραζιτίκος: [Parts III & IV]. 2/7/2024

Μετασχηματισμός Cramér, Συνάρτηση Βάθους του Tukey, thresholds για λογαριθμικώς κοίλα μέτρα.

Theorem: Let μ be log-concave probability measure on $\mathbb{R}$ and $\mu_n := \underbrace{\mu \otimes \dots \otimes \mu}_{n\text{-times}}$, Let $X_1, X_2, \dots, X_N$ independed from ~~chosen with~~ μ_n:

- If $N > e^{(1+\varepsilon)\cdot \mathbb{E}\Lambda^*_{\mu_n}}$, then $\mathbb{E}\mu_n(K_N) \xrightarrow[n\to+\infty]{} 1$ ($K_N := \mathrm{conv}\{X_1, X_2, \dots, X_N\}$)
- If $N < e^{(1-\varepsilon)\cdot \mathbb{E}\Lambda^*_{\mu_n}}$, then $\mathbb{E}\mu_n(K_N) \xrightarrow[n\to+\infty]{} 0$.

where $\Lambda_{\mu_n}(\xi) = \log \mathbb{E}\, e^{\langle X, \xi\rangle}$ and $\Lambda^*_{\mu_n}(x) = \sup_{\xi\in\mathbb{R}^n} \langle x, \xi\rangle - \Lambda_{\mu_n}(\xi)$

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*Reminder: μ is log-concave iff ($\forall A, B \in \mathcal{B}(\mathbb{R}^n)$) $(\mu(\lambda\cdot A + (1-\lambda)\cdot B) \geq \mu(A)^{\lambda}\cdot \mu(B)^{1-\lambda})$ $(\forall \lambda \in [0,1])$.

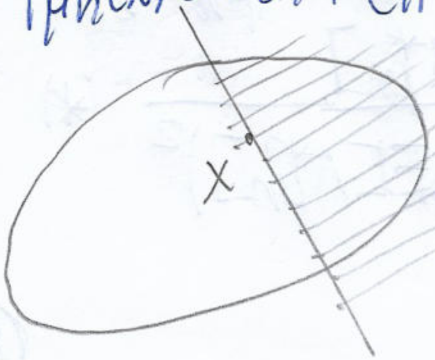
- **Lemma:** If ν any measure on $\mathbb{R}^n$ and A, B Borel.

(i) $\mathbb{E}\nu(K_N) \leq \nu(A) + N \sup_{x\notin A} q_{\mu_n}(x)$

(ii) $\mathbb{E}\nu(K_N) \geq \nu(B)\cdot\left(1 - 2\cdot\binom{N}{n}\cdot\left(1 - \inf_{x\in B} q_{\mu_n}(x)\right)^{N-n}\right)$

where $q_{\mu_n}(x)$ is the Tukey's halfspace depth

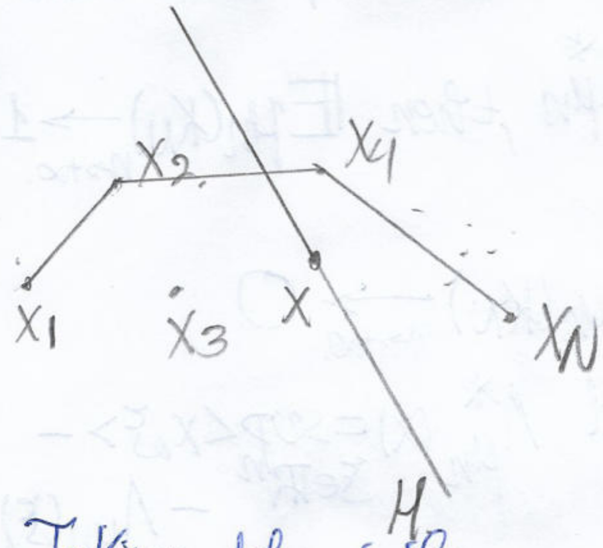
$q_{\mu_n}(x) := \inf(\mathbb{P}[\langle X, \xi\rangle \geq \langle x, \xi\rangle] = \inf_{x\in H \text{ (Halfspace)}} \mathbb{P}[X \in H]$.



Proof of the Lemma (i):

$$\mathbb{E}[v(K_N)] = \mathbb{E}[v(K_N \cap A)] + \mathbb{E}[v(K_N \setminus A)] \leq$$

$$\leq v(A) + \mathbb{E}\Big[\int_{\mathbb{R}^n \setminus A} \mathbb{1}_{K_N}(x) \cdot g(x)\, dv(x)\Big] = v(A) + \int_{\mathbb{R}^n \setminus A} \mathbb{P}[x \in K_N]\, dv(x).$$



- For any halfspace H that contains X, there is a X_i: $X_i \in H$. So: $\{x \in K_N\} \subseteq \bigcup \{X_i \in H\}$ $\Rightarrow \mathbb{P}[x \in K_N] \leq \sum \mathbb{P}[X_i \in H] =$ $= N \cdot \mathbb{P}[X_i \in H]$

Taking the infimum over all H, then $\mathbb{P}[x \in K_N] \leq N \cdot q_{\mu_n}(x)$ and we can have (i) statement. ■

2nd Step: Choose A, B

$$A := \{x \in \mathbb{R}^n / \Lambda^*_{\mu_n}(x) \leq (1 - \tfrac{\varepsilon}{8}) \cdot \mathbb{E}[\Lambda^*_{\mu_n}]\}$$

$$B := \{x \in \mathbb{R}^n / \Lambda^*_{\mu_n}(x) \leq (1 + \tfrac{\varepsilon}{8}) \cdot \mathbb{E}[\Lambda^*_{\mu_n}]\}$$

- We have that $\max\{\mu_n(A), \mu_n(B^c)\} \leq$

$$\leq \mu_n\big(x \in \mathbb{R}^n / |\Lambda^*_{\mu_n}(x) - \mathbb{E}[\Lambda^*_{\mu_n}]| \geq \tfrac{\varepsilon}{8} \cdot \mathbb{E}[\Lambda^*_{\mu_n}]\big)$$

$$\underset{\text{Chebyshev.}}{\leq} \frac{\mathbb{E}[|\Lambda^*(x) - \mathbb{E}[\Lambda^*]|^2]}{(\frac{\varepsilon}{8} \cdot \mathbb{E}[\Lambda^*])^2} = \frac{\mathbb{V}[\Lambda^*_{\mu_n}]}{(\frac{\varepsilon}{8} \cdot \mathbb{E}[\Lambda^*_{\mu_n}])^2} \quad (*)$$

[Fact: $\Lambda^*_{\mu_n}((x_1,\dots,x_n)) = \Lambda^*_\mu(x_1)+\dots+\Lambda^*_\mu(x_n)$]

Therefore, $(*) = \dfrac{n\cdot \mathbb{V}[\Lambda^*_\mu]}{\left(\frac{\varepsilon}{8}\cdot n \mathbb{E}[\Lambda^*_\mu]\right)^2} = \dfrac{1}{n\cdot\frac{\varepsilon^2}{8^2}}\cdot\dfrac{\mathbb{V}[\Lambda^*_\mu]}{(\mathbb{E}[\Lambda^*_\mu])^2}$

• Relation between q_{μ_n} and $\Lambda^*_{\mu_n}$

Fix $\xi\in\mathbb{R}^n$ $\mathbb{P}[\langle X,\xi\rangle \geq \langle x,\xi\rangle] \overset{\text{chernoff}}{\leq} \mathbb{E}[e^{\langle X,\xi\rangle - \langle x,\xi\rangle}]$

Taking inf $q_{\mu_n}(x) \leq e^{-\Lambda^*_{\mu_n}(x)} \Rightarrow$

$\Rightarrow \Lambda^*_{\mu_n}(x) \leq \ln\dfrac{1}{q_{\mu_n}(x)}$.

▶ **Thm:** If μ is probability measure on $\mathbb{R}$ which is log-concave (Continuous or discrete$^{(*)}$). Then $\forall\varepsilon\in(0,1)$ and $x\in\overline{\operatorname{supp}(\mu)}$, $\Lambda^*_\mu(x) \geq (1-\varepsilon)\cdot\ln\dfrac{1}{q_{\mu_n}(x)} + \log\dfrac{\varepsilon}{2^{1-\varepsilon}}$

The inequality is sharp.

Proof: • q_μ is log-concave:

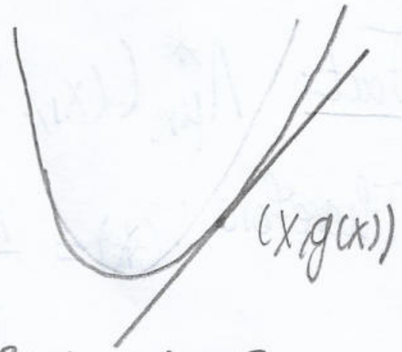
$\mathbb{P}[\langle X,\xi\rangle \geq \langle (1-\lambda)\cdot x+\lambda\cdot y,\xi\rangle] \geq \mathbb{P}[\langle X,\xi\rangle\geq\langle x,\xi\rangle]^{1-\lambda}\cdot$

$\cdot\,\mathbb{P}[\langle X,\xi\rangle \geq \langle y,\xi\rangle]^{\lambda} \geq q(x)^{1-\lambda}\cdot q(y)^{\lambda}$. Taking the inf we have the log-concavity.

• fix $x\in\operatorname{supp}(\mu)$ and $\varepsilon\in(0,1)$. Consider $g(y) := (1-\varepsilon)\ln\dfrac{1}{q_\mu(y)}$ (which is convex).

$(*)$: Discrete log-concavity: $(p(k))^2 \geq p(k+1)\cdot p(k-1)$.

Therefore, $g(x) = l(x)$ and
$g(y) \geq l(y) \quad \forall y \neq x.$



$l(y) = y \cdot t + \theta.$

$$\Rightarrow \mathbb{E}[e^{l(X)}] \leq \mathbb{E}[e^{g(X)}],$$

$$l(x) - \log \mathbb{E}[e^{l(X)}] \geq g(x) - \log \mathbb{E}[e^{g(X)}]$$

$$\Rightarrow x t + \theta - \log(\mathbb{E}[e^{tX+\theta}]) \geq g(x) - \log \mathbb{E}[e^{g(X)}]$$

$$\Rightarrow x \cdot t - \Lambda(t) \geq (1-\varepsilon) \cdot \ln \frac{1}{q_\mu(x)} - \log \mathbb{E}\left[\left(\frac{1}{q_\mu(X)}\right)^{1-\varepsilon}\right]$$

$$\Rightarrow \Lambda_\mu^*(x) \geq \ldots$$

Fact: If $F(x) = \mathbb{P}[X \leq x]$ and F is continuous then the r.v $F(X)$ is uniformly distributed in $[0,1]$.

$$\mathbb{P}[F(X) \leq x] = \mathbb{P}[X \leq F^{-1}(x)] = x.$$

Moreover $q(x) = \min\{F(x), 1-F(x)\}$, therefore

$$q(X) = \min\{U, 1-U\}, \quad U \sim \mathrm{unif}[0,1].$$

$$\mathbb{E}[q(X)^p] = \mathbb{E}[(\min\{U, 1-U\})^p] = \int_0^{1/2} u^p du + \int_{1/2}^{1} (1-u)^p du$$

$$\Rightarrow \mathbb{E}[q(X)^p] = \frac{1}{2^p \cdot (1+p)}$$

$$\Lambda_\mu^*(x) \geq (1-\varepsilon) \cdot \log \frac{1}{q_\mu(x)} + \log \frac{\varepsilon}{2^{1-\varepsilon}}, \quad \forall \varepsilon \in (0,1).$$

Let $A := \{x \in \mathbb{R}^n / \ q_X(x) < \frac{1}{2e}\}$, then choose, for $x \in A$,

$$\varepsilon_e = \left(\log \frac{1}{2q(x)}\right)^{-1} \in (0,1).$$

$$\forall x \in A: \ \Lambda^*(x) \geq \log\left(\frac{1}{2q(x) \cdot \log\left(\frac{1}{2q(x)}\right)}\right) - 1.$$

$\rightarrow$

$$\Rightarrow \mathbb{E}[\Lambda_\mu^*] \geq \mathbb{E}_A[\Lambda_\mu^*] \geq \mathbb{E}\left[\log \frac{1}{2q(X)\cdot \log\left(\frac{1}{2q(X)}\right)} - 1\right] \sim$$

~ 0.14

Upper Bound: $\mathbb{E}[\Lambda_\mu^{*2}] \leq \mathbb{E}\left[\log\left(\frac{1}{q(X)}\right)\right]^2 \sim 3{,}8.$

Thm: For all continuous probability measures: the one dimensional marginals are log-concave, we have that $\forall x$ and $\varepsilon \in (0,1)$.

$$\Lambda_\mu^*(x) \geq (1-\varepsilon)\cdot \log \frac{1}{q_\mu(x)} + \log \frac{\varepsilon}{2^{1-\varepsilon}}.$$

*: We can't prove this theorem in the discrete case.

$$\Lambda_\mu^*(x) = \sup_{\xi \in \mathbb{R}^n} (\langle x,\xi\rangle - \Lambda_\mu(\xi)) = \sup_{(t,\xi)\in \mathbb{R}\times S^{n-1}} (t\cdot\langle x,\xi\rangle -$$

$$- \Lambda(t\cdot\langle X,\xi\rangle) = \sup_{\xi\in S^{n-1}} \sup_t (t\cdot\langle x,\xi\rangle - \Lambda_{\langle X,\xi\rangle}(t)) =$$

$$= \sup_{\xi\in S^{n-1}} \Lambda^*_{\langle X,\xi\rangle}(\langle x,\xi\rangle).$$

- For fixed $\xi \in S^{n-1}$, $\Lambda^*_{\langle X,\xi\rangle}(\langle x,\xi\rangle) \geq$

$$(1-\varepsilon)\cdot\log\frac{1}{q_{\langle X,\xi\rangle}(\langle x,\xi\rangle)} + \log\frac{\varepsilon}{2^{1-\varepsilon}}.$$

$$N \cdot q_{\mu_n}(x) \le N \cdot e^{-\Lambda^*_{\mu_n}(x)} \le N \cdot e^{-(1-\frac{\varepsilon}{8}) \cdot \mathbb{E}[\Lambda^*_{\mu_n}]} \le$$

$$\le e^{(1-\varepsilon) \cdot \mathbb{E}[\Lambda^*_{\mu_n}] - (1-\frac{\varepsilon}{8}) \mathbb{E}[\Lambda^*_{\mu_n}]} = e^{-\frac{7\varepsilon}{8} \mathbb{E}[\Lambda^*_{\mu_n}]}$$

$$= e^{-\frac{7\varepsilon}{8} \cdot n \mathbb{E}[\Lambda^*_{\mu}]}.$$

The End !!!