SHARP BOUNDS ON p-NORMS FOR SUMS OF INDEPENDENT UNIFORM RANDOM VARIABLES, $0 < p < 1$

GIORGOS CHASAPIS, KEERTHANA GURUSHANKAR, AND TOMASZ TKOCZ

Abstract. We provide a sharp lower bound on the p-norm of a sum of independent uniform random variables in terms of its variance when $0 < p < 1$. We address an analogous question for p-Rényi entropy for p in the same range.

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1. Introduction and results

Moment comparison inequalities for sums of independent random variables, a.k.a. Khinchin-type inequalities, first established by Khinchin for Rademacher random variables (random signs) in his proof of the law of the iterated logarithm (see [14]), have been extensively studied ever since his work. Particularly challenging, interesting and conducive to new methods is the question of sharp constants in such inequalities. We only mention in passing several classical as well as recent references, [1, 9, 11, 15, 17, 18, 23]. This paper finishes the pursuit of sharp constants in $L_p - L_2$ Khinchin inequalities for sums of independent uniform random variables, addressing the range $0 < p < 1$. We are also concerned with a p-Rényi entropy analogue.

1.1. **Moments.** Let $U_1, U_2, \ldots$ be independent random variables uniform on $[-1, 1]$. As usual, $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$ is the p-norm of a random variable X. Given $p > -1$, let c_p and C_p be the best constants such that for every integer $n \geqslant 1$ and real numbers $a_1, \ldots, a_n$, we have

$$c_p \left(\sum_{j=1}^n a_j^2 \right)^{1/2} \leqslant \left\| \sum_{j=1}^n a_j U_j \right\|_p \leqslant C_p \left(\sum_{j=1}^n a_j^2 \right)^{1/2}, \tag{1}$$

or in other words, since $\|\sum a_j U_j\|_2 = \mathrm{Var}(\sum a_j U_j) = 3^{-1/2} \left(\sum a_j^2 \right)^2$, finding c_p and C_p amounts to extremising the p-norm of the sum $\sum a_j U_j$ subject to a fixed

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variance,

$$c_p = \inf \left\| \sum_{j=1}^n a_j U_j \right\|_p, \qquad C_p = \sup \left\| \sum_{j=1}^n a_j U_j \right\|_p,$$

where the infimum and supremum and taken over all integers $n \geqslant 1$ and unit vectors $a = (a_1, \dots, a_n)$ in $\mathbb{R}^n$.

For $p \geqslant 1$, the optimal constants c_p, C_p were found by Latała and Oleszkiewicz in [19] (see also [8] for an alternative approach and [1, 16] for generalisations in higher dimensions). They read

$$\begin{aligned} c_p &= \begin{cases} \lim_{n\to\infty} \left\| \frac{U_1+\cdots+U_n}{\sqrt{n}} \right\|_p = \|Z\|_p/\sqrt{3}, & 1 \leqslant p \leqslant 2, \\ \|U_1\|_p = (1+p)^{-1/p}, & p \geqslant 2, \end{cases} \\ C_p &= \begin{cases} \|U_1\|_p, & 1 \leqslant p \leqslant 2, \\ \|Z\|_p/\sqrt{3}, & p \geqslant 2, \end{cases} \end{aligned} \tag{2}$$

where Z here and throughout the text denotes a standard $N(0,1)$ Gaussian random variable. In fact stronger results are available (extremisers are known via Schur-convexity for each fixed n).

For $-1 < p < 0$, the behaviour is complicated by a phase transition (similar to the case of random signs as established by Haagerup in [9]). It has recently been proved in [4] that

$$c_p = \min\left\{ \|Z\|_p/\sqrt{3}, \|U_1+U_2\|_p/\sqrt{2} \right\} = \begin{cases} \|Z\|_p/\sqrt{3}, & -0.793.. < p < 0, \\ \|U_1+U_2\|_p/\sqrt{2}, & -1 < p < -0.793.., \end{cases}$$

and the limiting behaviour of c_p as $p \to -1^+$ recovers Ball's celebrated cube slicing inequality from [2].

The fact that

$$C_p = \|U_1\|_p, \qquad -1 < p < 1,$$

follows easily from unimodality and Jensen's inequality (see, e.g. Proposition 15 in [8]).

Thus what is unknown is the optimal value of c_p for $0 < p < 1$ and this paper fills out this gap. Our main result reads as follows.

Theorem 1. *For $0 < p < 1$, $c_p = \|Z\|_p/\sqrt{3}$, the best constant in* (1).

We record for future use that

$$\|Z\|_p^p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^p e^{-x^2/2} \mathrm{d}x = \frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{1+p}{2}\right).$$

1.2. **Rényi entropy.** For $p \in [0, \infty]$, the p-Rényi entropy of a random variable X with density f is defined as (see [25]),

$$h_p(X) = \frac{1}{1-p} \log \left(\int_{\mathbb{R}} f^p \right)$$

with $p \in \{0, 1, \infty\}$ defined by taking the limit: $h_0(f) = \log |\mathrm{supp}(f)|$ is the logarithm of the Lebesgue measure of the support of f, $h_1(f) = -\int f \log f$ is the Shannon entropy, and $h_\infty = -\log \|f\|_\infty$, where $\|f\|_\infty$ is the ∞-norm of f (with respect to Lebesgue measure). The question of maximising Rényi entropy under a variance constraint (or more generally, a moment constraint) for general distributions has been fully understood and leads to the notion of *relative entropy* that is of importance in information theory, providing a natural way of measuring distance to the extremal distributions (see [5, 13, 20, 22]). In analogy to Theorem 1, we provide an answer for p-Rényi entropies, $0 < p < 1$, for sums of uniforms under the variance constraints.

Theorem 2. *Let $0 < p < 1$. For every unit vector $a = (a_1, \ldots, a_n)$, we have*

$$h_p\left(U_1\right) \leqslant h_p\left(\sum_{j=1}^n a_j U_j\right) \leqslant h_p\left(Z/\sqrt{3}\right).$$

The lower bound is a simple consequence of the entropy power inequality. The upper bound is interesting in that the maximizer among *all* distributions of fixed variance is *not* Gaussian (rather, with density proportional to $(1+x^2)^{-1/(1-p)}$ for $\frac{1}{3} < p < 1$ and it does not exist for $p < \frac{1}{3}$, see e.g. [5]). It is derived from the $L_q - L_2$ Khinchin inequality for even q.

1.3. **Organisation of the paper.** In Section 2 we give an overview of the proof of Theorem 1 and show a reduction to two main steps: an integral inequality and an inductive argument. Then in Section 3 we gather all technical lemmas needed to accomplish these steps which is then done in Sections 4 and 5, respectively. Section 6 contains a short proof of Theorem 2.

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2. Proof of the main result

2.1. **Overview.** We follow an approach developed by Haagerup in [9], with major simplifications advanced later by Nazarov and Podkorytov in [24]. In essence, the argument begins with a Fourier-analytic integral representation for the power function $|\cdot|^p$ which allows to take advantage of independence and in turn, by virtue of the AM-GM inequality, to reduce the problem to establishing a certain integral inequality involving the Fourier transforms of the uniform and Gaussian distributions. Since this inequality holds only in a specific range of parameters, additional arguments are needed, mainly an induction on the number of summands n (similar problems were faced in e.g. [4, 24, 15]). In our case, this is further complicated by the fact that the base of the induction fails for large values of p (roughly for $p > 0.7$).

We point out that the main difference between the regimes $p > 1$ and $p < 1$ is that whilst for the former convexity type arguments allow for stronger comparison results and slick proofs (for a *fixed* n), for the latter Schur-convexity does not hold anymore (essentially due to the lack of convexity/concavity of the function $x \mapsto \mathbb{E}|U_1 + \sqrt{x}|^p$, see also [1], [8]) and the Fourier-analytic approach seems to be indispensable.

2.2. **Details.** The aforementioned Fourier-analytic formula reads as follows (it can be found for instance in [9], but we sketch its proof for completeness).

Lemma 3. *Let $0 < p < 2$ and $\kappa_p = \frac{2}{\pi}\Gamma(1+p)\sin\left(\frac{\pi p}{2}\right)$. For a random variable X in L_2 with characteristic function $\phi_X(t) = \mathbb{E}e^{itX}$, we have*

$$\mathbb{E}|X|^p = \kappa_p \int_0^\infty \frac{1 - \mathrm{Re}\phi_X(t)}{t^{p+1}} \mathrm{d}t.$$

Proof. A change of variables justifies the identity $|x|^p = \kappa_p \int_0^\infty \frac{1-\cos(tx)}{t^{p+1}}\mathrm{d}t$, $x \in \mathbb{R}$. It remains to apply it to X and take the expectation. □

We begin the proof of Theorem 1. Let $0 < p < 1$ and $c_p = \|Z\|_p/\sqrt{3}$. Let $a_1, \ldots, a_n$ be nonzero real numbers with $\sum_{j=1}^n a_j^2 = 1$. By the symmetry of the uniform distribution, without loss of generality we can assume that they are in fact positive. From Lemma 3, we obtain

$$\mathbb{E}\left|\sum_{j=1}^n a_j U_j\right|^p = \kappa_p \int_0^\infty \frac{1 - \prod_{j=1}^n \phi(a_j t)}{t^{1+p}} \mathrm{d}t,$$

where we have used independence and put $\phi(t) = \mathbb{E}e^{itU_1} = \frac{\sin t}{t}$ to be the characteristic function of the uniform distribution. We seek a sharp lower-bound on this expression (attained when $a_1 = \cdots = a_n = \frac{1}{\sqrt{n}}$ and $n \to \infty$, as anticipated by Theorem 1). By the AM-GM inequality,

$$\left|\prod_{j=1}^n \phi(a_j t)\right| \leqslant \sum_{j=1}^n a_j^2 |\phi(a_j t)|^{1/a_j^2}.$$

As a result,

$$\mathbb{E}\left|\sum_{j=1}^n a_j U_j\right|^p \geqslant \sum_{j=1}^n a_j^2 \mathcal{I}_p(1/a_j^2),$$

where we have set

$$\mathcal{I}_p(s) = \kappa_p \int_0^\infty \frac{1 - \left|\frac{\sin(t/\sqrt{s})}{t/\sqrt{s}}\right|^s}{t^{p+1}} \mathrm{d}t, \qquad s \geqslant 1.$$

Note that $\frac{\sin(t/\sqrt{s})}{t/\sqrt{s}} = 1 - \frac{t^2}{6s} + O(1/s^2)$ for a fixed t as $s \to \infty$ and consequently,

$$\mathcal{I}_p(\infty) = \lim_{s\to\infty} \mathcal{I}_p(s) = \kappa_p \int_0^\infty \frac{1 - e^{-t^2/6}}{t^{p+1}} \mathrm{d}t = \mathbb{E}|Z/\sqrt{3}|^p,$$

where the last equality follows from Lemma 3 because $e^{-t^2/6}$ is the characteristic function of $Z/\sqrt{3}$, $Z \sim N(0,1)$ (the exchange of the order of the limit and integration in the second equality can be easily justified by truncating the integral, see, e.g., (15) in [4]). In particular, if for some p and s_0,

$$\mathcal{I}_p(s) \geqslant \mathcal{I}_p(\infty), \qquad \text{for all } s \geqslant s_0, \tag{3}$$

then

$$\mathbb{E}\left|\sum_{j=1}^n a_j U_j\right|^p \geqslant \mathbb{E}|Z/\sqrt{3}|^p = c_p^p, \tag{4}$$

as long as $1/a_j^2 \geqslant s_0$ for each j. If (3) were true for all $0 < p < 1$ with $s_0 = 1$, then the proof of Theorem 1 would be complete. Unfortunately, that is not the case. In Section 4 we show the following result.

Theorem 4. *Inequality* (3) *holds for every* $0.6 < p < 1$ *with* $s_0 = 1$.

As a result, when $0.6 < p < 1$, (4) holds for arbitrary a_j and the proof of Theorem 1 is complete in this case. For smaller values of p, s_0 has to be increased.

Theorem 5. *Inequality* (3) *holds for every* $0 < p < 1$ *with* $s_0 = 2$.

This is proved in Section 4. Consequently, (4) holds provided that $a_j^2 \leqslant \frac{1}{2}$ for each j. To remove this restriction, we employ an inductive argument of Nazarov and Pokorytov from [24] developed for random signs and adapted to the uniform distribution in [4]. This works for $0 < p < 0.69$ and the proof of Theorem 1 is complete. This is done in Section 5.

3. Auxiliary lemmas

To show Theorems 4 and 5 and carry out the inductive argument, we first prove some technical lemmas.

3.1. Lemmas concerning the sinc function.

The zeroth spherical Bessel function (of the first kind) $j_0(x) = \frac{\sin x}{x} = \mathrm{sinc}(x)$ is sometimes referred to as the sinc function. As the characteristic function of a uniform random variable, it plays a major role in our approach. We shall need several elementary estimates.

Lemma 6. *For* $0 < t < \pi$*, we have* $\frac{\sin t}{t} < e^{-t^2/6}$.

Proof. This follows from the product formula, $\frac{\sin t}{t} = \prod_{n=1}^{\infty}\left(1 - \frac{t^2}{n^2\pi^2}\right)$. Since each term is positive for $0 < t < \pi$, the lemma follows by applying $1 + x \leqslant e^x$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. $\square$

Lemma 7. $\sup_{t\in\mathbb{R}} \left|\cos t - \frac{\sin t}{t}\right| < \frac{11}{10}$.

Proof. Since both $\cos t$ and $\frac{\sin t}{t}$ are even, it suffices to consider positive t. By the Cauchy-Schwarz inequality, we have $\left|\cos t - \frac{\sin t}{t}\right| \leqslant \sqrt{1 + \frac{1}{t^2}}$, so it suffices to consider $t < \frac{10}{\sqrt{21}}$. On $(0, \frac{\pi}{2})$, we have $\left|\cos t - \frac{\sin t}{t}\right| = \frac{\sin t}{t} - \cos t < 1 + 0 = 1$, so it remains to consider $\frac{\pi}{2} < t < \frac{10}{\sqrt{21}}$. Letting $t = \frac{\pi}{2} + x$, we have for $0 < x < \frac{10}{\sqrt{21}} - \frac{\pi}{2}$,

$$\left|\cos t - \frac{\sin t}{t}\right| = \frac{\sin t}{t} - \cos t = \frac{\cos x}{x + \pi/2} + \sin x < \frac{1}{x + \pi/2} + x.$$

Examining the derivative, the right hand side is clearly increasing, so it is upper bounded by its value at $x = \frac{10}{\sqrt{21}} - \frac{\pi}{2}$ which is $\frac{\sqrt{21}}{10} + \frac{10}{\sqrt{21}} - \frac{\pi}{2} < 1.07$. $\square$

Lemma 8. *Let $k \geqslant 0$ be an integer and let y_k be the value of the unique local maximum of $\left|\frac{\sin t}{t}\right|$ on $(k\pi, (k+1)\pi)$. Then*

$$\frac{1}{(k + 1/2)\pi} \leqslant y_k \leqslant \frac{1}{k\pi}.$$

Moreover, $y_1 < e^{-3/2}$.

Proof. The lower bound follows from taking $t = (k + 1/2)\pi$, whereas the upper bound follows from $|\sin t| \leqslant 1$ and $t > k\pi$. The bound on y_1 is equivalent to $\sin t < e^{-3/2}(t + \pi)$, $0 < t < \pi/2$. To show this in turn, it suffices to upper bound $\sin t$ by its tangent at, e.g., $t = 1.3$. $\square$

Lemma 9. *For $y \in (0, \frac{1}{30\pi})$, let $t = t_0$ be the unique solution to $\frac{\sin t}{t} = y$ on $(0, \pi)$. Then $t_0 > 0.98\pi$. Let $t = t_1$ be the larger of the two solutions to $\frac{|\sin t|}{t} = y$ on $(\pi, 2\pi)$. Then $t_1 > 1.97\pi$.*

Proof. Note that $\frac{\sin t_0}{t_0} = y < \frac{1}{30\pi} < \frac{\sin(0.98\pi)}{0.98\pi}$. Since $\frac{\sin t}{t}$ is decreasing on $(0, \pi)$, it follows that $t_0 > 0.98\pi$. Similarly, we check that $\frac{|\sin(1.97\pi)|}{1.97\pi} > \frac{1}{30\pi}$ to justify the claim about t_1. $\square$

Lemma 10. *For $0 < x < \pi$,*

$$\frac{1}{\sin^2 x} > \frac{1}{x^2} + \frac{1}{(\pi - x)^2}.$$

Proof. It is well known (and follows from $\sin(2x) = 2\sin x \cos x$) that

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos(x/2^k).$$

In particular, for $0 < x < \pi$, we have $\left(\frac{\sin x}{x}\right)^2 < \cos^2(x/2)$, hence

$$\sin^2 x\left(\frac{1}{x^2}+\frac{1}{(\pi-x)^2}\right) = \left(\frac{\sin x}{x}\right)^2 + \left(\frac{\sin(\pi-x)}{\pi-x}\right)^2$$
$$< \cos^2(x/2)+\cos^2(\pi/2-x/2) = 1.$$

□

Lemma 11. *Let $k \geqslant 1$ be an integer. On $((k-1)\pi, k\pi)$, we have*

(i) the function $\frac{|\sin t|}{t(t-(k-1)\pi)}$ is nonincreasing,

(ii) the function $\frac{|\sin t|}{t(k\pi-t)}$ is unimodal (first increases and then decreases).

Proof. (i) The derivative equals

$$\frac{|\sin t|}{t(t-(k-1)\pi)}\left(\cot(t)-\frac{1}{t}-\frac{1}{t-(k-1)\pi}\right)$$

which is negative on $((k-1)\pi, k\pi)$ because on this interval, $\cot(t) < \frac{1}{t-(k-1)\pi}$ (as, by periodicity, being equivalent to $\cot(t) < \frac{1}{t}$ on $(0,\pi)$, which is clear – recall that $\tan(x) > x$ on $(0, \frac{\pi}{2})$).

(ii) Here, the derivative reads

$$\frac{|\sin t|}{t(k\pi-t)}h(t), \qquad h(t) = \cot(t) - \frac{1}{t} + \frac{1}{k\pi - t}.$$

We shall argue that $h(t)$ is decreasing on $((k-1)\pi, k\pi)$. This suffices, since $h(t) > 0$ for t near $(k-1)\pi$ and $h(t) < 0$ for t near $k\pi$. Setting $t = (k-1)\pi + x$, we have

$$h'(t) = -\frac{1}{\sin^2 t}+\frac{1}{t^2}+\frac{1}{(k\pi-t)^2}$$
$$\leqslant -\frac{1}{\sin^2 x}+\frac{1}{x^2}+\frac{1}{(\pi-x)^2} < 0,$$

by Lemma 10. □

3.2. **Lemmas concerning sums of p-th powers.** Our computations require several technical bounds on various expressions involving sums of p-th powers.

Lemma 12. *Let $0 < p < 1$ and let $1 \leqslant m \leqslant 29$ be an integer. Set*

$$u_m(p) = B_m\left(b_{0,m}^p + 2\sum_{k=1}^m b_{k,m}^p\right)$$

with

$$B_m = \frac{20\log\big(\pi(m+3/2)\big)}{11\pi(m+3/2)}, \qquad b_{k,m} = \frac{1}{k+1}\sqrt{\frac{6}{\pi^2}\log\big(\pi(m+3/2)\big)}.$$

Then,

$$u_m(p) > 1.$$

Proof. Fix m. Plainly, $u_m(p)$ is a convex function (as a sum of convex functions). Thus, $u_m'(p) < u_m'(1)$ for $0 < p < 1$. We have,

$$u_m'(1) = B_m \left(b_{0,m} \log b_{0,m} + 2 \sum_{k=1}^{m} b_{k,m} \log b_{k,m} \right)$$

and Table 1 shows that each $u_m'(1)$ is negative, so each u_m is decreasing. Therefore, $u_m(p) > u_m(1)$, for $0 < p < 1$ and Table 1 shows that each $u_m(1)$ is greater than 1. This finishes the proof. $\square$

TABLE 1. Lower bounds on the values of $-u_m'(1)$ and $u_m(1)$.

m	1	2	4	4	5	6	7	8	9	10
$-u_m'(1)$	0.24	0.44	0.58	0.70	0.79	0.86	0.91	0.96	0.99	1.02
$u_m(1)$	1.06	1.27	1.36	1.40	1.41	1.41	1.40	1.38	1.36	1.34

m	11	12	13	14	15	16	17	18	19	20
$-u_m'(1)$	1.05	1.07	1.08	1.10	1.11	1.12	1.13	1.13	1.14	1.14
$u_m(1)$	1.32	1.29	1.27	1.25	1.23	1.21	1.19	1.17	1.15	1.14

m	21	22	23	24	25	26	27	28	29
$-u_m'(1)$	1.14	1.14	1.15	1.15	1.15	1.15	1.15	1.15	1.14
$u_m(1)$	1.12	1.10	1.09	1.07	1.06	1.04	1.03	1.02	1.00

Lemma 13. *For $0 < p < 1$, let*

$$\alpha_p = 2\pi^{-p+1}\left(3 - \frac{1}{1-p} + \frac{3}{2p}\right), \qquad \beta_p = \frac{2}{1-p} + \frac{1.05^p}{p}, \qquad \gamma_p = \frac{3\pi}{p},$$

$$\delta_p = \frac{1}{p}\left(\frac{30\pi}{6\log(30\pi)}\right)^{p/2}$$

and

$$h_p(y) = \delta_p y^{\frac{p}{2}-1} + \gamma_p y^p - \beta_p y^{p-1} - \alpha_p.$$

Then $h_p(y) > 0$ for every $0 < y < \frac{1}{30\pi}$.

Proof. Plainly, it suffices to show the following two claims,

$$h_p'(y) < 0, \qquad 0 < y < \frac{1}{30\pi}, \tag{5}$$

$$h_p\left(\frac{1}{30\pi}\right) > 0. \tag{6}$$

To prove (5), first we find

$$y^{2-\frac{p}{2}} h_p'(y) = -\left(1 - \frac{p}{2}\right)\delta_p + p\gamma_p y^{\frac{p}{2}+1} + (1-p)\beta_p y^{\frac{p}{2}}$$

which is clearly increasing in y, thus to show that it is negative, it suffices to prove that at $y = \frac{1}{30\pi}$, which in turn is equivalent to

$$2.1p + (1-p)1.05^p < \left(1 - \frac{p}{2}\right)\left(\frac{30\pi}{\sqrt{6\log(30\pi)}}\right)^p.$$

Crudely, $(1-p)1.05^p < 1.05^p < 1 + 0.05p$, by convexity, thus it suffices to show that

$$1 + 2.15p < \left(1 - \frac{p}{2}\right) A^p,$$

where we put $A = \frac{30\pi}{\sqrt{6\log(30\pi)}}$. Equivalently, after taking the logarithm, the inequality becomes

$$p \log A + \log\left(1 - \frac{p}{2}\right) - \log(1 + 2.15p) > 0.$$

Note that at $p = 0$ this becomes equality. We claim that the derivative of the left hand side is positive for $0 < p < 1$, which will finish the argument. The derivative is $\log A - \frac{1}{2-p} - \frac{2.15}{1+2.15p}$ which is clearly concave, thus it suffices to examine whether it is positive at the end-points $p = 0$ and $p = 1$, which respectively becomes $\log A > 2.65$ and $\log A > 1 + \frac{2.15}{3.15}$. Since $\log A = 2.89..$, both are clearly true.

It remains to show (6), that is that the following is positive for every $0 < p < 1$,

$$30^p \pi^{p-1} h_p\left(\frac{1}{30\pi}\right) = \underbrace{30 \frac{\left(\frac{30\pi}{\sqrt{6\log(30\pi)}}\right)^p - 1.05^p}{p}}_{L(p)} - \underbrace{\left(2\frac{30 - 30^p}{1-p} + 3\frac{30^p - 1}{p} + 6 \cdot 30^p\right)}_{R(p)}.$$

Both $L(p)$ and $R(p)$ are strictly increasing and convex on $(0, 1)$. This is clear for L, since its Taylor expansion at $p = 0$ has positive coefficients. Similarly for the term $\frac{30^p-1}{p}$ in $R(p)$. To see that $\frac{30-30^p}{1-p}$ is strictly increasing and convex, write it as $30\int_1^{30} u^p \frac{\mathrm{d}u}{u^2}$.

Case 1: $0 < p < 0.6$. By convexity, using a tangent line

$$L(p) \geqslant L(0.24) + L'(0.24)(p - 0.24) = \ell(p)$$

and a chord

$$R(p) \leqslant \frac{p}{0.6} R(0.6) + \frac{0.6 - p}{0.6} R(0^+) = r(p).$$

With hindsight, the tangent and the chord are chosen such that $\ell > r$ on $(0, 0.6)$, which can be checked directly by looking at the values of these linear functions at the end-points.

Case 2: $0.6 < p < 1$. Similarly, by convexity, using a tangent line

$$L(p) \geqslant L(0.8) + L'(0.8)(p - 0.8) = \tilde{\ell}(p)$$

and a chord

$$R(p) \leqslant \frac{1-p}{0.4} R(0.6) + \frac{p - 0.6}{0.4} R(1^-) = \tilde{r}(p).$$

Again, with hindsight, the tangent and the chord are chosen such that $\tilde{\ell} > \tilde{r}$ on $(0.6, 1)$. This completes the proof. $\square$

3.3. **Lemmas concerning the gamma function.** For the inductive part of our argument, we will later need bounds on the following function

$$\psi(p) = \frac{1+p}{\sqrt{\pi}} \left(\frac{4}{3}\right)^{p/2} \Gamma\left(\frac{1+p}{2}\right), \qquad 0 < p < 1.$$

Recall the Weierstrass' product formula, $\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$, where $\gamma = 0.57..$ is the Euler-Mascheroni constant. Writing $\sqrt{\pi}$ as $\Gamma(\frac{1}{2})$, we obtain

$$\psi(p) = e^{\frac{p}{2}(\log(4/3)-\gamma)} \prod_{n=1}^{\infty} \left(1 + \frac{p}{2n+1}\right)^{-1} e^{\frac{p}{2n}}. \tag{7}$$

Lemma 14. *For $0 < p < 0.69$, we have $\psi(p) < \frac{1}{2-(3/2)^{p/2}}$.*

Proof. We want to show that

$$f(p) = \log(2 - (3/2)^{p/2}) + \log \psi(p)$$

is negative on $(0, 0.69)$. By virtue of (7),

$$f''(p) = -\frac{1}{2} \log^2 \left(\frac{3}{2}\right) \frac{(3/2)^{p/2}}{(2 - (3/2)^{p/2})^2} + \sum_{n=1}^{\infty} \frac{1}{(2n+1+p)^2}$$

This is plainly a decreasing function. Using $\sum_{n=1}^{\infty} \frac{1}{(2n+1+p)^2} \geqslant \sum_{n=1}^{\infty} \frac{1}{(2n+2)^2} = \frac{\pi^2-6}{24}$, we get with $f''(0.9) > 0.007$, so f is strictly convex on $(0, 0.9)$. Checking that $f(0) = 0$ and $f(0.69) < -0.0001$ finishes the proof. □

Lemma 15. *For $0 < p < 1$, we have $\psi(p) < 1 + \frac{p(p+1)}{6}$.*

Proof. We want to show that

$$f(p) = -\log\left(1 + \frac{p(p+1)}{6}\right) + \log \psi(p)$$

is negative on $(0, 1)$. Since $f(0) = 0$, it suffices to show that $f'(p) < 0$ on $(0, 1)$. Using (7), we have

$$f'(p) = -\frac{2p+1}{p^2+p+6} + \frac{1}{2}(\log(4/3) - \gamma) + \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2n+1+p}\right).$$

Now, for $R(p) = -\frac{2p+1}{p^2+p+6} + \frac{1}{2}(\log(4/3) - \gamma)$, $R''(p) = \frac{(2p+1)(17-p^2-p)}{(p^2+p+6)^3} > 0$ on $(0, 1)$, so $R(p)$ is convex on $(0, 1)$. Let $S(p) = \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2n+1+p}\right)$. Plainly, this is a concave function. Thus, using tangents at $p = 0$ and $p = 1$, $S(p) \leqslant \min\{L_0(p), L_1(p)\}$ with $L_0(p) = S(0) + S'(0)p = (1 - \log 2) + (\frac{\pi^2}{8} - 1)p$ and $L_1(p) = S(1) + S'(1)(p - 1) = \frac{1}{2} + \frac{\pi^2-6}{24}(p - 1)$. We obtain the upper-bounds on $f'(p)$ by the convex functions $R(p) + L_0(p)$ and $R(p) + L_1(p)$. Examining the end-points we conclude that the former is negative on $(0, 0.5)$ and the latter is negative on $(0.4, 1)$. Thus $f'(p) < 0$ on $(0, 1)$, as desired. □

4. Integral inequality: proofs of Theorems 4 and 5

First observe that using the integral expression for $\mathcal{I}_p(\infty)$, inequality (3) becomes

$$0 \leqslant \mathcal{I}_p(s) - \mathcal{I}_p(\infty) = \kappa_p \int_0^\infty \frac{e^{-t^2/6} - \left|\frac{\sin(t/\sqrt{s})}{t/\sqrt{s}}\right|^s}{t^{p+1}} \mathrm{d}t$$

$$= \kappa_p s^{-p/2} \int_0^\infty \frac{e^{-st^2/6} - \left|\frac{\sin t}{t}\right|^s}{t^{p+1}} \mathrm{d}t. \tag{8}$$

To tackle such an inequality with an oscillatory integrand, we rely on the following extremely efficient and powerful lemma of Nazarov and Podkorytov from [24] (for the proof, see e.g. [15]).

Lemma 16 (Nazarov-Podkorytov, [24]). *Let $M > 0$ and $f, g : X \to [0, M]$ be any two measurable functions on a measure space (X, μ). Assume that the modified distribution functions*

$$F(y) = \mu(\{x \in X : f(x) < y\}) \qquad \textit{and} \qquad G(y) = \mu(\{x \in X : g(x) < y\})$$

of f and g respectively are finite for every $y \in (0, M)$. If there exists $y_* \in (0, M)$ such that $G(y) \geqslant F(y)$ for all $y \in (0, y_*)$, $G(y) \leqslant F(y)$ for all $y \in (y_*, M)$, then the function

$$s \mapsto \frac{1}{s y_0^s} \int_X (g^s - f^s)\, d\mu$$

is increasing on the set $\{s > 0 : g^s - f^s \in L^1(X, \mu)\}$.

In view of (3), (4) and (8), Theorems 4 and 5 immediately follow from the following lemma.

Lemma 17. *Let $f(t) = \left|\frac{\sin t}{t}\right|$, $g(t) = e^{-t^2/6}$, $t > 0$, and set*

$$H(p, s) = \int_0^\infty \frac{g(t)^s - f(t)^s}{t^{p+1}} \mathrm{d}t.$$

We have,

(a) $H(p, s) \geqslant 0$ for every $0 < p < 1$ and $s \geqslant 2$,

(b) $H(p, s) \geqslant 0$ for every $0.6 < p < 1$ and $s \geqslant 1$.

Proof. Fix $0 < p < 1$. We shall examine the modified distribution functions

$$F(y) = \mu(t > 0,\ f(t) < y),$$
$$G(y) = \mu(t > 0,\ g(t) < y), \qquad 0 < y < 1,$$

where $\mathrm{d}\mu(t) = t^{-p-1}\mathrm{d}t$. It suffices to show that

$(\star)$ $\quad G - F$ changes sign exactly once on $(0, 1)$ at some $y = y_*$ from $+$ to $-$.

Then Lemma 16 gives that

$$s \mapsto \frac{1}{s y_*^s} H(p, s)$$

is increasing on $(0,\infty)$. In particular, (a) and (b) result from the following claims whose proofs we defer until the end of this proof.

Claim A. $H(p,2) \geqslant 0$ for every $0 < p < 1$.

Claim B. $H(p,1) \geqslant 0$ for every $0.6 < p < 1$.

Towards $(\star)$, let $1 = y_0 > y_1 > y_2 > \dots$ be the consecutive maxima of f. On $(0,\pi)$, $f \leqslant g$ (Lemma 6), so $G - F < 0$ on $(y_1, 1)$. We plan to find $a \in (0, y_1)$ with the following two properties

(i) $(G-F)' < 0$ on (a, y_1),

(ii) $G - F > 0$ on $(0, a)$.

This clearly suffices to conclude $(\star)$.

Fix $m \in \{1, 2, \dots\}$ and $y \in (y_{m+1}, y_m)$. Plainly,

$$G(y) = \int_{\sqrt{-6\log y}}^{\infty} \frac{\mathrm{d}t}{t^{p+1}} = \frac{1}{p}\left(-6\log y\right)^{-p/2}.$$

Let $t_0^+ = t_0^+(y)$ be the unique solution to $f(t) = y$ on $(0,\pi)$ and for each $1 \leqslant k \leqslant m$, let $t_k^- < t_k^+$ be the unique solutions to $f(t) = y$ on $(k\pi, (k+1)\pi)$ ($t_k^\pm = t_k^\pm(y)$ are functions of y). We have,

$$F(y) = \mu(t_0^+, t_1^-) + \mu(t_1^+, t_2^-) + \dots + \mu(t_{m-1}^+, t_m^-) + \mu(t_m^+, \infty).$$

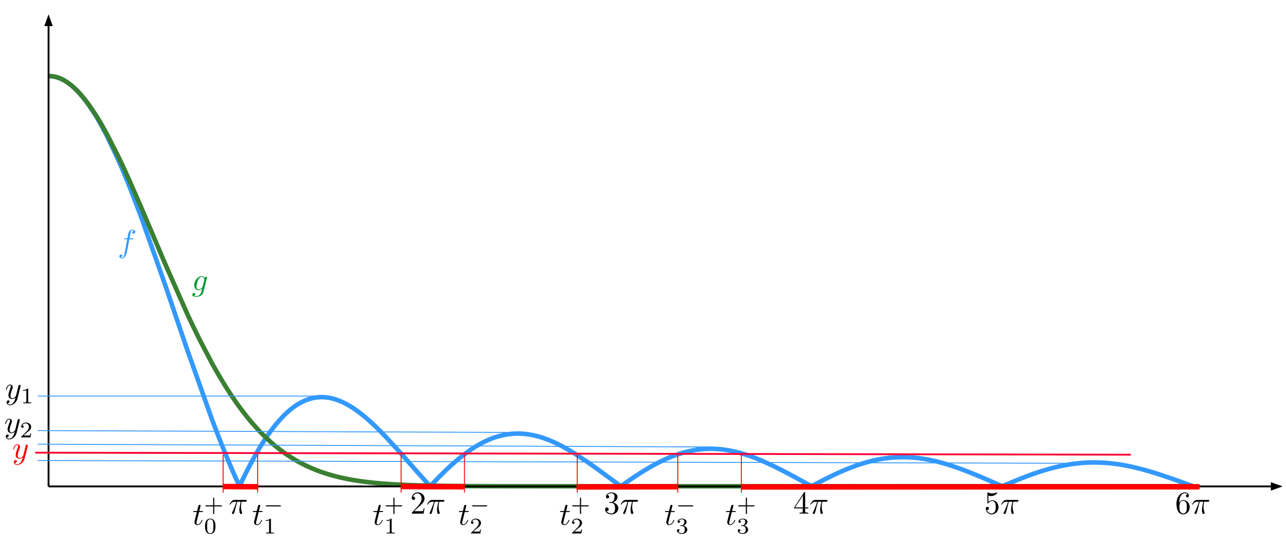

Figure 1. Functions f, g and the set $\{t > 0, f(t) < y\}$. Here $m = 3$, i.e. $y_3 < y < y_4$.

Condition (i). We have,

$$G'(y) = \frac{3}{y}\,(-6\log y)^{-p/2-1}$$

and

$$F'(y) = \sum_{t:f(t)=y} \frac{1}{t^{p+1}|f'(t)|}.$$

To lower bound $\frac{F'}{G'}$ in order to show that it is greater than 1, we lower bound F' and $\frac{1}{G'}$ separately as follows. First, using $|tf'(t)| = |\cos t - \frac{\sin t}{t}| < \frac{11}{10}$ for every $t > 0$ (Lemma 7), we have,

$$F'(y) > \frac{10}{11} \sum_{t:f(t)=y} t^{-p} > \frac{10}{11}\pi^{-p}\left(1 + 2\sum_{k=1}^{m}(k+1)^{-p}\right),$$

by crudely bounding $t_0^- < \pi$, $t_k^\pm < (k+1)\pi$. Second, since $y(-6\log y)^{p/2+1}$ is increasing on $(0, y_1)$ (it is increasing on $(0, e^{-1-p/2})$ and $e^{-1-p/2} > e^{-3/2} > y_1$), and $y_{m+1} > \frac{1}{\pi(m+3/2)}$ (Lemma 8),

$$\begin{aligned}\frac{1}{G'(y)} &= \frac{1}{3}y\,(-6\log y)^{p/2+1} > \frac{1}{3}y_{m+1}\,(-6\log y_{m+1})^{p/2+1} \\ &> \frac{1}{3}\frac{1}{\pi(m+3/2)}\left(6\log\Big(\pi(m+3/2)\Big)\right)^{p/2+1}.\end{aligned}$$

We obtain

$$\frac{F'(y)}{G'(y)} > \frac{10}{33}\frac{1}{\pi^{p+1}(m+3/2)}\left(6\log\Big(\pi(m+3/2)\Big)\right)^{p/2+1}\left(1 + 2\sum_{k=1}^{m}(k+1)^{-p}\right).$$

From Lemma 12 the right hand side is at least 1 for every $0 < p < 1$ and $1 \leqslant m \leqslant 29$. Therefore, to guarantee that Condition (i) holds, we can choose any $a \geqslant y_{30}$.

We set $a = y_{30}$ and argue next that Condition (ii) holds for every $y \in (0, a)$.

Condition (ii). We assume here that $m \geqslant 30$. Recall we have fixed $y \in (y_{m+1}, y_m)$. Since G is explicit, it suffices to upper bound F. We have,

$$\begin{aligned}F(y) &= \sum_{k=1}^{m}\int_{t_{k-1}^+}^{t_k^-}\frac{\mathrm{d}t}{t^{p+1}} + \int_{t_m^+}^{\infty}\frac{\mathrm{d}t}{t^{p+1}} \\ &\leqslant \sum_{k=1}^{m}(t_k^- - t_{k-1}^+)(t_{k-1}^+)^{-p-1} + \frac{1}{p}(t_m^+)^{-p}.\end{aligned}$$

For $k \geqslant 3$, we crudely estimate $t_{k-1}^+ \geqslant (k-1)\pi$, whereas for $k = 1, 2$, we have $t_0^+ > 0.98\pi$ and $t_1^+ > 1.97\pi$, thanks to Lemma 9. To upper bound the length $t_k^- - t_{k-1}^+$, note that with the aid of Figure 2,

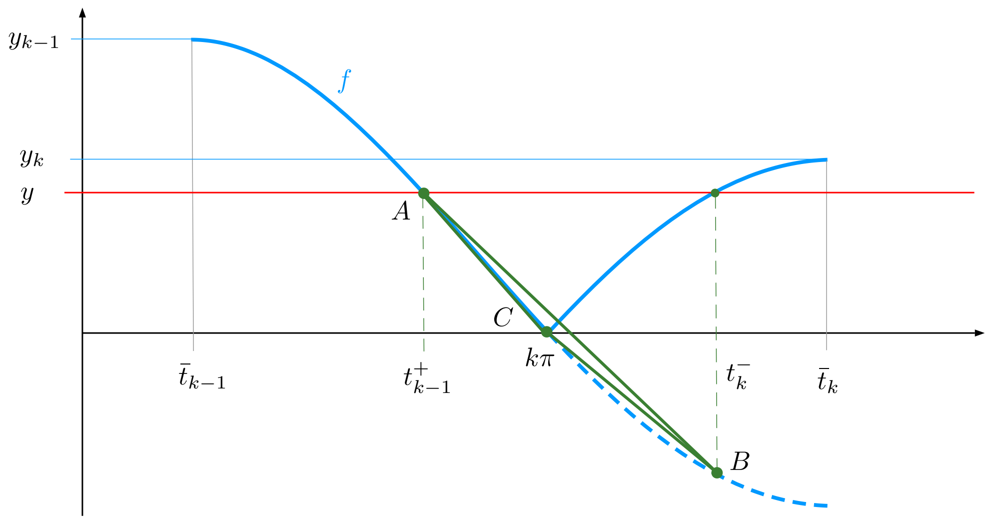

FIGURE 2. The slope of the segment AB is not smaller than the slope of either AC or BC.

$$\frac{2y}{t_k^- - t_{k-1}^+} = \frac{\left|\frac{\sin t_k^-}{t_k^-} - \frac{\sin t_{k-1}^+}{t_{k-1}^+}\right|}{t_k^- - t_{k-1}^+} = |\mathrm{slope}(AB)|$$
$$\geqslant \min\left\{|\mathrm{slope}(AC)|, |\mathrm{slope}(BC)|\right\}.$$

Let $\bar{t}_j \in (j\pi, (j+1)\pi)$ denote the point where $f(t)$ attains its local maximum y_j on $(j\pi, (j+1)\pi)$. Observe that

$$|\mathrm{slope}(BC)| = \frac{|\sin t_k^-|}{t_k^-(t_k^- - k\pi)} \geqslant \frac{y_k}{\bar{t}_k - k\pi} \geqslant \frac{y_k}{\pi},$$

where the first inequality follows from Lemma 11 (i) applied to $t_k^- < \bar{t}_k$. Similarly,

$$|\mathrm{slope}(AC)| = \frac{|\sin t_{k-1}^+|}{t_{k-1}^+(k\pi - t_{k-1}^+)} \geqslant \min\left\{\frac{y_{k-1}}{k\pi - \bar{t}_{k-1}}, \frac{1}{k\pi}\right\} \geqslant \min\left\{\frac{y_{k-1}}{\pi}, \frac{1}{k\pi}\right\},$$

where in the first inequality we use Lemma 11 (ii) to lower bound the function in question by the minimum of its values at the end-points $t = \bar{t}_{k-1}$ and $t = k\pi$. Finally, putting these two estimates together and using $y_k > \frac{1}{\pi(k+\frac{1}{2})}$, we obtain

$$|\mathrm{slope}(AB)| \geqslant \frac{1}{\pi^2(k+\frac{1}{2})}$$

and

$$t_k^- - t_{k-1}^+ \leqslant 2\pi^2 y \left(k + \frac{1}{2}\right),$$

which results in

$$F(y) < 2\pi^{-p+1} y \left(\frac{3}{2}0.98^{-p-1} + \frac{5}{2}1.97^{-p-1} + \sum_{k=3}^{m}\left(k+\frac{1}{2}\right)(k-1)^{-p-1}\right) + \frac{1}{p}(m\pi)^{-p}.$$

Since $y > y_{m+1} > \frac{1}{(m+\frac{3}{2})\pi}$, and $m \geqslant 30$, we have

$$\frac{1}{p}(m\pi)^{-p} < \frac{1}{p}\left(\frac{m+3/2}{m}\right)^p y^p \leqslant \frac{1}{p}1.05^p y^p.$$

Moreover, since $y < y_m < \frac{1}{m\pi}$, we have (crudely), $m-1 < \frac{1}{\pi y}$ and bounding the sum using the integral, we obtain

$$\begin{aligned}\sum_{k=3}^{m}\left(k+\frac{1}{2}\right)(k-1)^{-p-1} &= \sum_{k=2}^{m-1}\frac{k+\frac{3}{2}}{k^{p+1}} \\ &< \int_1^{m-1}\left(x^{-p}+\frac{3}{2}x^{-p-1}\right)\mathrm{d}x \\ &< \frac{(\pi y)^{p-1}-1}{1-p}+\frac{3(1-(\pi y)^p)}{2p}.\end{aligned}$$

Therefore, in order to have $F(y) < G(y)$, it suffices to guarantee that

$$\begin{aligned}2\pi^{-p+1}y\left(\frac{3}{2}0.98^{-p-1}+\frac{5}{2}1.97^{-p-1}+\frac{(\pi y)^{p-1}-1}{1-p}+\frac{3(1-(\pi y)^p)}{2p}\right)& \\ +\frac{1}{p}1.05^p y^p < \frac{1}{p}\left(-6\log y\right)^{-p/2}&\end{aligned}$$

holds for every $0 < p < 1$ and $0 < y < \frac{1}{30\pi}$. Since $-y\log y$ is increasing for $y < \frac{1}{e}$, we have $-\log y < \frac{\log(30\pi)}{30\pi}\frac{1}{y}$ for $0 < y < \frac{1}{30\pi}$. By monotonicity, for $0 < p < 1$, we have $\frac{3}{2}0.98^{-p-1}+\frac{5}{2}1.97^{-p-1} < \frac{3}{2}0.98^{-1-1}+\frac{5}{2}1.97^{-1} < 3$. It remains to use Lemma 13. This shows that Condition (ii) holds and the proof of the lemma is complete. It remains to show Claims A and B. $\square$

Proof of Claim A. By the integral representation for p-norm from Lemma 3,

$$\kappa_p H(p,2) = \mathbb{E}|U_1+U_2|^p - \mathbb{E}\left|\sqrt{\frac{2}{3}}Z\right|^p = \frac{2^{p+1}}{(p+1)(p+2)} - \frac{1}{\sqrt{\pi}}\left(\frac{4}{3}\right)^{p/2}\Gamma\left(\frac{1+p}{2}\right).$$

By Lemma 15, it suffices to prove that $2^{p+1} > (p+2)\left(1+\frac{p(p+1)}{6}\right)$ for all $0 < p < 1$. The 3rd derivative of the difference changes sign once on $(0,1)$ from $-$ to $+$. The 2nd derivative is negative at the end-points $p=0$ and $p=1$, so it is negative on $(0,1)$ and hence the difference is concave. It vanishes at the end-points $p=0$ and $p=1$, which finishes the argument. $\square$

Proof of Claim B. Our argument is split into two steps: first we show that $H(p,1)$ increases with p and then we estimate $H(0.6,1)$. For somewhat similar computations, but related to random signs, see Section 5 in [21]. In Step 1, to numerically evaluate the integrals in question, we will frequently use that given $0 < a < b$ and an integer m, integrals of the form $\int_a^b(\sin t)t^{-m}\mathrm{d}t$ can be efficiently estimated to an arbitrary precision by expressing them in terms of the trigonometric integral functions Si, Ci. The same applies to the integrals of the form $\int_a^b e^{-t^2}t^q\mathrm{d}t$ with $0 < a < b \leqslant \infty$ and real q, thanks to reductions to the incomplete gamma functions and the exponential integral E_1. In Step 2, all the numerical computations are reduced to integrals of the form $\int_a^b \frac{\mathrm{d}t}{t^q}$ which are explicit.

Step 1: $\frac{\partial}{\partial p}H(p,1) > 0$, $0.6 < p < 1$. We have,

$$\frac{\partial}{\partial p}H(p,1) = \int_0^\infty (-\log t)\frac{g(t)-f(t)}{t^{p+1}}\mathrm{d}t.$$

We break the integral into several regions. Recall $g > f$ on $(0,\pi)$, by Lemma 6. Thus, plainly,

$$\int_0^1 (-\log t)\frac{g(t)-f(t)}{t^{p+1}}\mathrm{d}t > 0.$$

Moreover, $g-f$ changes sign from $+$ to $-$ exactly once on $(\pi,4)$ at $t = 3.578...$ Let $t_0 = 3.57$. On $(1,t_0)$, using $t^{-p-1} = t^{1-p}t^{-2} \leqslant t_0^{1-p}t^{-2}$, we obtain

$$\int_1^{t_0} (-\log t)\frac{g(t)-f(t)}{t^{p+1}}\mathrm{d}t \geqslant t_0^{1-p}\int_1^{t_0} (-\log t)\frac{g(t)-f(t)}{t^2}\mathrm{d}t > -0.0297\cdot t_0^{1-p},$$

where in the last inequality we use $\log t \leqslant \log\frac{5}{2} + \frac{2}{5}(t-\frac{5}{2})$ (by concavity) and then estimate the resulting integrals. Now,

$$\int_{t_0}^\infty (-\log t)\frac{g(t)-f(t)}{t^{p+1}}\mathrm{d}t = \int_{t_0}^\infty (\log t)\frac{f(t)}{t^{p+1}}\mathrm{d}t - \int_{t_0}^\infty (\log t)\frac{g(t)}{t^{p+1}}\mathrm{d}t.$$

For $t > t_0$, $t^{-p-1} = t^{1-p}t^{-2} > t_0^{1-p}t^{-2}$ and for $k \geqslant 1$, $\log t \geqslant \ell_k(t)$ on $(k\pi,(k+1)\pi)$ with

$$\ell_k(t) = \frac{(k+1)\pi - t}{\pi}\log(k\pi) + \frac{t-k\pi}{\pi}\log((k+1)\pi),$$

thus

$$\int_{t_0}^\infty (\log t)\frac{f(t)}{t^{p+1}}\mathrm{d}t \geqslant t_0^{1-p}\left(\int_{t_0}^{2\pi}\ell_1(t)\frac{-\sin t}{t^3}\mathrm{d}t + \sum_{k=2}^n\int_{k\pi}^{(k+1)\pi}\ell_k(t)\frac{(-1)^k\sin t}{t^3}\mathrm{d}t\right).$$

For $n = 5$, this gives

$$\int_{t_0}^\infty (\log t)\frac{f(t)}{t^{p+1}}\mathrm{d}t > 0.0437\cdot t_0^{1-p}.$$

Finally, since $\log u \leqslant \frac{u}{e}$, $u > 0$, we have $\frac{\log t}{t^p} < \frac{1}{ep} < \frac{1}{0.6e} < 0.6132 < 0.6132\cdot t_0^{1-p}$, thus

$$\int_{t_0}^\infty (\log t)\frac{g(t)}{t^{p+1}}\mathrm{d}t \leqslant 0.6132\cdot t_0^{1-p}\int_{t_0}^\infty \frac{e^{-t^2/6}}{t}\mathrm{d}t < 0.0127\cdot t_0^{1-p}.$$

Putting these together yields

$$\frac{\partial}{\partial p}H(p,1) > (0.0437 - 0.0297 - 0.0127)t_0^{1-p} = 0.0013\cdot t_0^{1-p} > 0.$$

Step 2: $H(0.6,1) > 0$. We have,

$$H(0.6,1) = \int_0^\pi \frac{e^{-t^2/6} - \frac{\sin t}{t}}{t^{8/5}}\mathrm{d}t + \int_\pi^\infty \frac{e^{-t^2/6}}{t^{8/5}}\mathrm{d}t - \int_\pi^\infty \frac{|\sin t|}{t^{13/5}}\mathrm{d}t.$$

On $(0,\pi)$, we use Taylor's polynomials to bound the integrand,

$$e^{-t^2/6} - \frac{\sin t}{t} > \sum_{k=0}^7 \frac{(-t^2/6)^k}{k!} - \sum_{k=0}^6 \frac{(-1)^k t^{2k}}{(2k+1)!}.$$

Plugging this into the integral results in

$$\int_0^\pi \frac{e^{-t^2/6} - \frac{\sin t}{t}}{t^{8/5}} \mathrm{d}t > 0.0434.$$

Using the incomplete Gamma function,

$$\int_\pi^\infty \frac{e^{-t^2/6}}{t^{8/5}} \mathrm{d}t > 0.0184.$$

Finally,

$$\begin{aligned}\int_\pi^\infty \frac{|\sin t|}{t^{13/5}} \mathrm{d}t &= \int_0^\pi (\sin t) \left(\sum_{k=1}^\infty \frac{1}{(t+k\pi)^{13/5}} \right) \mathrm{d}t \\ &\leqslant \int_0^\pi (\sin t) \left(\sum_{k=1}^n \frac{1}{(t+k\pi)^{13/5}} \right) \mathrm{d}t + \int_{(n+1)\pi}^\infty \frac{\mathrm{d}t}{t^{13/5}}.\end{aligned}$$

We use Taylor's polynomial again, $\sin t \leqslant 1 - \frac{1}{2}(t-\pi/2)^2 + \frac{1}{24}(t-\pi/2)^4$. Choosing $n = 8$ gives

$$\int_\pi^\infty \frac{|\sin t|}{t^{13/5}} \mathrm{d}t < 0.0615.$$

Adding up these estimates yields $H(0.6, 1) > 0.0434 + 0.0184 - 0.0615 = 0.0003$. □

5. Inductive argument

As explained in Section 2, Theorem 5 gives the following corollary (we use homogeneity to rewrite (4) in an equivalent form, better suited for the ensuing arguments). Recall $c_p = \|Z\|_p/\sqrt{3}$ and define

$$\varphi_p(x) = (1+x)^{p/2}, \qquad x \geqslant 0.$$

Corollary 18. *Let $0 < p < 1$. For every $n \geqslant 2$ and real numbers $a_2, \ldots, a_n$ with $\sum_{j=2}^n a_j^2 \leqslant 1$, we have*

$$\mathbb{E}\left| U_1 + \sum_{j=2}^n a_j U_j \right|^p \geqslant c_p^p \cdot \varphi_p\left(\sum_{j=2}^n a_j^2 \right).$$

The goal here is to remove the restriction $\sum_{j=2}^n a_j^2 \leqslant 1$. The key idea from [24] is to replace φ_p with a pointwise *larger* function, thereby strengthening the inequality and to proceed by induction on n. We use the function from [24],

$$\Phi_p(x) = \begin{cases} \varphi_p(x), & x \geqslant 1, \\ 2\varphi_p(1) - \varphi_p(2-x), & 0 \leqslant x \leqslant 1. \end{cases}$$

Even though this function changes from being convex to concave at $x = 1$, it is designed to satisfy the following *extended convexity* property on $[0, 2]$, crucial for the proof.

Lemma 19 (Nazarov-Podkorytov, [24])**.** *For every $0 < p < 2$ and $a, b \in [0, 2]$ with $a + b \leqslant 2$, we have*
$$\frac{\Phi_p(a) + \Phi_p(b)}{2} \geqslant \Phi_p\left(\frac{a+b}{2}\right).$$

As in [4], in order to have certain algebraic identities, we run the argument for $\xi_1, \xi_2, \ldots,$ independent random vectors in $\mathbb{R}^3$ uniformly distributed on the centred unit Euclidean sphere S^2. Here $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ is the standard inner product and the resulting Euclidean norm in $\mathbb{R}^3$, respectively.

Theorem 20. *Let $0 < p < 0.69$. For every $n \geqslant 2$ and vectors $v_2, \ldots, v_n$ in $\mathbb{R}^3$, we have*
$$\mathbb{E}\left|\langle e_1, \xi_1 \rangle + \sum_{j=2}^{n} \langle v_j, \xi_j \rangle\right|^p \geqslant c_p^p \cdot \Phi_p\left(\sum_{j=2}^{n} \|v_j\|^2\right).$$
Here $e_1 = (1, 0, 0)$, the unit vector of the standard basis.

Since $\langle v_j, \xi_j \rangle$ has the same distribution as $\|v_j\| U_j$ (the Archimedes' hat-box theorem) and $\Phi_p \geqslant \varphi_p$, this gives Theorem 1 for $0 < p < 0.69$, thereby completing its proof. It remains to show Theorem 20, which is done by repeating almost verbatim the proof of Theorem 18 from [4]: the only difference is in the inductive base which holds here provided p is not too large. Indeed, when $n = 2$, we have
$$\mathbb{E}\,|\langle e_1, \xi_1 \rangle + \langle v_2, \xi_2 \rangle|^p = \frac{|1 + x|^{2+p} - |1 - x|^{2-p}}{2(1+p)(2+p)x},$$
where $x = \|v_2\|$. Arguing as in [4], it suffices to consider the case when $0 < x < 1$. Recalling the definition of c_p and Φ_p, the desired inequality $\mathbb{E}\,|\langle e_1, \xi_1 \rangle + \langle v_2, \xi_2 \rangle|^p \geqslant c_p^p \Phi_p(x^2)$ is handled in the following final lemma.

Lemma 21. *For every $0 < x < 1$ and $0 < p < 0.69$, we have*
$$\frac{(1+x)^{2+p} - (1-x)^{2+p}}{2(2+p)x} > \frac{1+p}{\sqrt{\pi}}\Gamma\left(\frac{1+p}{2}\right)\left(\frac{2}{3}\right)^{p/2}\left(2^{1+p/2} - (3 - x^2)^{p/2}\right).$$

Proof. We first observe that keeping only the first two terms in the binomial series expansion, we obtain
$$\frac{(1+x)^{2+p} - (1-x)^{2+p}}{2(2+p)x} = \sum_{k=0}^{\infty} \frac{1}{p+2}\binom{p+2}{2k+1} x^{2k} > 1 + \frac{p(p+1)}{6}x^2,$$
because all the terms are positive. It thus suffices to show that for every $0 < x < 1$ and $0 < p < 0.69$,
$$1 + \frac{p(p+1)}{6}x + \frac{1+p}{\sqrt{\pi}}\Gamma\left(\frac{1+p}{2}\right)\left(\frac{2}{3}\right)^{p/2}\left((3-x)^{p/2} - 2^{1+p/2}\right) > 0$$
(we have replaced x^2 by x). By the evident concavity in x, it suffices to check that the inequality holds at the end-points $x = 0$ and $x = 1$ which follows from Lemmas 15 and 14, respectively. □

The rest of the inductive argument is identical to the one in [4].

6. Rényi entropy: Proof of Theorem 2

For the lower bound,

$$h_p\left(\sum_j a_j U_j\right) \geqslant h_1\left(\sum_j a_j U_j\right) \geqslant h_1(U_1),$$

where the first inequality follows from the fact that $p \mapsto h_p(\cdot)$ is nonincreasing and the second one is justified by the entropy power inequality (see, e.g. Theorem 4 in [6]). It remains to notice that $h_p(U_1) = \log 2$ for every p.

Towards the upper bound, we first note that for nonnegative functions f and g, $0 < p < 1$, we have

$$\left(\int f^p\right)^{\frac{1}{p}} \left(\int g^p\right)^{\frac{p-1}{p}} \leqslant \int f g^{p-1}.$$

This follows directly from Hölder's inequality. Now, fix a unit vector a in $\mathbb{R}^n$, let f be the density of $\sum_j a_j U_j$ and $g(x) = (2\pi/3)^{-1/2} e^{-x^2/6}$, the density of $Z/\sqrt{3}$. In view of the above inequality, it suffices to show that

$$\int f g^{p-1} \leqslant \int g g^{p-1}.$$

Since

$$g(x)^{p-1} = (2\pi/3)^{\frac{1-p}{2}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1-p}{6}\right)^k x^{2k},$$

it suffices to show that for each positive integer k,

$$\mathbb{E}\left(\sum a_j U_j\right)^{2k} = \int x^{2k} f(x) \mathrm{d}x \leqslant \int x^{2k} g(x) \mathrm{d}x = \mathbb{E}\left(\frac{Z}{\sqrt{3}}\right)^{2k}.$$

This follows from the main result of [19], that $C_p = \|Z\|_p/\sqrt{3}$, $p > 1$, see (2). □

We finish by remarking that the problem of maximising $h_p(\sum a_j U_j)$ under a variance constraint for a *fixed* number of summands to the best of our knowledge remains wide open for $p \in (0, \infty)$. The case of Shannon entropy, $p = 1$, seems to be the most important and interesting, see Question 9 in [7], or Question 3 in [3], also comprehensively presenting many other related and tangential problems. The natural conjecture is that: $h_1(\sum_{j=1}^n a_j U_j) \leqslant h_1(\sum_{j=1}^n \frac{1}{\sqrt{n}} U_j)$, for every unit vector a in $\mathbb{R}^n$ (see 8.3.1 in [3] for a conceivable approach). The case $p = 0$ is of course trivial, whereas the case $p = \infty$ amounts to the cube-slicing inequalities: $h_\infty(\sum_{j=1}^n a_j U_j) \leqslant h_\infty(U_1)$ is due to Hadwiger and, independently, Hensley (see [10, 12]), $h_\infty(\sum_{j=1}^n a_j U_j) \geqslant h_\infty((U_1 + U_2)/\sqrt{2})$ is due to Ball (see [2]).

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Carnegie Mellon University; Pittsburgh, PA 15213, USA.

Email address: {gchasapi, kgurusha, ttkocz}@andrew.cmu.edu