SLICING ℓ_p-BALLS RELOADED: STABILITY, PLANAR SECTIONS IN ℓ_1

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Abstract. We show that the two-dimensional minimum-volume central section of the n-dimensional cross-polytope is attained by the regular $2n$-gon. We establish stability-type results for hyperplane sections of ℓ_p-balls in all the cases where the extremisers are known. Our methods are mainly probabilistic, exploring connections between negative moments of projections of random vectors uniformly distributed on convex bodies and volume of their sections.

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1. Introduction

Let $p > 0$. Let $B_p^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1\}$ be the ball in the standard ℓ_p^n norm. The problem of determining k-dimensional sections of B_p^n of maximal and minimal volume has attracted significant attention over the past few decades, notably prompting development of several important analytic, geometric and probabilistic techniques. It originated in the context of the sections of the cube from questions in geometry of numbers (see, e.g. [20, 46]).

1.1. **Known results.**

We begin by briefly recalling the known results. Let H_k be the hyperplane perpendicular to $e_1 + \ldots + e_k$, where $(e_j)_{1\leq j\leq n}$ is the standard basis of $\mathbb{R}^n$. The smallest hyperplane section of the cube B_∞^n is obtained by taking the hyperplane H_1, which was proved by Hadwiger in [19] and independently by Hensley in [20]. This has been generalized to sections of arbitrary dimension by Vaaler in [46]. In [3] Ball showed that H_2 gives the hyperplane section of the cube with the largest volume, see also [39] for a simpler proof. This important result led to the negative answer to the Busemann-Petty question in large dimensions, see [4]. The article [5] contains a study of maximal lower dimensional sections of the cube (the results are optimal if the dimension k of the subspace divides n or $k \geq n/2$). It is shown in [40] that H_2 is not a maximising subspace for the volume of hyperplane sections of B_p^n for $p \leq 24$. For a comprehensive survey of the results for the cube, we refer to Chapter 1 of [47]. For some recent related results, we also refer to [1, 2, 22, 27, 29, 31, 32].

Meyer and Pajor studied in [35] the same problem for B_p^n with finite p. They showed that for any dimension k, the set B_p^k obtained by taking the standard coordinate subspace

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$\mathrm{span}\{e_1, \ldots, e_k\}$ is the maximal section for $1 \le p \le 2$ and the minimal section for $p \ge 2$. For extensions to $p \in (0,1)$ see [7, 13]. In [35], Meyer and Pajor also found the minimal hyperplane section of B_1^n, which is given by taking the hyperplane H_n. Koldobsky in [24] extended this result to $p \in (0,2)$. Later on several works treated the complex case (see [26, 41]) as well as a further generalisation to *block subspaces* (see [17]). We emphasise the fact that in all of the cases, the known extremising subspaces are also known to be unique (modulo symmetries).

We mention in passing that the analogous, dual question for extremal *projections* of B_p^n has also been considered. The problem is related to certain Khinchin-type inequalities, as explained in [6, 9]. In particular, finding extremal projections of B_1^n is equivalent to deriving optimal constants in the classical Khinchin inequality, which was done by Szarek in [44], followed up by De, Diakonikolas and Serviedo who developed a stability version in [16]. The case $p \ge 2$ has been studied by Barthe and Naor in [9], where the authors showed that the smallest and the largest $(n-1)$-dimensional projections of B_p^n are those onto the hyperplanes H_1 and H_n, respectively. Koldobsky, Ryabogin and Zvavitch in [25] developed a Fourier analytic approach. Chakerian and Filliman in [14] found that the 2-dimensional orthogonal projections of the cube B_∞^n of maximal volume are attained by regular $2n$-gons (the same extremiser as in our Theorem 1) and, by McMullen's formula from [33], this also gives $(n-2)$-dimensional projections of maximal volume. See [21] for recent results on lower dimensional projections of the cross-polytope B_1^n. Paper [18] provides a different unified probabilistic approach to the volume and mean-width of central sections and projections and in addition to identifying the extremisers, also delivers Schur-convexity-type results.

1.2. **Our results.** It remains an open problem to determine k-dimensional sections of B_p^n of extremal volume: the minimal ones when $2 \le k \le n-2$, $0 < p < 2$ and maximal ones when $2 < p < \infty$, $2 \le k \le n-1$. This paper is twofold. First, we take on this question in the case of the cross-polytope and two-dimensional sections, so for $p = 1$ and $k = 2$. Second, we establish stability-type results for the hyperplane sections in all of the cases where the extremisers are known. Our bounds on deficits are sharp modulo multiplicative constants.

Cross-polytope. By vol_n we denote Lebesgue measure on $\mathbb{R}^n$, by vol_H, Lebesgue k-dimensional measure on a k-dimensional subspace H of $\mathbb{R}^n$. Often, instead of writing vol_H, we shall write vol_k, where k is the dimension of H, if it is clear what H is.

Our first main result is the following theorem about minimal volume two-dimensional central sections of the cross-polytope B_1^n.

Theorem 1. *Let $n \ge 2$. For every 2-dimensional subspace H of $\mathbb{R}^n$ one has*

$$\mathrm{vol}_H(B_1^n \cap H) \ge \frac{n^2 \sin^3\left(\frac{\pi}{2n}\right)}{\cos\left(\frac{\pi}{2n}\right)}.$$

Moreover, if the equality holds, then $B_1^n \cap H$ is isometric to a regular $2n$-gon in $\mathbb{R}^2$. The minimum is achieved for $H = T(\mathbb{R}^2)$, with $Tx = (\langle v_1, x\rangle, \ldots, \langle v_n, x\rangle)$ and $v_k = (\cos(\frac{k\pi}{n}), \sin(\frac{k\pi}{n}))$, $k = 1, \ldots, n$. The minimizing subspace H is unique, up to coordinate reflections and permutations.

In essence, our argument relies on convexity of certain functions which arise from the radial function of a planar embedding of the cross-section $B_1^n \cap H$, after leveraging the fact that it is a polygon and breaking it up into triangles.

Stability. For a vector $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$ denotes its Euclidean norm and recall that $(e_j)_{1\leq j\leq n}$ is the standard basis of $\mathbb{R}^n$. Our second main result concerns dimension-free refinements of the known results for hyperplane sections, providing stability of the unique extremising hyperplanes.

Theorem 2. *There is a positive constant c_p which depends only on p such that for every $n \geq 1$ and every unit vector $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have*

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap e_1^\perp)} \leq \Big(a_1^p + (1-a_1^2)^{p/2}\Big)^{-1/p}, \qquad 0 < p < 2, \tag{1}$$

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap (\frac{e_1+\cdots+e_n}{\sqrt{n}})^\perp)} \geq 1 + c_p \sum_{j=1}^n (a_j^2 - 1/n)^2, \qquad 0 < p < 2, \tag{2}$$

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap e_1^\perp)} \geq 1 + c_p|a - e_1|^2, \qquad 2 < p \leq \infty, \tag{3}$$

$$\frac{\mathrm{vol}_{n-1}(B_\infty^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_\infty^n \cap (\frac{e_1+e_2}{\sqrt{2}})^\perp)} \leq 1 - c_\infty \left| a - \frac{e_1+e_2}{\sqrt{2}} \right|. \tag{4}$$

Sharpness of these results is explained in detail in the sections devoted to their proofs. Briefly, the dependence on the right hand side of each of these inequalities on the deficit quantity $\delta = \delta(a)$ (which measures *how far* a is from the extremiser) is best possible, modulo the value of constants c_p.

Our proofs involve several different approaches building on various analytic and probabilistic formulae for the volume of sections in question. To a large extent, the probabilistic underpinning for these new geometric results is a connection to negative moments of weighted sums of independent random variables. To give a short overview: (1) simply follows from Schur convexity, its reversal, (2) is obtained from a formula involving negative moments combined with complete monotonicity allowing to invoke the Laplace transform to leverage independence, (3) for $2 < p < \infty$ relies on viewing the volume of sections as the ∞-norm of an appropriate probability density which is estimated using peakedness and additional probabilistic tools, e.g. the Berry-Esseen theorem, whereas (3) for $p = \infty$ follows from a more general stability result for an underlying Khinchin-type inequality, obtained thanks to negative moments, and, finally, (4) is established by a careful analysis of Ball's proof, souped-up with new insights gained from representations via negative moments allowing for certain self-improvements of Ball's inequality (in the spirit of [16] which establishes an analogous stability result for Szarek's $L_1 - L_2$ classical Khinchin inequality, with arguments based on discrete Fourier analysis). In a recent independent work [34], Melbourne and Roberto have addressed the stability of maximal hyperplane sections of the cube, obtaining a similar result to (4), with explicit values of the numerical constants involved. Their approach is somewhat different and relies on developing a stability version of Ball's integral inequality.

For the sake of simplicity of our arguments, we have not made any attempts to optimise the values of the involved multiplicative constants c_p (or for that matter even explicitly compute some values, except for the case of (3) when $p = \infty$).

1.3. **Organisation.** In Section 2, we recall well-known and develop some new formulae for volumes of sections which will be used throughout the paper. Our new result for the cross-polytope, Theorem 1, is proved in Section 3. Our stability results are proved in Sections 4 and 5. First, we deal with the cube and prove (3) for $p = \infty$ in Section 4.1, as well as (4) in Section 4.2. Then, we consider the case $0 < p < 2$ and show (1) in Section 5.1.1, followed by the proof of (2) in Section 5.1.2. Finally, we present the proof of (3) when $2 < p < \infty$ in Section 5.2. We gather some concluding comments and possible future directions in Section 6.

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2. Formulae for volumes of sections of convex sets

2.1. **Sections via linear embeddings.** In what follows by a *convex body* we mean a convex compact set with non-empty interior. Recall that there is a standard correspondence between symmetric convex bodies in $\mathbb{R}^n$ and norms on $\mathbb{R}^n$. The Minkowski function (gauge function) associated with a convex body K will be denoted by $\|\cdot\|_K$. We shall use the following standard lemma.

Lemma 3. *Let K be a convex body in $\mathbb{R}^n$ and let $T : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. Define $K_T = \{x \in \mathbb{R}^k : \|Tx\|_K \le 1\}$. Then $K \cap T(\mathbb{R}^k) = T(K_T)$. Moreover, if T is of full rank then*

$$\mathrm{vol}_{T(\mathbb{R}^k)}(K \cap T(\mathbb{R}^k)) = \sqrt{\det(T^*T)}\,\mathrm{vol}_k(K_T).$$

Proof. For the first part, let us show two inclusions. If $y \in K \cap T(\mathbb{R}^k)$, then $y \in K$ and $y = Tx$ for some $x \in \mathbb{R}^k$. It follows that $\|Tx\|_K \le 1$, so $x \in K_T$. Thus $y = Tx \in T(K_T)$. Now, if $y \in T(K_T)$, then $y = Tx$ for some x satisfying $\|Tx\|_K \le 1$. Thus $\|y\|_K \le 1$, so $y \in K$. Since clearly $y \in T(\mathbb{R}^k)$, it follows that $y \in K \cap T(\mathbb{R}^k)$.

For the second part, observe that one can treat $H = T(\mathbb{R}^k)$ as a manifold parametrized by T. Since vol_H is volume on this manifold, we have the well-known formula for the volume element, $\mathrm{d}\,\mathrm{vol}_H = \sqrt{\det((DT)^*(DT))}\,\mathrm{d}\,\mathrm{vol}_k$, where DT stands for the derivative of T. In our case $DT = T$ and so the assertion follows. □

A straightforward application of the above lemma to the case of K being the B_p^n ball yields the following corollary.

Corollary 4. *Suppose that H is an image of $\mathbb{R}^k$ under a linear map $T : \mathbb{R}^k \to \mathbb{R}^n$ of full rank, given by $Tx = (\langle v_1, x\rangle, \ldots, \langle v_n, x\rangle)$ for some vectors $v_1, \ldots, v_n \in \mathbb{R}^k$. Then*

$$\mathrm{vol}_H(B_p^n \cap H) = \det\left(\sum_{i=1}^n v_i \otimes v_i\right)^{1/2} \mathrm{vol}_k\left(\left\{x \in \mathbb{R}^k : \sum_{i=1}^n |\langle v_i, x\rangle|^p \le 1\right\}\right).$$

Here, as usual, $v \otimes v$ is the matrix $vv^\top$. Let us now assume that the map T is an isometric embedding. This means that $\langle x, y\rangle = \langle Tx, Ty\rangle = \langle x, T^*Ty\rangle$, which gives the condition $T^*T = I_{k\times k}$, where $I_{k\times k}$ stands for the $k \times k$ identity matrix. If the mapping is written in the form $Tx = (\langle v_1, x\rangle, \dots, \langle v_n, x\rangle)$, the condition $T^*T = I_{k\times k}$ rewrites as $\sum_{i=1}^n v_i \otimes v_i = I_{k\times k}$. Thus, finding extremal k dimensional sections of K is equivalent to solving the following problem.

Problem 1. *Maximize/minimize the volume of the set $K_T = \{x \in \mathbb{R}^k : \ \|Tx\|_K \le 1\}$ under the constrain $T^*T = I_{k\times k}$. In the case of $K = B_p^n$, maximize/minimize the volume of the set*

$$K_v = \left\{ x \in \mathbb{R}^k : \ \sum_{i=1}^n |\langle v_i, x\rangle|^p \le 1 \right\} \quad \text{over} \quad v_1, \dots, v_n \in \mathbb{R}^k, \ \sum_{i=1}^n v_i \otimes v_i = I_{k\times k}.$$

Remark 5. Since the condition $T^*T = I_{k\times k}$ ensures that the map is an isometric embedding, the set K_T in $\mathbb{R}^k$ in the above extremization problem is isometric to the section $K \cap T(\mathbb{R}^k)$.

2.2. **Sections via negative moments.** The goal of this section is to connect extremal-volume sections of convex bodies to sharp Khinchin-type inequalities for negative moments.

Lemma 6. *Let X be random vector with density g in $\mathbb{R}^n$. Let H be a codimension k subspace of $\mathbb{R}^n$ and let U be a $k \times n$ matrix whose rows $u_1, \dots, u_k$ form an orthonormal basis of $H^\perp$, the orthogonal complement of H. Then $f(x) = \int_{H+U^\top x} g$ is the density of the random vector UX in $\mathbb{R}^k$.*

Proof. For $x = (x_1, \dots, x_k)$ we have $U^\top x = \sum_{i=1}^k u_i x_i$. Since u_i span $H^\perp$, we get that $y \in H^\perp$ iff $y = U^\top x$ for some $x \in \mathbb{R}^k$. Moreover, since u_i are orthonormal, we get that $x \mapsto U^\top x$ is an isometric embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$, whose image is $H^\perp$. By Fubini's theorem f is measurable on $\mathbb{R}^k$.

Let us now take a measurable set $B \subseteq \mathbb{R}^k$. Note that $H = \{x \in \mathbb{R}^n : \langle x, u_i\rangle = 0, 1 \le i \le k\}$ and thus $H = \ker U$. Every point $y \in U^{-1}(B)$ can be written as $y = y_1 + y_2$, where $y_1 \in H$ and $y_2 \in H^\perp \cap U^{-1}(B)$. Since every point in $H^\perp$ is of the form $y_2 = U^\top z$ for $z \in \mathbb{R}^k$ and $U^\top z \in U^{-1}(B)$ iff $UU^\top z \in B$, which is just $z \in B$ as $UU^\top = I_{k\times k}$, we get that $U^{-1}(B) = H + U^\top B$. Thus, by Fubini's theorem we get

$$\mathbb{P}(UX \in B) = \mathbb{P}\left(X \in U^{-1}(B)\right) = \mathbb{P}\left(X \in H + U^\top B\right) = \int_B \left(\int_{H+U^\top x} g \right) \mathrm{d}x = \int_B f(x)\mathrm{d}x.$$

□

Corollary 7. *Let A be a measurable set in $\mathbb{R}^n$ of volume 1 and let X be a uniform random vector on A. Let H be a codimension k subspace of $\mathbb{R}^n$ and let U be a $k \times n$ matrix whose rows form an orthonormal basis of $H^\perp$, the orthogonal complement of H. Then*

$$f(x) = \mathrm{vol}_{n-k}(A \cap (H + U^\top x))$$

is the density of the random vector UX in $\mathbb{R}^k$. Moreover, if A is a convex body, then on its support the above function is the unique continuous version of the density of UX. This continuous version satisfies

$$f(0) = \mathrm{vol}_{n-k}(A \cap H)$$

if $0 \in \mathrm{int}\,\mathrm{supp}(f)$.

Proof. This is a special case of Lemma 6. If A is a convex body, then by Brunn-Minkowski inequality $f^{\frac{1}{n-k}}$ is concave on the interior of its support and therefore continuous. $\square$

Lemma 8. *Let X be a random vector in $\mathbb{R}^k$ with density f such that $\|f\|_\infty = f(0)$ and f is lower semi-continuous at 0. Let $\|\cdot\|$ be a norm on $\mathbb{R}^k$ with closed unit ball K. We have,*

$$f(0) = \lim_{q\to k-} \frac{k-q}{k\cdot \mathrm{vol}_k(K)}\mathbb{E}\|X\|^{-q}.$$

Proof. We first claim that

$$\int_{tK} \|x\|^{-q}\mathrm{d}x = \frac{k}{k-q}t^{k-q}\,\mathrm{vol}_k(K), \qquad \text{for} \quad t>0,\ 0<q<k. \tag{5}$$

Indeed, thanks to the homogeneity of volume, we have

$$\begin{aligned}
\int_{tK} \|x\|^{-q}\mathrm{d}x &= \int_{tK}\int_{\|x\|}^{\infty} qs^{-(q+1)}\mathrm{d}s\mathrm{d}x = \int_{tK}\left(\int_0^\infty qs^{-(q+1)}\mathbf{1}_{\|x\|\le s}\mathrm{d}s\right)\mathrm{d}x \\
&= \int_0^\infty qs^{-(q+1)}\left(\int_{tK}\mathbf{1}_{\|x\|\le s}\mathrm{d}x\right)\mathrm{d}s = \int_0^\infty qs^{-(q+1)}\left(\int_{\mathbb{R}^k}\mathbf{1}_{\|x\|\le \min(s,t)}\mathrm{d}x\right)\mathrm{d}s \\
&= \mathrm{vol}_k(K)\int_0^\infty qs^{-(q+1)}\min(s,t)^k\mathrm{d}s = \frac{k}{k-q}t^{k-q}\,\mathrm{vol}_k(K).
\end{aligned}$$

Take $M>0$. Using (5) with $t=M$, we get

$$\begin{aligned}
\frac{k-q}{k\cdot \mathrm{vol}_k(K)}\mathbb{E}\|X\|^{-q} &= \frac{k-q}{k\cdot \mathrm{vol}_k(K)}\int_{MK}\|x\|^{-q}f(x)\mathrm{d}x + \frac{k-q}{k\cdot \mathrm{vol}_k(K)}\int_{(MK)^c}\|x\|^{-q}f(x)\mathrm{d}x \\
&\le \frac{k-q}{k\cdot \mathrm{vol}_k(K)}\|f\|_\infty\int_{MK}\|x\|^{-q}\mathrm{d}x + \frac{k-q}{k\cdot \mathrm{vol}_k(K)}M^{-q} \\
&= \|f\|_\infty M^{k-q} + \frac{k-q}{k\cdot \mathrm{vol}_k(K)}M^{-q}.
\end{aligned}$$

Fix $\varepsilon>0$. Since $\|f\|_\infty = f(0)$ and f is lower semi-continuous at 0, the set $\{x\in\mathbb{R}^k, f(x) > \|f\|_\infty - \varepsilon\}$ contains a neighbourhood of 0, say δK for some $\delta>0$. Then,

$$\begin{aligned}
\frac{k-q}{k\cdot \mathrm{vol}_k(K)}\mathbb{E}\|X\|^{-q} &\ge \frac{k-q}{k\cdot \mathrm{vol}_k(K)}\int_{\delta K}\|x\|^{-q}f(x)\mathrm{d}x \\
&\ge \frac{k-q}{k\cdot \mathrm{vol}_k(K)}(\|f\|_\infty-\varepsilon)\int_{\delta K}\|x\|^{-q}\mathrm{d}x \\
&= (\|f\|_\infty - \varepsilon)\delta^{k-q}.
\end{aligned}$$

These two bounds show that as $q\to k-$, the $\liminf$ and $\limsup$ of $\frac{k-q}{k\cdot\mathrm{vol}_k(K)}\mathbb{E}\|X\|^{-q}$ are within ε of $\|f\|_\infty$. $\square$

Combining Corollary 7 and Lemma 8 yields a probabilistic formula for sections in terms of negative moments.

Corollary 9. *Let A be a symmetric convex body in $\mathbb{R}^n$ of volume 1 and let X be uniform on A. Let $\|\cdot\|$ be a norm in $\mathbb{R}^k$ with closed unit ball K. Let H be a codimension k subspace of*

$\mathbb{R}^n$ and let U be a $k \times n$ matrix whose rows form an orthonormal basis of $H^\perp$. Then

$$\mathrm{vol}_{n-k}(A \cap H) = \lim_{q \to n-} \frac{k-q}{k \cdot \mathrm{vol}_k(K)} \mathbb{E}\|UX\|^{-q}.$$

Proof. Since UX is log-concave and symmetric on $\mathbb{R}^k$, one gets $\|f\|_\infty = f(0)$. □

2.3. **Sections of the cube.** As a first application, we sketch how to obtain a convenient probabilistic formula for central section of the cube in terms of negative moments. It was derived first perhaps in [28] and later appeared in [10] as well as [31]. Our argument is different, more direct, bypassing the Fourier-analytic identities involving Bessel functions. It was recently presented in full detail in [15]. It is more convenient to treat the cube of unit volume, so we set

$$Q_n = \frac{1}{2} B_\infty^n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n.$$

Lemma 10 (König-Koldobsky, [28])**.** *For a unit vector $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_{n-1}\left(Q_n \cap a^\perp\right) = \mathbb{E}\left|\sum_{k=1}^n a_k \xi_k\right|^{-1},$$

where the ξ_k are uniform on S^2 in $\mathbb{R}^3$.

Proof. Let $U_1, \ldots, U_n$ be i.i.d. uniform on $[-1,1]$. From Corollary 9 applied with $k=1$ one gets

$$\mathrm{vol}_{n-1}\left(Q_n \cap a^\perp\right) = \lim_{q \to 1-} (1-q)\mathbb{E}\left|\sum_{k=1}^n a_k U_k\right|^{-q}.$$

It is therefore enough to show that for $q < 1$ one has

$$\mathbb{E}\left|\sum_{k=1}^n a_k \xi_k\right|^{-q} = (1-q)\mathbb{E}\left|\sum_{k=1}^n a_k U_k\right|^{-q}.$$

This can be shown by repeating Latała's argument leveraging rotational symmetry from Proposition 4 in [30]. It has also been written in full detail in Lemma 3 in [15]. □

Remark 11*.* The following alternative Fourier-analytic formula for the volume of central codimension 1 sections perhaps goes back to Pólya and is well known (see, e.g. [3])

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) = \frac{2}{\pi} \int_0^\infty \prod_{j=1}^n \frac{\sin(a_j t)}{a_j t} \mathrm{d}t.$$

2.4. **Sections of B_p^n via negative moments.** Let $p > 0$. Throughout the paper, we let

$$Y_1^{(p)}, Y_2^{(p)}, \ldots \quad \text{be i.i.d. random variables with density } e^{-\beta_p^p |x|^p},$$

where

$$\beta_p = 2\Gamma(1 + 1/p)$$

is chosen such that $\int_{\mathbb{R}} e^{-\beta_p^p |x|^p} \mathrm{d}x = 1$. We shall derive the following lemma.

Lemma 12. *Let H be a subspace in $\mathbb{R}^n$ of codimension k such that the rows of a $k\times n$ matrix U form an orthonormal basis of $H^\perp$. Let $v_1,\dots,v_n$ be the columns of U. Then*

$$\frac{\mathrm{vol}_{n-k}(B_p^n\cap H)}{\mathrm{vol}_{n-k}(B_p^{n-k})}=\lim_{q\to k-}\frac{k-q}{k\,\mathrm{vol}_k(B_2^k)}\mathbb{E}\Big|\sum_{j=1}^n Y_j^{(p)}v_j\Big|^{-q}.$$

Proof. Let $v_1,\dots,v_n$ be the columns of U. Note that

$$\sum_{j=1}^n v_jv_j^\top=I_{k\times k}.$$

We take $X=(X_1,\dots,X_n)$ to be uniform on B_p^n. Then $X/\mathrm{vol}_n(B_p^n)^{1/n}$ is uniform on $\tilde{B}_p^n=B_p^n/\mathrm{vol}_n(B_p^n)^{1/n}$, which has volume 1. Using Corollary 9 with the Euclidean norm $|\cdot|$ gives

$$\frac{\mathrm{vol}_{n-k}\left(B_p^n\cap H\right)}{(\mathrm{vol}_n(B_p^n))^{n-k}}=\mathrm{vol}_{n-k}\left(\tilde{B}_p^n\cap H\right)=\lim_{q\to k-}\frac{\mathrm{vol}_n(B_p^n)^{\frac{q}{n}}(k-q)}{k\,\mathrm{vol}_k(B_2^k)}\mathbb{E}\left|\sum_{j=1}^n X_jv_j\right|^{-q}.$$

We shall now use two important facts:

(a) (Barthe, Guédon, Mendelson, Naor, [8]) Let $Y_1,\dots,Y_n$ be i.i.d. random variables with densities $\beta_p^{-1}e^{-|x|^p}$ and write $Y=(Y_1,\dots,Y_n)$. Define $S=\left(\sum_{j=1}^n|Y_j|^p\right)^{1/p}$. Let $\mathcal{E}$ be an exponential random variable with density $e^{-t}\mathbf{1}_{\{t>0\}}$, independent of the Y_j. Then the random vector $\frac{Y}{(S^p+\mathcal{E})^{1/p}}$ is uniformly distributed on B_p^n.

(b) (Schechtman, Zinn, see [43] and Rachev, Rüschendorf, [42]) With the above notation S and Y/S are independent.

In [8] Barthe, Guédon, Mendelson and Naor observed that using (a) and (b) one gets

$$\mathbb{E}\Big|\sum_{j=1}^n X_jv_j\Big|^{-q}=\mathbb{E}\Big|\frac{1}{(S^p+\mathcal{E})^{1/p}}\sum_{j=1}^n Y_jv_j\Big|^{-q}=\mathbb{E}\Big|\frac{S}{(S^p+\mathcal{E})^{1/p}}\Big|^{-q}\mathbb{E}\Big|\sum_{j=1}^n\frac{Y_j}{S}v_j\Big|^{-q}.$$

It follows that $\mathbb{E}\Big|\frac{S}{(S^p+\mathcal{E})^{1/p}}\Big|^{-q}$ is finite. Thus

$$e^{-1}\mathbb{E}|S|^{-q}=\mathbb{E}|S|^{-q}\mathbf{1}_{\mathcal{E}>1}\le\mathbb{E}\Big|\frac{S}{(S^p+\mathcal{E})^{1/p}}\Big|^{-q}<\infty.$$

Then, again by independence of S and Y/S, we have

$$\mathbb{E}\Big|\sum_{j=1}^n\frac{Y_j}{S}v_j\Big|^{-q}\mathbb{E}|S|^{-q}=\mathbb{E}\Big|\sum_{j=1}^n Y_jv_j\Big|^{-q}$$

and therefore

$$\begin{aligned}\mathbb{E}\Big|\sum_{j=1}^n X_jv_j\Big|^{-q}&=\frac{1}{\mathbb{E}|S|^{-q}}\mathbb{E}\Big|\frac{S}{(S^p+\mathcal{E})^{1/p}}\Big|^{-q}\mathbb{E}\Big|\sum_{j=1}^n Y_jv_j\Big|^{-q}\\&=c_1(p,q,n)\mathbb{E}\Big|\sum_{j=1}^n Y_jv_j\Big|^{-q}=c_2(p,q,n)\mathbb{E}\Big|\sum_{j=1}^n Y_j^{(p)}v_j\Big|^{-q},\end{aligned}$$

where $c_i(p,q,n) > 0$ is independent of $v_1, \dots, v_n$. As a result one gets

$$\mathrm{vol}_{n-k}(B_p^n \cap H) = c_3(k,p,n) \lim_{q \to k-} \frac{k-q}{k\,\mathrm{vol}_k(B_2^k)} \mathbb{E}\Big|\sum_{j=1}^n Y_j^{(p)} v_j\Big|^{-q}.$$

Taking $v_j = e_j$ for $1 \le i \le k$ and $v_j = 0$ for $k+1 \le j \le n$ and using Lemma 8 we obtain

$$\mathrm{vol}_{n-k}(B_p^{n-k}) = c_3(k,p,n) \lim_{q \to k-} \frac{k-q}{k\,\mathrm{vol}_k(B_2^k)} \mathbb{E}\Big|(Y_1^{(p)}, \dots, Y_k^{(p)})\Big|^{-q} = c_3(k,p,n).$$

□

Corollary 13. *Let $p > 0$. For a unit vector $a \in \mathbb{R}^n$, we have*

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^{n-1})} = f_a(0),$$

where f_a is the density of $\sum_{j=1}^n a_j Y_j^{(p)}$.

Proof. This formula follows by combining Lemma 12 with Lemma 8. The correctness of the normalization constant can be checked by plugging in $a = e_1$. □

As an application, we show how to obtain the following theorem of Meyer and Pajor from [35]. The main idea of exploiting Kanter's peakedness from [23] comes from the original proof of Meyer and Pajor. In addition to illustrating our approach via negative moments, which we will build upon later, we hope this proof might be of independent interest.

Theorem 14 (Meyer-Pajor, [35])**.** *Let $1 \le k \le n$ and let H be a subspace in $\mathbb{R}^n$ of codimension k. Then the following function*

$$p \mapsto \mathrm{vol}_{n-k}(B_p^n \cap H) / \mathrm{vol}_{n-k}(B_p^{n-k}).$$

is nondecreasing on $(0,\infty)$.

Proof. For $\beta > \alpha$ the random variable $Y_j^{(\beta)}$ is more peaked than $Y_j^{(\alpha)}$ (see [23] and [35]). Thus for every vectors $v_1, \dots, v_n$ in $\mathbb{R}^k$, $\sum_{j=1}^n Y_j^{(\beta)} v_j$ is more peaked than $\sum_{j=1}^n Y_j^{(\alpha)} v_j$. Consequently, for a norm $\|\cdot\|$ on $\mathbb{R}^k$ and $0 < q < k$,

$$\mathbb{E}\left\|\sum_{j=1}^n Y_j^{(\beta)} v_j\right\|^{-q} \ge \mathbb{E}\left\|\sum_{j=1}^n Y_j^{(\alpha)} v_j\right\|^{-q}. \tag{6}$$

Thus, the function $\alpha \mapsto \mathbb{E}\left\|\sum_{j=1}^n Y_j^{(\alpha)} v_j\right\|^{-q}$ is nondecreasing on $(0,\infty)$. Using this together with Lemma 12, we get that

$$p \mapsto \frac{\mathrm{vol}_n(B_p^n \cap H)}{\mathrm{vol}_{n-k}(B_q^{n-k})} = \lim_{q \to k-} \frac{k-q}{k\,\mathrm{vol}_k(B_2^k)} \mathbb{E}\Big|\sum_{j=1}^n Y_j^{(p)} v_j\Big|^{-q}$$

is nondecreasing. □

2.5. **Sections of B_p^n via Gaussian mixtures.** In the sequel we shall need one more formula in the special case of B_p^n with $0 < p < 2$. This formula was mentioned in [18] (a hyperplane case) and [38] (a general case). We sketch a slightly different argument below, based again on negative moments, for simplicity for hyperplane sections.

We first need some notation. For $\alpha \in (0,1)$, let g_α be the density of a standard positive α-stable random variable, that is a positive random variable W_α with the Laplace transform $\mathbb{E}e^{-uW_\alpha} = e^{-u^\alpha}$, $u > 0$. Let $V_1, \ldots, V_n$ be i.i.d. positive random variables with density proportional to $t^{-3/2} g_{p/2}(t^{-1})$ and set $R_i = \sqrt{V_i/2}$. Take G_i to be standard Gaussian random variables, independent of the V_j. According to Lemma 23(a) from [18], the random variables $R_i G_i$ have densities $\beta_p^{-1} e^{-|x|^p}$. We also let $\bar{V}_j = (\mathbb{E}V_j^{-1/2})^2 V_j$ be normalised so that $\mathbb{E}\bar{V}_j^{-1/2} = 1$.

Lemma 15 (Eskenazis-Nayar-Tkocz, [18])**.** *Let $0 < p < 2$. For a unit vector $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$, we have*

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^{n-1})} = \mathbb{E}\left(\sum_{j=1}^n a_j^2 \bar{V}_j\right)^{-1/2}. \tag{7}$$

Proof. Using Lemma 12 and the above Gaussian mixture representation for the $Y_j^{(p)}$,

$$\begin{aligned}\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^{n-1})} &= \lim_{q \to 1-} \frac{1-q}{2} \mathbb{E}\Big|\sum_{j=1}^n a_j Y_j^{(p)}\Big|^{-q} \\ &= \kappa_p \lim_{q \to 1-} (1-q) \mathbb{E}\Big|\sum_{j=1}^n a_j \sqrt{V_j} G_j\Big|^{-q}\end{aligned}$$

for a positive constant κ_p which depends only on p (resulting from rescalings of the random variables involved). Since $\sum_{j=1}^n a_j \sqrt{V_j} G_j$ has the same distribution as $\sqrt{\sum a_j^2 V_j} G_1$ and $(1-q)\mathbb{E}|G_1|^{-q}$ converges to $\sqrt{\frac{2}{\pi}}$ (twice the density at 0) as $q \to 1-$, after further rescalings, we obtain

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^{n-1})} = \kappa_p' \mathbb{E}\left(\sum_{j=1}^n a_j^2 \bar{V}_j\right)^{-1/2}.$$

Plugging in $a = e_1$ shows that $\kappa_p' = 1$. $\square$

Remark 16. The above expectation is finite due to the fact that $\mathbb{E}W_\alpha^r < \infty$ iff $r < \alpha$. Indeed,

$$\int_0^\infty t^{q-3/2} g_{p/2}(t^{-1}) \mathrm{d}t = \int_0^\infty t^{-q-1/2} g_{p/2}(t) \mathrm{d}t = \mathbb{E}W_{p/2}^{-q-1/2}$$

thus $\mathbb{E}V_1^q < \infty$ as long as $-q - 1/2 < p/2$, that is $q > -\frac{p+1}{2}$. The above fact can be deduced from the asymptotic formulas (see, e.g. [36])

$$g_\alpha(t) \sim_{t \to \infty} M_\alpha t^{-(1+\alpha)}, \qquad g_\alpha(t) \sim_{t \to 0^+} K_\alpha t^{-\frac{2-\alpha}{2(1-\alpha)}} \exp(A_\alpha t^{-\frac{\alpha}{1-\alpha}}).$$

3. Two-dimensional central sections of the cross-polytope

This proof was kindly communicated to us by Fedor Nazarov. Recall that our goal is to minimize the volume of the set $K_v = \{x \in \mathbb{R}^2 : \sum_{i=1}^n |\langle v_i, x\rangle| \leq 1\}$ under the constraint $\sum_{i=1}^n v_i \otimes v_i = I_{2\times 2}$. In general, the set K_v is a convex symmetric $2k$-gon, $k \leq n$. We point out that some of the vectors v_i might be zero, and some of them may be parallel. While studying the geometry of K_v, one can assume that the vectors v_i are non-parallel, since if for some $a_1, \ldots, a_l$, $i_1, \ldots, i_l$ and v one has $v_{i_1} = a_1 v, \ldots, v_{i_l} = a_l v$, then considering only one vector $\tilde{v} = \sum_{j=1}^l |a_{i_j}| v$ instead of the vectors v_{i_j} will result in the same set. However, this operation in general affects the constraint $\sum_{i=1}^n v_i \otimes v_i = I_{2\times 2}$.

Let $\rho : S^1 \to (0, \infty)$, given by $\rho(\theta) = (\sum_{i=1}^n |\langle v_i, \theta\rangle|)^{-1}$, be the radial function of K_v. One can assume that in our configuration there are at least two non-parallel vectors (otherwise the resulting set is an infinite strip and so its volume is infinite; in this case $\sum_{i=1}^n v_i \otimes v_i$ is of rank one, and the constraint is not satisfied). It is not hard to check that under this assumption the vertices of K_v correspond exactly to directions θ perpendicular to v_i for some non-zero v_i (that is, to the changes of sign of $\langle v_i, \theta\rangle$). Indeed, for points x on the boundary of K_v one has $\sum_{i=1}^n |\langle v_i, x\rangle| = 1$. If in a small neighborhood of x all the signs of $\langle v_i, x\rangle$ are fixed, this is a linear equation and the set of solutions is a line. This corresponds to 1-dimensional faces of K_v. If on the other hand x satisfies $\langle v_i, x\rangle = 0$ for some non-zero $v_i = (a, b)$ (if there are vectors parallel to v_i we join them together as above), then within a small ball around $x = (s_0, t_0)$ there is a part of the boundary being a subset of the line of the form $\{(s, t) : as + bt + As + Bt = 1\}$ and a part being a subset of the line of the form $\{(s, t) : -as - bt + As + Bt = 1\}$. These two lines intersect each other at x. We shall show that they are non-parallel. If they were parallel, they would have to coincide and thus we would have $a + A = -a + A$ and $b + B = -b + B$, which gives $a = b = 0$, contradiction. Thus x is an intersection of two non-parallel parts of the boundary and thus is a vertex of K_v. A simple consequence of these observations is that K_v has at most $2n$ vertices.

Suppose that the boundary of K_v consists of segments F_j, $j = 1, \ldots, k$. Let C_j be the corresponding segments of S^1, that is $\theta \in C_j$ if $\rho(\theta)\theta \in F_j$, and let $T_j = \mathrm{conv}(0, F_j)$ be the corresponding triangle in K_v. We define $A_j = \frac{1}{2}\int_{C_j} \rho^2$ and $I_j = \int_{C_j} \rho^{-1}$. Suppose that the angle of T_j at vertex $O = 0$ has measure $2\beta_j$, where $\beta_j \in (0, \pi/2)$. Note that $\sum_{j=1}^k \beta_j = \pi$. We shall need the following elementary lemma.

Lemma 17. *We have* $A_j I_j^2 \geq \frac{4\sin^3\beta_j}{\cos\beta_j}$.

Proof. Let OLR be one of our triangles T_j and let 2β be the measure of the angle at vertex O. Let h be the height of OLR perpendicular to LR and let l be the bisector of $\angle LOR$. The directed angle from h to l will be denoted by α. Let θ be the directed angle on S^1, where $\theta = 0$ corresponds to points on h. Clearly $\rho(\theta) = h/\cos\theta$. We have

$$I_j = \int_{\alpha-\beta}^{\alpha+\beta} \frac{\cos\theta}{h} \mathrm{d}\theta = \frac{1}{h}[\sin(\alpha+\beta) - \sin(\alpha-\beta)],$$

$$A_j = \frac{1}{2}h^2 \int_{\alpha-\beta}^{\alpha+\beta} \frac{1}{\cos^2\theta} \mathrm{d}\theta = \frac{1}{2}h^2[\tan(\alpha+\beta) - \tan(\alpha-\beta)].$$

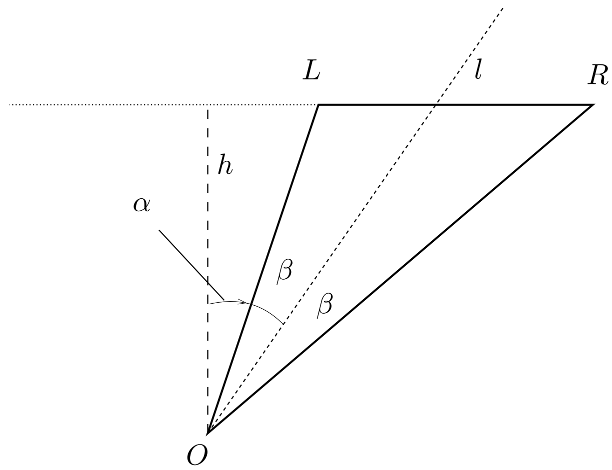

FIGURE 1. One *piece* of K_v: triangle OLR.

Thus,

$$\begin{aligned}A_j I_j^2 &= \frac{1}{2}\left[\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} - \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}\right]\cdot[\sin(\alpha+\beta)-\sin(\alpha-\beta)]^2 = \frac{2\sin(2\beta)\cdot\sin^2\beta\cos^2\alpha}{\cos(\alpha+\beta)\cos(\alpha-\beta)}\cdot\\ &= \frac{4\sin^3\beta\cos\beta\cos^2\alpha}{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta} = \frac{4\sin^3\beta}{\cos\beta}\cdot\frac{1}{1-\tan^2\alpha\tan^2\beta} \geq \frac{4\sin^3\beta}{\cos\beta}.\end{aligned}$$

□

Lemma 18. *The function $\psi(x) = \frac{\sin x}{(\cos x)^{1/3}}$ is strictly convex on $[0, \pi/2)$. In particular, the function $[0, \pi/2) \ni x \mapsto \psi(x)/x$ is non-decreasing and thus the sequence $a_n = \frac{n\sin\left(\frac{\pi}{2n}\right)}{\cos^{1/3}\left(\frac{\pi}{2n}\right)}$ is non-increasing.*

Proof. Observe that $\psi'(x) = \cos^{2/3}x + \frac{1}{3}\sin^2 x\cos^{-4/3}x = \frac{2}{3}\cos^{2/3}x + \frac{1}{3}\cos^{-4/3}x$. It suffices to show that this function is strictly increasing. Taking $y = \cos^{2/3}x$ we see that this is equivalent to showing that $f(y) = 2y + y^{-2}$ is strictly decreasing $(0, 1)$. This is true since $f'(y) = 2(1 - y^{-3}) < 0$ for $y \in (0, 1)$.

The second part follows from the monotonicity of the slopes of convex functions and the fact that $\psi(0) = 0$. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. We shall solve Problem 1. Assume that $\sum_{i=1}^n v_i \otimes v_i = I_{2\times 2}$ and that K_v is a convex symmetric $2k$-gon, where $k \leq n$. Note that

$$\int_{S^1}\rho(\theta)^{-1}\mathrm{d}\theta = \sum_{i=1}^n\int_{S^1}|\langle v_i, \theta\rangle|\mathrm{d}\theta = 4\sum_{i=1}^n|v_i| \leq 4\sqrt{n}\sqrt{\sum_{i=1}^n|v_i|^2} = 4\sqrt{2n},$$

where in the last equality we use $\sum_{i=1}^n |v_i|^2 = \operatorname{tr}\left(\sum_{i=1}^n v_i \otimes v_i\right)$. Moreover, using Hölder's inequality, Lemma 17 and Lemma 18, we get

$$\begin{aligned}|K_v|^{\frac{1}{3}}(4\sqrt{2n})^{\frac{2}{3}} &= |K_v|^{\frac{1}{3}}\left(\int_{S^1}\rho(\theta)^{-1}\mathrm{d}\theta\right)^{\frac{2}{3}} = \left(\sum_{j=1}^{2k}A_j\right)^{\frac{1}{3}}\left(\sum_{j=1}^{2k}I_j\right)^{\frac{2}{3}} \\ &\geq \sum_{j=1}^{2k}A_j^{\frac{1}{3}}I_j^{\frac{2}{3}} \geq 4^{\frac{1}{3}}\sum_{j=1}^{2k}\frac{\sin\beta_j}{\cos^{1/3}\beta_j} \\ &\geq 4^{\frac{1}{3}}\cdot 2k\frac{\sin\left(\frac{1}{2k}\sum_{j=1}^{2k}\beta_j\right)}{\cos^{1/3}\left(\frac{1}{2k}\sum_{j=1}^{2k}\beta_j\right)} = 2\cdot 4^{\frac{1}{3}}\cdot\frac{k\sin\left(\frac{\pi}{2k}\right)}{\cos^{1/3}\left(\frac{\pi}{2k}\right)} \geq 2\cdot 4^{\frac{1}{3}}\cdot\frac{n\sin\left(\frac{\pi}{2n}\right)}{\cos^{1/3}\left(\frac{\pi}{2n}\right)}.\end{aligned}$$

We arrive at $|K_v| \geq \frac{n^2\sin^3\left(\frac{\pi}{2n}\right)}{\cos\left(\frac{\pi}{2n}\right)}$.

We now show that this bound is achieved for K_v being a regular $2n$-gon. Let us consider $v_k = \sqrt{\frac{2}{n}}(\cos(\frac{k\pi}{n}), \sin(\frac{k\pi}{n}))$ for $k = 1, \ldots, n$. It is easy to verify that $\sum_{i=1}^n v_i \otimes v_i = I_{2\times 2}$. As we already mentioned, the vertices of K_v correspond to the directions perpendicular to v_i. Since v_i are equally spaced on the upper half-circle, we get that K_v is a regular $2n$-gon. Clearly $|v_1| = \ldots = |v_n|$, $\beta_1 = \ldots = \beta_{2n}$, $I_1 = \ldots = I_{2n}$ and $A_1 = \ldots = A_{2n}$. Thus, one has equalities in all the inequalities in the above proof, so $|K_v| = n^2\sin^3\left(\frac{\pi}{2n}\right)/\cos\left(\frac{\pi}{2n}\right)$. Conversely, it is easy to see that the only possibility of having equalities in all the estimates of the proof is to have the set $\{v_1, -v_1, \ldots, v_n, -v_n\}$ equally spaced on the circle. Thus, in the extremal case the only freedom of choosing v_i is to apply rotations to all the vectors v_i (which does not change the section $B_1^n \cap T(\mathbb{R}^2)$, as it corresponds to replacing T with $T \circ U$ for some orthogonal transformation U of $\mathbb{R}^2$), permuting some of the vectors (which corresponds to applying permutations of coordinates in $\mathbb{R}^n$, under which H changes), and reflecting some of the vectors v_i (which corresponds to applying coordinate reflections in $\mathbb{R}^n$ which again changes H). Thus, up to coordinate reflections and permutations, there is only one minimal two-dimensional section of B_1^n. The fact that the section of minimal volume is isometric to a regular $2n$-gon in $\mathbb{R}^2$ follows from Remark 5. $\square$

4. Cube slicing

4.1. **Minimal hyperplane cube sections.** Prior to Vaaler's work [46], Hadwiger in [19] and independently Hensley in [20] established that the minimal hyperplane sections of the cube are attained for coordinate subspaces. A different simple proof was later given in [3] (which was based on a direct minimisation of $\|f\|_\infty$ over even unimodal probability densities with fixed variance). Our method involving negative moments offers another simple approach with the advantage that it is well-suited to give a stability result. First we establish a robust version of a relevant Khinchin inequality.

Theorem 19. *Let $0 < p < 2$ and let $\xi_1, \ldots, \xi_n$ be i.i.d. random vectors in $\mathbb{R}^d$ uniform on S^{d-1}, $d \geq 3$. For every $n \geq 1$ and real numbers $a_1, \ldots, a_n$ such that $a_1^2 + \cdots + a_n^2 = 1$, we* *have*

$$\mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^{-p} \geq 1 + \frac{p(p+2)(2d-p-4)}{9d^2}\left(1 - \sum_{j=1}^n a_j^4\right).$$

Proof. First we remark that a sharp inequality without the remainder term is a simple consequence of convexity. Indeed, for any $p > 0$ we have

$$\mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^{-p} = \mathbb{E}\left(\left|\sum_{j=1}^n a_j\xi_j\right|^2\right)^{-p/2} \geq \left(\mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^2\right)^{-p/2} = 1. \tag{8}$$

To control the error in this estimate, a natural idea presents itself: we write

$$\left|\sum_{j=1}^n a_j\xi_j\right|^2 = 1 + S$$

with

$$S = 2\sum_{i<j} a_i a_j \langle \xi_i, \xi_j\rangle$$

and seek a refinement of the pointwise bound $(1+x)^{-p/2} \geq 1 - \frac{p}{2}x$, $x > -1$ (resulting just from convexity) which gives (8), in view of the fact that $S > -1$ a.s. and $\mathbb{E}S = 0$. We shall use the following lemma, the proof of which we defer for now (for simplicity, we did not try to optimise the numerical constants).

Lemma 20. *For every $p > 0$ and $x > -1$, we have*

$$(1+x)^{-p/2} \geq 1 - \frac{p}{2}x + \frac{p(p+2)}{9}x^2 - \frac{p(p+2)(p+4)}{72}x^3.$$

This lemma yields

$$\mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^{-p} = \mathbb{E}(1+S)^{-p/2} \geq 1 + \frac{p(p+2)}{9}\mathbb{E}S^2 - \frac{p(p+2)(p+4)}{72}\mathbb{E}S^3.$$

To compute $\mathbb{E}S^2$ and $\mathbb{E}S^3$, first note that thanks to rotational invariance and independence, for $i < j$,

$$\mathbb{E}\langle \xi_i, \xi_j\rangle^2 = \mathbb{E}\langle \xi_i, e_1\rangle^2 = \frac{1}{d}$$

and for $i < j < k$,

$$\mathbb{E}\langle \xi_i, \xi_j\rangle\langle \xi_j, \xi_k\rangle\langle \xi_i, \xi_k\rangle = \mathbb{E}\langle \xi_i, \xi_j\rangle\langle \xi_j, e_1\rangle\langle \xi_i, e_1\rangle = \mathbb{E}\langle \xi_j, e_1\rangle^2\langle \xi_i, e_1\rangle^2 = \frac{1}{d^2}.$$

Thus, using symmetry,

$$\mathbb{E}S^2 = 4\sum_{i<j} a_i^2a_j^2\mathbb{E}\langle \xi_i, \xi_j\rangle^2 = \frac{4}{d}\sum_{i<j} a_i^2a_j^2$$

and

$$\mathbb{E}S^3 = 8\cdot 6\sum_{i<j<k} a_i^2a_j^2a_k^2\mathbb{E}\langle \xi_i, \xi_j\rangle\langle \xi_j, \xi_k\rangle\langle \xi_i, \xi_k\rangle = \frac{48}{d^2}\sum_{i<j<k} a_i^2a_j^2a_k^2.$$

Introducing, $s_l = \sum_{i=1}^n a_i^{2l}$, $l = 1, 2, \ldots$, we have $s_1 = 1$ and using Newton identities for symmetric functions, we express $2\sum_{i<j} a_i^2 a_j^2 = 1 - s_2$, $6\sum_{i<j<k} a_i^2 a_j^2 a_k^2 = 1 - 3s_2 + 2s_3$. Moreover, $s_3 \le s_2$. As a result,

$$\mathbb{E}S^2 = \frac{2}{d}(1 - s_2),$$
$$\mathbb{E}S^3 = \frac{8}{d^2}(1 - 3s_2 + 2s_3) \le \frac{8}{d^2}(1 - s_2).$$

Therefore,

$$\mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^{-p} \ge 1 + \frac{2p(p+2)}{9d}(1 - s_2) - \frac{p(p+2)(p+4)}{9d^2}(1 - s_2)$$
$$= 1 + \frac{p(p+2)(2d-p-4)}{9d^2}(1 - s_2).$$

□

Now we are able to deduce a stability result for minimal hyperplane sections of the cube, (3) for $p = \infty$. For convenience, we restate this here.

Theorem 21. *Let $a = (a_1, \ldots, a_n)$ be a unit vector in $\mathbb{R}^n$ with $a_1 \ge a_2 \ge \cdots \ge 0$. Then,*

$$\mathrm{vol}_{n-1}\left(Q_n \cap a^\perp\right) \ge 1 + \frac{1}{54}|a - e_1|^2.$$

Proof. Note that under the assumption on a,

$$\frac{1}{2}|a - e_1|^2 = \frac{1}{2}\left((1-a_1)^2 + \sum_{i=2}^n a_i^2\right) = 1 - a_1 \le 1 - a_1^2 = 1 - \sum_i a_1^2 a_i^2 \le 1 - \sum_i a_i^4.$$

Thus the assertion follows immediately from Theorem 19 applied to $p = 1$ and $d = 3$, in view of Lemma 10. □

Remark 22. The dependence on $\delta(a) = 1 - \sum_{j=1}^n a_j^4$ in Theorem 19 modulo a constant factor is best possible: there are examples of unit vectors a with $\delta(a) \to 0$ for which $\mathbb{E}|\sum a_j\xi_j|^{-p} - 1 = O_{p,d}(\delta(a))$. For instance, take $a = (\sqrt{1-\varepsilon}, \sqrt{\varepsilon}, 0, \ldots, 0)$ with $\varepsilon < \frac{1}{16}$. Since for $0 < p < 2$ and $x \in [-\frac{1}{2}, 1]$ one has $(1+x)^{-\frac{p}{2}} \le 1 - \frac{p}{2}x + 8x^2$ (use Taylor formula with Lagrange remainder), it follows that

$$\mathbb{E}|\sum a_j\xi_j|^{-p} = \mathbb{E}(1 + 2\sqrt{\varepsilon(1-\varepsilon)}\,\langle \xi_1, \xi_2\rangle)^{-p/2} \le 1 + 32\varepsilon(1-\varepsilon)\mathbb{E}\,\langle \xi_1, \xi_2\rangle^2 = 1 + \frac{32\varepsilon(1-\varepsilon)}{d}.$$

Since $1 - \sum_{j=1}^n a_j^4 = 2\varepsilon(1-\varepsilon)$, we get $\mathbb{E}|\sum a_j\xi_j|^{-p} \le 1 + \frac{16}{d}(1 - \sum_{j=1}^n a_j^4)$.

In particular, the same remark applies to Theorem 21 as well.

It remains to prove the point-wise inequality we used.

Proof of Lemma 20. From the Taylor formula with Lagrange reminder for the function $(1+x)^{-\frac{p}{2}}$ one gets that for $x \le \frac{2}{p+4}$

$$(1+x)^{-p/2} - 1 + \frac{p}{2}x \ge \frac{p(p+2)}{8}x^2 - \frac{p(p+2)(p+4)}{48}x^3 \ge \frac{p(p+2)}{9}x^2 - \frac{p(p+2)(p+4)}{72}x^3.$$

We now show how to treat the case $x \geq 0$. Define

$$\psi(x) = (1+x)^{-p/2} - 1 + \frac{p}{2}x - \frac{p(p+2)}{9}x^2 + \frac{p(p+2)(p+4)}{72}x^3.$$

Our goal is to prove that $\psi(x) \geq 0$ for $x \geq 0$. Note that $\psi(0) = \psi'(0) = 0$. Thus it suffices to show that for $x \geq 0$ we have $\psi''(x) \geq 0$. This is equivalent to $(1+x)^{-\frac{p+4}{2}} \geq \frac{8}{9} - \frac{1}{3}(p+4)x$. Define $\alpha = \frac{1}{2}(p+4)$. Our inequality reads $(1+x)^{-\alpha} \geq \frac{8}{9} - \frac{2}{3}\alpha x$. We shall verify this for arbitrary $\alpha, x > 0$. Let $t = \alpha x$. Rewriting gives $(1+\frac{t}{\alpha})^{-\alpha} \geq \frac{8}{9} - \frac{2}{3}t$. We have $(1+\frac{t}{\alpha})^{-\alpha} \geq e^{-t}$ (take the logarithm and use the inequality $\ln(1+y) \leq y$) and thus it is enough to show that $e^{-t} \geq \frac{8}{9} - \frac{2}{3}t$ for $t > 0$. The function $h(t) = e^{-t} - \frac{8}{9} + \frac{2}{3}t$ has a minimum for $t = \ln(\frac{3}{2})$. It is enough to verify that $\frac{2}{3} \geq \frac{8}{9} - \frac{2}{3}\ln(\frac{3}{2})$. This is $\ln(\frac{3}{2}) \geq \frac{1}{3}$ which is true. $\square$

4.2. **Maximal hyperplane cube sections.** Our goal here is to prove (4). We recall two formulae (see Lemma 10 and Remark 11),

$$\mathrm{vol}_{n-1}(Q_n \cap a^{\perp}) = \mathbb{E}\left|\sum_{j=1}^{n} a_j \xi_j\right|^{-1} \tag{9}$$

$$= \frac{2}{\pi}\int_0^{\infty} \prod_{j=1}^{n} \frac{\sin(a_j t)}{a_j t}\mathrm{d}t, \tag{10}$$

as well as the fact that

$$\|a\|_{\mathrm{Bus}} = \frac{|a|}{\mathrm{vol}_{n-1}(Q_n \cap a^{\perp})} \tag{11}$$

defines a norm on $\mathbb{R}^n$, thanks to Busemann's theorem (see [12], or, e.g. Theorem 3.9 in [37]). It follows that the function $a \mapsto \mathrm{vol}_{n-1}(Q_n \cap a^{\perp})$ is 2-Lipschitz on the unit sphere.

Lemma 23. *For every unit vectors a, b in $\mathbb{R}^n$, we have*

$$\left|\mathrm{vol}_{n-1}(Q_n \cap a^{\perp}) - \mathrm{vol}_{n-1}(Q_n \cap b^{\perp})\right| \leq 2|a-b|.$$

Proof. Letting $F(a) = \mathrm{vol}_{n-1}(Q_n \cap a^{\perp})$, by the triangle inequality we have

$$\frac{|F(a)-F(b)|}{F(a)F(b)} = |\|a\|_{\mathrm{Bus}} - \|b\|_{\mathrm{Bus}}| \leq \|a-b\|_{\mathrm{Bus}} = \frac{|a-b|}{F(a-b)}.$$

Using that $1 \leq F(x) \leq \sqrt{2}$ for every vector x concludes the proof. $\square$

We will also need the following observation.

Lemma 24. *Let X and Y be two independent rotationally invariant random vectors in $\mathbb{R}^3$. Then*

$$\mathbb{E}|X+Y|^{-1} = \mathbb{E}\min\left\{|X|^{-1}, |Y|^{-1}\right\} \leq \min\{\mathbb{E}|X|^{-1}, \mathbb{E}|Y|^{-1}\}.$$

In particular,

$$\mathrm{vol}_{n-1}(Q_n \cap a^{\perp}) \leq \min\{|a_j|^{-1}\}.$$

Proof. Since X and Y are rotationally invariant, their distributions can be written as $|X|\xi_1$ and $|Y|\xi_2$, where ξ_1, ξ_2 are uniform on S^2. By conditioning it suffices to verify the identity

$\mathbb{E}_{\xi_1,\xi_2}|r\xi_1 + s\xi_2|^{-1} = \min(r,s)^{-1}$. Note that by rotation invariance $\langle \xi_1, \xi_2\rangle$ has the same distribution as $\langle \xi_1, e_1\rangle$, that is a uniform distribution on $[-1,1]$. Therefore

$$\begin{aligned}\mathbb{E}_{\xi_1,\xi_2}|r\xi_1 + s\xi_2|^{-1} &= \mathbb{E}_{\xi_1,\xi_2}(|r\xi_1 + s\xi_2|^2)^{-1/2} = \frac{1}{2}\int_{-1}^{1}(r^2+s^2+2rsu)^{-1/2}\mathrm{d}u \\ &= \frac{(r^2+s^2+2rsu)^{1/2}}{2rs}\Big|_{-1}^{1} = \frac{|r+s|-|r-s|}{2rs} = \frac{\min\{r,s\}}{rs} = \min\{r^{-1},s^{-1}\}^{-1}.\end{aligned}$$

To prove the second part it suffices to take $X = \sum_{j=1}^{n-1} a_j\xi_j$, $Y = a_n\xi_n$ and use the inequality $\mathbb{E}|X+Y|^{-1} \leq \mathbb{E}|Y|^{-1}$. □

Since the maximal section has volume $\sqrt{2}$, that is $\mathrm{vol}_{n-1}(Q_n \cap (\frac{e_1+e_2}{\sqrt{2}})^\perp) = \sqrt{2}$, our stability result (4) for maximal sections of the cube can be equivalently stated as follows

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \leq \sqrt{2} - c_0\left|a - \frac{e_1+e_2}{\sqrt{2}}\right|, \tag{12}$$

for every n and every unit vector a in $\mathbb{R}^n$ with $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$, for some universal constant c_0.

The proof involves different arguments, depending on whether a is *close* to the extremiser or not and whether its largest coordinate is *large* or not. We assume throughout that a is a unit vector in $\mathbb{R}^n$ with $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ and set

$$\delta(a) = \left|a - \frac{e_1+e_2}{\sqrt{2}}\right|^2 = 2 - \sqrt{2}(a_1+a_2).$$

For vectors a *close* to the extremiser, we have the following *local* stability result (it is to some extent in the spirit of Lemma 3.7 from [16]).

Lemma 25. *There are universal constants $\delta_0 \in (0, \frac{1}{\sqrt{2}})$ and $c_0 > 0$ such that* (12) *holds for every a with $\delta(a) \leq \delta_0$.*

For vectors a *away* from the extremiser with largest coordinate sufficiently close to $\frac{1}{\sqrt{2}}$, we prove the following lemma.

Lemma 26. *Let δ_0 be the constant from Lemma 25. There are positive universal constants γ_0, c_1 such that*

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \leq \sqrt{2} - c_1 \tag{13}$$

holds for every a with $\delta(a) > \delta_0$ and $a_1 \leq \frac{1}{\sqrt{2}} + \gamma_0$.

The remaining case is straightforward: taking these two lemmas for granted, it is very easy to prove (12).

Proof of (12). In view of Lemmas 25 and 26, it remains to consider the case when $a_1 > \frac{1}{\sqrt{2}} + \gamma_0$. From Lemma 24, we have

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \leq \frac{1}{a_1} < \frac{1}{1/\sqrt{2}+\gamma_0} < \sqrt{2} - \gamma_0 < \sqrt{2} - \frac{\gamma_0}{\sqrt{2}}\sqrt{\delta(a)},$$

because $\delta(a) < 2$, so in this case (12) also holds. $\square$

It remains to prove the lemmas.

Proof of Lemma 25. The idea is to argue that Ball's inequality $\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \sqrt{2}$ allows for a self-improvement near the extremiser. We shall assume that $n \ge 3$ and $a_1^2 + a_2^2 < 1$ (the case $n = 2$ can be analysed directly). A starting point is formula (9), combined with Lemma 24,

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) = \mathbb{E}_{X,Y} \min\left\{|X|^{-1}, |Y|^{-1}\right\},$$

where we apply it to $X = a_1\xi_1 + a_2\xi_2$ and $Y = \sum_{j=3}^n a_j\xi_j$. By Ball's inequality,

$$\mathbb{E}_Y|Y|^{-1} \le \sqrt{2}(1 - a_1^2 - a_2^2)^{-1/2}.$$

Thus, thanks to the independence of X and Y and the simple inequality

$$\mathbb{E}_Y \min\left\{|X|^{-1}, |Y|^{-1}\right\} \le \min\left\{|X|^{-1}, \mathbb{E}_Y|Y|^{-1}\right\},$$

we obtain

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \mathbb{E}_X \min\left\{|X|^{-1}, \sqrt{2}(1 - a_1^2 - a_2^2)^{-1/2}\right\}$$

Note that $|X|$ has the same distribution as $(a_1^2+a_2^2+2a_1a_2U)^{1/2}$, where U is a random variable uniform on $[-1, 1]$. To evaluate $\mathbb{E}_X$, observe that $|X|^{-1} < \sqrt{2}(1 - a_1^2 - a_2^2)^{-1/2}$ corresponds to $U > u_0$, where

$$u_0 = \frac{1 - 3(a_1^2 + a_2^2)}{4a_1a_2},$$

We need to consider two cases. Let $\delta = \delta(a)/2$, that is

$$a_1 + a_2 = \sqrt{2}(1 - \delta).$$

Case 1: $u_0 \le -1$. Then

$$\mathbb{E}_X \min\left\{|X|^{-1}, \sqrt{2}(1 - a_1^2 - a_2^2)^{-1/2}\right\} = \mathbb{E}|X|^{-1} = \min(a_1, a_2)^{-1} = a_1^{-1}.$$

Given $a_1 + a_2 = \sqrt{2}(1 - \delta)$, the condition $u_0 \le -1$ implies that $a_1 \ge \bar{a}_1$, where $\bar{a}_1$ is the larger of the two solutions to the quadratic equation

$$1 - 3(a_1^2 + (\sqrt{2}(1 - \delta) - a_1)^2) = -4a_1(\sqrt{2}(1 - \delta) - a_1).$$

This yields

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \frac{1}{\bar{a}_1} = \sqrt{2}\left(1 - \delta + \sqrt{\frac{\delta}{5}}\sqrt{2 - \delta}\right)^{-1} \le \sqrt{2} - c_0\sqrt{\delta}$$

for a universal constant $c_0 > 0$, provided that δ is sufficiently small.

Case 2: $u_0 > -1$. It is clear that for all δ sufficiently small, $u_0 < 1$ (in fact since $a_1 + a_2 \le \sqrt{2}(a_1^2 + a_2^2) \le \sqrt{2}$, the equality $a_1 + a_2 = \sqrt{2}(1 - \delta)$ for small δ implies that both numbers

a_1, a_2 are close to $\frac{1}{\sqrt{2}}$ and thus u_0 is close to -1). Then

$$
\begin{aligned}
&\mathbb{E}_X \min\left\{|X|^{-1}, \sqrt{2}(1-a_1^2-a_2^2)^{-1/2}\right\} \\
&\quad = \frac{1}{2}(u_0+1)\sqrt{2}(1-a_1^2-a_2^2)^{-1/2} + \frac{1}{2}\int_{u_0}^{1}(a_1^2+a_2^2+2a_1a_2u)^{-1/2}\mathrm{d}u \\
&\quad = \frac{u_0+1}{\sqrt{2(1-a_1^2-a_2^2)}} + \frac{a_1+a_2-\sqrt{a_1^2+a_2^2+2a_1a_2u_0}}{2a_1a_2}.
\end{aligned}
$$

Plugging in u_0 and rewriting in terms of $s = a_1 + a_2$, $\rho = a_1^2 + a_2^2$ results with an upper bound on $\mathrm{vol}_{n-1}(Q_n \cap a^\perp)$ by

$$
h(s,\rho) = \frac{s}{s^2-\rho} + \frac{2s^2-1-3\rho}{2\sqrt{2}(s^2-\rho)\sqrt{1-\rho}}.
$$

Note that $\frac{s^2}{2} \le \rho < 1$. We claim that for every $1 \le s \le \sqrt{2}$, function $\rho \mapsto h(s,\rho)$ is decreasing on $(\frac{s^2}{2}, 1)$. Thus,

$$
\begin{aligned}
\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le h(s, s^2/2) &= \frac{2}{s} - \frac{\sqrt{1-s^2/2}}{\sqrt{2}s^2} \\
&= \sqrt{2}(1-\delta)^{-2}\left(1-\delta-\frac{\sqrt{\delta}}{2\sqrt{2}}\sqrt{2-\delta}\right) \\
&< \sqrt{2} - c_0\sqrt{\delta}
\end{aligned}
$$

for a universal constant $c_0 > 0$ and all sufficiently small δ.

To prove that $\rho \mapsto h(s,\rho)$ is decreasing on $(\frac{s^2}{2}, 1)$, we fix $1 \le s \le \sqrt{2}$ and compute the derivative

$$
\frac{\partial h}{\partial \rho} = -\frac{2\sqrt{2} - 3\sqrt{2}\rho(1+\rho) - 8(1-\rho)^{3/2}s + 3\sqrt{2}s^2(1+\rho) - 2\sqrt{2}s^4}{8(1-\rho)^{3/2}(s^2-\rho)^2}.
$$

Note that the numerator

$$
\tilde{h}(s,\rho) = 2\sqrt{2} - 3\sqrt{2}\rho(1+\rho) - 8(1-\rho)^{3/2}s + 3\sqrt{2}s^2(1+\rho) - 2\sqrt{2}s^4
$$

is a concave function of $\rho \in (s^2/2, 1)$, as a sum of concave functions. It suffices to show that the values at the endpoints are non-negative. At $\rho = 1$, we have

$$
\tilde{h}(s,1) = -2\sqrt{2}(s^4 - 3s^2 + 2) = 2\sqrt{2}(s^2-1)(2-s^2) \ge 0.
$$

At $\rho = s^2/2$, we get

$$
\tilde{h}(s, s^2/2) = \frac{2-s^2}{2\sqrt{2}}\left(5s^2 + 4 - 8s\sqrt{2-s^2}\right) \ge \frac{2-s^2}{2\sqrt{2}}\left(5s^2 - 4\right) \ge 0,
$$

by $2s\sqrt{2-s^2} \le s^2 + (2-s^2) = 2$. $\square$

Proof of Lemma 26. Assume that $\delta(a) > \delta_0$. In particular,

$$
a_2 \le \frac{1}{2}(a_1+a_2) = \frac{2-\delta(a)}{2\sqrt{2}} < \frac{1}{\sqrt{2}} - \frac{\delta_0}{2\sqrt{2}}. \tag{14}
$$

The argument is now split into two cases: when $a_1 \le \frac{1}{\sqrt{2}}$, we employ (10) and use Ball's approach to show that savings simply come from a_2 being small, whilst when $a_1 > \frac{1}{\sqrt{2}}$, provided a_1 is *close* to $\frac{1}{\sqrt{2}}$, we employ Busemann's theorem to reduce this case to the previous one.

Case 1: $a_1 \le \frac{1}{\sqrt{2}}$. For $s \ge 2$, we define

$$\Psi(s) = \frac{2}{\pi}\sqrt{s}\int_0^\infty \left|\frac{\sin t}{t}\right|^s \mathrm{d}t.$$

To establish his cube-slicing result, Ball showed in [3] that

$$\Psi(s) < \Psi(2) = \sqrt{2}, \qquad s > 2.$$

Moreover, since $\frac{\sin(t\sqrt{s})}{t/\sqrt{s}} = 1 - \frac{t^2}{6s} + O(s^{-2})$ as $s \to \infty$,

$$\lim_{s\to\infty} \Psi(s) = \sqrt{\frac{6}{\pi}} < \sqrt{2}.$$

In particular, by continuity, for every $s_0 > 2$, there is $0 < \theta_0 < 1$ such that

$$\Psi(s) \le \theta_0\sqrt{2}, \qquad s \ge s_0. \tag{15}$$

As in [3], applying Hölder's inequality in (10) yields

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \prod_{j=1}^n \Psi(a_j^{-2})^{a_j^2}.$$

Letting $s_0 = 2\,(1-\delta_0/2)^{-2}$, from (14), we know that $a_j^{-2} \ge s_0$ for each $j \ge 2$, thus (15) applied to each $j \ge 2$ and $\Psi(a_1^{-2}) \le \sqrt{2}$ give

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \theta_0^{1-a_1^2}\sqrt{2} \le \theta_0^{1/2}\sqrt{2} = \sqrt{2} - c_1.$$

Case 2: $\frac{1}{\sqrt{2}} < a_1$. We argue that there are positive universal constants γ_0, c_2 such that if additionally $a_1 < \frac{1}{\sqrt{2}} + \gamma_0$, then $\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \sqrt{2} - c_2$. To this end, we modify a and consider the unit vector

$$b = \left(\frac{1}{\sqrt{2}}, \sqrt{a_1^2 + a_2^2 - \frac{1}{2}}, a_3, \dots, a_n\right).$$

Note that $b_1 \ge b_2$ and since $b_2 \ge a_2$, also $b_2 \ge b_3 \ge \dots \ge b_n$. Moreover, crudely,

$$\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2 = \frac{a_1^2 - \frac{1}{2}}{\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} + a_2} \le \sqrt{a_1^2 - \frac{1}{2}} \le \sqrt{2\gamma_0},$$

thus

$$|a-b|^2 = \left(a_1 - \frac{1}{\sqrt{2}}\right)^2 + \left(\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2\right)^2 < \gamma_0^2 + 2\gamma_0.$$

Lemma 23 yields

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \mathrm{vol}_{n-1}(Q_n \cap b^\perp) + 2\sqrt{\gamma_0^2 + 2\gamma_0}.$$

If $\delta(b) > \delta_0$, then Case 1 applied to b gives

$$\mathrm{vol}_{n-1}(Q_n \cap b^\perp) < \sqrt{2} - c_1.$$

Otherwise, observing that

$$\begin{aligned}\delta(b) &= \delta(a) - \sqrt{2}\left(\frac{1}{\sqrt{2}} + \sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_1 - a_2\right)\\ &> \delta_0 - \sqrt{2}\left(\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2\right)\\ &> \delta_0 - 2\sqrt{\gamma_0},\end{aligned}$$

Lemma 25 applied to b gives

$$\mathrm{vol}_{n-1}(Q_n \cap b^\perp) < \sqrt{2} - c_0\sqrt{\delta_0 - 2\sqrt{\gamma_0}}.$$

In any case, choosing γ_0 sufficiently small (depending on the values of c_0, c_1, δ_0), we can ensure that

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \leq \sqrt{2} - c_2$$

with a positive universal constant c_2. □

Remark 27. The dependence on $\delta(a)$ in (12) (modulo the universal constant c_0) is best possible: if we consider $a_\varepsilon = \left(\sqrt{\frac{1}{2}+\varepsilon}, \sqrt{\frac{1}{2}-\varepsilon}, 0, \dots, 0\right)$ with $\varepsilon \to 0$, then $\delta(a) = \varepsilon^2 + O(\varepsilon^4)$ and $\mathrm{vol}_{n-1}(Q_n \cap a^\perp) = a_1^{-1} = \sqrt{2} - \sqrt{2\delta(a)} + o(\sqrt{\delta(a)})$.

5. Hyperplane sections of B_p^n, $0 < p < \infty$

5.1. **Case** $0 < p < 2$.

As remarked in [18], formula (7) immediately yields the Schur-convexity of the function

$$(b_1, \dots, b_n) \mapsto \mathrm{vol}_{n-1}(B_p^n \cap (\sqrt{b_1}, \dots, \sqrt{b_n})^\perp)$$

on $\mathbb{R}_+^n$, in particular asserting that the subspaces of minimal and maximal volume cross-section are $(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^\perp$ and $(1, 0, \dots, 0)$. Moreover, the formula allows to obtain stability results for these extremisers, which has not been observed before.

5.1.1. *Case $0 < p < 2$: maximal sections.* Thanks to Schur-convexity the case of maximal sections is straightforward.

Proof of (1). By (7) and Schur-convexity,

$$\begin{aligned}\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^{n-1})} &= \mathbb{E}\left(\sum_{j=1}^n a_j^2 \bar{V}_j\right)^{-1/2} \leq \mathbb{E}\left(a_1^2\bar{V}_1 + (1-a_1^2)\bar{V}_2\right)^{-1/2}\\ &= \frac{\mathrm{vol}_1(B_p^2 \cap (a_1, \sqrt{1-a_1^2})^\perp)}{\mathrm{vol}_1(B_p^1)},\end{aligned}$$

which is exactly the right hand side of (1). □

Remark 28. The bound is clearly optimal as it is attained in the case of vectors with at most two nonzero coordinates. Moreover, the right hand side of (1) in terms of $\delta = \delta(a) = |a - e_1|$ is asymptotic to $1 - \frac{1}{p}\delta^p$ as $\delta \to 0^+$.

5.1.2. *Case* $0 < p < 2$*: minimal sections.* Here our goal is to establish (2). We begin with a relevant stability result for negative moments. We rely on the fact that $x \mapsto x^{-q}$, $q > 0$ is completely monotone, which allows to use simple convexity properties of log-moment generating functions.

Lemma 29. *Let Y be a nonnegative random variable and $\Lambda(u) = \log \mathbb{E} e^{-uY}$, $u \geq 0$. For every nonnegative real numbers $b_1, \dots, b_n$ with $B = \sum_{j=1}^n b_j$, we have*

$$\sum_{j=1}^n \Lambda(b_j) \geq n\Lambda(B/n) + c\sum_{j=1}^n (b_j - B/n)^2, \tag{16}$$

where

$$c = \frac{1}{4} \sup_{0<\alpha<\beta<\gamma} e^{-L(\alpha+\gamma)}(\beta-\alpha)^2 \mathbb{P}\left(Y < \alpha\right)\mathbb{P}\left(\beta < Y < \gamma\right)$$

with $L = \max_{j \leq n} b_j$.

Proof. By Taylor's theorem with Lagrange's reminder,

$$\Lambda(b_j) = \Lambda(B/n) + (b_j - B/n)\Lambda'(B/n) + \frac{1}{2}(b_j - B/n)^2\Lambda''(\theta_j),$$

for some θ_j between b_j and B/n. Adding these inequalities over $j \leq n$ gives (16) with $c = \frac{1}{2}\inf_{(0, \max_j b_j)} \Lambda''$. Let Y_1, Y_2 be independent copies of Y. Crudely, $\mathbb{E}e^{-uY_1} \leq 1$, so for $0 < \alpha < \beta < \gamma$,

$$\begin{aligned}
\Lambda''(u) &= \frac{1}{2}\frac{1}{(\mathbb{E}e^{-uY_1})^2}\mathbb{E}(Y_2 - Y_1)^2 e^{-uY_1}e^{-uY_2} \\
&\geq \frac{1}{2}\mathbb{E}(Y_2 - Y_1)^2 e^{-uY_1}e^{-uY_2}\mathbf{1}_{\{Y_1<\alpha\}}\mathbf{1}_{\{\beta<Y_2<\gamma\}} \\
&\geq \frac{1}{2}(\beta-\alpha)^2 e^{-u(\alpha+\gamma)}\mathbb{P}\left(Y_1 < \alpha\right)\mathbb{P}\left(\beta < Y_2 < \gamma\right),
\end{aligned}$$

which proves (16). □

Theorem 30. *Let $q > 0$. Let Y be a nonnegative random variable which is not constant a.s. with $\mathbb{E}Y < \infty$. Let $Y_1, Y_2, \dots$ be its i.i.d. copies. For every $b_1, \dots, b_n \geq 0$ with $\sum_{j=1}^n b_j = 1$, we have*

$$\mathbb{E}\left(\sum_{j=1}^n b_j Y_j\right)^{-q} \geq \mathbb{E}\left(\sum_{j=1}^n \frac{1}{n}Y_j\right)^{-q} + c_{q,Y}\sum_{j=1}^n (b_j - 1/n)^2, \tag{17}$$

for some positive constant $c_{q,Y}$ which depends only on q and the distribution of Y.

Proof. Using $x^{-q} = \Gamma(q)^{-1}\int_0^\infty e^{-tx}t^{q-1}\mathrm{d}t$, $x > 0$, we have

$$\mathbb{E}\left(\sum_{j=1}^n b_j Y_j\right)^{-q} = \Gamma(q)^{-1}\int_0^\infty \exp\left(\sum_{j=1}^n \Lambda(tb_j)\right)t^{q-1}\mathrm{d}t, \tag{18}$$

where $\Lambda(u) = \log \mathbb{E}e^{-uY}$. We apply Lemma 29 to the numbers tb_j which add up to t. It is clear that under our assumptions on Y, the constant c from Lemma 29 satisfies $c \geq c_1 e^{-c_2 t}$, for some positive constants $c_1, c_2 > 0$ which depend only on the distribution of Y. Thus, from (16), we get

$$\mathbb{E}\left(\sum_{j=1}^n b_j Y_j\right)^{-q} \geq \Gamma(q)^{-1} \int_0^\infty \exp\left(n\Lambda(t/n) + c_1 e^{-c_2 t} t^2 \delta\right) t^{q-1} \mathrm{d}t$$

with $\delta = \sum_{j=1}^n (b_j - 1/n)^2$. Using $\exp\left(c_1 e^{-c_2 t} t^2 \delta\right) \geq c_1 e^{-c_2 t} t^2 \delta + 1$, we obtain

$$\mathbb{E}\left(\sum_{j=1}^n b_j Y_j\right)^{-q} \geq \mathbb{E}\left(\sum_{j=1}^n \frac{1}{n} Y_j\right)^{-q} + \delta \cdot c_1 \Gamma(q)^{-1} \int_0^\infty \exp\left(n\Lambda(t/n)\right) e^{-c_2 t} t^{q+1} \mathrm{d}t$$

By the convexity of Λ, the sequence $(n\Lambda(t/n))_n$ is nonincreasing with the limit $-t\mathbb{E}Y$, hence

$$\int_0^\infty \exp\left(n\Lambda(t/n)\right) e^{-c_2 t} t^{q+1} \mathrm{d}t \geq \int_0^\infty e^{-(c_2 + \mathbb{E}Y)t} t^{q+1} \mathrm{d}t,$$

which gives (17).

$\square$

We are ready to establish the desired stability results for minimal sections.

Proof of (2). Let

$$A_{n,p} = \mathbb{E}\left(\sum_{j=1}^n \frac{1}{n} \bar{V}_j\right)^{-1/2}.$$

From (7) and (17) applied to the $\bar{V}_j$ and $q = \frac{1}{2}$, we have

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^\perp)} = \frac{1}{A_{n,p}} \mathbb{E}\left(\sum_{j=1}^n a_j \bar{V}_j\right)^{-1/2} \tag{19}$$

$$\geq 1 + \frac{c_p}{A_{n,p}} \sum_{j=1}^n (a_j^2 - 1/n)^2$$

with a positive constant c_p which depends only on p (through the distribution of $\bar{V}_1$). It remains to note that thanks to Schur-convexity, the sequence $A_{n,p}$ is nonincreasing, thus $A_{n,p} \leq A_{1,p} = \mathbb{E}\bar{V}_1^{-1/2} = 1$. $\square$

Remark 31. The sequence $A_{n,p}$ is in fact bounded below as well, namely by

$$\lim_{n\to\infty} A_{n,p} \geq \mathbb{E}\left[\lim_{n\to\infty} \left(\sum_{j=1}^n \frac{1}{n} \bar{V}_j\right)^{-1/2}\right] = \left(\mathbb{E}\bar{V}_1\right)^{-1/2}.$$

Moreover, as $n \to \infty$, we have

$$A_{n,p} = c_0(p) + \frac{c_1(p)}{n} + O(n^{-3/2}) \tag{20}$$

for some constants $c_0(p), c_1(p)$ which depend only on p. This is justified by first noting that $A_{n,p} = g_n(0)$, where here $g_n(x)$ is the density of $\frac{1}{\sqrt{n}}\sum_{j=1}^n Y_j$ (plug in $a = e_1$ in (19) and recall Corollary 13) and then evoking the Edgeworth expansion for g_n (see, e.g. Theorem 3.2 in [11] and classical references therein).

Remark 32. The dependence on $\delta_n(a) = \sum_{j=1}^n (a_j^2 - 1/n)^2$ in (2) modulo a constant factor is best possible, in the following two scenarios.

1) As $n \to \infty$, there are unit vectors a in $\mathbb{R}^n$ with $\delta_n = \delta_n(a) \to 0$ such that the left hand side of (2) is in fact of the order $1 + c(p)\cdot\delta_n + o(\delta_n)$. Consider $a = (\frac{1}{\sqrt{n-1}}, \dots, \frac{1}{\sqrt{n-1}}, 0)$ in $\mathbb{R}^n$. Then $\delta_n = \delta_n(a) = (n-1)\left(\frac{1}{n-1} - \frac{1}{n}\right)^2 + \frac{1}{n^2} = \frac{1}{n^2} + O\left(\frac{1}{n^3}\right)$ and, using (20),

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap (\frac{1}{\sqrt{n-1}}, \dots, \frac{1}{\sqrt{n-1}}, 0)^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^\perp)} = \frac{A_{n-1,p}}{A_{n,p}} = 1 + \frac{c(p)}{n^2} + O\left(\frac{1}{n^{5/2}}\right).$$

2) For a fixed n, there are unit vectors a in $\mathbb{R}^n$ with $\delta = \delta_n(a) \to 0$ such that the left hand side of (2) is of the order $1 + c(p,n)\delta + o(\delta)$. For simplicity, let n be a fixed even integer. Let $\varepsilon \to 0^+$ and consider

$$a_\varepsilon = (\underbrace{\sqrt{\frac{1}{n} + \varepsilon}, \dots, \sqrt{\frac{1}{n} + \varepsilon}}_{n/2}, \underbrace{\sqrt{\frac{1}{n} - \varepsilon}, \dots, \sqrt{\frac{1}{n} - \varepsilon}}_{n/2}).$$

Then $\delta_\varepsilon = \delta_n(a_\varepsilon) = n\varepsilon^2$ and with

$$X = \bar{V}_1 + \dots + \bar{V}_{n/2}, \quad Y = \bar{V}_{n/2+1} + \dots + \bar{V}_n,$$

which are i.i.d., we have

$$\begin{aligned}\frac{\mathrm{vol}_{n-1}(B_p^n \cap a_\varepsilon^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^\perp)} &= \frac{1}{A_{n,p}}\mathbb{E}\left(\frac{X+Y}{n} + \varepsilon(X-Y)\right)^{-1/2} \\ &= \frac{1}{A_{n,p}}\mathbb{E}\left[\left(\frac{X+Y}{n}\right)^{-1/2}\left(1 + \varepsilon n\frac{X-Y}{X+Y}\right)^{-1/2}\right].\end{aligned}$$

Since $\left|\varepsilon n \frac{X-Y}{X+Y}\right| \le \varepsilon n < \frac{1}{2}$, for sufficiently small ε, using $(1+x)^{-1/2} \le 1 - \frac{1}{2}x + x^2$, $x > -\frac{1}{2}$, we can thus upper bound the right hand side by

$$\frac{1}{A_{n,p}}\mathbb{E}\left[\left(\frac{X+Y}{n}\right)^{-1/2}\left(1 - \frac{1}{2}\varepsilon n\frac{X-Y}{X+Y} + \varepsilon^2 n^2\left(\frac{X-Y}{X+Y}\right)^2\right)\right] = 1 + c(p,n)\varepsilon^2,$$

where we use that $\left|\frac{X-Y}{X+Y}\right| \le 1$ to guarantee the existence of the expectations involved and symmetry to conclude that term linear in ε vanishes.

5.2. **Case $2 < p < \infty$.** Here we prove (3). We use the formula from Corollary 13, that for a unit vector $a \in \mathbb{R}^n$, we have

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^{n-1})} = f_a(0),$$

where f_a is the density of $\sum_{j=1}^n a_j Y_j$, $Y_1, Y_2, \ldots$ are i.i.d. random variables, each with density $\exp(-\beta_p^p |x|^p)$, where $\beta_p = 2\Gamma(1+1/p)$.

Lemma 33. *Let $2 < p < \infty$. For every $u_0 > 0$, there is $c > 0$ depending only on u_0 and p such that for every $0 < u < u_0$, we have*

$$(1+u)^{1/2} \int_{\mathbb{R}} \exp\left\{-\beta_p^p u^{p/2} |x|^p - \pi x^2\right\} \mathrm{d}x \geq 1 + cu. \tag{21}$$

Proof. Fix $2 < p < \infty$ and $u_0 > 0$. Using $\exp(-t) \geq 1 - t$, we obtain

$$\int_{\mathbb{R}} \exp\left\{-\beta_p^p u^{p/2} |x|^p - \pi x^2\right\} \mathrm{d}x \geq 1 - A_p u^{p/2}$$

with $A_p = \beta_p^p \int_{\mathbb{R}} |x|^p e^{-\pi x^2} \mathrm{d}x$. Thus it is clearly possible to choose sufficiently small $u_1 > 0$ and $c > 0$ which depend only on p such that (21) holds for all $0 < u < u_1$. Moreover, a change of variables $x = u^{-1/2} y$ yields

$$\int_{\mathbb{R}} \exp\left\{-\beta_p^p u^{p/2} |x|^p - \pi x^2\right\} \mathrm{d}x = u^{-1/2} \mathbb{E} \exp\left\{-\pi u^{-1} Y^2\right\},$$

where Y is a random variable with density $\exp(-\beta_p^p |x|^p)$ which is *more* peaked than a Gaussian random variable G with density $\exp(-\pi x^2)$. Thus, for every $u > 0$,

$$\int_{\mathbb{R}} \exp\left\{-\beta_p^p u^{p/2} |x|^p - \pi x^2\right\} \mathrm{d}x > u^{-1/2} \mathbb{E} \exp\left\{-\pi u^{-1} G^2\right\} = (1+u)^{-1/2}.$$

Thus, by continuity, the infimum of left hand side of (21) over $u_1 < u < u_0$ is strictly larger than 1. Decreasing c if necessary allows to finish the argument. □

Proof of (3). We use different arguments, depending on whether the vector a is close or not to the minimising one e_1. With hindsight, fix θ_p to be a positive sufficiently small constant which depends only on p such that

$$\begin{aligned} &(2\pi \mathbb{E} Y_1^2)^{-1/2} \exp(-0.28\theta_p (\mathbb{E}|Y_1|^3)(\mathbb{E} Y_1^2)^{-5/2}) \\ &\qquad\qquad - (0.56\theta_p (\mathbb{E}|Y_1|^3)(\mathbb{E} Y_1^2)^{-3/2})^{1/2} > 1. \end{aligned} \tag{22}$$

Such a choice is possible since $2\pi \mathbb{E} Y_1^2 < 1$ for $p > 2$, as explained later in the proof.

Case 1: $a_1 > \theta_p$. Here the starting point is a formula obtained from writing $f_a(0)$ as the convolution of the densities $\frac{1}{a_j} \exp(-\beta_p^p |x_j/a_j|^p)$ and changing the variables $y_j = x_j/a_j$, leading to

$$f_a(0) = \frac{1}{a_1} \mathbb{E} \exp\left\{-\beta_p^p \left|\sum_{j=2}^n b_j Y_j\right|^p\right\},$$

with $b_j = \frac{a_j}{a_1}$. Let

$$u = \sum_{j=2}^n b_j^2 = \frac{1 - a_1^2}{a_1^2}.$$

Note that our assumption $a_1 \geq \theta_p$ is equivalent to $u \leq \theta_p^{-2} - 1$. Since Y_j is *more* peaked than a Gaussian with density $\exp(-\pi x^2)$, we get

$$\mathbb{E} \exp\left\{-\beta_p^p \left|\sum_{j=2}^n b_j Y_j\right|^p\right\} \geq \int_{\mathbb{R}} \exp\left\{-\beta_p^p \left(\sum_{j=2}^n b_j^2\right)^{p/2} |x|^p - \pi x^2\right\} \mathrm{d}x.$$

Note that $\frac{1}{a_1} = \sqrt{1+u}$. Lemma 33 applied with $u_0 = \theta_p^{-2} - 1$ thus yields

$$f_a(0) \geq 1 + c_p u = 1 + c_p \frac{1 - a_1^2}{a_1^2} \geq 1 + c_p(1 - a_1).$$

with a positive constant c_p which depends only on p.

Case 2: $a_1 \leq \theta_p$. Since in this case

$$\rho = \sum_{j=1}^n \mathbb{E}|a_j Y_j|^3 \leq a_1 \mathbb{E}|Y_1|^3 \sum_{j=1}^n a_j^2 \leq \theta_p \mathbb{E}|Y_1|^3,$$

we can use the Berry-Esseen theorem to argue that $f_a(0)$ is *large*. Let

$$\sigma_p = (\mathbb{E}Y_1^2)^{1/2}.$$

We have (see, e.g. [45] which provides the current best value of the numerical constant in the Berry-Esseen theorem),

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\sum_{j=1}^n a_j Y_j \leq x\right) - \mathbb{P}\left(Z_p \leq x\right)\right| \leq 0.56 \sigma_p^{-3} \rho,$$

where Z_p is a Gaussian random variable with variance σ_p. Let ϕ_p denote the density of Z_p. Crucially, peakedness yields

$$\phi_p(0) = \frac{1}{\sqrt{2\pi}\sigma_p} > \frac{1}{\sqrt{2\pi}\sigma_2} = 1,$$

since $p > 2$. Thanks to the symmetry and monotonicity of the densities involved, in particular we obtain that for every $\delta > 0$,

$$\delta f_a(0) \geq \int_0^\delta f_a(x)\mathrm{d}x \geq \int_0^\delta \phi_p(x)\mathrm{d}x - \varepsilon_p$$

with $\varepsilon_p = 0.56\theta_p \sigma_p^{-3} \mathbb{E}|Y_1|^3$. Letting, say $\delta = \varepsilon_p^{1/2}$ and using $\delta^{-1}\int_0^\delta \phi_p(x)\mathrm{d}x > \phi_p(\delta) = \phi_p(0)e^{-\delta^2/(2\sigma_p^2)}$, we see that θ_p chosen sufficiently small according to (22) guarantees that

$$f_a(0) \geq \varepsilon_p^{-1/2} \int_0^{\varepsilon_p^{1/2}} \phi_p(x)\mathrm{d}x - \varepsilon_p^{1/2} \geq \phi_p(0)e^{-\varepsilon_p/(2\sigma_p^2)} - \varepsilon_p^{1/2} = 1 + c_p$$

with a positive constant c_p which depends only on p. This gives $f_p(0) \geq 1 + c_p$, which finishes the proof. □

Remark 34. It can be seen again by taking vectors with exactly two nonzero coordinates that the dependence on $\delta(a) = |a - e_1|^2$ in (3) modulo a constant factor is best possible.

For instance, take $\varepsilon \to 0$ and consider $a_\varepsilon = (\sqrt{1-\varepsilon}, \sqrt{\varepsilon}, 0, \ldots, 0)$. Then $\delta_\varepsilon = \delta(a_\varepsilon) = 2(1-\sqrt{1-\varepsilon}) = \varepsilon + O(\varepsilon^2)$ and

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a_\varepsilon^\perp)}{\mathrm{vol}_{n-1}(B_p^{n-1})} = \Big((1-\varepsilon)^{p/2} + \varepsilon^{p/2}\Big)^{-1/p} = 1 + \frac{1}{2}\varepsilon + O(\varepsilon^{p/2}) = 1 + \frac{1}{2}\delta_\varepsilon + o(\delta_\varepsilon),$$

since $p > 2$.

6. Conclusion

Our result of Theorem 1 confirms the intuition that the (unknown) extremal subspaces for minimal-volume central sections of B_p^n, $0 < p < 2$, are conceivably *as symmetric as possible*. Note that in the case of the corresponding question for maximal-volume sections and $p > 2$, the situation is more delicate, at least for *large* p, as suggested by Ball's results (even in the hyperplane case).

It has been elusive how to extend the arguments from Section 3 to other values of p than $p = 1$, or higher dimensions k than $k = 2$. We conjecture that when $k = 2$, the minimising subspace H is the same as in Theorem 1 for all $0 < p < 2$.

Theorem 2 deals only with the case of hyperplane sections. It would be of interest to ask for corresponding stability results for lower dimensional sections. We believe that (at least some of) our methods are robust enough to yield satisfactory answers. Another challenging and intriguing question is that of a sharp dependence on p of the constants c_p in Theorem 2.

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