IMPROVED HÖLDER AND REVERSE HÖLDER INEQUALITIES FOR GAUSSIAN RANDOM VECTORS.

WEI-KUO CHEN, NIKOS DAFNIS, AND GRIGORIS PAOURIS

Abstract. We propose algebraic criteria that yield sharp Hölder types of inequalities for the product of functions of Gaussian random vectors with arbitrary covariance structure. While our lower inequality appears to be new, we prove that the upper inequality gives an equivalent formulation for the geometric Brascamp-Lieb inequality for Gaussian measures. As an application, we retrieve the Gaussian hypercontractivity as well as its reverse and we present a generalization of the sharp Young and reverse Young inequalities. From the latter, we recover several known inequalities in literatures including the Prékopa-Leindler and Barthe inequalities.

1. Introduction and main results

Let (X_1, X_2) be a centered bivariate normal random vector and f_1, f_2 be any nonnegative measurable functions on $\mathbb{R}$. What are the good upper and lower bounds for the expectation $\mathbb{E}f_1(X_1)f_2(X_2)$? Suppose that p_1 and p_2 are Hölder's conjugate exponents,

$$\frac{1}{p_1} + \frac{1}{p_2} = 1. \tag{1.1}$$

The Hölder and reverse Hölder inequalities state that regardless the covariance between X_1 and X_2, one always has

$$\mathbb{E}f_1(X_1)f_2(X_2) \le (\mathbb{E}f_1(X_1)^{p_1})^{\frac{1}{p_1}}(\mathbb{E}f_2(X_2)^{p_2})^{\frac{1}{p_2}} \tag{1.2}$$

if $p_1, p_2 \ge 1$ and

$$\mathbb{E}f_1(X_1)f_2(X_2) \ge (\mathbb{E}f_1(X_1)^{p_1})^{\frac{1}{p_1}}(\mathbb{E}f_2(X_2)^{p_2})^{\frac{1}{p_2}} \tag{1.3}$$

if $0 < p_1 < 1$ and $p_2 < 0$. In this paper, we are interested in searching improved two-sided bounds that are related to the covariance of (X_1, X_2) and can be easily used as Hölder's inequalities.

Our main result is stated as follows. Recall that a real symmetric $N \times N$ matrix A is called positive definite (semi-definite) and denoted by $A > 0$ (≥ 0) if the usual inner product $\langle Ax, x\rangle > 0$ (≥ 0) for all nonzero $x \in \mathbb{R}^N$. For two real symmetric $N \times N$ matrices A, B, we say $B > A$ if $B - A > 0$ and $B \ge A$ if $B - A \ge 0$.

Theorem 1. *Let $m, n_1, \ldots, n_m$ be positive integers and let $N = n_1 + \cdots + n_m$. Suppose that X_i is a n_i-dimensional random vector for $1 \le i \le m$ such that their joint law,*

$$\mathbf{X} := (X_1, \ldots, X_m),$$

forms a centered jointly N-dimensional Gaussian random vector with covariance matrix $T = (T_{ij})_{1\le i,j\le m}$, where T_{ij} is the covariance matrix between X_i and X_j for $1 \le i, j \le m$. Let P be the block diagonal matrix,

$$P = \operatorname{diag}(p_1T_{11}, \ldots, p_mT_{mm}).$$

For any set of nonnegative measurable functions f_i on $\mathbb{R}^{n_i}$ for $1 \le i \le m$, the following statements hold.

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(i) *If* $T \le P$, *then*

$$\mathbb{E}\prod_{i=1}^m f_i(X_i) \le \prod_{i=1}^m \Big(\mathbb{E} f_i(X_i)^{p_i}\Big)^{\frac{1}{p_i}}. \tag{1.4}$$

(ii) *If* $T \ge P$, *then*

$$\mathbb{E}\prod_{i=1}^m f_i(X_i) \ge \prod_{i=1}^m \Big(\mathbb{E} f_i(X_i)^{p_i}\Big)^{\frac{1}{p_i}}. \tag{1.5}$$

Here the right-hand sides of (1.4) and (1.5) adapt the convention that $\infty\cdot 0 = 0$ whenever such situation occurs, which will remain in force throughout the rest of the paper.

Remark 1. Suppose that $0 < \mathbb{E}f_i(X_i)^{p_i} < \infty$ for $1 \le i \le m$ and at least one of f_i's is not equal to a constant almost everywhere. Then we get strict inequalities in (1.4) if $T < P$ and in (1.5) if $T > P$. To see this, take $T > P$ for instance. This allows us to find $q_1, \dots, q_m$ with $q_1 > p_1, \dots, q_m > p_m$ such that $Q := \mathrm{diag}(q_1T_{11}, \dots, q_mT_{mm})$ satisfies $T > Q > P$. From Jensen's inequality, $(\mathbb{E}f_i(X_i)^{p_i})^{1/p_i} \le (\mathbb{E}f_i(X_i)^{q_i})^{1/q_i}$ and this inequality is strict if f_i is not a.s. a constant. So (1.5) yields

$$\prod_{i=1}^m \mathbb{E}f_i(X_i) \ge \prod_{i=1}^m (\mathbb{E}f_i(X_i)^{q_i})^{1/q_i} > \prod_{i=1}^m (\mathbb{E}f_i(X_i)^{p_i})^{1/p_i}.$$

Remark 2. If inequality (1.4) (resp. (1.5)) holds for all nonnegative $f_1, \dots, f_m$, then we get that $T \le P$ (resp. $T \ge P$). This can be seen by using the test functions $f_i(x_i) = e^{\langle \alpha_i, x_i\rangle}$ for $\alpha_i \in \mathbb{R}^{n_i}$. A direct computation gives that for $\alpha = (\alpha_1, \dots, \alpha_m)$,

$$\mathbb{E}\prod_{i=1}^m f_i(X_i) = \exp\frac{1}{2}\langle T\alpha, \alpha\rangle \quad \text{and} \quad \prod_{i=1}^m (\mathbb{E}f_i(X_i)^{p_i})^{\frac{1}{p_i}} = \exp\frac{1}{2}\langle P\alpha, \alpha\rangle. \tag{1.6}$$

Thus, if (1.4) holds for all nonnegative functions, then we get that $\langle T\alpha, \alpha\rangle \le \langle P\alpha, \alpha\rangle$ for any $\alpha \in \mathbb{R}^N$ and so $T \le P$. Similarly, if (1.5) holds true for all nonnegative functions, then $T \ge P$.

Remark 3. If $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathrm{Ker}(T - P)$ for $\alpha_i \in \mathbb{R}^{n_i}$, then equalities hold in (1.4) and (1.5) when $f_i(x_i) = e^{\langle \alpha_i, x_i\rangle}$ for $1 \le i \le m$.

Let us now illustrate how our theorem recovers Hölder and reverse Hölder inequalities. Let (X_1, X_2) be a centered non-degenerate bivariate normal random vector with covariance matrix $T = (T_{ij})_{1\le i,j\le 2}$. Suppose that $p_1, p_2 \neq 1$ satisfy Hölder's condition (1.1). Set $P = \mathrm{diag}(p_1T_{11}, p_2T_{22})$. Note that (1.1) implies

$$(p_1 - 1)(p_2 - 1) = 1 - (\, p_1 + p_2 - p_1p_2 \,) = 1. \tag{1.7}$$

Thus, for $p_1, p_2 > 1$, we have that

$$P - T = \begin{pmatrix} T_{11}(p_1 - 1) & T_{12} \\ T_{12} & T_{22}(p_2 - 1) \end{pmatrix} \ge 0$$

since using (1.7) gives $\det(P - T) = \det(T) \ge 0$. This shows that Theorem 1(i) implies Hölder inequality. Similarly, if $p_1, p_2 < 1$, (1.7) yields $T - P \ge 0$ and then Theorem 1(ii) implies the reverse Hölder inequality.

To see why Theorem 1 improves Hölder's bounds in general, assume $\det(T) > 0$ and again $p_1, p_2 \neq 1$ satisfy (1.1). Let f_1 and f_2 be any two nonnegative measurable functions such that at least one of them is not equal to a constant a.e. and $0 < (\mathbb{E}f_1(X_1)^{p_1})^{1/p_1}(\mathbb{E}f_2(X_2)^{p_2})^{1/p_2} < \infty$. First we consider the case $p_1, p_2 > 1$. Observe that for any $q_1 \in [1, p_1)$ and $q_2 \in [1, p_2)$, we have $Q - T \ge 0$ if and only if $\det(Q - T) \ge 0$, where $Q := \mathrm{diag}(q_1T_{11}, q_2T_{22})$. Write

$$\det(Q - T) = \det(T) - \varepsilon_Q T_{11}T_{22},$$

where $\varepsilon_Q := q_1 + q_2 - q_1 q_2$. Note that $\varepsilon_Q \to 0$ when $q_1 \uparrow p_1$ and $q_2 \uparrow p_2$. Since $\det(T) > 0$, there exist exponents $q_1 \in [1, p_1)$ and $q_2 \in [1, p_2)$ such that $T \leq Q < P$, which implies from Theorem 1(i) and then Jensen's inequality that

$$\mathbb{E} f_1(X_1) f_2(X_2) \leq \left(\mathbb{E} f_1(X_1)^{q_1}\right)^{\frac{1}{q_1}} \left(\mathbb{E} f_2(X_1)^{q_2}\right)^{\frac{1}{q_2}} < \left(\mathbb{E} f_1(X_1)^{p_1}\right)^{\frac{1}{p_1}} \left(\mathbb{E} f_2(X_1)^{p_2}\right)^{\frac{1}{p_2}}.$$

Similarly, if $p_1, p_2 < 1$, there exist $q_1 \in (p_1, 1]$ and $q_2 \in (p_2, 1]$ such that $T \geq Q > P$, which implies from Theorem 1(ii) and again Jensen's inequality that

$$\mathbb{E} f_1(X_1) f_2(X_2) \geq \left(\mathbb{E} f_1(X_1)^{q_1}\right)^{\frac{1}{q_1}} \left(\mathbb{E} f_2(X_1)^{q_2}\right)^{\frac{1}{q_2}} > \left(\mathbb{E} f_1(X_1)^{p_1}\right)^{\frac{1}{p_1}} \left(\mathbb{E} f_2(X_1)^{p_2}\right)^{\frac{1}{p_2}}.$$

In other words, the exponents q_1, q_2 in each case improve Hölder's bounds.

Example 1. Assume that $m = 2$ and X_1, X_2 are standard Gaussian with $\mathbb{E} X_1 X_2 = t$ for $0 \leq t \leq 1$. The simplest Hölder's types of bounds for $\mathbb{E} f_1(X_1) f_2(X_2)$ can be obtained as follows. Note that $(1-t) I_2 \leq T \leq (1+t) I_2$. Theorem 1 gives that for $q_t := 1 - t$ and $p_t := 1 + t$,

$$\left(\mathbb{E} f_1(X_1)^{q_t}\right)^{\frac{1}{q_t}} \left(\mathbb{E} f_2(X_2)^{q_t}\right)^{\frac{1}{q_t}} \leq \mathbb{E} f_1(X_1) f_2(X_2) \leq \left(\mathbb{E} f_1(X_1)^{p_t}\right)^{\frac{1}{p_t}} \left(\mathbb{E} f_2(X_2)^{p_t}\right)^{\frac{1}{p_t}} \quad (1.8)$$

for any nonnegative measurable functions f_1, f_2. In particular, if $t = 0$, then X_1, X_2 are independent and the three quantities in (1.8) are the same; if $t = 1$, the left-hand side is the Jensen inequality and the right-hand side gives the Cauchy-Schwartz inequality. To see the sharpness of (1.8), note that $(1,1) \in \operatorname{Ker}(T - (1+t) I_2)$ and $(1,-1) \in \operatorname{Ker}(T - (1-t) I_2)$. From Remark 3, $f_1(x) = f_2(x) = e^x$ give the left-hand sided equality of (1.8), while the functions $f_1(x) = e^x$ and $f_2(x) = e^{-x}$ give the equality for the other side.

Inequality (1.4) is strongly related to the famous Brascamp-Lieb inequality, firstly proved by Brascamp and Lieb in [17] and later fully generalized by Lieb in [32]. It says that if $m \geq n$, $p_1, \ldots, p_m \geq 1$ with $\sum_{i=1}^m n_i p_i^{-1} = n$ and U_i is a surjective linear map from $\mathbb{R}^n$ to $\mathbb{R}^{n_i}$ for $1 \leq i \leq m$, then for any set of nonnegative $f_i \in L_{p_i}(\mathbb{R}^{n_i})$ for $1 \leq i \leq m$, the ratio

$$\frac{\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(U_i x) dx}{\prod_{i=1}^m \|f_i\|_{p_i}} \quad (1.9)$$

is maximized by centered Gaussian functions, i.e., functions of the form

$$f_i(x_i) = \exp(-\langle A_i x_i, x_i \rangle),$$

where A_i is a $n_i \times n_i$-dimensional real symmetric and positive definite matrix. Their approach was based on a tensorization argument and the Brascamp- Lieb-Luttinger rearrangement inequality [16]. Several different proofs of this inequality have appeared later using different tools, see [3, 6, 12, 13, 29]. For more information about the Brascamp-Lieb inequalities as well as their generalizations in non-Euclidean settings, we refer to [7, 8, 19, 20, 21, 32].

Among various formulations, Ball first put forward the geometric form of the Brascamp-Lieb inequality in [1] and used it to derive sharp inequalities for convex bodies in $\mathbb{R}^n$ (see [2]). Later it was generalized by Barthe [3] and in the recent paper [7], Bennett, Carbery, Christ and Tao showed that by a clever change of variables, one can retrieve the initial Brascamp-Lieb inequality by this geometric form.

For the purpose of our discussion, we shall also consider an equivalent version of the geometric Brascamp-Lieb inequality (Theorem 4 in Section 2.2), where the underlying measures are Gaussian. We will show that Theorem 1(i) is indeed another formulation of this inequality. While the connection between the upper bound (1.4) and the Brascamp-Lieb inequality can be completely clarified, the lower bound (1.5) appears to be new to the authors as it is by no means clear which known equalities will imply (1.5).

In this paper, we will present two applications from Theorem 1 with proper chosen covariance matrices and exponents. The first is Nelson's Gaussian hypercontractivity and its reverse form. The second is the Lebesgue version of Theorem 1 that provides a generalization of the sharp Young and reverse Young inequalities, for which we now formulate.

Theorem 2. *Let $n, m \in \mathbb{N}$, $n_1, \dots, n_m \le n$ and $p_1, \dots, p_m$ be real numbers such that*

$$\sum_{i=1}^{m} \frac{n_i}{p_i} = n, \tag{1.10}$$

Assume that U_i is a $n_i \times n$ matrix with rank n_i for $1 \le i \le m$. Set $N = \sum_{i=1}^{m} n_i$. Let U be the $N \times n$ matrix with block rows $U_1, \dots, U_m$, i.e., $U^* = (U_1^*, \dots, U_m^*)$. Let B be a $n \times n$ real symmetric and positive definite matrix. Set

$$P = \operatorname{diag}\left(p_1 I_{n_1}, \dots, p_m I_{n_m}\right),$$
$$D_{UBU^*} = \operatorname{diag}\left(U_1 B U_1^*, \dots, U_m B U_m^*\right).$$

For nonnegative $f_i \in L_{p_i}(\mathbb{R}^{n_i})$ for $i \le m$, the following statements hold.

(i) *If*

$$UBU^* \le P D_{UBU^*}, \tag{1.11}$$

then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(U_i\, x)\, dx \le \left(\frac{\det(B)}{\prod_{i=1}^{m} \det(U_i B U_i^*)^{\frac{1}{p_i}}} \right)^{\frac{1}{2}} \prod_{i=1}^{m} \|f_i\|_{p_i}. \tag{1.12}$$

The equality holds if $f_i(x_i) = \exp\left(-p_i^{-1} \left\langle (U_i B U_i^*)^{-1} x_i, x_i \right\rangle\right)$ for $i \le m$.

(ii) *If*

$$UBU^* \ge P D_{UBU^*}, \tag{1.13}$$

then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(U_i\, x)\, dx \ge \left(\frac{\det(B)}{\prod_{i=1}^{m} \det(U_i B U_i^*)^{\frac{1}{p_i}}} \right)^{\frac{1}{2}} \prod_{i=1}^{m} \|f_i\|_{p_i}. \tag{1.14}$$

As we will see in Sections 5 and 6, (1.12) is indeed the Brascamp-Lieb inequality, only now the original geometric condition in Ball's geometric Brascamp-Lieb inequality is equivalently replaced by the algebraic inequality (1.11). However, once again the authors do not know which known inequalities in the literatures are equivalent to (1.14). Neither do they know the form of the f_i's that yields the equality in (1.14) in general.

There are several consequences that can be drawn from Theorem 2. They all start from a generalization of Barthe's lemma. In [4], Barthe used measure transportation techniques to give a simple proof of the sharp Young and reverse Young inequalities. Later in [5], he generalized the argument and derived a reverse form of the Brascamp-Lieb inequality (1.9), known as Barthe's inequality. The core of [4, 5] was played by a moment inequality stated in Lemma 1 [4], which we call Barthe's lemma. Using Theorem 2, we will establish a generalization of his lemma (see Theorem 5 in Section 5) that yields a two-sided moment inequality. Incidentally, similar results are discovered independently in a recent work of Barthe and Wolff [9] using again measure transportation methods.

The power of our result could be borne upon the fact that it indeed implies several inequalities as:

(I) The Prékopa-Leindler inequality
(II) The sharp Young and reverse Young inequalities.
(III) The Brascamp-Lieb and Barthe inequalities.
(IV) An entropy inequality (see Subsection 6.3 below).

For a comprehensive overview about the connections among these and other known inequalities in literatures, we refer to the survey paper of Gardner [24].

The rest of the paper is organized as follows. Section 2 is devoted to proving Theorem 1, where two different proofs are presented. The first is based on the Gaussian integration by parts formula combining with an iteration argument and the second uses the Ornstein-Uhlenbeck semi-group techniques. In Section 3, we investigate the connection between Theorem 1(i) and the Brascamp-Lieb inequality, we study the geometry of the eligible exponents in Theorem 1(i) and we prove Nelson's hypercontractivity and its reverse. In Section 4, we prove Theorem 2 and explain how it generalizes the sharp Young and reverse Young inequalities. In Section 5, we prove the generalized Barthe's lemma. Finally, its applications are given in Section 6, where we deduce inequalities (I), (II), (III) and (IV).

2. Proofs of Theorem 1

In this section, we will present two fundamental proofs for Theorem 1. Let $m, n_1, \ldots, n_m$ be positive integers and let $N = n_1 + \cdots + n_m$. Recall $\mathbf{X}, X_1, \ldots, X_m$ and the matrices T, P from the statement of Theorem 1. We denote $X_i = (X_{i1}, \ldots, X_{in_i})$. An application of change of variables suggests that in both proofs, we may assume without loss of generality, $T_{11} = I_{n_1}, \ldots, T_{mm} = I_{n_m}$. In other words, each X_i is a n_i-dimensional standard Gaussian random vector. For notational convenience, we use γ_k to denote the k-dimensional standard Gaussian measure on $\mathbb{R}^k$ for $k \geq 1$.

2.1. First proof: the Gaussian integration by parts.

We begin with the formulation of the Gaussian integration by parts formula. Let $Y, Z_1, \ldots, Z_N$ be centered jointly Gaussian random variables. For a real-valued function F defined on $\mathbb{R}^N$ with uniformly bounded first partial derivatives, this formula reads

$$\mathbb{E} Y F(Z_1, \ldots, Z_N) = \sum_{i=1}^N \mathbb{E} Y Z_i \cdot \mathbb{E} \frac{\partial F}{\partial x_i}(Z_1, \ldots, Z_N).$$

This formula has been playing a fundamental role in understanding the behavior of the highly correlated Gaussian random variables arising from modeling various scientific phenomenon, such as the mean field spin glass models [35]. The argument that we are about to present below has already been applied to quantify the error estimate in similar inequalities. We refer to [22] along this direction. Set $A = T - P$. We now state a lemma that is the real ingredient of the matter.

Lemma 1. *Let $L_1, \ldots, L_m$ be real-valued functions defined respectively on $\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m}$ and their first four partial derivatives be uniformly bounded. Define for $u \in [0,1]$,*

$$\phi(u) = \log \mathbb{E} \exp \sum_{i=1}^m L_i(\sqrt{u} X_i)$$

and

$$\phi_i(u) = \frac{1}{p_i} \log \mathbb{E} \exp p_i L_i(\sqrt{u} X_i)$$

for $1 \leq i \leq m$, where ϕ_i should be read as $\mathbb{E} L_i(\sqrt{u} X_i)$ when $p_i = 0$. If $A \leq 0$, then

$$\phi(u) \leq \sum_{i=1}^m \phi_i(u) + K u^2; \tag{2.1}$$

if $A \geq 0$, then

$$\phi(u) \geq \sum_{i=1}^m \phi_i(u) - K u^2. \tag{2.2}$$

Here K is some positive constant depending only on the supremum norms of the first four partial derivatives of $L_1, \dots, L_m$.

Proof. For clarity, we adapt the notation $x_i = (x_{i1}, \dots, x_{in_i}) \in \mathbb{R}^{n_i}$ and $\partial_{x_{ij}} L_i(x_i)$ standards for the partial derivative of U_i with respect to x_{ij}. Using the relation $\mathbb{E}X_{ij}X_{ij'} = \delta_{j,j'}$ for all $1 \le i \le m$ and every $1 \le j, j' \le n_i$, a direct computation using Gaussian integration by parts and $T_{ii} = I_{n_i}$ yields

$$
\begin{aligned}
&\phi'(u) \\
&= \frac{1}{2\sqrt{u}} \frac{\mathbb{E}\sum_{i=1}^m \sum_{j=1}^{n_i} X_{ij}\partial_{x_{ij}} L_i(\sqrt{u}X_i) \exp \sum_{k=1}^m L_k(\sqrt{u}X_k)}{\exp \phi(u)} \\
&= \frac{1}{2\exp\phi(u)} \Bigg(\sum_{i=1}^m \sum_{j=1}^{n_i} \mathbb{E}\partial^2_{x_{ij}} L_i(\sqrt{u}X_i) \exp \sum_{k=1}^m L_k(\sqrt{u}X_k) \ + \\
&\sum_{i,i'=1}^m \sum_{j=1}^{n_i} \sum_{j'=1}^{n_{i'}} \mathbb{E}X_{ij}X_{i'j'} \mathbb{E}\partial_{x_{ij}} L_i(\sqrt{u}X_i) \partial_{x_{i'j'}} L_{i'}(\sqrt{u}X_i') \exp \sum_{k=1}^m L_k(\sqrt{u}X_k) \Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_i'(u) &= \frac{1}{2\sqrt{u}} \frac{\mathbb{E}\sum_{j=1}^{n_i} X_{ij}\partial_{x_{ij}} L_i(\sqrt{u}X_i) \exp p_i L_i(\sqrt{u}X_i)}{\exp p_i \phi_i(u)} \\
&= \frac{1}{2} \sum_{j=1}^{n_i} \frac{\mathbb{E}(\partial^2_{x_{ij}} L_i(\sqrt{u}X_i) + p_i(\partial_{x_{ij}} L_i(\sqrt{u}X_i))^2) \exp p_i L_i(\sqrt{u}X_i)}{\exp p_i \phi_i(u)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\phi'(0) - \sum_{i=1}^n \phi_i'(0) &= \frac{1}{2} \sum_{i,i'=1}^m \sum_{j=1}^{n_i} \sum_{j'=1}^{n_{i'}} \mathbb{E}X_{ij}X_{i'j'} \partial_{x_{ij}} L_i(0) \partial_{x_{i'j'}} L_{i'}(0) \\
&\quad - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^{n_i} p_i (\partial_{x_{ij}} L_i(0))^2 \\
&= \frac{1}{2} \langle AV, V \rangle, \qquad (2.3)
\end{aligned}
$$

where

$$
V = (\partial_{x_{11}} L_1(0), \dots, \partial_{x_{1n_1}} L_1(0), \dots, \partial_{x_{m1}} L_m(0), \dots, \partial_{x_{mn_m}} L_m(0)).
$$

One may also perform a similar computation as above to represent the second derivatives of $\phi_1, \dots, \phi_m, \phi$ in terms of the first four partial derivatives of $L_1, \dots, L_m$ using Gaussian integration by parts formula. From the uniformly boundedness of the first four partial derivatives of $L_1, \dots, L_m$, we have that

$$
\sup_{0 \le u \le 1} \left| \phi''(u) - \sum_{i=1}^m \phi_i''(0) \right| \le K
$$

for some fixed positive constant K. Using this, (2.3) and $\phi(0) = \sum_{i=1}^m \phi_i(0)$, we conclude from the mean value theorem that if $A \le 0$, then (2.1) follows since

$$
\begin{aligned}
\phi(u) - \sum_{i=1}^m \phi_i(u) &\le \phi(0) - \sum_{i=1}^m \phi_i(0) + \left(\phi'(0) - \sum_{i=1}^m \phi_i'(0) \right) u + Ku^2 \\
&= 0 + \frac{1}{2} \langle AV, V \rangle u + Ku^2 \\
&\le Ku^2.
\end{aligned}
$$

Similarly, if $A \ge 0$, we also obtain (2.2) and this completes our proof. ☐

Proof of Theorem 1: To avoid triviality, we will assume that each f_i is not identically zero and each T_{ii} is not a zero matrix. Our arguments will be divided into two major parts. First, we consider the case that $f_1 = \exp L_1, \ldots, f_m = \exp L_m$, where $L_1, \ldots, L_m$ are defined respectively on $\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m}$ with uniformly bounded partial derivatives of any orders. Define

$$\phi(u, x_1, \ldots, x_m) = \log \mathbb{E} \exp \sum_{i=1}^{m} L_i(x_i + \sqrt{u} X_i)$$

and

$$\phi_i(u, x_i) = \frac{1}{p_i} \log \mathbb{E} \exp p_i L_i(x_i + \sqrt{u} X_i),$$

for $u \in [0,1]$ and $x_i = (x_{i1}, \ldots, x_{in_i}) \in \mathbb{R}^{n_i}$, where ϕ_i is read as $\mathbb{E} L_i(x_i + \sqrt{u} X_i)$ when $p_i = 0$. We prove (1.4) first. Note that since the first four partial derivatives of $L_1, \ldots, L_m$ are uniformly bounded, one can use the Gaussian integration by parts formula as we have done in Lemma 1 to obtain a constant $K > 0$ independent of $u, x_1, \ldots, x_m$ such that the first four partial derivatives of $\phi_1(u, x_1 + \cdot), \ldots, \phi_m(u, x_m + \cdot)$ are uniformly bounded by K. Let K' be the constant obtained by applying (2.1) to $\phi_1(u, x_1 + \cdot), \ldots, \phi_m(u, x_m + \cdot)$ instead of $L_1, \ldots, L_m$, i.e. K' satisfies that

$$\begin{aligned} &\log \mathbb{E} \exp \sum_{i=1}^{m} \phi_i(u, x_i + \sqrt{v} X_i) \\ &\leq \sum_{i=1}^{m} \frac{1}{p_i} \log \mathbb{E} \exp p_i \phi_i(u, x_i + \sqrt{v} X_i) + K' v^2. \end{aligned} \tag{2.4}$$

Note that K' only depends on K.

We claim that for every $M \in \mathbb{N}$,

$$\phi\left(\frac{j}{M}, x_1, \ldots, x_m\right) \leq \sum_{i=1}^{m} \phi_i\left(\frac{j}{M}, x_i\right) + \frac{K' j}{M^2} \tag{2.5}$$

for all $1 \leq j \leq M$. Since $\phi_i(0, \cdot) = L_i(\cdot)$ for $1 \leq i \leq m$, the base case $j = 1$ follows by letting $u = 0$ and $v = 1/M$ in (2.4). Suppose that our claim holds for some $1 \leq j \leq M-1$. Write

$$\phi\left(\frac{j+1}{M}, x_1, \ldots, x_m\right) = \log \mathbb{E} \exp \phi\left(\frac{j}{M}, x_1 + \frac{X_1}{\sqrt{M}}, \ldots, x_m + \frac{X_m}{\sqrt{M}}\right).$$

Using the induction hypothesis and then (2.4) with $u = j/M$ and $v = 1/M$, we have

$$\begin{aligned} &\phi\left(\frac{j+1}{M}, x_1, \ldots, x_m\right) \\ &\leq \log \mathbb{E} \exp \sum_{i=1}^{m} \phi_i\left(\frac{j}{M}, x_i + \frac{X_i}{\sqrt{M}}\right) + \frac{K' j}{M^2} \\ &\leq \sum_{i=1}^{m} \frac{1}{p_i} \log \mathbb{E} \exp p_i \phi_i\left(\frac{j}{M}, x_i + \frac{X_i}{\sqrt{M}}\right) + \frac{K'}{M^2} + \frac{K' j}{M^2} \\ &= \sum_{i=1}^{m} \phi_i\left(\frac{j+1}{M}, x_i\right) + \frac{K'(j+1)}{M^2}. \end{aligned}$$

This completes the proof of our claim. Now, letting $j = M$ and $x_1, \ldots, x_m$ be all equal to the zero vectors in (2.5) and $M \to \infty$, we obtain (1.4) in the case that $f_1 = \exp L_1, \ldots, f_m = \exp L_m$. One may argue similarly to obtain (1.5) in such case as well.

Next we consider the general case that $f_1,\dots,f_m$ are nonnegative measurable. Note that in the following, $(\mathbb{E}f(Y)^p)^{1/p}$ will always be read as $\exp\mathbb{E}\log f(Y)$ whenever $p=0$ and the latter is well-defined. First, we assume that for every $1\le i\le m$,

$$\mathbb{E}f_i(X_i)^{p_i}<\infty \text{ if } p_i\neq 0 \text{ and } \mathbb{E}\log f_i(X_i)>-\infty \text{ if } p_i=0. \tag{2.6}$$

Under this assumption, from the monotone convergence theorem, it suffices to assume that $1/2\le f_1,\dots,f_m\le 1$. Let

$$f_{i,j}=\frac{1}{2}1_{([-j,j]^{n_i})^c}+f_j1_{[-j,j]^{n_i}}.$$

Since $f_{i,j}\uparrow f_i$ as $j\to\infty$, we can further assume by the monotone convergence theorem that $f_i=1/2$ on $([-1,1]^{n_i})^c$ and $1/2\le f_i\le 1$ on $[-1,1]^{n_i}$. Now we use mollifier function to construct a sequence of smooth functions $(g_{i,j})_{j\ge1}$ that satisfies $g_{i,j}=1/2$ on $([-3/2,3/2]^{n_i})^c$, $1/2\le g_{i,j}\le 1$ on $[-3/2,3/2]^{n_i}$, and converges to f_i a.e. with respect to the Lebesgue measure. Therefore, with these constructions,

$$\begin{aligned}\lim_{j\to\infty}\Big(\mathbb{E}g_{i,j}(X_i)^{p_i}\Big)^{\frac{1}{p_i}}&=\Big(\mathbb{E}f_i(X_i)^{p_i}\Big)^{\frac{1}{p_i}},\\ \lim_{j\to\infty}\mathbb{E}\prod_{i=1}^m g_{i,j}(X_i)&=\mathbb{E}\prod_{i=1}^m f_i(X_i).\end{aligned}\tag{2.7}$$

Take $L_{i,j}=\log g_{i,j}$. Then each $L_{i,j}$ has uniformly bounded derivatives of any orders and $g_{i,j}=\exp L_{i,j}$. By the first part of our argument and (2.7), we obtain (1.4) and (1.5).

To finish the proof, it remains to deal with the case that (2.6) does not hold for all $1\le i\le m$. Let

$$\begin{aligned}I&=\{i:p_i>0,\mathbb{E}f_i(X_i)^{p_i}=\infty\},\\ I'&=\{i:p_i>0,\mathbb{E}f_i(X_i)^{p_i}<\infty\},\\ J&=\{i:p_i\le 0,\ \mathbb{E}\log f_i(X_i)=-\infty \text{ if } p_i=0 \text{ and } \mathbb{E}f_i(X_i)^{p_i}=\infty \text{ if } p_i<0\},\\ J'&=\{i:p_i\le 0,\ \mathbb{E}\log f_i(X_i)>-\infty \text{ if } p_i=0 \text{ and } \mathbb{E}f_i(X_i)^{p_i}<\infty \text{ if } p_i<0\}.\end{aligned}$$

Note that $I\cup I'\cup J\cup J'=\{1,\dots,m\}$ and $I\cup J\neq\emptyset$. In the case that $P\ge T$, we have $p_1,\dots,p_m\ge 1$. This means that $I\neq\emptyset$ and $J=\emptyset=J'$. So

$$\prod_{i=1}^m(\mathbb{E}f_i(X_i)^{p_i})^{1/p_i}=\prod_{i\in I}(\mathbb{E}f_i(X_i)^{p_i})^{1/p_i}\cdot\prod_{i\in I'}(\mathbb{E}f_i(X_i)^{p_i})^{1/p_i}=\infty,$$

which clearly gives (1.4). Suppose that $P\le T$. If $J\neq\emptyset$, noting that $(\mathbb{E}f_i(X_i)^{p_i})^{1/p_i}=0$ for all $i\in J$, it follows that

$$\prod_{i=1}^m(\mathbb{E}f_i(X_i)^{p_i})^{1/p_i}=\prod_{i\in I\cup I'\cup J'}(\mathbb{E}f_i(X_i)^{p_i})^{1/p_i}\cdot\prod_{i\in J}(\mathbb{E}f_i(X_i)^{p_i})^{1/p_i}=0$$

and this yields (1.5). Suppose that $J=\emptyset$. Then $I\neq\emptyset$ and $\{1,\dots,m\}=I\cup I'\cup J'$. Note that $\mathbb{E}(f_i(X_i)\wedge M)^{p_i}<\infty$ for all $M>0$ and each $i\in I$. Applying the proceeding case (2.6) to $(f_i\wedge M)_{i\in I}$ and $(f_i)_{i\in I'\cup J'}$ gives

$$\mathbb{E}\prod_{i\in I}(f_i(X_i)\wedge M)\cdot\prod_{i\in I'\cup J'}f_i(X_i)\ge\prod_{i\in I}\Big(\mathbb{E}(f_i(X_i)\wedge M)^{p_i}\Big)^{\frac{1}{p_i}}\cdot\prod_{i\in I'\cup J'}\Big(\mathbb{E}f_i(X_i)^{p_i}\Big)^{\frac{1}{p_i}}.$$

From the monotone convergence theorem, letting $M\uparrow\infty$ leads to

$$\prod_{i=1}^m\mathbb{E}f_i(X_i)\ge\prod_{i\in I}\Big(\mathbb{E}f_i(X_i)^{p_i}\Big)^{\frac{1}{p_i}}\cdot\prod_{i\in I'\cup J'}\Big(\mathbb{E}f_i(X_i)^{p_i}\Big)^{\frac{1}{p_i}}=\infty,$$

which gives (1.5). This completes our proof. □

2.2. **Second proof: the Ornstein-Uhlenbeck semigroup.** In the second proof, we will adapt the ideas from [6] and [21]. As in the first proof, we will continue to assume that $T_{ii} = I_{n_i}$ for $1 \le i \le m$. Consider the Ornstein-Uhlenbeck semigroup operator $(P_t)_{t\ge 0}$ defined on $f : \mathbb{R}^n \to \mathbb{R}$ as

$$P_t f(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y)d\gamma_n(y) \tag{2.8}$$

with generator

$$L = \Delta - \langle id_n, \nabla\rangle.$$

From the definitions of P_t and L, we have

$(P1)$ $P_t f \to \int_{\mathbb{R}^n} f\, d\gamma$ a.s. as $t \to \infty$,
$(P2)$ $P_0 f = f$

and the integration by parts formula

$$\int_{\mathbb{R}^n} Lf g\, d\gamma_n = -\int_{\mathbb{R}^n} \langle \nabla f, \nabla g\rangle\, d\gamma_n. \tag{2.9}$$

Moreover, the $g(t,x) = P_t f(x)$ satisfies the PDE

$$\frac{\partial g}{\partial t}(t,x) = \Delta g(t,x) - \langle x, \nabla g(t,x)\rangle = Lg(t,x),$$

and $F^{(t)}(x) = F(t,x) := \log P_t f(x)$ satisfies

$$\frac{\partial F^{(t)}}{\partial t}(x) = LF^{(t)}(x) + |\nabla F^{(t)}(x)|^2, \tag{2.10}$$

where $|x|$ stands for the Euclidean norm of the vector x. Our first goal is to prove the following

Theorem 3. *Let $m \ge n$ and $n_1, \dots, n_m \le n$ be positive integers and set $N = n_1 + \cdots + n_m$. For every $i = 1, \dots, m$, consider the $n_i \times n$ matrices U_i with $U_i U_i^* = I_{n_i}$. Set the $N \times n$ matrix U consisting of block rows $U_1, \dots, U_m$ and the $N \times N$ diagonal matrix D with nonzero entries,*

$$D = \operatorname{diag}\Big(d_1 I_{n_1}, \dots, d_m I_{n_m}\Big).$$

For nonnegative Lebesgue measurable functions f_i on $\mathbb{R}^{n_i}$ for $1 \le i \le m$, we have

(*i*) *if $UU^* \le D^{-1}$, then*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(U_i x)\, d\gamma_n(x) \le \prod_{i=1}^m \Big(\int_{\mathbb{R}^{n_i}} f_i(x_i)^{1/d_i}\, d\gamma_{n_i}(x_i)\Big)^{d_i}; \tag{2.11}$$

(*ii*) *if $UU^* \ge D^{-1}$, then*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(U_i x)\, d\gamma_n(x) \ge \prod_{i=1}^m \Big(\int_{\mathbb{R}^{n_i}} f_i(x_i)^{1/d_i}\, d\gamma_{n_i}(x_i)\Big)^{d_i}. \tag{2.12}$$

Proof. As we have discussed in the first proof of Theorem 1 or referring to the approximation procedure in [6], we may assume without loss of generality that f_i's are smooth and uniformly bounded from above and away from zero on $\mathbb{R}^n$. For $1 \le i \le m$, set

$$F_i^{(t)}(x_i) = \log P_t f_i(x_i), \quad x_i \in \mathbb{R}^{n_i}.$$

For $t \in [0, \infty)$, we consider

$$a(t) := \int_{\mathbb{R}^n} \prod_{i=1}^m P_t f_i(U_i x)^{d_i}\, d\gamma_n(x) = \int_{\mathbb{R}^n} \exp\Big(\sum_{i=1}^m d_i F_i^{(t)}(U_i x)\Big)\, d\gamma_n(x).$$

Note that from $(P1)$ and $(P2)$,

$$\lim_{t\to\infty} a(t) = \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i \, d\gamma_{n_i}\right)^{d_i} \quad \text{and } a(0) = \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(U_i x)^{d_i} \, d\gamma_n(x).$$

Thus, it is enough to show that, under condition $UU^* \leq D^{-1}$ (resp. $UU^* \geq D^{-1}$), $a(t)$ is increasing (resp. decreasing). To do so, we compute its derivative:

$$a'(t) = \int_{\mathbb{R}^n} \sum_{i=1}^m d_i \big(LF_i^{(t)}(U_i x) + \big|\nabla F_i^{(t)}(U_i x)\big|^2\big)$$
$$\exp\left(\sum_{i=1}^m d_i F_i^{(t)}(U_i x)\right) d\gamma_n(x),$$

where we used (2.10) in dimension n_i for all $i = 1, \ldots, m$. Let t be fixed. Set $F_i = F_i^{(t)}$ and $H_i = F_i \circ U_i$. Since $U_i U_i^* = I_{n_i}$, we have that

$$\begin{aligned} LH_i(x) &= \Delta H_i(x) - \langle x, \nabla H_i(x)\rangle \\ &= \Delta F_i(U_i x) - \langle U_i x, \nabla F_i(U_i x)\rangle \\ &= LF_i(U_i x). \end{aligned}$$

Thus, we can use the n-dimensional integration by parts formula (2.9) for the functions $H_i(x)$ and $G(x) := \exp\big(\sum_{i=1}^m d_i F_i(U_i x)\big)$ to get that for every $1 \leq i \leq m$,

$$\begin{aligned} &\int_{\mathbb{R}^n} LF_i(U_i x)\, G(x)\, d\gamma_n(x) \\ &= \int_{\mathbb{R}^n} LH_i(x)\, G(x)\, d\gamma_n(x) \\ &= -\int_{\mathbb{R}^n} \langle \nabla H_i(x), \nabla G(x)\rangle \, d\gamma_n(x) \\ &= -\int_{\mathbb{R}^n} \sum_{j=1}^m d_j \left\langle \nabla F_i(U_i x) U_i U_j^* \nabla F_j(U_j x)\right\rangle \exp\left(\sum_{i=1}^m d_i F_i(U_i x)\right) d\gamma_n(x). \end{aligned}$$

It follows that

$$a'(t) = \int_{\mathbb{R}^n} \Big(-\sum_{i=1}^m \sum_{j=1}^m d_i d_j \left\langle \nabla F_i(U_i x), U_i U_j^* \nabla F_j(U_j x)\right\rangle$$
$$+ \sum_{i=1}^m d_i \big|\nabla F_i(U_i x)\big|^2\Big) \exp\left(\sum_{i=1}^m d_i F_i(U_i x)\right) d\gamma_n(x)$$

or equivalently,

$$a'(t) = \int_{\mathbb{R}^n} \left(-\left|\sum_{i=1}^m d_i U_i^* \nabla F_i(U_i x)\right|^2 + \sum_{i=1}^m d_i \big|\nabla F_i(U_i x)\big|^2\right)$$
$$\exp\left(\sum_{i=1}^m d_i F_i(U_i x)\right) d\gamma_n(x).$$

This implies that the proof will be complete if we show that

(i) $UU^* \leq D^{-1}$ if and only if

$$\left|\sum_{i=1}^m d_i\, U_i^* \xi_i\right|^2 \leq \sum_{i=1}^m d_i \big|\xi_i\big|^2, \ \forall\, \xi_i \in \mathbb{R}^{n_i}. \tag{2.13}$$

(*ii*) $UU^* \geq D^{-1}$ if and only if

$$\left|\sum_{i=1}^{m} d_i\, U_i^* \xi_i\right|^2 \geq \sum_{i=1}^{m} d_i \big|\xi_i\big|^2, \ \forall\, \xi_i \in \mathbb{R}^{n_i}. \tag{2.14}$$

To check (2.14), we write $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^N$ with $\xi_i \in \mathbb{R}^{n_i}$, and then we have that

$$\begin{aligned} UU^* \geq D^{-1} &\Leftrightarrow \langle UU^* x, x\rangle \geq \langle D^{-1}x, x\rangle, \ \forall\, x \in \mathbb{R}^N \\ \big(x = D\xi\big) &\Leftrightarrow \langle UU^* D\xi, D\xi\rangle \geq \langle \xi, D\xi\rangle, \ \forall\, \xi \in \mathbb{R}^N \\ &\Leftrightarrow |U^* D\xi|^2 \geq \langle \xi, D\xi\rangle, \ \forall\, \xi \in \mathbb{R}^N \\ &\Leftrightarrow \left|\sum_{i=1}^{m} d_i\, U_i^* \xi_i\right|^2 \geq \sum_{i=1}^{m} d_i \big|\xi_i\big|^2, \ \forall\, \xi_i \in \mathbb{R}^{n_i}. \end{aligned}$$

The verification of (2.13) is identical. □

Next, we will show how Theorem 1 can be obtained from Theorem 3. First, we need a standard linear algebra fact.

Lemma 2. *Let n, N be positive integers. Let T be a $N \times N$ symmetric and positive semi-definite matrix with* $\operatorname{rank}(T) = n$*. Then there exists a $N \times n$ matrix $U = U(T)$ with* $\operatorname{rank}(U) = \operatorname{rank}(T) = n$ *such that $T = UU^*$. Moreover, U is unique up to an orthogonal transformation.*

Proof. The existence of U is guaranteed from the singular value decomposition of T. More precisely, let us order the eigenvalues $\lambda_1, \dots, \lambda_N$ of $T^2 = T^*T = TT^*$ such that $\lambda_1 \geq \cdots \geq \lambda_n > 0 = \lambda_{n+1} = \cdots = \lambda_N$ and let $v_1, \dots, v_N$ be the corresponding eigenvectors. Consider the $N \times N$ matrices $V = (v_1, \dots, v_N)$ and $L = \operatorname{diag}(\lambda_1, \dots, \lambda_n, 0, \dots, 0)$. Then from the singular value decomposition, we have that

$$T = VLV^* = (V\sqrt{L})(V\sqrt{L})^* =: V_L\, V_L^*,$$

where $V_L := V\sqrt{L}$. We write

$$\begin{aligned} V_L &= \begin{pmatrix} v_1 & \dots & v_N \end{pmatrix} \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}, 0, \dots, 0) \\ &= \begin{pmatrix} \sqrt{\lambda_1} v_1 & \dots & \sqrt{\lambda_n} v_n & \mathbb{O}_{N\times 1} & \dots & \mathbb{O}_{N\times 1} \end{pmatrix} \\ &= \begin{pmatrix} u_1 & \mathbb{O}_{1\times(N-n)} \\ \vdots & \vdots \\ u_N & \mathbb{O}_{1\times(N-n)} \end{pmatrix} =: \begin{pmatrix} U & \mathbb{O}_{N\times(N-n)} \end{pmatrix}, \end{aligned}$$

and so

$$T = V_L\, V_L^* = \Big(\langle u_i, u_j\rangle\Big) = U\, U^*,$$

where U is the $N \times n$ matrix with rows $u_1, \dots, u_N$.

For the uniqueness of U, we need to show that if V is a $N \times n$ matrix with $VV^* = T = UU^*$, then $\Phi U^* = V^*$ for some $\Phi \in O(n)$, orthogonal transformation in $\mathbb{R}^n$. If we write $v_1, \dots, v_N$ for the rows of V we have that $\mathbb{R}^n = \operatorname{span}\{u_1, \dots, u_N\} = \operatorname{span}\{v_1, \dots, v_N\}$. Define the linear transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi u_i = v_i$ for all $i = 1, \dots, N$ or equivalently $\Phi U^* = V^*$. With this construction, one clearly sees that $\Phi \in O(n)$. Indeed, by definition,

$$\langle \Phi u_i, \Phi u_j\rangle = \langle v_i, v_j\rangle = \langle u_i, u_j\rangle$$

for $1 \leq i, j \leq N$ and so

$$\langle \Phi x, \Phi x\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \langle \Phi u_i, \Phi u_j\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \langle u_i, u_j\rangle = \langle x, x\rangle$$

for every $x = \sum_{i=1}^{N} a_i u_i \in \mathbb{R}^n$. This completes our proof. □

We are now ready to complete the second proof of our main result:

Second proof of Theorem 1. Without loss of generality we can assume that p_i's are non-zero. Recall that we have assumed that $T_{11} = I_{n_1}, \ldots, T_{mm} = I_{n_m}$. Let $n = \text{rank}(T)$. From Lemma 2, there exists a $N \times n$ matrix U such that $T = UU^*$. We denote by $u_1^i, \ldots, u_{n_i}^i$ the rows of U_i and by U the $N \times n$ matrix with block rows $U_1, \ldots, U_m$. Since

$$T = UU^* = (U_i U_j^*)_{i,j \le m},$$

we have that $U_i U_i^* = T_{ii} = I_{n_i}$ for $1 \le i \le m$. On the other hand, observe that $\big(X_{ij} : 1 \le i \le m, 1 \le j \le n_i\big)$ and $\big(\langle Z, u_j^i \rangle : 1 \le i \le m, 1 \le j \le n_i\big)$ are identically distributed, where Z is a n-dimensional standard Gaussian random vector. So

$$\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} U_1 Z \\ \vdots \\ U_m Z \end{pmatrix} = UZ.$$

Thus, we have that

$$\mathbb{E} \prod_{i=1}^m f_i(X_i) = \mathbb{E} \prod_{i=1}^m f_i(U_i Z) = \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(U_i x) d\gamma_n(x)$$

and Theorem 1 follows immediately from Theorem 3. □

Actually, it is easy to show that Theorem 1 implies Theorem 3. Indeed, if U and D are as in Theorem 3, then $T = UU^*$ and $P = D^{-1}$ satisfy the assumptions of Theorem 1. Working as in the previous proof, Theorem 3 follows.

3. The Brascamp-Lieb inequality, the geometry of eligible exponents and Gaussian hypercontractivity

This section will be concentrated on Theorem 1(i). The equivalence between this bound and the geometric form of the Brascamp-Lieb inequality for Gaussian measures will first be established. Next, we turn to the study of some geometric properties of the eligible exponents in Theorem 1(i). We close this section by showing that theorem 1 generalize the Gaussian hypercontractivity and its reverse form.

3.1. **Connection to the Brascamp-Lieb inequality.** The main objective of this subsection is to show that Theorem 1(i) is a reformulation of the geometric Brascamp-Lieb inequality for Gaussian measures, which is stated below.

Theorem 4. *Assume that $n \le m$ and $n_1, \ldots, n_m \le n$ are positive integers. For every $i = 1, \ldots, m$, consider the $n_i \times n$ matrices U_i with $U_i U_i^* = I_{n_i}$ and $p_i > 0$ such that*

$$U^* P^{-1} U = I_n, \tag{3.1}$$

where $P := diag(p_1 I_{n_1}, \ldots, p_m I_{n_m})$. Then for measurable function $f_i : \mathbb{R}^{n_i} \to [0, \infty)$ for $i = 1, \ldots, m$, one has that

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(U_i x)\, d\gamma_n(x) \le \prod_{i=1}^m \|f_i\|_{L_{p_i}(\gamma_{n_i})}, \tag{3.2}$$

where the notation γ_k means the k-dimensional standard Gaussian measure on $\mathbb{R}^k$.

Note that as we have seen from Subsection 2.2, Theorem 3(i) is equivalent to Theorem 1(i). To attend our goal, it suffices to establish the equivalence between Theorem 3(i) and Theorem 4. The argument that the second implies the first is simple. Indeed, assuming that the assumptions $U_iU_i^* = I_{n_i}$ for $i \le m$ and (3.2) holds for some $p_1, \dots, p_m > 0$, one sees that

$$\begin{aligned} U^*P^{-1}U = I_n &\Rightarrow U^*P^{-1}U \le I_n \\ &\Leftrightarrow \lambda_1(U^*P^{-1}U) \le 1 \\ &\Leftrightarrow \lambda_1(P^{-1/2}UU^*P^{-1/2}) \le 1 \\ &\Leftrightarrow P^{-1/2}UU^*P^{-1/2} \le I_N \\ &\Leftrightarrow UU^* \le P, \end{aligned} \tag{3.3}$$

where $\lambda_1(A) := \|A\|_{OP}$, the largest eigenvalue of the real symmetric matrix A. Consequently, the assumptions of Theorem 3(i) are satisfied by U_i's and $d_i := p_i^{-1}$ for $i \le m$ and thus (3.2) follows from (2.11).

As for the reverse direction, recall U and D from Theorem 3(i) and set $P = D^{-1}$. Then $UU^* \le P$ is equivalent to $\|A\|_{OP} \le 1$ for $A := U^*P^{-1}U$. Let $\lambda_1, \dots, \lambda_n \ge 0$ be the eigenvalues of A listed in non-increasing order and $\theta_1 \dots, \theta_n$ be the corresponding orthonormal eigenvectors. Let k be the largest integer such that $\lambda_1 = \cdots = \lambda_k$. Consider the decomposition of the identity matrix I_n,

$$\sum_{i=1}^{m} \frac{1}{p_i\lambda_1} U_i^*U_i + \sum_{i=k+1}^{n} \left(1 - \frac{\lambda_i}{\lambda_1}\right) \theta_i\theta_i^* = I_n, \tag{3.4}$$

where the validity of this identity can be easily checked by showing that both sides agree on $\theta_1, \dots, \theta_n$. If $\lambda_1 < 1$, this equation may as well be written as

$$\sum_{i=1}^{m} \frac{1}{p_i} U_i^*U_i + \sum_{i=k+1}^{n} \left(1 - \frac{\lambda_i}{\lambda_1}\right) \theta_i\theta_i^* + \sum_{i=1}^{m} \frac{1}{p_i}\left(\frac{1}{\lambda_1} - 1\right) U_i^*U_i = I_n.$$

Note that the coefficient terms in the last two equations are all positive since $\lambda_1 = \|A\|_{OP} \le 1$. To sum up, there exists some $\nu \in \mathbb{N} \cup \{0\}$ such that there are $k_j \times n$ matrix B_j with $B_jB_j^* = I_{k_j}$ and $b_j > 0$ for $1 \le j \le \nu$ satisfying

$$\sum_{i=1}^{m} \frac{1}{p_i} U_i^*U_i + \sum_{j=1}^{\nu} \frac{1}{b_j} B_j^*B_j = I_n, \tag{3.5}$$

where $\sum_{j=1}^{0} b_j^{-1} B_j^*B_j$ is read as the n-dimensional zero matrix. For given nonnegative measurable function f_i on $\mathbb{R}^{n_i}$ for $i \le m$, we set $g_1 = f_1, \dots, g_m = f_m, g_{m+1} = 1, \dots, g_{m+\nu} = 1$. Since p_i's and b_j's satisfy (3.5), we may apply these g_i's to (3.2) to obtain (2.13) by noting that $\|g_i\|_{L_{b_i}(\gamma_{k_i})} = 1$ for all $m+1 \le i \le m+\nu$. This completes our argument.

3.2. **The geometry of the eligible exponents.** Let $n \le N$ and U be a $N \times n$ matrix with $\mathrm{rank}(U) = n$. Assume that U has as block-rows the $n_i \times n$ matrices U_i, $1 \le i \le m$, with $U_iU_i^* = I_{n_i}$. Then we define

$$\mathcal{C}(U) = \left\{(c_1, \dots, c_m) :\ UU^* \le C^{-1}\right\} \tag{3.6}$$

where $C = \mathrm{diag}(c_1 I_{n_1}, \dots, c_m I_{n_m})$. By the discussion in the last subsection we have that if $(c_1, \dots, c_m) \in \mathcal{C}(U)$ then $c_1, \dots, c_m$ satisfy (3.5). On the other hand if an m-tuple $c_1, \dots, c_m$ satisfies (3.5) then trivially, $U^*CU = \sum_{i=1}^{m} c_iU_i^*U_i \le I_n$ and by (3.3) this means that $(c_1, \dots, c_m) \in \mathcal{C}(U)$. Thus, we have proved that

$$(c_1, \dots, c_m) \in \mathcal{C}(U) \quad \text{if and only if} \quad (c_1, \dots, c_m) \text{ satisfies (3.5)} \tag{3.7}$$

Next we gather some interesting properties for the set $\mathcal{C}(U)$.

Proposition 1. *Let U be the $N \times n$ matrix defined in Definition 3.6.*

(*i*) *Let V be a matrix with the same size as U and $UU^* = VV^*$, then $\mathcal{C}(U) = \mathcal{C}(V)$.*

(*ii*) *Define the* $n_i \times N$ *matrix* $R_i = (\mathbf{0}_{n_i\times n_1}, \ldots, I_{n_i}, \ldots, \mathbf{0}_{n_i\times n_m})$*, for* $1 \le i \le m$*. For* $\sigma \subseteq \{1, \ldots, m\}$*, set the* $\left(\sum_{i\in\sigma} n_i\right) \times N$ *matrix* $\Xi_\sigma := ([R_i^*]; i \in \sigma)^*$*, i.e. the matrices* R_i*,* $i \in \sigma$ *are the block-rows of* Ξ_σ*. Then*

$$P_\sigma \mathcal{C}(U) \subseteq \mathcal{C}(\Xi_\sigma U),$$

where P_σ *denotes the projection from* $\mathbb{R}^m$ *to* $\mathbb{R}^\sigma$ *through* $P_\sigma(c) = (c_i)_{i\in\sigma}$*.*

(*iii*) $C(U)$ *is a convex subset of* $\mathbb{R}^m$ *and*

$$\left\{x \in [0,\infty)^m : \sum_{i=1}^m x_i \le 1\right\} \subseteq \mathcal{C}(U) \subseteq [0,1]^m.$$

(*iv*) *If* $(c_1, \ldots, c_m) \in \mathcal{C}(U)$ *and* $\lambda_1, \ldots, \lambda_m \in [0,1]$*, then*

$$(\lambda_1 c_1, \ldots, \lambda_m c_m) \in \mathcal{C}(U). \tag{3.8}$$

Proof. (*i*) From Lemma 2, we have that $V^* = \Phi U^*$ for some $\Phi \in O(n)$. Let $(c_1, \ldots, c_m) \in \mathcal{C}(U)$. By the definition of $\mathcal{C}(U)$, this means that for some

$$\mathrm{B} = \begin{pmatrix} [\leftarrow \mathcal{B}_1 \rightarrow] \\ \vdots \\ [\leftarrow \mathcal{B}_\nu \rightarrow] \end{pmatrix}$$

and $L = \operatorname{diag}(b_1 I_{k_1}, \ldots, b_\nu I_{k_\nu})$, we have that $U^* C\, U + \mathrm{B}^* L\, \mathrm{B} = I_n$. Taking Φ and Φ^*, we write equivalently

$$\Phi U^* C\, U \Phi^* + \Phi \mathrm{B}^* L\, \mathrm{B} \Phi^* = \Phi I_n \Phi^*$$

or

$$V^* C\, V + \mathrm{B}_\Phi^* L\, \mathrm{B}_\Phi = I_n,$$

where

$$\mathrm{B}_\Phi := \mathrm{B}\Phi^* = \begin{pmatrix} [\leftarrow \mathcal{B}_1 \Phi^* \rightarrow] \\ \vdots \\ [\leftarrow \mathcal{B}_\nu \Phi^* \rightarrow] \end{pmatrix}.$$

This gives that $(c_1, \ldots, c_N) \in \mathcal{C}(V)$. Thus, we have proved that $\mathcal{C}(U) \subseteq \mathcal{C}(V)$. The same argument gives also the other inclusion and the claim follows.

As for (*ii*), let $x = (x_i)_{i\in\sigma} \in P_\sigma \mathcal{C}(U)$. This means that for some $c = (c_1, \ldots, c_m) \in \mathcal{C}(U)$, we have that $x_i = c_i$ for all $i \in \sigma$. Recall from the definition of $\mathcal{C}(U)$ that the equation (3.5) holds and it can be rewritten as

$$\sum_{i\in\sigma} x_i\, \mathcal{U}_i^* \mathcal{U}_i + \sum_{i\notin\sigma} c_i\, \mathcal{U}_i^* \mathcal{U}_i + \sum_{j=1}^{\nu} b_j\, \mathcal{B}_j^* \mathcal{B}_j = I_n.$$

Note that $\Xi_\sigma U$ has as block rows, the matrices $\mathcal{U}_i$ for $i \in \sigma$. The last equation guarantees that $x \in \mathcal{C}(\Xi_\sigma U)$.

For (*iii*), assume that $(c_1, \ldots, c_m)$, $(\hat{c}_1, \ldots, \hat{c}_m) \in C(U)$ and $\lambda \in [0,1]$. Then there exist b_j's, $\hat{b}_j$'s, $\mathcal{B}_j$'s and $\hat{\mathcal{B}}_j$'s such that

$$\sum_{i=1}^m c_i\, \mathcal{U}_i^* \mathcal{U}_i + \sum_{j=1}^{\nu_1} b_j\, \mathcal{B}_j^* \mathcal{B}_j = I_n \text{ and } \sum_{i=1}^m \hat{c}_i \mathcal{U}_i^* \mathcal{U}_i + \sum_{j=1}^{\nu_2} \hat{b}_j\, \hat{\mathcal{B}}_j^* \hat{\mathcal{B}}_j = I_n.$$

Consequently,

$$\begin{aligned} &\textstyle\sum_{i=1}^m (\lambda c_i + (1-\lambda)\hat{c}_i)\, U_i^* U_i + \sum_{j=1}^{\nu_1} \lambda b_j\, B_j^* B_j + \sum_{j=1}^{\nu_2} (1-\lambda) \hat{b}_j' \hat{B}_j^* \hat{B}_j \\ &= \lambda \Big(\textstyle\sum_{i=1}^m c_i U_i^* U_i + \sum_{j=1}^{\nu_1} b_j B_j^* B_j \Big) + (1-\lambda) \Big(\textstyle\sum_{i=1}^m \hat{c}_i\, U_i^* U_i + \sum_{j=1}^{\nu_2} \hat{b}_j \hat{B}_j^* \hat{B}_j \Big) \\ &= \lambda I_n + (1-\lambda) I_n \\ &= I_n, \end{aligned}$$

which means that $\lambda c_i + (1-\lambda)\hat{c}_i \in \mathcal{C}(U)$ and this shows the convexity of $\mathcal{C}(U)$. For the second part of the assertion (iii), since $\mathcal{U}_i^*\mathcal{U}_i$ is a projection from $\mathbb{R}^n$ to $\mathbb{R}^{n_i}$ for $1 \leq i \leq m$, we have that

$$\|U^*CU\|_{OP} = \left\|\sum_{i=1}^m c_i \mathcal{U}_i^*\mathcal{U}_i\right\|_{OP} \leq \sum_{i=1}^m c_i \|\mathcal{U}_i^*\mathcal{U}_i\|_{OP} \leq \sum_{i=1}^m c_i \tag{3.9}$$

for all $N \times N$ diagonal matrix $C = \mathrm{diag}(c_1 I_{n_1}, \ldots, c_m I_{m_m})$. From (3.3), we have that $c = (c_1, \ldots, c_m) \in \mathcal{C}(U) \Leftrightarrow \|U^*CU\|_{OP} \leq 1$, and so, by (3.9)

$$\left\{x \in [0,\infty)^m : \sum_{i=1}^m x_i \leq 1\right\} \subseteq \mathcal{C}(U).$$

On the other hand, from (3.3) again, we have that

$$(c_1, \ldots, c_m) \in \mathcal{C}(U) \Leftrightarrow UU^* - C^{-1} \leq 0 \Leftrightarrow \left\langle (UU^* - C^{-1}), x, x\right\rangle \leq 0, \ \forall x \in \mathbb{R}^N.$$

Taking the vectors $\mathbf{x}_i = (0, \ldots, 0, x_i, 0, \ldots, 0) \in \mathbb{R}^N$, $1 \leq i \leq m$, for any non-zero $x_i \in \mathbb{R}^{n_i}$, we get that $c_i \leq 1$ and this shows that $\mathcal{C}(U) \subset [0,1]^m$.

Finally, (iv) can be be easily verified by rewriting the equation (3.5) as

$$\sum_{i=1}^m \lambda_i c_i U_i^* U_i + \sum_{i=1}^m (1-\lambda_i) c_i U_i^* U_i + \sum_{j=1}^{\nu} b_j B_j^* B_j = I_n.$$

□

We are now ready to discuss the geometry of the eligible exponents in Theorem 1(i). Note that we consider only its normalized version, i.e. we assume that $T_{ii} = I_{n_i}$ for every $1 \leq i \leq m$. Nevertheless, as one could see by a simple change of variables, this simpler version of Theorem 1, is just an equivalent reformulation of the initial general statement.

Let $\mathbf{X}$ be the Gaussian random vector in $\mathbb{R}^N$, with covariance matrix $T = (T_{ij})_{i,j \leq m}$, as in Theorem 1, with $T_{ii} = I_{n_i}$. We define $\mathcal{C}(\mathbf{X})$ in $\mathbb{R}^m$ to be the set of all vectors $(1/p_1, \ldots, 1/p_m) \in [0,\infty)^m$ such that

$$\mathbb{E}\prod_{j=1}^m f_j(X_j) \leq \prod_{i=1}^m \|f_i\|_{L^{p_i}(\gamma_{n_i})}, \quad \forall\ f_i, \ \ 1 \leq i \leq m. \tag{3.10}$$

We also define $\mathcal{B}(\mathbf{X})$ as the set of all vectors $(1/p_1, \ldots, 1/p_m) \in \mathcal{C}(\mathbf{X}) \cap (0,1)^m$, with the following property: For every $1 \leq i \leq m$, if $q > p_i$, then there exist $f_1, \ldots, f_m$ measurable functions, such that

$$\mathbb{E}\prod_{i=1}^m f_i(X_i) > \prod_{j \neq i} \left(\mathbb{E} f_j(X_j)^{p_j}\right)^{1/p_j} \ \left(\mathbb{E} f_i(X_i)^q\right)^{1/q}.$$

If $(1/p_1, \ldots, 1/p_m) \in \mathcal{C}(\mathbf{X})$, we say that $(p_1, \ldots, p_m)$ are *eligible exponents* in Theorem 1(i). Respectively, if $(1/p_1, \ldots, 1/p_m) \in \mathcal{B}(\mathbf{X})$, we say that $(p_1, \ldots, p_m)$ is a choice of *best possible exponents* in Theorem 1(i). By Lemma 2, there exists a matrix U such that $T = UU^*$ and we set $\mathcal{C}(T) = \mathcal{C}(U)$. Observe that $\mathcal{C}(T)$ is well-defined by Proposition 1(i). Finally, by Remark 2 we have that $\mathcal{C}(\mathbf{X}) = \mathcal{C}(U) = \mathcal{C}(T)$. Moreover, we know that $\mathcal{C}(\mathbf{X})$ is a convex set of $[0,\infty)^m$ that satisfies

$$\left\{y \in [0,\infty)^m : \sum_{i=1}^m y_i \leq 1\right\} \subseteq \mathcal{C}(\mathbf{X}) \subseteq (0,1]^m.$$

Since $\mathcal{C}(\mathbf{X})$ has the property (3.8), it can be extended to an 1-unconditional convex body $\tilde{\mathcal{C}}(\mathbf{X})$ in $\mathbb{R}^m$ in the obvious way: $(c_1, \ldots, c_m) \in \tilde{\mathcal{C}}(\mathbf{X})$ if and only if $(|c_1|, \ldots, |c_m|) \in \mathcal{C}(\mathbf{X})$. In this case we have that

$$B_1^m \subseteq \tilde{\mathcal{C}}(\mathbf{X}) \subseteq B_\infty^m, \tag{3.11}$$

where

$$B_1^m = \{x \in \mathbb{R}^m : \sum_{i \le m} |x_i| \le 1\} \quad \text{and} \quad B_\infty^m = \{x \in \mathbb{R}^m : \max_{i \le m} |x_i| \le 1\}.$$

The associated norm in $\mathbb{R}^m$, is given by

$$\|c\|_{\tilde{\mathcal{C}}(X)} = \|U\,|\mathrm{C}|\,U^*\|_{op}$$

for every $c = (c_1, \dots, c_m) \in \mathbb{R}^m$, where $|\mathrm{C}| := \mathrm{diag}\big(|c_1| I_{n_1}, \dots, |c_m| I_{n_m}\big)$. Moreover, one can show that if $(1/p_1, \dots, 1/p_m) \in \partial\,\mathcal{C}(\mathbf{X}) \cap (0,1)^m$, then there exists an $\mathrm{a} = (\mathrm{a}_1, \dots, \mathrm{a}_m)$ in $\mathbb{R}^N$ where $\mathrm{a}_i \in \mathbb{R}^{n_i}$, and $f_i : \mathbb{R}^{n_1} \to \mathbb{R}$ of the form $f_i(x_i) = \exp(\langle \mathrm{a}_i, x_i \rangle)$, $x_i \in \mathbb{R}^{n_i}$, such that equality holds in (1.4). Indeed, by Remark 3, it is enough to show that $\langle (P - T)\mathrm{a}, \mathrm{a} \rangle = 0$. First note that

$$(c_1, \dots, c_m) \in \partial\,\mathcal{C}(\mathbf{X}) \Leftrightarrow \|\sqrt{C}\,T\sqrt{C}\|_{op} = 1 \Leftrightarrow \lambda_1(\sqrt{C}\,T\sqrt{C}) = 1.$$

Let $\mathrm{v}_1 \in \mathbb{R}^N$ be the normal eigenvector of $\lambda_1 = \lambda_1(\sqrt{C}\,T\sqrt{C})$. Then, for $\mathrm{a} = \sqrt{C}\,\mathrm{v}_1 \in \mathbb{R}^N$ we have that

$$\langle T\mathrm{a}, \mathrm{a} \rangle = \langle \sqrt{C}\,T\sqrt{C}\,\mathrm{v}_1, \mathrm{v}_1 \rangle = \lambda_1 \langle \mathrm{v}_1, \mathrm{v}_1 \rangle = 1 = \langle C^{-1}\mathrm{a}, \mathrm{a} \rangle.$$

Finally, note that in particular we have shown that $\partial\mathcal{C}(\mathbf{X}) \cap (0,1)^m \subseteq \mathcal{B}(\mathbf{X})$. Also $\mathcal{B}(\mathbf{X}) \subseteq \partial\mathcal{C}(\mathbf{X}) \cap (0,1)^m$ by Hölder's inequality, and thus we have proved the following

Proposition 2. *Let $\mathbf{X}$ be the Gaussian random vector in $\mathbb{R}^N$, with covariance matrix $T = (T_{ij})_{i,j \le m} \neq I_N$, as in Theorem 1, with $T_{ii} = I_{n_i}$, and $\mathcal{B}(\mathbf{X})$ as defined above. Then for every $c = (c_1, \dots, c_m) \in \mathbb{R}^m$ and $C = \mathrm{diag}(c_1 I_{n_1}, \dots, c_m I_{n_m})$, we have that*

$$c \in \mathcal{B}(\mathbf{X}) \Leftrightarrow c \in \partial\,\mathcal{C}(\mathbf{X}) \cap (0,1)^m \Leftrightarrow \|\sqrt{C}T\sqrt{C}\|_{op} = 1.$$

Moreover, for every $c = (c_1, \dots, c_m) \in \mathcal{B}(\mathbf{X})$, there exist functions $f_i(x_i) = \exp(\langle \alpha_i, x_i \rangle)$, $x_i \in \mathbb{R}^{n_i}$ such that one has equality in (1.4) *for $(p_1, \dots, p_m) = (1/c_1, \dots, 1/c_m)$.*

Let us close the discussion, considering again, the simplest case where $m = 2$ and $n_1 = n_2 = 1$, i.e. $\mathbf{X} = (X_1, X_2)$, where X_1 and X_2 are two standard Gaussian random variables with $\mathbb{E}X_1X_2 = t \in [0,1]$. A direct computation shows that the set of all eligible exponents is

$$\mathcal{C}(\mathbf{X}) = \Big\{ (x,y) \in [0,1]^2 : \Big(\frac{1}{x} - 1\Big)\Big(\frac{1}{y} - 1\Big) \ge t^2 \Big\}$$

and

$$\|(x,y)\|_{\tilde{\mathcal{C}}(X)} = \frac{\sqrt{(|x| + |y|)^2 - 4(1 - t^2)\,|xy|}\; + |x| + |y|}{2}.$$

Moreover, the couple of exponents (p_1, p_2) with $p_1, p_2 \ge 1$ is best possible in (1.4) if and only if $(1/p_1, 1/p_2)$ lies on $\mathcal{B}(\mathbf{X}) = \partial\,\mathcal{C}(\mathbf{X}) \cap (0,1)^2$ or equivalently, if and only if

$$(p_1 - 1)(p_2 - 1) = t^2 p_1 p_2. \tag{3.12}$$

3.3. **Gaussian hypercontractivity inequalities.** Recall the Ornstein-Uhlenbeck semigroup operators $(P_t)_{t \ge 0}$ form (2.8). The Gaussian hypercontractivity, discovered by Nelson [28], states that if $p, q > 1$ and $t \ge 0$ satisfy $(p-1)(q-1)^{-1} \ge e^{-2t}$, then

$$\|P_t f\|_{L_q(\gamma_n)} \le \|f\|_{L_p(\gamma_n)} \tag{3.13}$$

for any measurable $f : \mathbb{R}^n \to \mathbb{R}$. See also [11, 14, 17, 26] for various approaches to this inequality. Later, Borell [15] proved a reverse hypercontractivity inequality for the Bernoulli probability measure. His result was recently extended by Mossel, Oleszkiewicz and Sen in [27] to a more general class of probability measures satisfying log-Sobolev inequalities of certain type. In the special case of the Gaussian measure, their result states that if $p, q < 1$ and $t \ge 0$ with $(1-p)(1-q)^{-1} \ge e^{-2t}$, then

$$\|P_t f\|_{L_q(\gamma_n)} \ge \|f\|_{L_p(\gamma_n)} \tag{3.14}$$

for any measurable f on $\mathbb{R}^n$.

In this subsection we show that Theorem 1 generalizes those two results. To recover (3.13) and (3.14) from Theorem 1, consider two n-dimensional standard Gaussian random vectors X and Y such that their joint law has the $2n \times 2n$ covariance matrix

$$T = \begin{pmatrix} I_n & e^{-t} I_n \\ e^{-t} I_n & I_n \end{pmatrix}, \ t \geq 0. \tag{3.15}$$

For arbitrary measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}$,

$$\mathbb{E} g(X) f(Y) = \mathbb{E} g(X) P_t f(X). \tag{3.16}$$

Indeed, note that since Gaussian random vector is characterized by its mean and covariance, (X, Y) has the same joint distribution as $(X, e^{-t}X + \sqrt{1 - e^{-2t}} Z)$, where Z is an independent copy of X. A standard computation using conditional expectation yields

$$\mathbb{E} g(X) f(Y) = \mathbb{E}\Big(\mathbb{E}\left(g(X) f(e^{-t}X + \sqrt{1 - e^{-2t}} Z) \Big| X\right)\Big) = \mathbb{E} g(X) P_t f(X).$$

For a real number $r \neq 1$, let r' be the Hölder conjugate exponent of r. Let $p, q \in \mathbb{R}$ and $q \neq 1$, consider the $2n \times 2n$ diagonal matrix,

$$P_o = \begin{pmatrix} q' I_n & \mathbb{O} \\ \mathbb{O} & p I_n \end{pmatrix}.$$

A direct computation shows that, for any $p, q > 1$,

$$T \leq P_o \Leftrightarrow \frac{p-1}{q-1} \geq e^{-2t} \tag{3.17}$$

and for any $p, q < 1$,

$$T \geq P_o \Leftrightarrow \frac{1-p}{1-q} \geq e^{-2t}. \tag{3.18}$$

Recall the duality relations

$$\|f\|_{L_r(\gamma_n)} = \sup_{\|g\|_{L_{r'}(\gamma_n)} \leq 1} \mathbb{E} f(X) g(X), \ \ r > 1 \tag{3.19}$$

and for f nonnegative,

$$\|f\|_{L_r(\gamma_n)} = \inf_{g > 0, \|g\|_{L_{r'}(\gamma_n)} \geq 1} \mathbb{E} f(X) g(X), \ \ r < 1. \tag{3.20}$$

Suppose that f is a measurable function on $\mathbb{R}^n$. If $p, q > 1$ with $(p-1)/(q-1) \geq e^{-2t}$, then (3.17) implies $T \leq P_o$ and thus from (3.19) and Theorem 1 (i),

$$\begin{aligned} \|P_t f\|_{L_q(\gamma_n)} &= \sup_{\|g\|_{L_{q'}(\gamma_n)} \leq 1} \mathbb{E} g(X) P_t f(X) \\ &= \sup_{\|g\|_{L_{q'}(\gamma_n)} \leq 1} \mathbb{E} g(X) f(Y) \\ &\leq \sup_{\|g\|_{L_{q'}(\gamma_n)} \leq 1} \|g\|_{L_{q'}(\gamma_n)} \|f\|_{L_p(\gamma_n)} \\ &= \|f\|_{L_p(\gamma_n)}, \end{aligned}$$

which gives (3.13). Proceeding similarly by using (3.18), (3.20) and Theorem 1 (ii) yields the reverse inequality (3.14)

Conversely, one may retrieve the special case of Theorem 1, for $m = 2$, $n_1 = n_2 = n$ and the $2n \times 2n$ covariance matrix T is as in (3.15), using the hypercontractivity and reverse hypercontractivity inequalities (3.13) and (3.14).

Indeed, suppose for example, that $T \geq P = \mathrm{diag}(qI_n, pI_n)$. Thus $q, p < 1$ and applying (3.18) to $T \geq P$, one sees that $(1-p)/(1-q') \geq e^{-2t}$. This allows us to use the pair p, q' in (3.14) and combining this with reverse Hölder's inequality yields

$$\mathbb{E}g(X)f(Y) = \mathbb{E}g(X)P_t f(X) \geq \|g\|_{L_q(\gamma_n)}\|P_t f\|_{L_{q'}(\gamma_n)} \geq \|g\|_{L_q(\gamma_n)}\|f\|_{L_p(\gamma_n)}.$$

This gives Theorem 1 (i). A similar argument, using (3.19) instead of (3.20) also shows (ii) of Theorem 1.

We note here that the connection between Theorem 1(i) and the Gaussian hypercontractivity is rather classical. We refer to [17] and to Theorem 13.8.1 in [25].

4. Theorem 2: A Generalization of the sharp Young and reverse Young inequalities.

4.1. **The sharp Young and reverse Young inequalities.**

The sharp Young and reverse Young inequalities states that for nonnegative measurable functions f_1 and f_2 on $\mathbb{R}^n$, if $p, q, r > 0$ satisfy $p^{-1} + q^{-1} = 1 + r^{-1}$, then we have respectively,

$$\|f_1 * f_2\|_r \leq C^n \|f_1\|_p \|f_2\|_q \text{ for } p, q, r \geq 1 \tag{4.1}$$

and

$$\|f_1 * f_2\|_r \geq C^n \|f_1\|_p \|f_2\|_q \text{ for } p, q, r \leq 1, \tag{4.2}$$

where, $C := C_p C_q / C_r$, where $C_u^2 = |u|^{1/u}/|u'|^{1/u'}$ for $1/u + 1/u' = 1$.

The sharp Young inequality (4.1) was proved by Beckner in [11] and shortly after, by Brascamp and Lieb in [17]. In the late paper Brascamp and Lieb proved a generalization of (4.1), the so-called Brascamp-Lieb inequality. In addition they introduced, the reverse inequality (4.2). In this section, we prove Theorem 2 and we show how it generalizes both sharp Young and reverse sharp Young inequalities. These fundamental inequalities have many applications in analysis. As it was noticed by Brascamp and Lieb in [17], from the sharp reverse Young inequality one can retrieve the Prékopa-Leindler inequality [30, 33]. On the other hand, Lieb in [31] showed that sharp Young inequality implies Shannon's entropy power inequality. Furthermore, an argument that the sharp reverse Young inequality interpolates between the Shannon entropy power inequality ($r \to 1-$) and the Brunn-Minkowski inequality ($r \to 0+$), is presented in Chapter 17 of the book [23].

Let us first explain how (4.1) and (4.2) can be recovered, directlly from our theorem. Set the matrices

$$U = \begin{pmatrix} I_n & -I_n \\ \mathbb{O} & I_n \\ I_n & \mathbb{O} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} c_3(1-c_3)I_n & (1-c_2)(1-c_3)I_n \\ (1-c_2)(1-c_3)I_n & c_2(1-c_2)I_n \end{pmatrix},$$

$$B_2 = \begin{pmatrix} c_3(1+c_3)I_n & (c_2-1)(1+c_3)I_n \\ (c_2-1)(1+c_3)I_n & c_2(c_2-1)I_n \end{pmatrix},$$

where $c_1 = p^{-1}, c_2 = q^{-1}$ and $c_3 = |r'|^{-1}$, and let

$$P := \mathrm{diag}(pI_n, qI_n, r'I_n).$$

One can check that if $p,q,r \geq 1$, then $UB_1U^* \leq PD_{UB_1U^*}$ and if $0 < p,q,r \leq 1$, $UB_2U^* \geq PD_{UB_2U^*}$. In either case, a direct computation gives

$$\left(\frac{\det B_i}{(\det U_1 B_i U_1^*)^{\frac{1}{p}}(\det U_2 B_i U_2^*)^{\frac{1}{q}}(\det U_3 B_i U_3^*)^{\frac{1}{r'}}}\right)^{\frac{1}{2}} = C^n.$$

Then, for any nonnegative measurable functions f_1, f_2, g on $\mathbb{R}^n$, we apply Theorem 2(i) with U, B_1, P and Theorem 2(ii) with U, B_2, P,. Finally, the duality relations (3.19) and (3.20) lead to (4.1) and (4.2), respectively. The proceeding argument can be found in [29], see also [7].

4.2. **Proof of Theorem 2.** Before we give the proof of Theorem 2, let us comment on its assumptions. Note first that the additional assumption (1.10) that appears in Theorem 2 is actually a necessary condition due to the homogeneity of the Lebesgue measure. Moreover, in the following lemma, we shall see that under this homogeneity condition, the assumption (1.11) is equivalent with (4.5).

Lemma 3. *In the setting of Theorem 2, the following are equivalent*

$$\sum_{i=1}^{n} \frac{n_i}{p_i} = n \;\; \textit{and} \;\; UBU^* \leq PD_{UBU^*}, \tag{4.3}$$

$$B^{-1} = U^*(PD_{UBU^*})^{-1}U. \tag{4.4}$$

$$B^{-1} = \sum_{i=1}^{m} \frac{1}{p_i} U_i^* \left(U_i B U_i^*\right)^{-1} U_i \tag{4.5}$$

Proof. Clearly (4.4) and (4.5) are equivalent. Let's see first how (4.4) implies (4.3). Write $B = \Sigma\Sigma^*$ and $C_B^{-1} := PD_{UBU^*}$. Then (4.4) can be written equivalently as

$$\big(U\Sigma\big)^* C_B \big(U\Sigma\big) = \left(\sqrt{C_B}U\Sigma\right)^* \left(\sqrt{C_B}U\Sigma\right) = I_n \tag{4.6}$$

and so $\left(\sqrt{C_B}U\Sigma\right)\left(\sqrt{C_B}U\Sigma\right)^* \leq I_N$ or equivalently $UBU^* \leq PD_{UBU^*}$. The homogeneity condition $\sum_{i=1}^m n_i/p_i = n$ follows by taking trace in (4.6). Indeed, if we set $U_{i\Sigma} := U_i\Sigma$, a direct computation shows that

$$\begin{aligned}
(U\Sigma)^* C_B (U\Sigma) &= \sum_{i=1}^{m} c_i (U_i\Sigma)^* \left(U_i B U_i^*\right)^{-1} (U_i\Sigma) \\
&= \sum_{i=1}^{m} c_i \, U_{i\Sigma}^* \left(U_{i\Sigma} U_{i\Sigma}^*\right)^{-1} U_{i\Sigma} = \sum_{i=1}^{m} c_i \left(\widetilde{U}_{i\Sigma}\right)^* \widetilde{U}_{i\Sigma}
\end{aligned}$$

where, $\widetilde{U}_{i\Sigma} := \left(U_{i\Sigma} U_{i\Sigma}^*\right)^{-1/2} U_{i\Sigma}$. Note that $\widetilde{U}_{i\Sigma}\left(\widetilde{U}_{i\Sigma}\right)^* = I_{n_i}$, and thus,

$$n = \mathrm{tr}(I_n) = \mathrm{tr}\Big(\left(U\Sigma\right)^* C_B \left(U\Sigma\right)\Big) = \mathrm{tr}\Big(\sum_{i=1}^{m} c_i \left(\widetilde{U}_{i\Sigma}\right)^* \widetilde{U}_{i\Sigma}\Big) = \sum_{i=1}^{m} c_i n_i = \sum_{i=1}^{m} n_i p_i. \tag{4.7}$$

To see why (4.3) implies (4.4) recall that $UBU^* \leq PD_{UBU^*}$ can be written equivalently as $\left(\sqrt{C_B}U\Sigma\right)\left(\sqrt{C_B}U\Sigma\right)^* \leq I_N$ which implies that

$$\big(U\Sigma\big)^* C_B \big(U\Sigma\big) \leq I_n \tag{4.8}$$

To complete the proof we have to show that equality holds in (4.8). Indeed, note that if A_1, A_2 are two positive definite matrices with $A_1 \leq A_2$ and $\mathrm{tr}(A_1) = \mathrm{tr}(A_2)$ then $A_1 = A_2$. Thus, under the homogeneity condition, $\sum_{i=1}^m n_i/p_i = n$ we get that

$$\mathrm{tr}\Big(\left(U\Sigma\right)^* C_B \left(U\Sigma\right)\Big) = \sum_{i=1}^{m} n_i p_i = n = \mathrm{tr}(I_n),$$

and so equality holds in (4.8). □

Remark 4. In [29], Lehec proved a reformulation of the Brascamp-Lieb inequality, which states that (1.12) holds true under the assumption (4.5). As an immediate consequence of Lemma 3, Theorem 2(i) is exactly the Brascamp-Lieb inequality. We refer to [29] for more details.

We close this section with the proof of Theorem 2.

Proof of Theorem 2. We prove (i) first. Note that under the given assumptions, we have that $\mathrm{rank}(U_i) = n_i \leq n = \mathrm{rank}(B)$, for every $i \leq m$. Thus, the $n_i \times n_i$ matrix $B_i := U_i B U_i^*$ has full rank n_i. Consider a Gaussian random vector

$$X = (X_i, \dots, X_m) \sim N(0, UBU^*),$$

where $X_i \sim N(0, B_i)$. Using assumption (1.11), we apply Theorem 1(i) to get that

$$\mathbb{E} \prod_{i=1}^m f_i(X_i) \leq \prod_{i=1}^m \Big(\mathbb{E} f_i(X_i)^{p_i} \Big)^{1/p_i}. \tag{4.9}$$

Write $B = \Sigma\Sigma^*$ for some nonsingular matrix Σ. Then the covariance matrix of X_i can be written as $B_i = U_i B U_i^* = (U_i\Sigma)(U_i\Sigma)^*$. Thus, by the change of variables $y = \Sigma x$, we have that

$$\begin{aligned}\mathbb{E} \prod_{i=1}^m f_i(X_i) &= \int_{\mathbb{R}^n} \prod_{i=1}^m f_i\big(U_i \Sigma x\big) d\gamma_n(x) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} \det(B)^{\frac{1}{2}}} \int_{\mathbb{R}^n} \prod_{i=1}^m f_i\big(U_i x\big) \exp\left(-\frac{1}{2}\langle x, B^{-1}x\rangle\right) dx.\end{aligned}$$

On the other hand,

$$\mathbb{E} f_i(X_i)^{p_i} = \frac{1}{(2\pi)^{\frac{n_i}{2}} \det(B_i)^{\frac{1}{2}}} \int_{\mathbb{R}^{n_i}} f_i(x_i)^{p_i} \exp\left(-\frac{1}{2}\langle x_i, B_i^{-1}x_i\rangle\right) dx_i.$$

Finally, taking σB instead of B for $\sigma > 0$ and using the homogeneity condition (1.10), we can write (4.9) equivalently as

$$\begin{aligned}&\int_{\mathbb{R}^n} \prod_{i=1}^m f_i\big(U_i x\big) \exp\left(-\frac{1}{2\sigma}\langle x, B^{-1}x\rangle\right) dx \\ &\leq \left(\frac{\det(B)}{\prod_{i=1}^m \det(B_i)^{\frac{1}{p_i}}}\right)^{\frac{1}{2}} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x_i)^{p_i} \exp\left(-\frac{1}{2\sigma}\langle x_i, B_i^{-1}x_i\rangle\right) dx_i\right)^{\frac{1}{p_i}}.\end{aligned}$$

Letting $\sigma \to +\infty$, we get (1.12).

For the equality case, using lemma 3, and taking the functions

$$f_i(x_i) = \exp(-p_i^{-1}\left\langle B_i^{-1}x_i, x_i\right\rangle),$$

a direct computation gives the equality in (1.12). To prove (ii), one may proceed similarly by using Theorem 1(ii). □

5. A generalization of Barthe's lemma.

We begin this section by recalling the following result of F. Barthe from [4].

Proposition 3. *Assume that* $p, q, r > 1$ *with* $1/p + 1/q = 1 + 1/r$ *and set* $c = \sqrt{\frac{r'}{q'}}$ *and* $s = \sqrt{\frac{r'}{p'}}$. *For any* f, g, F, G *continuous and positive functions in* $L^1(\mathbb{R})$ *satisfying* $\int f = \int F$ *and* $\int g = \int G$, *we have that*

$$\begin{aligned} &\left(\int \left(\int f^{\frac{1}{p}}(cx - sy)\, g^{\frac{1}{q}}(sx + cy)\, dx\right)^r dy\right)^{\frac{1}{r}} \\ &\qquad \leq \int \left(\int F^{\frac{r}{p}}(cX - sY)\, G^{\frac{r}{q}}(sX + cY)\, dY\right)^{\frac{1}{r}} dX. \end{aligned} \tag{5.1}$$

As we have mentioned in the introduction, starting from this lemma, Barthe presented a simplified proof for both the sharp Young and reverse Young inequalities ([4]). Later he further showed in [5], that a generalization of this lemma to more than two functions, can been used to prove the rank 1 case of both, the Brascamp-Lieb and Barthe inequalities. In this section, using Theorem 2, we will derive a more general form of his lemma, from wich one can retrieve the general case (rank > 1) of the Brascamp-Lieb and Barthe inequalities.

For notational convenience, we set the following two conditions, that we will use throughout this section.

($A1$) Let $m, n, n_1, \ldots, n_m$ be positive integers. Denote by U_i a $n_i \times n$ matrix with $\operatorname{rank}(U_i) = n_i$ for $i \leq m$. Set $N = \sum_{i=1}^m n_i$, and let U be the $N \times n$ matrix with block rows $U_1, \ldots, U_m$.

($A2$) Let $c_1, \ldots, c_m$ be positive numbers and A be a $n \times n$-dimensional real symmetric and positive definite matrix. Set $A_i = U_i A U_i^*$ for $i \leq m$ and suppose that

$$U^* C_A U = A^{-1}, \tag{5.2}$$

where $C_A := \operatorname{diag}\big(c_1 A_1^{-1}, \ldots, c_m A_m^{-1}\big)$.

Remark 5. Since $\operatorname{rank}(U_i) = n_i$ and A is symmetric positive definite, it implies that $n_i \leq n$, A_i^{-1} exists and C_A is well-defined. Moreover, from (5.2), Lemma 3 implies the homogeneity condition

$$\sum_{i=1}^m c_i n_i = n$$

together with $UAU^* \leq C_A^{-1}$. Thus, $N \geq n$.

Remark 6. The assumption (5.2) ensures that there exists a $N \times (N - n)$ matrix W with $\operatorname{rank}(W) = N - n$ such that

$$\sqrt{C_A} U A U^* \sqrt{C_A} + W W^* = I_N. \tag{5.3}$$

In other words, the row vectors of the $N \times N$ matrix $(\sqrt{C_A} U \sqrt{A}, W)$ form an orthonormal basis of $\mathbb{R}^N$. To see this, recall that for any real matrix M, M^*M and MM^* have the same non-zero eigenvalues with the same algebraic multiplicities. If we set $M = \sqrt{C_A} U \sqrt{A}$, then (5.2) reads $M^*M = I_n$ and thus, $I_N - \sqrt{C_A} U A U^* \sqrt{C_A} = I_N - MM^*$ is symmetric positive semi-definite and has rank $N - n$. Lemma 2 guarantees then, the existence of W.

Theorem 5. *Assume that* ($A1$) *and* ($A2$) *hold and* W *satisfies* (5.3). *For* $\rho > 0$, *set*

$$\Gamma_\rho = \frac{(\det(A))^{\frac{1}{2}}}{\rho^{\frac{N-n}{2\rho}}} \prod_{i=1}^m \frac{(c_i p_i)^{\frac{n_i}{2p_i}}}{(\det(A_i))^{\frac{1}{2p_i}}}, \tag{5.4}$$

where

$$p_i := \frac{1}{c_i(1 + \frac{1 - c_i}{\rho c_i})}. \tag{5.5}$$

Let f_i be nonnegative measurable function on $\mathbb{R}^{n_i}$ for $i \le m$ and define

$$F(x,y) = \prod_{i=1}^m f_i\Big(U_i x + \frac{1}{\sqrt{c_i}}\sqrt{A_i} W_i y\Big), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^{N-n}.$$

(i) For $\rho \ge 1$,

$$\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^{N-n}} F^\rho(x,y)\,dy\Big)^{\frac{1}{\rho}} dx \ge \Gamma_\rho \prod_{i=1}^m \|f_i\|_{p_i} \ge \Big(\int_{\mathbb{R}^{N-n}} \Big(\int_{\mathbb{R}^n} F(x,y)\,dx\Big)^\rho dy\Big)^{\frac{1}{\rho}}. \quad (5.6)$$

(ii) For $0 < \rho \le 1$,

$$\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^{N-n}} F^\rho(x,y)\,dy\Big)^{\frac{1}{\rho}} dx \le \Gamma_\rho \prod_{i=1}^m \|f_i\|_{p_i} \le \Big(\int_{\mathbb{R}^{N-n}} \Big(\int_{\mathbb{R}^n} F(x,y)\,dx\Big)^\rho dy\Big)^{\frac{1}{\rho}}. \quad (5.7)$$

Moreover, one has equality in (5.6) *and* (5.7) *if $\rho = 1$ (for any choice of the f_i's) or if*

$$f_i(x) = \exp\big(-c_i \langle x, A_i^{-1} x\rangle\big), \ \ x \in \mathbb{R}^{n_i} \qquad 1 \le i \le m$$

(for any choice of $\rho > 0$).

The verification of the equality cases in (5.6) and (5.7) can be easily carried out with a routine computation and will be omitted. To show (5.6) and (5.7), we will apply Theorem 2 with suitable chosen matrices. The proof consists in two parts. In the first part we consider the "geometric" form, where $A = I_n$ and $U_i U_i^* = I_{n_i}$ for $i \le m$, while in the second part we deal with the general form of the theorem.

The core of our argument is based on a idea of Brascamp and Lieb from [17], where the authors proved that, the Precopa-Leindler inequality can be retrieved from their reverse sharp Young inequality. Theorem 5 is the result of our effort to generalize this proof of Brascamp and Lieb, in order to retrieve Barthe's inequality, from our Theorem 2(ii).

First, we recall a standard result for positive definite matrices, for the proof of which we refer to Theorem 1.3.3 in the book [10]

Lemma 4. *Let k, d be positive integers, A and B be two $k \times k$ and $d \times d$ real symmetric and positive definite matrices respectively and X be a $d \times k$ matrix. Then*

$$\begin{pmatrix} A & X^* \\ X & B \end{pmatrix} \ge 0 \ \Leftrightarrow \ B - XA^{-1}X^* \ge 0. \quad (5.8)$$

Proof of Theorem 5. We recall some standard notations. For any positive integers k, d, we denote the $k \times d$-dimensional zero matrix by $\mathbb{O}_{k\times d}$ or simply $\mathbb{O}$ whenever there is no ambiguity. For any real number r, r' stands for the Hölder conjugate exponent. Since for $\rho = 1$, Theorem 5 holds trivially true by Fubini's Theorem, we may assume without loss of generality that $\rho \ne 1$. The first part of our argument runs as follows.

Part I : $A = I_n$ and $U_i U_i^* = I_{n_i}$ for $i \le m$.

In this case, we can rewrite (5.2) and (5.3) as

$$U^* C U = I_n, \quad (5.9)$$

$$\sqrt{C} U U^* \sqrt{C} + WW^* = I_N, \quad (5.10)$$

where $C := C_{I_n} = \mathrm{diag}(c_1 I_{n_1}, \dots, c_m I_{n_m})$. Set g_i by $g_i(\sqrt{c_i} x_i) = f_i(x_i)$ for $x_i \in \mathbb{R}^{n_i}$ and define

$$G(x) = \int_{\mathbb{R}^{N-n}} \prod_{i=1}^m g_i^\rho\big(\sqrt{c_i} U_i x + W_i y\big) dy, \qquad x \in \mathbb{R}^n,$$

and

$$\tilde{G}(y) = \int_{\mathbb{R}^n} \prod_{i=1}^m g_i\big(\sqrt{c_i} U_i x + W_i y\big)\,dx, \qquad y \in \mathbb{R}^{N-n}.$$

First, we prove the left-hand side of (5.6). We set $r := 1/\rho$, and using the duality relation (3.20) we write

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{N-n}} F^\rho(x,y)\,dy\right)^{\frac{1}{\rho}}\,dx &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{N-n}}\prod_{i=1}^m g_i^\rho\big(\sqrt{c_i}U_i x + W_i y\big)\,dy\right)^{\frac{1}{\rho}}\,dx\\ &= \|G\|_r^r\\ &= \left(\inf_{\|H\|_{r'}=1}\int_{\mathbb{R}^n} H(x)G(x)\,dx\right)^r\\ &= \left(\inf_{\|H\|_{r'}=1}\int_{\mathbb{R}^N} H(V_0 z)\prod_{i=1}^m g_i^\rho\big(V_i z\big)\,dz\right)^{1/\rho}\end{aligned}$$

where $V_0 = \big(I_n\ \mathbb{O}_{n\times(N-n)}\big)$, and $V_i = \big(\sqrt{c_i}U_i\ W_i\big)$, $1\le i\le m$.

In what follows, we prove that the following inequality holds true

$$\int_{\mathbb{R}^N} H(V_0 z)\prod_{i=1}^m g_i^\rho\big(V_i z\big)\,dz \ \ge\ \Gamma_\rho^\rho\prod_{i=1}^m \|f_i\|_{p_i}^\rho, \tag{5.11}$$

for every nonnegative H such that $\|H\|_{r'}=1$, and this will prove the left hand side of (5.6). To do so, we use Theorem 2(ii) for the following choice of matrices:

$$V = \begin{pmatrix} V_0\\ V_1\\ \vdots\\ V_m\end{pmatrix} = \begin{pmatrix} I_n & \mathbb{O}_{n\times(N-n)}\\ \sqrt{C}U & W\end{pmatrix} \tag{5.12}$$

and

$$B = \begin{pmatrix} I_n & \mathbb{O}\\ \mathbb{O} & \frac{1}{\rho}I_{N-n}\end{pmatrix},\quad Q = \operatorname{diag}\big(r'I_n, q_1 I_{n_1},\dots,q_m I_{n_m}\big), \tag{5.13}$$

where $q_i := p_i/\rho$.

A straightforward computation gives that

$$VBV* = \begin{pmatrix} I_n & U^*\sqrt{C}\\ \sqrt{C}U & R_\rho\end{pmatrix}$$

where $R_\rho = \sqrt{C}UU^*\sqrt{C} + \frac{1}{\rho}WW^*$. So, using the identity (5.10) and the assumption that $U_iU_i^* = I_{n_i}$, we get

$$\begin{aligned} D_{VBV^*} &:= \operatorname{diag}\Big(V_0BV_0^*, V_1BV_1^*,\dots,V_mBV_m^*\Big)\\ &= \operatorname{diag}\left(I_n, \Big(c_1+\frac{1-c_1}{\rho}\Big)I_{n_1},\dots,\Big(c_m+\frac{1-c_m}{\rho}\Big)I_{n_m}\right)\\ &= \operatorname{diag}\left(I_n, \frac{1}{p_1}I_{n_1},\dots,\frac{1}{p_m}I_{n_m}\right)\end{aligned}$$

In order to apply Theorem 2(ii), for this set of matrices, we need to check its assumptions. Recall first that $\sum_{i=1}^m c_i n_i = n$ and $\sum_{i=1}^m n_i = N$, and so

$$\frac{n}{r'} + \sum_{i=1}^m \frac{n_i}{q_i} = n(1-\rho) + \rho\sum_{i=1}^m n_i c_i + \sum_{i=1}^m n_i(1-c_i) = N,$$

and thus the homogeneity condition (1.10) holds true. Moreover we need to check that

$$VBV^* - QD_{VBV^*} = \begin{pmatrix} (1-r')I_n & U^*\sqrt{C}\\ \sqrt{C}U & \Delta_\rho\end{pmatrix} \ge 0, \tag{5.14}$$

where

$$\begin{aligned}\Delta_\rho &= \sqrt{C}UU^*\sqrt{C} + \frac{1}{\rho}WW^* - \operatorname{diag}\Big(\frac{q_1}{p_1}I_{n_1}, \dots, \frac{q_m}{p_m}I_{n_m}\Big)\\ &= \sqrt{C}UU^*\sqrt{C} + WW^* - \Big(1-\frac{1}{\rho}\Big)WW^* - \frac{1}{\rho}I_N\\ &= \Big(1-\frac{1}{\rho}\Big)\big(I_N - WW^*\big) = \Big(1-\frac{1}{\rho}\Big)\sqrt{C}UU^*\sqrt{C}\end{aligned}$$

Note finally, that using the identity (5.10) again, we get

$$\Delta_\rho - \sqrt{C}U\Big((1-r')\,I_n\Big)^{-1}U^*\sqrt{C} = \Delta_\rho - \Big(1-\frac{1}{\rho}\Big)\sqrt{C}UU^*\sqrt{C} = 0.$$

and so by lemma 4 we have that (5.14) holds also true.

So by applying Theorem 2(ii), we get that

$$\begin{aligned}\int_{\mathbb{R}^N} H(V_0z)\prod_{i=1}^m g_i^\rho\big(V_iz\big)\,dz &\geq \left(\frac{\det(B)}{\det(V_0BV_0^*)^{\frac{1}{r'}}\prod_{i=1}^m\det(V_iBV_i^*)^{\frac{1}{q_i}}}\right)^{\frac{1}{2}}\prod_{i=1}^m\|g_i^\rho\|_{q_i}\\ &= \left(\frac{\rho^{-(N-n)}}{1^{\frac{1}{r'}}\prod_{i=1}^m p_i^{-\frac{\rho n_i}{p_i}}}\right)^{\frac{1}{2}}\prod_{i=1}^m c_i^{\frac{\rho n_i}{2p_i}}\|f_i\|_{p_i}^\rho\\ &= \Gamma_\rho^\rho\prod_{i=1}^m\|f_i\|_{p_i}^\rho,\end{aligned}$$

where in the last equality we used that $\sum_{i=1}^m c_in_i = n$, $\sum_{i=1}^m n_i = N$ and the change of variables $\|g_i\|_{p_i/\rho} = c_i^{\rho n_i/2p_i}\|f_i\|_{p_i}^\rho$. This proves (5.11), and thus the left-hand side of (5.6).

We now turn to the right-hand side of (5.6). Using the duality relation (3.19) we have that

$$\begin{aligned}\left(\int_{\mathbb{R}^{N-n}}\left(\int_{\mathbb{R}^n}F(x,y)\,dx\right)^\rho dy\right)^{\frac{1}{\rho}} &= \left(\int_{\mathbb{R}^{N-n}}\left(\int_{\mathbb{R}^n}\prod_{i=1}^m g_i\big(\sqrt{c_i}U_ix + W_iy\big)\,dx\right)^\rho dy\right)^{\frac{1}{\rho}}\\ &= \|\tilde{G}\|_\rho\\ &= \sup_{\|H\|_{\rho'}=1}\int_{\mathbb{R}^{N-n}}H(y)\tilde{G}(y)\,dx\\ &= \sup_{\|H\|_{\rho'}=1}\int_{\mathbb{R}^N}H(\tilde{V}_0z)\prod_{i=1}^m g_i\big(V_iz\big)\,dz\end{aligned}$$

where $\tilde{V}_0 = \big(\mathbb{O}_{(N-n)\times n}\ \ I_{N-n}\big)$, and $V_i = \big(\sqrt{c_i}U_i\ \ W_i\big)$, $1 \leq i \leq m$, as before. Similarly to the previous case, we will show that the following inequality holds true

$$\int_{\mathbb{R}^N}H(\tilde{V}_0z)\prod_{i=1}^m g_i\big(V_iz\big)\,dz \,\leq\, \Gamma_\rho\prod_{i=1}^m\|f_i\|_{p_i}, \tag{5.15}$$

for every H such that $\|H\|_{\rho'} = 1$, and this will prove the right-hand side of (5.6). Now we will use Theorem 2(i) for the matrices:

$$\tilde{V} = \begin{pmatrix}\tilde{V}_0\\ V_1\\ \vdots\\ V_m\end{pmatrix} = \begin{pmatrix}\mathbb{O}_{(N-n)\times n} & I_{N-n}\\ \sqrt{C}U & W\end{pmatrix} \tag{5.16}$$

and

$$\tilde{B} = \begin{pmatrix} \rho I_n & \mathbb{O} \\ \mathbb{O} & I_{N-n} \end{pmatrix}, \quad P = \mathrm{diag}\big(\rho' I_{N-n}\,,\, p_1 I_{n_1}\,,\, \dots\,,\, p_m I_{n_m}\big). \tag{5.17}$$

This time we have that

$$\tilde{V}\tilde{B}\tilde{V}^* = \begin{pmatrix} I_{N-n} & W^* \\ W & \tilde{R}_\rho \end{pmatrix}$$

where $\tilde{R}_\rho = \rho\sqrt{C}UU^*\sqrt{C} + WW^*$. Using the identity (5.10) and the fact $U_iU_i^* = I_{n_i}$, we get

$$\begin{aligned} D_{\tilde{V}\tilde{B}\tilde{V}^*} &:= \mathrm{diag}\Big(\tilde{V}_0\tilde{B}\tilde{V}_0^*, \tilde{V}_1\tilde{B}\tilde{V}_1^*, \dots, \tilde{V}_m\tilde{B}\tilde{V}_m^*\Big) \\ &= \mathrm{diag}\Big(I_{N-n}, \rho\Big(c_1 + \frac{1-c_1}{\rho}\Big)I_{n_1}, \dots, \rho\Big(c_m + \frac{1-c_m}{\rho}\Big)I_{n_m}\Big) \\ &= \mathrm{diag}\Big(I_{N-n}, \frac{\rho}{p_1}I_{n_1}, \dots, \frac{\rho}{p_m}I_{n_m}\Big) \end{aligned}$$

To apply Theorem 2(i), for this set of matrices, we check its assumptions. Note first that

$$\frac{N-n}{\rho'} + \sum_{i=1}^m \frac{n_i}{p_i} = (N-n)\frac{\rho-1}{\rho} + \sum_{i=1}^m n_i c_i + \sum_{i=1}^m n_i \frac{1-c_i}{\rho} = N,$$

and thus the homogeneity condition (1.10) holds true. We need also to check that

$$\tilde{V}\tilde{B}\tilde{V}^* - PD_{\tilde{V}\tilde{B}\tilde{V}^*} = \begin{pmatrix} (1-\rho')I_{N-n} & W^* \\ W & \tilde{\Delta}_\rho \end{pmatrix} \le 0, \tag{5.18}$$

where

$$\begin{aligned} \tilde{\Delta}_\rho &= \rho\,\sqrt{C}UU^*\sqrt{C} + WW^* - \mathrm{diag}\Big(\rho\, I_{n_1}, \dots, \rho\, I_{n_m}\Big) \\ &= \sqrt{C}UU^*\sqrt{C} + WW^* - (1-\rho)\,\sqrt{C}UU^*\sqrt{C} - \rho\, I_N \\ &= (1-\rho)\,\big(I_N - \sqrt{C}UU^*\sqrt{C}\big) = (1-\rho)\; WW^*. \end{aligned}$$

Note finally, that using the identity (5.10) again, we get

$$\tilde{\Delta}_\rho - W\Big((1-\rho')\, I_n\Big)^{-1} W^* = \tilde{\Delta}_\rho - (1-\rho)\, WW^* = 0.$$

and so by lemma 4 we have that (5.18) holds also true. Applying now Theorem 2(i), we get that

$$\begin{aligned} \int_{\mathbb{R}^N} H(\tilde{V}_0 z) \prod_{i=1}^m g_i\big(V_i z\big)\, dz &\le \left(\frac{\det(\tilde{B})}{\det(\tilde{V}_0\tilde{B}\,\tilde{V}_0^*)^{\frac{1}{\rho'}} \prod_{i=1}^m \det(V_i\tilde{B}\,V_i^*)^{\frac{1}{p_i}}} \right)^{\frac{1}{2}} \prod_{i=1}^m \|g_i\|_{p_i} \\ &= \left(\frac{\rho^n}{\prod_{i=1}^m \left(\frac{\rho}{p_i}\right)^{\frac{n_i}{p_i}}} \right)^{\frac{1}{2}} \prod_{i=1}^m c_i^{\frac{n_i}{2p_i}} \|f_i\|_{p_i} \\ &= \Gamma_\rho \prod_{i=1}^m \|f_i\|_{p_i}, \end{aligned}$$

where in the last equality we used that $\sum_{i=1}^m c_i n_i = n$, $\sum_{i=1}^m n_i = N$ and the change of variables $\|g_i\|_{p_i} = c_i^{n_i/2p_i}\|f_i\|_{p_i}$. This proves (5.15), and thus the right-hand side of (5.6).

The proof of (5.7) for $0 < \rho \le 1$ is identical and is omitted.

Part II : The general case.

Now we are not assuming that $A = I_n$ and $U_i U_i^* = I_{n_i}$. In order to reduce the general case to the previous case, we set $\widetilde{U}_i = \sqrt{A_i} U_i \sqrt{A}$ and let $\widetilde{U}$ be the $N \times n$ matrix with block rows $\widetilde{U}_1, \ldots, \widetilde{U}_m$. Then, by the assumptions $(A1)$ and $(A2)$, one sees that

$$\widetilde{U}_i \widetilde{U}_i^* = I_{n_i}, \ \forall i \leq m,$$
$$\widetilde{U}^* C \widetilde{U} = I_n,$$
$$\sqrt{C}\widetilde{U}\widetilde{U}^* \sqrt{C} + WW^* = I_N,$$

where $C := \mathrm{diag}(c_1 I_{n_1}, \ldots, c_n I_{n_m})$. We define $h_i(x) := f_i(\sqrt{A_i} x)$, $x \in \mathbb{R}^{n_i}$, and using the first case, we can apply the left-hand side of (5.6), to $h_i' s$ and $\widetilde{U}$, and get that if for example $\rho \geq 1$,

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{N-n}} \prod_{i=1}^m h_i^\rho \Big(\widetilde{U}_i x + \frac{1}{\sqrt{c_i}} W_i y \Big) dy \right)^{\frac{1}{\rho}} dx \geq \frac{1}{\rho^{\frac{N-n}{2\rho}}} \prod_{i=1}^m (c_i p_i)^{\frac{n_i}{2p_i}} \|h_i\|_{p_i}. \tag{5.19}$$

The change of variables $x \mapsto \sqrt{A} x$ leads to

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{N-n}} \prod_{i=1}^m h_i^\rho \Big(\widetilde{U}_i x + \frac{1}{\sqrt{c_i}} W_i y \Big) dy \right)^{\frac{1}{\rho}} dx = \frac{1}{\det(A)^{\frac{1}{2}}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{N-n}} F^\rho \, dy \right)^{\frac{1}{\rho}} dx \tag{5.20}$$

and again, $x_i \mapsto \sqrt{A_i}\, x_i$ gives

$$\|h_i\|_{p_i} = \left(\int_{\mathbb{R}^{n_i}} f_i^{p_i} \big(\sqrt{A_i}\, x_i \big) \, dx_i \right)^{\frac{1}{p_i}} = \frac{1}{\det(A_i)^{\frac{1}{2p_i}}} \|f_i\|_{p_i}. \tag{5.21}$$

Combining (5.19), (5.20) and (5.21) implies

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{N-n}} F^\rho(x,y) \, dy \right)^{\frac{1}{\rho}} dx \geq \frac{\det(A)^{\frac{1}{2}}}{\rho^{\frac{N-n}{2\rho}}} \prod_{i=1}^m \left(\frac{(c_i p_i)^{n_i}}{\det(A_i)} \right)^{\frac{1}{2p_i}} \|f_i\|_{p_i} = \Gamma_\rho \prod_{i=1}^m \|f_i\|_{p_i}.$$

The proof for the other inequalities is identical and will be omitted. □

6. Applications of Theorem 5

6.1. **Convolution inequalities.** Recall the notations from $(A1)$ and $(A2)$. Assume that $m = 2$, $n_1 = n_2 = n$, $N = 2n$ and for any $\lambda \in [0,1]$, consider the following trivial decomposition of the identity in $\mathbb{R}^n$

$$\lambda\, I_n + (1-\lambda)\, I_n = I_n.$$

Set $c_1 = \lambda$, $c_2 = 1 - \lambda$, $U_1 = U_2 = A = I_n$ and $W_1 = \sqrt{1-\lambda} I_n$, $W_2 = -\sqrt{\lambda} I_n$. Then a direct computation shows that (5.2) and (5.3) hold and thus, Theorem 5 reads

Proposition 4. *Let f_1, f_2 be nonnegative measurable functions on $\mathbb{R}^n$ and $\lambda \in [0,1]$. Define the function*

$$F_\lambda(x,y) = f_1\Big(x + \sqrt{\frac{1-\lambda}{\lambda}} y \Big) f_2 \left(x - \sqrt{\frac{\lambda}{1-\lambda}} y \right), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$$

and for $\rho > 0$, set

$$p_1 = \frac{\rho}{(\rho-1)\lambda + 1}, \qquad p_2 = \frac{\rho}{(\rho-1)(1-\lambda)+1},$$

and

$$\Im_\rho = \left(\frac{\lambda^{\frac{1}{p_1}} (1-\lambda)^{\frac{1}{p_2}} \ \, p_1^{\frac{1}{p_1}} p_2^{\frac{1}{p_2}}}{\rho^{\frac{2}{\rho}}} \right)^{\frac{n}{2}}.$$

(i) *If* $\rho \geq 1$, *then*

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} F_\lambda^\rho(x,y)\,dy\right)^{\frac{1}{\rho}} dx \;\geq\; \Im_\rho\,\|f_1\|_{p_1}\,\|f_2\|_{p_2} \;\geq\; \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} F_\lambda(x,y)\,dx\right)^{\rho} dy\right)^{\frac{1}{\rho}}. \tag{6.1}$$

(ii) *If* $0 \leq \rho \leq 1$, *then*

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} F_\lambda^\rho(x,y)\,dy\right)^{\frac{1}{\rho}} dx \;\leq\; \Im_\rho\,\|f_1\|_{p_1}\,\|f_2\|_{p_2} \;\leq\; \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} F_\lambda(x,y)\,dx\right)^{\rho} dy\right)^{\frac{1}{\rho}}. \tag{6.2}$$

Sharp Young and reverse Young inequalities. By a change of variables, Proposition 4 can also be read as

Proposition 5. *Let* f_1, f_2 *be non-negative measurable functions on* $\mathbb{R}^n$. *For any* $\lambda \in (0,1)$, *let* p_1, p_2 *and* $\Im_\rho$ *be as in Proposition 4 and set*

$$\Im'_\rho = \frac{\Im_\rho}{\big(\lambda(1-\lambda)\big)^{\frac{n}{2\rho}}}.$$

(i) *If* $\rho \geq 1$ *then*

$$\|f_1 * f_2\|_\rho \;\leq\; \Im'_\rho\,\|f_1\|_{p_1}\,\|f_2\|_{p_2} \;\leq\; \|(f_1^\rho * f_2^\rho)^{\frac{1}{\rho}}\|_1. \tag{6.3}$$

(ii) *If* $0 \leq \rho \leq 1$, *then*

$$\|f_1 * f_2\|_\rho \;\geq\; \Im'_\rho\,\|f_1\|_{p_1}\,\|f_2\|_{p_2} \;\geq\; \|(f_1^\rho * f_2^\rho)^{\frac{1}{\rho}}\|_1. \tag{6.4}$$

This is indeed, a reformulation of the sharp Young and reverse Young inequalities. To see this, suppose that $p, q, r > 0$ satisfy $p^{-1} + q^{-1} = 1 + r^{-1}$. Choose $\rho = r$ and $\lambda = r'/q'$ in Proposition 5, where r', q' are conjugate exponents of r, q, respectively. Then $p_1 = p$, $p_2 = q$ and $\Im'_\rho = C^n$, where C is defined in (4.1) and (4.2). If $p, q, r \geq 1$, the left-hand side of (6.3) gives (4.1), while if $0 < p, q, r < 1$, the left-hand side of (6.4) gives (4.2).

Prekopa-Leindler inequality. Letting $\rho \to \infty$ in Proposition 4, the right-hand side of (6.1) gives Hölder's inequality. As for the left-hand side, Prékopa-Leindler inequality [30, 33], which is the functional form of the Brunn-Minkowski inequality, the cornerstone of the Brunn-Minkowski theory. For more information on this subject, we refer the reader to the book [34] and the survey paper of Gardner [24].

Theorem 6 (Prékopa-Leindler's inequality)**.** *Let* f, g, h *be three nonnegative measurable functions in* $\mathbb{R}^n$ *and* $\lambda \in [0,1]$ *such that*

$$h(\lambda x + (1-\lambda)y) \geq f^\lambda(x) g^{1-\lambda}(y),\ \forall x, y \in \mathbb{R}^n. \tag{6.5}$$

Then

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{\lambda}\left(\int_{\mathbb{R}^n} g(x)dx\right)^{1-\lambda}. \tag{6.6}$$

Proof. Applying the left-hand side inequality of (6.1) to $f_1 := f^\lambda$ and $f_2 := g^{1-\lambda}$ and sending ρ to ∞, we get that

$$\int_{\mathbb{R}^n} \mathrm{essup}_{y\in\mathbb{R}^n} f^\lambda\left(x + \sqrt{\frac{1-\lambda}{\lambda}}y\right) g^{1-\lambda}\left(x - \sqrt{\frac{\lambda}{1-\lambda}}y\right) dx \;\; \geq \left(\int f\right)^{\lambda}\left(\int g\right)^{1-\lambda},$$

Notice that from (6.5),

$$\begin{aligned}&\mathrm{essup}_{y\in\mathbb{R}^n} f^\lambda\left(x+\sqrt{\frac{1-\lambda}{\lambda}}y\right)g^{1-\lambda}\left(x-\sqrt{\frac{\lambda}{1-\lambda}}y\right)\\ &\le \mathrm{essup}_{y\in\mathbb{R}^n} h\left(\lambda\left(x+\sqrt{\frac{1-\lambda}{\lambda}}y\right)+(1-\lambda)\left(x-\sqrt{\frac{\lambda}{1-\lambda}}y\right)\right)=h(x).\end{aligned}$$

This gives (6.6). □

Remark 7. The proof of Theorem 6 actually gives the essential supremum strengthened version of the Prékopa-Leindler's inequality, proved by Brascamp and Lieb in [18], which also avoids problems of measurability. We refer to the Appendix of [18] and Section 9 in [24] for more details.

6.2. **Brascamp-Lieb and Barthe inequalities.** Theorem 5, without the restriction $m=2$, leads to the Brascamp-Lieb and Barthe inequalities by letting again ρ in (5.6) tend to infinity.

Theorem 7. *Assume that (A1) and (A2) in Section 5 hold. Then*

(i) (Brascamp-Lieb's inequality). *For any nonnegative measurable functions f_i on $\mathbb{R}^{n_i}$, $i\le m$, we have that*

$$\int_{\mathbb{R}^n}\prod_{i=1}^m f_i(U_i x)\,dx \le \left(\frac{\det(A)}{\prod_{i=1}^m \det(A_i)^{c_i}}\right)^{\frac{1}{2}}\prod_{i=1}^m \|f_i\|_{\frac{1}{c_i}}. \tag{6.7}$$

The equality holds if

$$f_i(x_i)=\exp(-c_i\langle A_i^{-1}x_i,x_i\rangle),\quad i\le m$$

(ii) (Barthe's inequality). *For any nonnegative measurable functions f_i on $\mathbb{R}^{n_i}$, $i\le m$ and f on $\mathbb{R}^n$ that satisfy*

$$f\left(\sum_{i=1}^m c_i U_i^* x_i\right)\ge\prod_{i=1}^m f_i(x_i),\quad x_i\in\mathbb{R}^{n_i} \tag{6.8}$$

we have that

$$\prod_{i=1}^m \|f_i\|_{\frac{1}{c_i}}\le\left(\frac{\det(A)}{\prod_{i=1}^m \det(A_i)^{c_i}}\right)^{\frac{1}{2}}\int_{\mathbb{R}^n} f(x)dx. \tag{6.9}$$

The equality holds if

$$f_i(x_i)=\exp(-c_i\langle A_i x_i,x_i\rangle/2),\quad i\le m,$$

and

$$f(x)=\exp(-\langle Ax,x\rangle/2).$$

Remark 8. Theorem 7 actually retrieves the recent work of Lehec in [29] and differs from the initial statements of the Brascamp-Lieb and Barthe inequalities. However, the author in [29] provides an argument on how the initial statements can always be recovered from Theorem 7.

Proof of Theorem 7. The statements that equalities can be realized by the given functions in both cases can be easily verified by direct computations. We will omit this part of the argument. Recall W from (5.3). To show (6.7), sending ρ in the right-hand side of (5.6) to infinity, one gets

$$\sup_{y\in\mathbb{R}^{N-n}}\int_{\mathbb{R}^n}\prod_{i=1}^m f_i\Big(U_ix+\frac{1}{\sqrt{c_i}}\sqrt{A_i}W_iy\Big)dx \\ \le \Big(\frac{\det(A)}{\prod_{i=1}^m\det(A_i)^{c_i}}\Big)^{\frac12}\prod_{i=1}^m\|f_i\|_{\frac{1}{c_i}}.$$

Note that here we have used the condition (5) in the limit $\lim_{\rho\to\infty}\Gamma_\rho$. The inequality (6.7) then follows by observing that

$$\int_{\mathbb{R}^n}\prod_{i=1}^m f_i(U_ix)dx \le \sup_{y\in\mathbb{R}^{N-n}}\int_{\mathbb{R}^n}\prod_{i=1}^m f_i\Big(U_ix+\frac{1}{\sqrt{c_i}}\sqrt{A_i}W_iy\Big)dx.$$

As for (6.9), define $g_i:\mathbb{R}^{n_i}\to\mathbb{R}_+$ by $g_i(x_i)=f_i(A_i^{-1}x_i)$. Note $\|g_i\|_p=\det(A_i)^{1/p}\|f_i\|_p$. Applying the left-hand side of (5.6) for the functions g_i's and then passing to the limit $\rho\to\infty$ by using (5), we obtain

$$\int_{\mathbb{R}^n}\sup_{y\in\mathbb{R}^{N-n}}\prod_{i=1}^m f_i\left(\frac{1}{\sqrt{c_i}}A_i^{-\frac12}(\sqrt{c_i}A_i^{-\frac12}U_ix+W_iy)\right)dx \\ \ge \det(A)^{\frac12}\prod_{i=1}^m\det(A_i)^{\frac{1}{2p_i}}\prod_{i=1}^m\|f_i\|_{p_i}$$

and by the change of variable $x=Az$,

$$\int_{\mathbb{R}^n}\sup_{y\in\mathbb{R}^{N-n}}\prod_{i=1}^m f_i\left(\frac{1}{\sqrt{c_i}}A_i^{-\frac12}(\overline{U}_i\sqrt{A}z+W_iy)\right)dz \\ \ge \frac{\prod_{i=1}^m\det(A_i)^{\frac{1}{2p_i}}}{\det(A)^{\frac12}}\prod_{i=1}^m\|f_i\|_{p_i},$$

where $\overline{U}_i:=\sqrt{c_i}A_i^{-1/2}U_i\sqrt{A}$. From (6.8), (6.9) will be valid if the following holds

$$\sup_{y\in\mathbb{R}^{N-n}}\prod_{i=1}^m f_i\left(\frac{1}{\sqrt{c_i}}A_i^{-\frac12}(\overline{U}_i\sqrt{A}z+W_iy)\right)=\sup_{(\xi_1,\dots,\xi_m):\sum_{i=1}^m c_iU_i^*\xi_i=z}\prod_{i=1}^m f_i(\xi_i). \tag{6.10}$$

To show this identity, we first claim that for functions $F_i:\mathbb{R}^{n_i}\to\mathbb{R}_+$ for $i\le m$, we have

$$\sup_{y\in\mathbb{R}^{N-n}}\prod_{i=1}^m F_i\left(V_i(x,y)\right)=\sup_{(a_1,\dots,a_m):\sum_{i=1}^m\overline{U}_i^*a_i=x}\prod_{i=1}^m F_i(a_i). \tag{6.11}$$

Recalling (5.3), if we set $V_i=(\overline{U}_i,W_i)$ for $i\le m$, then the rows of $V_1,\dots,V_m$ form an orthonormal basis of $\mathbb{R}^N$. Suppose that $a=(a_1,\dots,a_m)\in\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}$ satisfies $\sum_{i=1}^m\overline{U}_i^*a_i=x$. If we set $y=\sum_{i=1}^m W_ia_i\in\mathbb{R}^{N-n}$, then $V_i(x,y)=a_i$. This proves $\ge$ in (6.11). The proof of $\le$ is similar and this completes the proof of (6.11). Finally, applying (6.11) to

$$F_i(x_i)=f_i\Big(\frac{1}{\sqrt{c_i}}A_i^{-1}x_i\Big)$$

for $x_i \in \mathbb{R}^{n_i}$, we have that

$$\begin{aligned}
&\sup_{y\in\mathbb{R}^{N-n}} \prod_{i=1}^m f_i\left(\frac{1}{\sqrt{c_i}} A_i^{-\frac{1}{2}} (\overline{U}_i \sqrt{A} z + W_i y)\right) \\
&= \sup_{y\in\mathbb{R}^{N-n}} \prod_{i=1}^m f_i\left(\frac{1}{\sqrt{c_i}} A_i^{-\frac{1}{2}} V_i(\sqrt{A} z, y)\right) \\
&= \sup_{(a_1,\dots,a_m):\sum_{i=1}^m \overline{U}_i^* a_i = \sqrt{A} z} \prod_{i=1}^m f_i\left(\frac{1}{\sqrt{c_i}} A_i^{-\frac{1}{2}} a_i\right) \\
&= \sup_{(\xi_1,\dots,\xi_m):\sum_{i=1}^m c_i U_i^* \xi_i = z} \prod_{i=1}^m f_i(\xi_i),
\end{aligned}$$

where the last equality used change of variables $\xi_i = c_i^{-1/2} A_i^{-1/2} a_i$. This gives (6.10) and we are done. □

6.3. **An entropy inequality.** Finally we derive entropy inequalities for probability density functions. Let f be a positive measurable function in $\mathbb{R}^n$. We define the entropy of f by

$$\operatorname{Ent}(f) := \int_{\mathbb{R}^n} f(x)\log f(x)\,dx - \left(\int f(x)dx\right)\log\int f(x)dx,$$

whenever this quantity makes sense. Note that if $g(p) := \|f\|_p$, then

$$g'(1) = \operatorname{Ent}(f). \tag{6.12}$$

Let F and Γ_ρ be defined as in Theorem 5 and set the functions G_1, G_2, G_3 on $[0,\infty)$ by

$$\begin{aligned}
G_1(\rho) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{N-n}} F^\rho(x,y)\,dy\right)^{\frac{1}{\rho}} dx, \\
G_2(\rho) &= \Gamma_\rho \prod_{i=1}^m \|f_i\|_{p_i}, \\
G_3(\rho) &= \left(\int_{\mathbb{R}^{N-n}}\left(\int_{\mathbb{R}^n} F(x,y)\,dx\right)^\rho dy\right)^{\frac{1}{\rho}}.
\end{aligned}$$

Note that Fubini's theorem implies that $G_1(1) = G_2(1) = G_3(1)$ and Theorem 5 states that

$$G_1(\rho) \le G_2(\rho) \le G_3(\rho), \quad \text{if } \rho \le 1,$$

and

$$G_1(\rho) \ge G_2(\rho) \ge G_3(\rho), \quad \text{if } \rho \ge 1.$$

Putting all these together gives

$$G_3'(1) \le G_2'(1) \le G_1'(1), \tag{6.13}$$

which leads to the following entropy inequalities.

Proposition 6. *Assume that (A1) and (A2) hold and W satisfies (5.3). For any probability density g_i on $\mathbb{R}^{n_i}$, $i \le m$, set*

$$G(x,y) = \prod_{i=1}^m g_i\left(\sqrt{c_i} U_i x + W_i y\right) \tag{6.14}$$

Then

$$D_1 \operatorname{Ent}\Big(\int_{\mathbb{R}^n} G(x,\cdot)\,dx\Big) \;\le\; \sum_{i=1}^m (1-c_i)\operatorname{Ent}(g_i) + D_2 \;\le\; D_1 \int_{\mathbb{R}^n} \operatorname{Ent}\big(G(x,\cdot)\big)\,dx, \tag{6.15}$$

where

$$D_1 := \left(\frac{\prod_{i=1}^m \det(A_i)}{\det(A)}\right)^{\frac{1}{2}} \quad \text{and} \quad D_2 := \frac{1}{2}\sum_{i=1}^m (1-c_i)\log\det(A_i).$$

Proof. The idea of the proof is to compute the derivatives of G_1, G_2 and G_3 at $\rho = 1$. One shall see that they lead to the three quantities in (6.15) and the inequalities are preserved through (6.13). Note first that from (6.12), we have that $G_1'(1) = \int_{\mathbb{R}^n} \mathrm{Ent}\big(G(x,\cdot)\big)\,dx$ and $G_3'(1) = \mathrm{Ent}\big(\int_{\mathbb{R}^n} G(x,\cdot)\,dx\big)$. As for $G_2'(1)$, recalling that Γ_ρ from (5) and defining $\Omega_\rho = \prod_{i=1}^m \|f_i\|_{p_i}$, we get directly by definition that

$$\Gamma_1 = \left(\det(A)\prod_{i=1}^m \frac{c_i^{n_i}}{\det(A_i)}\right)^{\frac{1}{2}}, \ \Omega_1 = \prod_{i=1}^m \|f_i\|_1,$$

and a quite tedious computation yields

$$\begin{aligned}
\left.\frac{d\Gamma_\rho}{d\rho}\right|_{\rho=1} &= \det(A)^{\frac{1}{2}}\left(\prod_{i=1}^m \frac{c_i^{n_i/2}}{\sqrt{\det(A_i)}}\right)\left(\sum_{i=1}^m \frac{(1-c_i)n_i}{2}\left[\frac{\log\det(A_i)}{n_i} - \log c_i\right]\right),\\
\left.\frac{d\Omega_\rho}{d\rho}\right|_{\rho=1} &= \left(\prod_{i=1}^m \|f_i\|_1\right)\left(\sum_{i=1}^m (1-c_i)\frac{\mathrm{Ent}(f_i)}{\|f_i\|_1}\right).
\end{aligned}$$

Combining these all together gives

$$\begin{aligned}
G_2'(1) = &\left(\prod_{i=1}^m \|f_i\|_1\right)\left(\det(A)\prod_{i=1}^m \frac{c_i^{n_i}}{\det(A_i)}\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=1}^m \frac{n_i(1-c_i)}{2}\left(\frac{2\mathrm{Ent}(f_i)}{n_i\|f_i\|_1} + \frac{\log\det(A_i)}{n_i} - \log c_i\right)\right).
\end{aligned}$$

Set $f_i(x) := g_i(\sqrt{c_i}x)$. Observe that

$$c_i^{\frac{n_i}{2}}\int f_i = \int g_i = 1 \quad \text{and} \quad c_i^{\frac{n_i}{2}}\mathrm{Ent}(f_i) = \mathrm{Ent}(g_i) + \frac{n_i}{2}\log c_i.$$

So

$$\frac{2\mathrm{Ent}(f_i)}{n_i\|f_i\|_1} = \log c_i + \frac{2}{n_i}\mathrm{Ent}(g_i)$$

and thus,

$$\begin{aligned}
G_2'(1) &= \left(\frac{\det(A)}{\prod_{i=1}^m \det(A_i)}\right)^{\frac{1}{2}}\left(\sum_{i=1}^m (1-c_i)\mathrm{Ent}(g_i) + \frac{1}{2}\sum_{i=1}^m (1-c_i)\log\det(A_i)\right)\\
&= D_1^{-1}\sum_{i=1}^m (1-c_i)\mathrm{Ent}(g_i) + D_1^{-1}D_2.
\end{aligned}$$

Using our computations for $G_1'(1)$, $G_2'(1)$ and $G_3'(1)$, (6.13) completes our proof. $\square$

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W.-K. Chen, Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA

E-mail address: wkchen@math.uchicago.edu

N. Dafnis, Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

E-mail address: nikdafnis@gmail.com

G. Paouris, Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

E-mail address: grigorios.paouris@gmail.com