

Further unifying two approaches to the hyperplane conjecture

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Abstract

We compare and combine two approaches that have been recently introduced by Dafnis and Paouris [DP] and by Klartag and Milman [KM] with the aim of providing bounds for the isotropic constants of convex bodies. By defining a new hereditary parameter for all isotropic log-concave measures, we are able to show that the method in [KM], and the apparently stronger conclusions it leads to, can be extended in the full range of the “weaker” assumptions of [DP]. The new parameter we define is related to the highest dimension $k \leq n - 1$ in which one can always find marginals of an n -dimensional isotropic measure which have bounded isotropic constant.

1 Introduction

The purpose of this note is to compare two recent approaches to the hyperplane conjecture that have been introduced by Dafnis and Paouris in [DP] and by Klartag and Milman in [KM]; these are based on two fruitful techniques initially developed by Paouris (see [Pa1], [Pa2]) and by Klartag (see [K1]), namely the study of the L_q -centroid bodies and the use of the logarithmic Laplace transform of a measure. In [KM], Klartag and Milman were the first to observe that a combination of aspects of the two techniques can lead to better bounds for the isotropic constant problem in many interesting cases. Here we propose further combining their method with the approach in [DP]; this enables us to extend the range in which the former could be applied, and also to slightly improve the bounds that the latter can give us. The gluing ingredient in this paper is a variant of the main parameter in [DP], and is related to the highest dimension $k \leq n - 1$ in which we can find marginals of an n -dimensional isotropic measure which have bounded isotropic constant. Our results show some type of equivalence between the two approaches in question, and the bounds that they can provide for the isotropic constant problem, which might be improved through the study of the new parameter.

Let us now turn to the details. The hyperplane conjecture is one of the most well-known problems in Asymptotic Geometric Analysis. It asks whether the isotropic constant of every logarithmically-concave measure can be bounded by a quantity independent of the dimension of the measure. The notion of the isotropic constant, originally defined for convex bodies (see [B1]), has been generalised in the

setting of log-concave measures as follows: if μ is a log-concave measure on \mathbb{R}^n with density f_μ with respect to the Lebesgue measure, we set

$$\|\mu\|_\infty := \sup_{x \in \mathbb{R}^n} f_\mu(x)$$

and we define the isotropic constant of μ by

$$(1.1) \quad L_\mu := \left(\frac{\|\mu\|_\infty}{\int_{\mathbb{R}^n} f_\mu(x) dx} \right)^{\frac{1}{n}} [\det \text{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\text{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\text{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x) dx}{\int_{\mathbb{R}^n} f_\mu(x) dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x) dx}{\int_{\mathbb{R}^n} f_\mu(x) dx} \frac{\int_{\mathbb{R}^n} x_j f_\mu(x) dx}{\int_{\mathbb{R}^n} f_\mu(x) dx}.$$

We say that a log-concave measure μ on \mathbb{R}^n is isotropic (and we write $\mu \in \mathcal{I}\mathcal{L}_{[n]}$) if μ is a centered probability measure, i.e. a probability measure with barycentre at the origin, and if $\text{Cov}(\mu)$ is the identity matrix. Since every log-concave measure μ has an affine image which is isotropic, and since from the definition (1.1) of L_μ we see that the isotropic constant is an affine invariant, the hyperplane conjecture reduces to the question whether there exists an absolute constant C such that $L_n \leq C$ for all $n \geq 1$, where

$$L_n := \sup_{\mu \in \mathcal{I}\mathcal{L}_{[n]}} L_\mu = \sup_{\mu \in \mathcal{I}\mathcal{L}_{[n]}} \|\mu\|_\infty^{1/n}.$$

The first upper bound for L_n was given by Bourgain in [B2], $L_n \ll \sqrt[n]{n} \log n$, and a few years ago Klartag [K1] improved that bound to $L_n \ll \sqrt[n]{n}$; a second proof of the latter inequality is given in [KM]. More detailed information on isotropic log-concave measures (or more briefly in this paper, isotropic measures) is provided in the next section.

In [DP] Dafnis and Paouris observed that a way to obtain new bounds for L_n is to study the behaviour of the function $q \mapsto I_q(\mu)$, $q \in (-n, 0)$, where

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q f_\mu(x) dx \right)^{1/q}.$$

For every n -dimensional isotropic log-concave measure μ and every $\delta \geq 1$, they set

$$(1.2) \quad q_{-c}(\mu, \delta) := \max\{1 \leq p \leq n-1 : I_{-p}(\mu) \geq \delta^{-1} I_2(\mu) = \delta^{-1} \sqrt{n}\}.$$

Then the main theorem in [DP] states that for every $\delta \geq 1$,

$$(1.3) \quad L_n \leq C \delta \sup_{\mu \in \mathcal{I}\mathcal{L}_{[n]}} \sqrt{\frac{n}{q_{-c}(\mu, \delta)}} \log\left(\frac{en}{q_{-c}(\mu, \delta)}\right),$$

where C is an absolute constant. In their proofs they use a formula for the negative moments $I_q(\mu)$ when q is an integer (see the next section for details); this formula

is taken from [Pa2], where it is also shown that $I_{-p}(\mu) \gg \sqrt{n}/\|\mu\|_\infty^{1/n}$ for every log-concave probability measure μ and every $p \leq n-1$, and thus that

$$(1.4) \quad \inf_{\mu \in \mathcal{IL}_{[n]}} q_{-c}(\mu, c_0^{-1}L_n) = n-1$$

for some small enough absolute constant $c_0 > 0$.

The approach of Klartag and Milman in [KM] makes use of another parameter for log-concave probability measures,

$$q_*(\mu) := \sup\{1 \leq p \leq n : k_*(Z_p(\mu)) \geq p\},$$

which was introduced by Paouris in [Pa1]. Recall that if μ is a probability measure on \mathbb{R}^n , then $Z_p(\mu)$ is the L_p -centroid body of μ , namely the convex body with support function

$$h_{Z_p(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^p d\mu(x) \right)^{1/p}, \quad y \in \mathbb{R}^n,$$

and $k_*(Z_p(\mu))$ is the dual Dvoretzky dimension of $Z_p(\mu)$ (see [Pa1] for properties of the parameter $q_*(\mu)$). Klartag and Milman define a ‘‘hereditary’’ variant of $q_*(\mu)$ by setting

$$(1.5) \quad q_*^H(\mu) := n \inf_k \inf_{E \in G_{n,k}} \frac{q_*(\pi_E \mu)}{k},$$

where $\pi_E \mu$ is the marginal of μ with respect to the subspace E . Then they prove that

$$|Z_p(\mu)|^{1/n} \geq c \sqrt{\frac{p}{n}} [\det \text{Cov}(\mu)]^{\frac{1}{2n}} = c \sqrt{\frac{p}{n}}$$

for every isotropic measure μ on \mathbb{R}^n , for every $p \leq q_*^H(\mu)$. In particular, this implies that

$$(1.6) \quad L_\mu \simeq \frac{1}{|Z_n(\mu)|^{1/n}} \leq \frac{1}{|Z_{q_*^H(\mu)}(\mu)|^{1/n}} \leq C \sqrt{\frac{n}{q_*^H(\mu)}}$$

(see the next section as to why the first two relations hold).

Here we define two more hereditary parameters, which we will show are more or less equivalent, and we discuss how the results from [DP] and [KM] can be extended to hold for every p up to these parameters. The first one is an obvious hereditary variant of $q_{-c}(\mu, \delta)$ following the definition of $q_*^H(\mu)$; set

$$q_{-c}^H(\mu, \delta) := n \inf_k \inf_{E \in G_{n,k}} \frac{[q_{-c}(\pi_E \mu, \delta)]}{k}$$

(note that the use of integer parts in the definition is not of essence, but will allow us to state some results in a more precise way). For the second parameter, we first define

$$(1.7) \quad r_{\frac{1}{2}}(\mu, A) := \max\{1 \leq k \leq n-1 : \exists E \in G_{n,k} \text{ such that } L_{\pi_E \mu} \leq A\}$$

for every log-concave probability measure μ on \mathbb{R}^n and every $A \geq 1$, then as previously we set

$$r_{\sharp}^H(\mu, A) := n \inf_k \inf_{E \in G_{n,k}} \frac{r_{\sharp}(\pi_E \mu, A)}{k}$$

(we agree that $r_{\sharp}(\pi_{\mathbb{R}\theta} \mu, A) = q_{-c}(\pi_{\mathbb{R}\theta} \mu, A) = 1$ for all 1-dimensional marginals).

The following theorem holds for every n -dimensional isotropic measure μ .

Theorem 1.1. *There exist absolute constants $C_1, C_2 > 0$ such that for every isotropic measure μ on \mathbb{R}^n and every $A \geq 1$,*

$$(1.8) \quad r_{\sharp}^H(\mu, A) \leq q_{-c}^H(\mu, C_1 A) \leq r_{\sharp}^H(\mu, C_2 A).$$

Moreover, for every $p \leq r_{\sharp}^H(\mu, A)$ we have that

$$(1.9) \quad |Z_p(\mu)|^{1/n} \geq \frac{c}{A} \sqrt{\frac{p}{n}},$$

where $c > 0$ is an absolute constant.

Remark. Note that, as in (1.6), Theorem 1.1 implies that

$$(1.10) \quad L_{\mu} \leq CA \sqrt{\frac{n}{r_{\sharp}^H(\mu, A)}} \leq CA \sqrt{\frac{n}{q_{-c}^H(\mu, \frac{C_1}{C_2} A)}}$$

(to be precise, the second inequality of (1.10) makes sense once we assume that A is larger than some $A_0 \simeq 1$).

Recall that the main result of [Pa2] states that if μ is an isotropic measure on \mathbb{R}^n then

$$(1.11) \quad q_{-c}(\mu, \delta_0) \gg q_*(\mu) \geq c_1 \sqrt{n},$$

where $c_1 > 0$ is an absolute constant and $\delta_0 \simeq 1$. Since every marginal $\pi_E \mu$ of an isotropic measure μ is also isotropic, (1.11) implies that $q_{-c}(\pi_E \mu, \delta_0) \geq c_1 \sqrt{k}$ for every $E \in G_{n,k}$, and hence that

$$(1.12) \quad q_{-c}^H(\mu, \delta_0) \gg q_*^H(\mu) \geq c_1 \sqrt{n}.$$

Then Theorem 1.1 tells us that $r_{\sharp}^H(\mu, A_1)$ as well is at least of the order of \sqrt{n} for some $A_1 \simeq 1$ and every isotropic measure μ on \mathbb{R}^n . Note that, since (1.10) holds true for every constant $A \geq A_0 \simeq 1$, replacing $q_{-c}(\mu, A)$ by $q_{-c}^H(\mu, A)$ one can remove the logarithmic term in (1.3), and slightly improve the bounds for L_n that the approach of Dafnis and Paouris can give us (in those cases of course that the estimates we have for the two parameters are of the same order, as for example in (1.11) and (1.12)).

On the other hand, the example of the suitably normalised uniform measure on B_1^n , the unit ball of ℓ_1^n , shows that there exist isotropic log-concave measures μ on \mathbb{R}^n for which $q_*(\mu) \simeq \sqrt{n}$, and hence $q_*^H(\mu) \simeq \sqrt{n}$. It could be that, even for those measures, $q_{-c}^H(\mu, \delta_0)$ is much larger than \sqrt{n} , and actually if the hyperplane conjecture is correct, we see from (1.4) that $q_{-c}^H(\mu, \delta_1)$ has to be of the order of n for some $\delta_1 \simeq c_0^{-1}L_n \simeq 1$. This shows that the choice of the parameters $r_{\sharp}^H(\mu, \cdot)$ and $q_{-c}^H(\mu, \cdot)$ should permit us to extend the range of p with which the method of Klartag and Milman can be applied. Moreover, the parameter $r_{\sharp}(\mu, A)$, which by definition (1.7) is the highest dimension $k \leq n-1$ in which we can find marginals of μ with isotropic constant bounded above by A , seems worth studying in its own right. Thus, in Section 4 we list a few things that we already know about the isotropic constant of marginals. Our main observation there is the following

Proposition 1.2. *There exist isotropic measures μ on \mathbb{R}^n with $L_{\mu} \simeq L_n$ such that for every $\lambda \in (0, 1)$ and every positive integer $k = \lambda n$, we have that*

$$(1.13) \quad L_{\pi_E \mu} \geq C^{-\frac{1}{\lambda}} L_{\mu}$$

for every subspace $E \in G_{n,k}$, where $C \geq 1$ is an absolute constant.

The rest of the paper is organised as follows. In Section 2 we recall the background material that we need. Theorem 1.1 is proved in Section 3, and a few final remarks about it, including Proposition 1.2, are discussed in Section 4.

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2 Background material

2.1 Notation and preliminaries

We work in \mathbb{R}^n , which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote the corresponding Euclidean norm by $\|\cdot\|_2$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1} . The Grassmann manifold $G_{n,k}$ of k -dimensional subspaces of \mathbb{R}^n is equipped with the Haar probability measure $\nu_{n,k}$. Let $k \leq n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from \mathbb{R}^n onto F by Proj_F . We also define $B_F := B_2^n \cap F$ and $S_F := S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$ (or $a \ll b$), we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$ (or $a \leq c_1 b$). Also if $K, L \subseteq \mathbb{R}^n$, we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

A convex body K in \mathbb{R}^n is a compact convex subset of \mathbb{R}^n with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$. We say that K is centered if the barycentre of K is at the origin; recall that the barycentre of K is the vector

$$(2.1) \quad \text{bar}(K) := \frac{1}{|K|} \int_K x dx = \frac{\int_{\mathbb{R}^n} x \mathbf{1}_K(x) dx}{\int_{\mathbb{R}^n} \mathbf{1}_K(x) dx}.$$

The support function of a convex body K is defined by

$$h_K(y) := \max\{\langle x, y \rangle : x \in K\},$$

and the mean width of K is

$$w(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta).$$

Also, for each $-\infty < q < \infty$, $q \neq 0$, we define the q -mean width of K by

$$w_q(K) := \left(\int_{S^{n-1}} h_K^q(\theta) d\sigma(\theta) \right)^{1/q}.$$

If the origin is an interior point of K , the polar body K° of K is defined as follows:

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.$$

Since the reciprocal of the support function of K is the radial function of K° , i.e. $h_{K^\circ}^{-1}(y) = \max\{r > 0 : ry \in K^\circ\}$ for all $y \neq 0$, integration in polar coordinates and Santaló's inequality show that

$$(2.2) \quad w_{-n}(K) = \frac{|B_2^n|^{1/n}}{|K^\circ|^{1/n}} \geq \frac{|K|^{1/n}}{|B_2^n|^{1/n}}$$

for every centered convex body K .

For basic facts from the Brunn-Minkowski theory, the asymptotic theory of finite dimensional normed spaces and the theory of isotropic convex bodies, we refer to the books [S], [MS] and [Pi] and to the online notes [G].

We write $\mathcal{P}_{[n]}$ for the class of all Borel probability measures on \mathbb{R}^n which are absolutely continuous with respect to the Lebesgue measure. The density of $\mu \in \mathcal{P}_{[n]}$ is denoted by f_μ . A measure μ on \mathbb{R}^n is called logarithmically-concave (or log-concave) if

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for any Borel subsets A and B of \mathbb{R}^n and any $\lambda \in (0, 1)$. A function $f : \mathbb{R}^n \rightarrow [0, \infty)$ is called log-concave if $\log f$ is concave on its support $\{f > 0\}$. It is known that if a probability measure μ on \mathbb{R}^n is log-concave and n -dimensional (by that we mean $\mu(H) < 1$ for every hyperplane H of \mathbb{R}^n), then $\mu \in \mathcal{P}_{[n]}$ and its density f_μ is log-concave. Note that if K is a convex body in \mathbb{R}^n , then the Brunn-Minkowski

inequality implies that $\mathbf{1}_K$ is the density of a log-concave measure. As in (2.1), we define the barycentre

$$\text{bar}(\mu) := \frac{\int_{\mathbb{R}^n} x f_\mu(x) dx}{\int_{\mathbb{R}^n} f_\mu(x) dx}$$

for every finite measure μ with density f_μ , and we say that μ is centered if $\text{bar}(\mu) = 0$. We have already mentioned in the Introduction that we denote the class of n -dimensional isotropic log-concave measures by $\mathcal{IL}_{[n]}$: these are the centered, log-concave probability measures μ on \mathbb{R}^n with the property that $\text{Cov}(\mu)$ is the identity matrix. It is well-known that every log-concave probability measure can be made isotropic by an affine transformation; see e.g. [G, Proposition 1.1.1] for the argument in the setting of convex bodies.

For every $\mu \in \mathcal{P}_{[n]}$ we define the marginal of μ with respect to the k -dimensional subspace E setting

$$\pi_E(\mu)(A) := \mu(\text{Proj}_E^{-1}(A)) = \mu(A + E^\perp)$$

for all Borel subsets of E . The density of $\pi_E\mu$ is the function

$$(2.3) \quad f_{\pi_E\mu}(x) = \int_{x+E^\perp} f_\mu(y) dy, \quad x \in E.$$

It is easily checked that if μ is centered, log-concave or isotropic, then $\pi_E\mu$ is respectively also centered, log-concave or isotropic. In particular, if $\mu \in \mathcal{IL}_{[n]}$ then

$$\det \text{Cov}(\pi_F\mu) = \det \text{Cov}(\mu) = 1$$

for every $1 \leq k \leq n$ and every $F \in G_{n,k}$.

If μ is a probability measure on \mathbb{R}^n , we define the L_q -centroid body $Z_q(\mu)$, $q \geq 1$, to be the centrally symmetric convex body with support function

$$h_{Z_q(\mu)}(y) := \left(\int |\langle x, y \rangle|^q d\mu(x) \right)^{1/q}, \quad y \in \mathbb{R}^n.$$

Note that a log-concave probability measure μ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$. From Hölder's inequality it follows that $Z_1(\mu) \subseteq Z_p(\mu) \subseteq Z_q(\mu)$ for all $1 \leq p \leq q < \infty$. Using Borell's lemma (see [MS, Appendix III]), one can check that inverse inclusions also hold:

$$(2.4) \quad Z_q(\mu) \subseteq c \frac{q}{p} Z_p(\mu)$$

for all $1 \leq p < q$, where c is an absolute constant. In particular, if μ is isotropic, then $R(Z_q(\mu)) := \max\{h_{Z_q(\mu)}(\theta) : \theta \in S^{n-1}\} \leq cq$.

We will use two basic formulas for the L_q -centroid bodies which were obtained in [Pa1] and [Pa2]. First, for every probability measure μ on \mathbb{R}^n , every $1 \leq k \leq n$ and every subspace $E \in G_{n,k}$, we have

$$(2.5) \quad \text{Proj}_E(Z_q(\mu)) = Z_q(\pi_E(\mu)).$$

Furthermore, if μ is centered and log-concave, then

$$(2.6) \quad [f_\mu(0)]^{1/n} \cdot |Z_n(\mu)|^{1/n} \simeq 1.$$

From a result of Fradelizi [F] we also know that, when μ is centered and log-concave,

$$(2.7) \quad \|\mu\|_\infty^{1/n} \leq e [f_\mu(0)]^{1/n},$$

therefore for the measures $\mu \in \mathcal{IL}_{[n]}$ (2.6) becomes

$$(2.8) \quad L_\mu \cdot |Z_n(\mu)|^{1/n} \simeq 1.$$

2.2 Basic tools and relations

We now recall some basic relations that were established in [DP] and [Pa2] and in [KM] and involve the main objects that are used to prove the key results in those articles. The first one is a formula relating the negative moments of the Euclidean norm with respect to a centered, log-concave probability measure μ on \mathbb{R}^n to negative mean widths of the L_q -centroid bodies of μ . Recall that the quantity $I_q(\mu)$ is defined for every $q \in (-n, \infty)$, $q \neq 0$, by

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q f(x) dx \right)^{1/q}.$$

In [Pa2] it is proven that

$$(2.9) \quad I_{-k}(\mu) = c_{n,k} \left(\int_{G_{n,k}} f_{\pi_E \mu}(0) d\nu_{n,k}(E) \right)^{-1/k}$$

for every positive integer $k \leq n-1$, where

$$c_{n,k} = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \simeq \sqrt{n}.$$

Complementally, it is shown that

$$(2.10) \quad w_{-k}(Z_k(\mu)) \simeq \sqrt{k} \left(\int_{G_{n,k}} |\text{Proj}_E(Z_k(\mu))|^{-1} d\nu_{n,k}(E) \right)^{-1/k}.$$

Since $\text{Proj}_E(Z_q(\mu)) = Z_q(\pi_E(\mu))$, we have from (2.6) that

$$|\text{Proj}_E(Z_k(\mu))|^{-1/k} \simeq f_{\pi_E \mu}(0)^{1/k}.$$

Therefore, for every positive integer $k \leq n-1$,

$$(2.11) \quad I_{-k}(\mu) \simeq \sqrt{\frac{n}{k}} w_{-k}(Z_k(\mu)).$$

We now turn our attention to the tools and relations that are used in the arguments of [KM]. The primary tool there, which was introduced by Klartag for the first time in arguments related to the slicing problem (see [K1]), is the logarithmic Laplace transform of the measure μ . Recall that for any finite Borel measure μ on \mathbb{R}^n , its logarithmic Laplace transform is defined by

$$\Lambda_\mu(\xi) := \log \left(\int_{\mathbb{R}^n} e^{\langle x, \xi \rangle} \frac{d\mu(x)}{\mu(\mathbb{R}^n)} \right), \quad \xi \in \mathbb{R}^n.$$

Through Λ_μ we can define a whole family of probability measures μ_x whose L_q -centroid bodies almost coincide with the corresponding L_q -centroid body of μ . Indeed, consider first the symmetrised level-sets of the logarithmic Laplace transform of μ , namely the bodies

$$\Lambda_p(\mu) := \{x \in \mathbb{R}^n : \Lambda_\mu(x) \leq p \text{ and } \Lambda_\mu(-x) \leq p\}, \quad p \geq 0.$$

As is proven in [KM, Lemma 2.3], when μ is a centered, log-concave probability measure, it holds that

$$(2.12) \quad \Lambda_p(\mu) \simeq p(Z_p(\mu))^\circ$$

for every $p \geq 1$ (a dual version of this was first observed by Latała and Wojtaszczyk in [LW]). When μ is log-concave, we also have that $\{\Lambda_\mu < \infty\}$ is an open set, and that Λ_μ is C^∞ -smooth and strictly-convex in this open set (see e.g. [K2, Section 2]). For every $x \in \{\Lambda_\mu < \infty\}$, we denote by μ'_x the probability measure whose density is proportional to the function $e^{\langle z, x \rangle} f_\mu(z)$, where f_μ is the density of the measure μ . In other words, μ'_x is the measure with density

$$f_{\mu'_x}(z) := \frac{e^{\langle z, x \rangle} f_\mu(z)}{\int_{\mathbb{R}^n} e^{\langle z, x \rangle} d\mu(z)}.$$

It is straightforward to check that the barycentre and the covariance matrix of μ'_x are exactly the first and second derivatives of Λ_μ at x :

$$\text{bar}(\mu'_x) = \nabla \Lambda_\mu(x) \quad \text{and} \quad \text{Cov} \mu'_x = \text{Hess} \Lambda_\mu(x).$$

We now write μ_x for the centered probability measure with density $f_{\mu_x}(z) := f_{\mu'_x}(z + \text{bar}(\mu'_x))$. One of the key observations in [KM] is that, whenever $x \in \frac{1}{2}\Lambda_p(\mu)$, we have

$$\Lambda_q(\mu) \simeq \Lambda_q(\mu_x) \quad \text{for every } q \geq p,$$

or equivalently, because of (2.12),

$$(2.13) \quad Z_q(\mu) \simeq Z_q(\mu_x) \quad \text{for every } q \geq p.$$

The other fundamental relation that Klartag and Milman arrive at is the following: if μ is a centered, log-concave probability measure on \mathbb{R}^n , then for every

$p \in [1, n]$ we have that

$$(2.14) \quad |Z_p(\mu)|^{1/n} \simeq \sqrt{\frac{p}{n}} \left(\frac{1}{|\frac{1}{2}\Lambda_p(\mu)|} \int_{\frac{1}{2}\Lambda_p(\mu)} \det \text{Cov}(\mu_x) dx \right)^{\frac{1}{2n}} \\ \simeq \sqrt{\frac{p}{n}} \inf_{x \in \frac{1}{2}\Lambda_p(\mu)} [\det \text{Cov}(\mu_x)]^{\frac{1}{2n}}.$$

An initial conclusion we can draw from this is that if $x_0 \in \frac{1}{2}\Lambda_p(\mu)$ is such that

$$[\det \text{Cov}(\mu_{x_0})]^{\frac{1}{2n}} \simeq \inf_{x \in \frac{1}{2}\Lambda_p(\mu)} [\det \text{Cov}(\mu_x)]^{\frac{1}{2n}},$$

then, using (2.13) as well, we get that

$$|Z_p(\mu_{x_0})|^{1/n} \simeq \sqrt{\frac{p}{n}} [\det \text{Cov}(\mu_{x_0})]^{\frac{1}{2n}}.$$

The aim of course is to show a similar relation for the measure μ instead of μ_{x_0} , and to accomplish this we need to be able to prove that

$$(2.15) \quad [\det \text{Cov}(\mu_{x_0})]^{\frac{1}{2n}} \geq \frac{1}{A} [\det \text{Cov}(\mu)]^{\frac{1}{2n}}$$

for as small a constant $A \geq 1$ as possible. In the next section we will carefully revisit the final steps of the argument in [KM] and we will explain why we can establish (2.15) for every $p \leq r_{\sharp}^H(\mu, cA)$ (where $c > 0$ is a constant independent of the measure μ , the dimension n or the parameter A).

3 Proof of Theorem 1.1

The first thing we have to show is that if μ is an isotropic measure on \mathbb{R}^n and $p \leq r_{\sharp}^H(\mu, A)$, then

$$|Z_p(\mu)|^{1/n} \geq \frac{c}{A} \sqrt{\frac{p}{n}} [\det \text{Cov}(\mu)]^{\frac{1}{2n}} = \frac{c}{A} \sqrt{\frac{p}{n}}$$

for some absolute constant $c > 0$. In order to do that, we recall that given (2.14) we have to show that

$$[\det \text{Cov}(\mu_x)]^{\frac{1}{2n}} \geq \frac{c'}{A}$$

for every $x \in \frac{1}{2}\Lambda_p(\mu)$. We denote the eigenvalues of $\text{Cov}(\mu_x)$ by $\lambda_1^x \leq \lambda_2^x \leq \dots \leq \lambda_n^x$, and we write E_k for the k -dimensional subspace which is spanned by eigenvectors corresponding to the first k eigenvalues of $\text{Cov}(\mu_x)$. We start with the following lemma which is essentially the same as [KM, Lemma 5.2] (we include its proof for the reader's convenience).

Lemma 3.1. *For every two integers $1 \leq s \leq k \leq n$ we have that*

$$(3.1) \quad \sqrt{\lambda_k^x} \geq c_1 \sup_{F \in G_{E_k, s}} |Z_s(\pi_F \mu_x)|^{1/s},$$

where $c_1 > 0$ is an absolute constant.

Proof. Note that

$$(3.2) \quad \lambda_k^x = \max_{\theta \in S_{E_k}} \int_{E_k} \langle z, \theta \rangle^2 d\pi_{E_k} \mu_x(z) = \sup_{F \in G_{E_k, s}} \max_{\theta \in S_F} \int_F \langle z, \theta \rangle^2 d\pi_F \mu_x(z).$$

This is because, for every subspace F of E_k and every $\theta \in S_F \subseteq S_{E_k}$, we have that

$$\int_F \langle z, \theta \rangle^2 d\pi_F \mu_x(z) = \int_{\mathbb{R}^n} \langle z, \theta \rangle^2 d\mu_x(z) = \int_{E_k} \langle z, \theta \rangle^2 d\pi_{E_k} \mu_x(z),$$

while λ_k^x is the largest eigenvalue of $\text{Cov}(\pi_{E_k} \mu_x)$.

On the other hand, since μ_x is a centered, log-concave probability measure, which means that so are its s -dimensional marginals $\pi_F \mu_x$, we get from (2.6) and (2.7) that

$$(3.3) \quad |Z_s(\pi_F \mu_x)|^{1/s} \simeq \frac{1}{\|\mathbf{f}_{\pi_F \mu_x}\|_\infty^{1/s}} = \frac{[\det \text{Cov}(\pi_F \mu_x)]^{\frac{1}{2s}}}{L_{\pi_F \mu_x}}.$$

Since $L_\nu \geq c$ for any isotropic measure ν , for some universal constant $c > 0$, it follows that

$$|Z_s(\pi_F \mu_x)|^{1/s} \leq c' [\det \text{Cov}(\pi_F \mu_x)]^{\frac{1}{2s}} \leq c' \max_{\theta \in S_F} \sqrt{\int_F \langle z, \theta \rangle^2 d\pi_F \mu_x(z)}$$

for every $F \in G_{E_k, s}$, which combined with (3.2) gives us (3.1). \square

To bound the right-hand side of (3.1) by an expression that involves $\det \text{Cov}(\mu)$, we have to compare the volume of $Z_s(\pi_F \mu_x)$ to that of $Z_s(\pi_F \mu)$ (we are able to do that because of (2.13)). The right choice of s is prompted by the following lemma.

Lemma 3.2. *Recall that for some fixed $x \in \frac{1}{2}\Lambda_p(\mu)$ and every integer $k \leq n$, we denote by E_k the k -dimensional subspace which is spanned by eigenvectors corresponding to the first k eigenvalues of $\text{Cov}(\mu_x)$. For convenience, we also set $s_k^x := r_\#(\pi_{E_k} \mu, A)$. Then*

$$(3.4) \quad \sup_{F \in G_{E_k, s_k^x}} |Z_{s_k^x}(\pi_F \mu)|^{1/s_k^x} \geq \frac{c_2}{A} [\det \text{Cov}(\mu)]^{\frac{1}{2n}} = \frac{c_2}{A},$$

where $c_2 > 0$ is an absolute constant.

Proof. As in (3.3), we can write

$$|Z_{s_k^x}(\pi_F \mu)|^{1/s_k^x} \geq \frac{c_2}{\|f_{\pi_F \mu}\|_{\infty}^{1/s_k^x}} = \frac{c_2 [\det \text{Cov}(\pi_F \mu)]^{\frac{1}{2s_k^x}}}{L_{\pi_F \mu}}$$

for some absolute constant $c_2 > 0$ and for every $F \in G_{E_k, s_k^x}$. Remember that since μ is isotropic, $[\det \text{Cov}(\pi_F \mu)]^{1/(2s_k^x)} = [\det \text{Cov}(\mu)]^{1/(2n)} = 1$. Moreover, by the definition of $s_k^x = r_{\sharp}(\pi_{E_k} \mu, A)$, there is at least one s_k^x -dimensional subspace of E_k , say F_0 , such that the marginal $\pi_{F_0}(\pi_{E_k} \mu) \equiv \pi_{F_0} \mu$ has isotropic constant bounded above by A . Combining all of these, we get

$$\sup_{F \in G_{E_k, s_k^x}} |Z_{s_k^x}(\pi_F \mu)|^{1/s_k^x} \geq |Z_{s_k^x}(\pi_{F_0} \mu)|^{1/s_k^x} \geq \frac{c_2}{A}$$

as required. \square

Observe now that in order to compare $Z_{s_k^x}(\pi_F \mu_x)$ and $Z_{s_k^x}(\pi_F \mu)$ for every $F \in G_{E_k, s_k^x}$, we have two cases to consider:

- (i) if $p \leq s_k^x = r_{\sharp}(\pi_{E_k} \mu, A)$, then by (2.13) we have that $Z_{s_k^x}(\mu_x) \simeq Z_{s_k^x}(\mu)$, and therefore for every $F \in G_{E_k, s_k^x}$,

$$Z_{s_k^x}(\pi_F \mu_x) = \text{Proj}_F(Z_{s_k^x}(\mu_x)) \simeq \text{Proj}_F(Z_{s_k^x}(\mu)) = Z_{s_k^x}(\pi_F \mu)$$

as well;

- (ii) if $s_k^x < p$, then using (2.4) and (2.13) we can write

$$Z_{s_k^x}(\pi_F \mu_x) \geq c_0 \frac{s_k^x}{p} Z_p(\pi_F \mu_x) \geq c'_0 \frac{s_k^x}{p} Z_p(\pi_F \mu) \geq c'_0 \frac{s_k^x}{p} Z_{s_k^x}(\pi_F \mu)$$

for some absolute constants $c_0, c'_0 > 0$. We also recall that since

$$p \leq r_{\sharp}^H(\mu, A) = n \inf_k \inf_{E \in G_{n,k}} \frac{r_{\sharp}(\pi_E \mu, A)}{k} \leq \frac{n}{k} r_{\sharp}(\pi_{E_k} \mu, A),$$

it holds that $s_k^x/p = r_{\sharp}(\pi_{E_k} \mu, A)/p \geq k/n$.

To summarise the above, we see that in any case and for every $F \in G_{E_k, s_k^x}$,

$$(3.5) \quad Z_{s_k^x}(\pi_F \mu_x) \geq c''_0 \min\left\{1, \frac{s_k^x}{p}\right\} Z_{s_k^x}(\pi_F \mu) \geq c''_0 \frac{k}{n} Z_{s_k^x}(\pi_F \mu),$$

where $c''_0 > 0$ is a small enough absolute constant. We now have everything we need to bound $|Z_p(\mu)|^{1/n}$ from below.

Theorem 3.3. *Let μ be an n -dimensional isotropic measure and let $A \geq 1$. Then, for every $p \in [1, r_{\sharp}^H(\mu, A)]$, we have that*

$$(3.6) \quad |Z_p(\mu)|^{1/n} \geq \frac{c}{A} \sqrt{\frac{p}{n}},$$

where $c > 0$ is an absolute constant.

Proof. Combining Lemmas 3.1 and 3.2 with (3.5), we see that for every $p \in [1, r_{\sharp}^H(\mu, A)]$ and for every $x \in \frac{1}{2}\Lambda_p(\mu)$,

$$[\det \text{Cov}(\mu_x)]^{1/2} = \prod_{k=1}^n \sqrt{\lambda_k^x} \geq \prod_{k=1}^n \frac{c}{A} \frac{k}{n} = \frac{c^n}{A^n} \frac{n!}{n^n}.$$

If we take n -th roots, the theorem then follows from (2.14). \square

It remains to establish the first conclusion of Theorem 1.1. The key step is the following consequence of Theorem 3.3.

Corollary 3.4. *There exists a positive absolute constant C_1 such that, for every n -dimensional isotropic measure μ and every $A \geq 1$,*

$$(3.7) \quad r_{\sharp}^H(\mu, A) \leq \lfloor q_{-c}(\mu, C_1 A) \rfloor.$$

In other words, for every $p \leq \lceil r_{\sharp}^H(\mu, A) \rceil$ we have that

$$(3.8) \quad I_{-p}(\mu) \geq \frac{1}{C_1 A} I_2(\mu) = \frac{1}{C_1 A} \sqrt{n}.$$

Proof. Set $p_A := r_{\sharp}^H(\mu, A)$ and observe that

$$|Z_{\lceil p_A \rceil}(\mu)|^{1/n} \geq |Z_{p_A}(\mu)|^{1/n} \geq \frac{c'}{A} \sqrt{\frac{\lceil p_A \rceil}{n}}.$$

By Hölder's and Santaló's inequalities, this gives us that

$$w_{-\lceil p_A \rceil}(Z_{\lceil p_A \rceil}(\mu)) \geq w_{-n}(Z_{\lceil p_A \rceil}(\mu)) \geq \frac{|Z_{\lceil p_A \rceil}(\mu)|^{1/n}}{\omega_n^{1/n}} \geq \frac{c''}{A} \sqrt{\lceil p_A \rceil}.$$

Since $r_{\sharp}^H(\mu, A) \leq r_{\sharp}(\mu, A) \leq n - 1$ by definition, we have $\lceil p_A \rceil \leq n - 1$, and thus we can use (2.11) to conclude that

$$I_{-\lceil p_A \rceil}(\mu) \geq \frac{1}{C_1 A} \sqrt{n}$$

for some absolute constant $C_1 > 0$. This completes the proof. \square

Proof of (1.8). For the left-hand side inequality we apply Corollary 3.4 for every marginal $\pi_E\mu$ of μ ; we get that

$$r_{\sharp}^H(\pi_E\mu, A) \leq \lfloor q_{-c}(\pi_E\mu, C_1A) \rfloor.$$

In addition, we observe that

$$(3.9) \quad r_{\sharp}^H(\mu, A) = n \inf_k \inf_{F \in G_{n,k}} \frac{r_{\sharp}^H(\pi_F\mu, A)}{k} \\ \leq n \inf_{s \leq \dim E} \inf_{F \in G_{E,s}} \frac{r_{\sharp}^H(\pi_F\mu, A)}{s} = \frac{n}{\dim E} r_{\sharp}^H(\pi_E\mu, A),$$

which means that for every integer k , for every subspace $E \in G_{n,k}$,

$$r_{\sharp}^H(\mu, A) \leq \frac{n}{k} r_{\sharp}^H(\pi_E\mu, A) \leq \frac{n}{k} \lfloor q_{-c}(\pi_E\mu, C_1A) \rfloor,$$

or equivalently that $r_{\sharp}^H(\mu, A) \leq q_{-c}^H(\mu, C_1A)$.

For the other inequality of (1.8) we will use (2.9): if k is an integer such that

$$I_{-k}(\mu) \simeq \sqrt{n} \left(\int_{G_{n,k}} f_{\pi_E\mu}(0) d\nu_{n,k}(E) \right)^{-1/k} \geq \frac{1}{C_1A} I_2(\mu) = \frac{1}{C_1A} \sqrt{n},$$

namely if $k \leq \lfloor q_{-c}(\mu, C_1A) \rfloor$, then there must exist at least one $E \in G_{n,k}$ such that $f_{\pi_E\mu}(0) \leq (C_1'A)^k$ for some absolute constant C_1' (depending only on C_1). Since $\pi_E\mu$ is isotropic, we have

$$L_{\pi_E\mu} = \|f_{\pi_E\mu}\|_{\infty}^{1/k} \leq e(f_{\pi_E\mu}(0))^{1/k} \leq C_2A.$$

This means that

$$r_{\sharp}(\mu, C_2A) \geq \lfloor q_{-c}(\mu, C_1A) \rfloor,$$

and the same will hold for every marginal $\pi_F\mu$ of μ . The inequality now follows from the definitions of $r_{\sharp}^H(\mu, C_2A)$ and $q_{-c}^H(\mu, C_1A)$. \square

4 Further remarks

As we mentioned in the Introduction, Theorem 1.1 enables us to remove the logarithmic term in (1.3) in those cases that the lower bounds we know for the parameters $q_{-c}(\mu, \delta)$ and $q_{-c}^H(\mu, \delta)$ are of the same order (this can happen if for example we know that

$$\inf_{\mu \in \mathcal{IL}_{[n]}} q_{-c}(\mu, \delta) \geq h_{\delta}(n)$$

for some function h_{δ} such that $h_{\delta}(n)/n$ is decreasing in n). An improvement to those bounds could come from the study of the parameter $r_{\sharp}(\mu, A)$; actually, it becomes clear from our results that the hyperplane conjecture is equivalent to the

seemingly weaker condition that every isotropic measure μ on \mathbb{R}^n has marginals of dimension proportional to n with bounded isotropic constant. Although we are nowhere near establishing such a property, and the only estimate we currently have for $r_{\sharp}(\mu, A)$ for an arbitrary measure μ comes from (1.11) (since it's always true that $r_{\sharp}(\mu, A) \geq \lfloor q_{-c}(\mu, cA) \rfloor$ for some small absolute constant $c > 0$), we already know a few interesting things about the isotropic constant of marginals.

First, recall that by Hölder's and Santaló's inequalities and by (2.11), we have

$$(4.1) \quad \begin{aligned} I_{-k}(\mu) &\geq c_1 \sqrt{\frac{n}{k}} w_{-k}(Z_k(\mu)) \geq c_1 \sqrt{\frac{n}{k}} w_{-n}(Z_k(\mu)) \\ &\geq c_1 \sqrt{\frac{n}{k}} \frac{|Z_k(\mu)|^{1/n}}{\omega_n^{1/n}} \geq c_2 \frac{n}{\sqrt{k}} |Z_k(\mu)|^{1/n} \end{aligned}$$

for every integer $k \leq n-1$, for every centered, log-concave probability measure μ on \mathbb{R}^n , where $c_1, c_2 > 0$ are absolute constants. From Hölder's inequality and Borell's lemma, we also have that the inclusions $Z_{n-1}(\mu) \subseteq Z_n(\mu) \subseteq 2Z_{n-1}(\mu)$ hold, and thus, by (2.6) and Fradelizi's result (2.7), $|Z_{n-1}(\mu)|^{1/n} \simeq \|\mu\|_{\infty}^{-1/n}$. It follows that

$$(4.2) \quad I_{-p}(\mu) \geq I_{-(n-1)}(\mu) \gg \sqrt{n} |Z_{n-1}(\mu)|^{1/n} \gg \sqrt{n} \|\mu\|_{\infty}^{1/n}$$

for every $p \leq n-1$ (as we mentioned in the Introduction, an alternative proof of (4.2) can be found in [Pa2]). But then, in the cases that μ is isotropic, which means that so are all its marginals, we get by (2.9) and (2.7) that

$$(4.3) \quad \begin{aligned} I_{-k}(\mu) &\simeq \sqrt{n} \left(\int_{G_{n,k}} f_{\pi_E \mu}(0) d\nu_{n,k}(E) \right)^{-1/k} \\ &= \sqrt{n} \left(\int_{G_{n,k}} [(f_{\pi_E \mu}(0))^{1/k}]^k d\nu_{n,k}(E) \right)^{-1/k} \\ &\simeq \sqrt{n} \left(\int_{G_{n,k}} L_{\pi_E \mu}^k d\nu_{n,k}(E) \right)^{-1/k} \end{aligned}$$

for every integer $k \leq n-1$. Combining this with (4.2) we conclude that

$$\left(\int_{G_{n,k}} L_{\pi_E \mu}^k d\nu_{n,k}(E) \right)^{1/k} \leq C_0 \|\mu\|_{\infty}^{1/n} = C_0 L_{\mu}$$

and

$$(4.4) \quad \nu_{n,k}(\{E \in G_{n,k} : L_{\pi_E \mu} \leq C_1 L_{\mu}\}) \geq 1 - e^{-k}$$

for some absolute constants C_0, C_1 (even better estimates for the measure of the sets in (4.4) are obtained by Dafnis and Paouris [DP2] in the setting of isotropic convex bodies).

Secondly, we have Proposition 1.2 which gives a lower bound for the isotropic constant of marginals in cases of measures with maximal isotropic constant. For its proof, we will consider isotropic measures which are uniformly distributed in convex bodies. Recall that in such cases we have a centered, convex body K with the property that

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 \mathbf{1}_K(x) dx = |K|$$

for every $\theta \in S^{n-1}$ (it is known that every convex body in \mathbb{R}^n can be brought to such a position), and then our measure $\mu \equiv \mu_K$ is defined to have probability density

$$f_\mu(x) := |K|^{-1} \cdot \mathbf{1}_K(x).$$

From the definitions it is clear that $\mu \in \mathcal{IL}_{[n]}$ and $L_\mu = |K|^{-1/n}$. We denote the subclass of such isotropic measures by $\mathcal{IK}_{[n]}$ and we recall that

$$L_n = \sup_{\mu \in \mathcal{IL}_{[n]}} L_\mu \leq C \sup_{\mu_K \in \mathcal{IK}_{[n]}} L_{\mu_K}$$

for some absolute constant C .

Proof of Proposition 1.2. Let $\alpha \in (0, 1]$ and let $\mu \in \mathcal{IK}_{[n]}$ be an isotropic measure with $L_\mu \geq \alpha L_n$. Let K be the support of μ (that means that the measure μ has density $f_\mu = |K|^{-1} \cdot \mathbf{1}_K$), and let \mathcal{E}_K be an M -ellipsoid of K , namely an ellipsoid such that $|\mathcal{E}_K| = |K|$ and $N(K, \mathcal{E}_K) \leq e^{b_0 n}$ for some absolute constant b_0 , where $N(A, B)$ is the minimum number of translates of the non-empty set $B \subseteq \mathbb{R}^n$ that we need so as to cover the set $A \subseteq \mathbb{R}^n$ (for the existence of such an ellipsoid see e.g. [Pi, Chapter 7]). The idea of working with bodies that have maximal isotropic constant and their M -ellipsoids comes from [BKM]. Recall that by the Rogers-Shephard inequality we have

$$(4.5) \quad |K| \leq |K \cap E^\perp| |\text{Proj}_E(K)| \leq \binom{n}{k} |K|$$

(and the same with \mathcal{E}_K instead of K) for every $E \in G_{n,k}$. We begin by applying the left-hand side inequality with $F \in G_{n,n-k}$: we see that for every such subspace,

$$|\text{Proj}_F(K)| \geq \frac{1}{|K|^{-1} |K \cap F^\perp|}.$$

But by definition

$$|K|^{-1} |K \cap F^\perp| = |K|^{-1} \int_{F^\perp} \mathbf{1}_K(y) dy = \int_{F^\perp} f_\mu(y) dy = f_{\pi_F \mu}(0) \leq (L_{\pi_F \mu})^{n-k}.$$

Since $L_{\pi_F \mu} \leq L_{(n-k)} \leq b_1 L_n$ for some absolute constant b_1 (see [BKM]), it follows that

$$\min_{F \in G_{n,n-k}} |\text{Proj}_F(K)| \geq \frac{1}{(b_1 L_n)^{n-k}} \geq \left(\frac{\alpha}{b_1 L_\mu} \right)^{n-k}.$$

Note that $N(\text{Proj}_F(K), \text{Proj}_F(\mathcal{E}_K)) \leq N(K, \mathcal{E}_K) \leq e^{b_0 n}$, and thus

$$\min_{F \in G_{n, n-k}} |\text{Proj}_F(\mathcal{E}_K)| \geq e^{-b_0 n} \min_{F \in G_{n, n-k}} |\text{Proj}_F(K)|.$$

But then, by the right-hand side inequality of (4.5) we see that

$$(4.6) \quad \max_{E \in G_{n, k}} |\mathcal{E}_K \cap E| = \max_{F \in G_{n, n-k}} |\mathcal{E}_K \cap F^\perp| \\ \leq \binom{n}{n-k} e^{b_0 n} \left(\frac{b_1 L_\mu}{\alpha} \right)^{n-k} |\mathcal{E}_K| = \binom{n}{k} e^{b_0 n} \left(\frac{b_1 L_\mu}{\alpha} \right)^{n-k} |K|.$$

Recall now that every ellipsoid \mathcal{E} has the property that

$$\max_{H \in G_{n, s}} |\text{Proj}_H(\mathcal{E})| = \max_{H \in G_{n, s}} |\mathcal{E} \cap H|$$

for all $1 \leq s \leq n$, therefore by (4.6) we have that

$$\max_{E \in G_{n, k}} |\text{Proj}_E(K)| \leq e^{b_0 n} \max_{E \in G_{n, k}} |\text{Proj}_E(\mathcal{E}_K)| \leq \binom{n}{k} e^{2b_0 n} (\alpha^{-1} b_1 L_\mu)^{n-k} |K|.$$

We need one final application of the left-hand side inequality of (4.5) to deduce that

$$\min_{E \in G_{n, k}} |K \cap E^\perp| \geq \binom{n}{k}^{-1} e^{-2b_0 n} (\alpha^{-1} b_1 L_\mu)^{-(n-k)},$$

or equivalently that

$$(4.7) \quad \min_{E \in G_{n, k}} ((L_\mu)^n |K \cap E^\perp|) \geq \binom{n}{k}^{-1} \frac{(e^{-2b_0} \alpha b_1^{-1})^n}{(\alpha b_1^{-1})^k} (L_\mu)^k.$$

But since $(L_\mu)^n = |K|^{-1}$ and $|K|^{-1} |K \cap E^\perp| = f_{\pi_E \mu}(0) \leq (L_{\pi_E \mu})^k$, we can rewrite inequality (4.7) as

$$\min_{E \in G_{n, k}} (L_{\pi_E \mu})^k \geq \left(\frac{en}{k} \right)^{-k} \frac{(e^{-2b_0} \alpha b_1^{-1})^n}{(\alpha b_1^{-1})^k} (L_\mu)^k,$$

and then, if we take k -th roots, it will follow that

$$(4.8) \quad \min_{E \in G_{n, k}} L_{\pi_E \mu} \geq \frac{k}{en} \frac{\alpha^{\frac{n}{k}} (e^{-2b_0} b_1^{-1})^{\frac{n}{k}}}{\alpha b_1^{-1}} L_\mu \\ = \lambda \alpha^{\frac{1}{\lambda} - 1} \frac{e^{-1} b_1}{(e^{2b_0} b_1)^{1/\lambda}} L_\mu$$

as required. Note that the above hold for every isotropic measure $\mu \in \mathcal{IK}_{[n]}$ with $L_\mu \geq \alpha L_n$. \square

Proposition 1.2 points perhaps to some limitations of the two methods we have discussed. This is because, by (4.8) and (4.3), we can write

$$I_{-k}(\mu) \leq \alpha^{1-\frac{1}{\lambda}} C_0^{\frac{1}{\lambda}} \frac{\sqrt{n}}{L_\mu}$$

for all positive integers $k = \lambda n \leq n - 1$ and for all isotropic measures $\mu \in \mathcal{IK}_{[n]}$ with $L_\mu \geq \alpha L_n$, where C_0 is an absolute constant. In the other direction, we have (4.2) for every $\mu \in \mathcal{IL}_{[n]}$, and also a corresponding inequality for the volume of $Z_p(\mu)$; indeed, as Klartag and Milman show in [KM], from (2.14) and the way the bodies $\Lambda_p(\mu)$ are defined, we see that

$$\frac{|Z_p(\mu)|^{1/n}}{\sqrt{p}} \gg \frac{|Z_q(\mu)|^{1/n}}{\sqrt{q}}$$

for all $1 \leq p < q \leq n$ and every centered, log-concave probability measure μ , whence it follows that

$$(4.9) \quad |Z_p(\mu)|^{1/n} \gg \sqrt{\frac{p}{n}} |Z_n(\mu)|^{1/n} \simeq \frac{\sqrt{p}}{\sqrt{n} L_\mu}$$

for every $1 \leq p \leq n$ and $\mu \in \mathcal{IL}_{[n]}$ (this generalises a similar inequality of Lutwak, Yang and Zhang [LYZ] for convex bodies of volume 1). The above can be summarised as follows:

$$(4.10) \quad c_1 \frac{\sqrt{n}}{L_\mu} \leq \frac{n}{\sqrt{p}} |Z_p(\mu)|^{1/n} \leq c_2 I_{-p}(\mu) \leq C_3^{\frac{n}{p}} \frac{\sqrt{n}}{L_\mu}$$

for every $1 \leq p \leq n - 1$ and for all isotropic measures $\mu \in \mathcal{IK}_{[n]}$ with $L_\mu \simeq L_n$, where $c_1 > 0$ and c_2, C_3 are absolute constants (the second inequality holds true due to (4.1)); obviously, (4.10) is optimal (up to the value of the constants) for p proportional to n .

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