

Small ball probability estimates, ψ_2 -behavior and the hyperplane conjecture

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Abstract

We introduce a method which leads to upper bounds for the isotropic constant. We prove that a positive answer to the hyperplane conjecture is equivalent to some very strong small probability estimates for the Euclidean norm on isotropic convex bodies. As a consequence of our method, we obtain an alternative proof of the result of J. Bourgain that every ψ_2 -body has bounded isotropic constant, with a slightly better estimate: If K is a convex body in \mathbb{R}^n such that $\|\langle \cdot, \theta \rangle\|_q \leq \beta \|\langle \cdot, \theta \rangle\|_2$ for every $\theta \in S^{n-1}$ and every $q \geq 2$, then $L_K \leq C\beta\sqrt{\log \beta}$, where $C > 0$ is an absolute constant.

1 Introduction

A convex body K in \mathbb{R}^n is called isotropic if it has volume $|K| = 1$, center of mass at the origin, and its inertia matrix is a multiple of the identity. Equivalently, if there is a constant $L_K > 0$ such that

$$(1.1) \quad \int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1} . It is not hard to see that for every convex body K in \mathbb{R}^n there exists an affine transformation T of \mathbb{R}^n such that $T(K)$ is isotropic. Moreover, this isotropic image is unique up to orthogonal transformations; consequently, one may define the isotropic constant L_K as an invariant of the affine class of K .

The isotropic constant is closely related to the hyperplane conjecture (also known as the slicing problem) which asks if there exists an absolute constant $c > 0$ such that $\max_{\theta \in S^{n-1}} |K \cap \theta^\perp| \geq c$ for every convex body K of volume 1 in \mathbb{R}^n with center of mass at the origin. This is because, by Brunn's principle, for any convex body K in \mathbb{R}^n and any $\theta \in S^{n-1}$, the function $t \mapsto |K \cap (\theta^\perp + t\theta)|^{\frac{1}{n-1}}$ is concave on its support, and this implies that

$$(1.2) \quad \int_K \langle x, \theta \rangle^2 dx \simeq |K \cap \theta^\perp|^{-2}.$$

Using this relation one can check that an affirmative answer to the slicing problem is equivalent to the following statement: There exists an absolute constant $C > 0$ such that $L_K \leq C$ for every convex body K . We refer to the article [16] of Milman and Pajor for background information about isotropic convex bodies.

The isotropic constant and the hyperplane conjecture can be studied in the more general setting of log-concave measures. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$. We say that f is isotropic if f has center of mass at the origin and

$$(1.3) \quad \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^2 f(x)dx = 1$$

for every $\theta \in S^{n-1}$. It is well-known that the hyperplane conjecture for convex bodies is equivalent to the following statement: There exists an absolute constant $C > 0$ such that, for every isotropic log-concave function f on \mathbb{R}^n ,

$$(1.4) \quad f(0)^{1/n} \leq C.$$

It is known that $L_K \geq L_{B_2^n} \geq c > 0$ for every convex body K in \mathbb{R}^n (we use the letters c, c_1, C etc. to denote absolute constants). Bourgain proved in [3] that $L_K \leq c\sqrt[n]{n} \log n$ and, a few years ago, Klartag [8] obtained the estimate $L_K \leq c\sqrt[n]{n}$.

The approach of Bourgain in [3] is to reduce the problem to the case of convex bodies that satisfy a ψ_2 -estimate (with constant $\beta = O(\sqrt[n]{n})$). We say that K satisfies a ψ_2 -estimate with constant β if

$$(1.5) \quad \|\langle \cdot, y \rangle\|_{\psi_2} \leq \beta \|\langle \cdot, y \rangle\|_2$$

for all $y \in \mathbb{R}^n$. Bourgain proved in [4] that, if (1.5) holds true, then

$$(1.6) \quad L_K \leq C\beta \log \beta.$$

The purpose of this paper is to introduce a different method which leads to upper bounds for L_K . We prove that a positive answer to the hyperplane conjecture is equivalent to some very strong small probability estimates for the Euclidean norm on isotropic convex bodies; for $-n < p \leq \infty$, $p \neq 0$, we define

$$(1.7) \quad I_p(K) := \left(\int_K \|x\|_2^p dx \right)^{1/p}$$

and, for $\delta \geq 1$, we consider the parameter

$$(1.8) \quad q_{-c}(K, \delta) := \max\{p \geq 1 : I_2(K) \leq \delta I_{-p}(K)\}.$$

Then, the hyperplane conjecture is equivalent to the following statement:

There exist absolute constants $C, \xi > 0$ such that, for every isotropic convex body K in \mathbb{R}^n ,

$$q_{-c}(K, \xi) \geq Cn.$$

The main results of [22] and [23] show that there exists a parameter $q_* := q_*(K)$ (related to the L_q -centroid bodies of K) with the following properties: (i) $q_*(K) \geq c\sqrt{n}$, (ii) $q_{-c}(K, \xi) \geq q_*(K)$ for some absolute constant $\xi \geq 1$, and hence, $I_2(K) \leq \xi I_{-q_*}(K)$. The question that arises is to understand what happens with $I_{-p}(K)$ when p lies in the interval $[q_*, n]$, where there are no general estimates available up to now. In the case where K is a ψ_2 -body, one has $q_* \simeq n$ and the problem is automatically resolved.

The main idea in our approach is to start from an extremal isotropic convex body K in \mathbb{R}^n with maximal isotropic constant $L_K \simeq L_n := \sup\{L_K : K \text{ is a convex body in } \mathbb{R}^n\}$. Building on ideas from the work [5] of Bourgain, Klartag and Milman, we construct a second isotropic convex body K_1 which is also extremal and, at the same time, is in α -regular M -position in the sense of Pisier (see [24]). Then, we use the fact that small ball probability estimates are closely related to estimates on covering numbers. This gives the estimate

$$(1.9) \quad L_{K_1} I_{-c \frac{n}{(2-\alpha)t^\alpha}}(K_1) \leq Ct\sqrt{n},$$

for $t \geq C(\alpha)$, where $c, C > 0$ are absolute constants. The construction of K_1 from K can be done inside any subclass of isotropic log-concave measures which is stable under the operations of taking marginals or products. This leads us to the definition of a *coherent* class of probability measures (see Section 4): a subclass \mathcal{U} of the class of probability measures \mathcal{P} is called coherent if it satisfies two conditions:

1. If $\mu \in \mathcal{U}$ is supported on \mathbb{R}^n then, for all $k \leq n$ and $F \in G_{n,k}$, $\pi_F(\mu) \in \mathcal{U}$.
2. If $m \in \mathbb{N}$ and $\mu_i \in \mathcal{U}$, $i = 1, \dots, m$, then $\mu_1 \otimes \dots \otimes \mu_m \in \mathcal{U}$.

It should be noted that the class of isotropic convex bodies is not coherent. This is the reason for working with the more general class of log-concave measures. The basic tools that enable us to pass from one language to the other come from K. Ball's bodies and are described in Section 2.

Our main result is the following:

Theorem 1.1. *Let \mathcal{U} be a coherent subclass of isotropic log-concave measures and let $n \geq 2$ and $\delta \geq 1$. Then,*

$$(1.10) \quad \sup_{\mu \in \mathcal{U}_{[n]}} f_\mu(0)^{\frac{1}{n}} \leq C\delta \sup_{\mu \in \mathcal{U}_{[n]}} \sqrt{\frac{n}{q_{-c}(\mu, \delta)} \log \left(\frac{en}{q_{-c}(\mu, \delta)} \right)},$$

where $C > 0$ is an absolute constant and $\mathcal{U}_{[n]}$ denotes the subclass of n -dimensional measures in \mathcal{U} .

In Section 4, for every $\alpha \in (1, 2]$ we introduce a coherent class of isotropic log-concave measures which is equivalent to the class of ψ_α -class. Then, Theorem 1.1 has the following consequence:

Theorem 1.2. *Let $\alpha \in (1, 2]$ and $\beta_\alpha > 0$. Let μ be an isotropic ψ_α log-concave measure with constant β_α in \mathbb{R}^n . Then,*

$$(1.11) \quad f_\mu(0)^{\frac{1}{n}} \leq C \sqrt{n^{\frac{2-\alpha}{2}} \beta_\alpha^\alpha} \sqrt{\log \left(n^{\frac{2-\alpha}{2}} \beta_\alpha^\alpha \right)},$$

where $C > 0$ is an absolute constant. In particular, if μ is ψ_2 with constant $\beta_2 > 0$, then

$$(1.12) \quad f_\mu(0)^{\frac{1}{n}} \leq C \beta_2 \sqrt{\log \beta_2}.$$

Moreover, for every isotropic log-concave measure μ ,

$$(1.13) \quad f_\mu(0)^{\frac{1}{n}} \leq C n^{\frac{1}{4}} \sqrt{\log n}.$$

From Theorem 1.2 we immediately deduce two facts:

1. If a convex body K satisfies a ψ_2 -estimate with constant β , then

$$L_K \leq C \beta \sqrt{\log \beta}.$$

2. For every isotropic convex body K in \mathbb{R}^n ,

$$L_K \leq C \sqrt[4]{n} \sqrt{\log n}.$$

The first fact slightly improves Bourgain's estimate from [4]. The second one is weaker than Klartag's $\sqrt[4]{n}$ -bound in [8]; nevertheless, our method has the advantage that it can take into account any additional information on the ψ_α behavior of K .

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2 Background material

2.1 Basic notation. We work in \mathbb{R}^n , which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1} . The Grassmann manifold $G_{n,k}$ of k -dimensional subspaces of \mathbb{R}^n is equipped with the Haar probability measure $\mu_{n,k}$. Let $k \leq n$ and $F \in G_{n,k}$. We will denote by P_F the orthogonal projection from \mathbb{R}^n onto F .

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. In order to facilitate reading, we will denote by $\bar{c}, \eta, \kappa, \xi, \tau$ etc. some (absolute) positive constants that appear in more than one places.

Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Also if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

2.2 Probability measures. We denote by $\mathcal{P}_{[n]}$ the class of all probability measures in \mathbb{R}^n which are absolutely continuous with respect to the Lebesgue measure. We write \mathcal{A}_n for the Borel σ -algebra in \mathbb{R}^n . The density of $\mu \in \mathcal{P}_{[n]}$ is denoted by f_μ .

The subclass $\mathcal{SP}_{[n]}$ consists of all symmetric measures $\mu \in \mathcal{P}_{[n]}$; μ is called symmetric if f_μ is an even function on \mathbb{R}^n .

The subclass $\mathcal{CP}_{[n]}$ consists of all $\mu \in \mathcal{P}_{[n]}$ that have center of mass at the origin; so, $\mu \in \mathcal{CP}_{[n]}$ if

$$(2.1) \quad \int_{\mathbb{R}^n} \langle x, \theta \rangle d\mu(x) = 0$$

for all $\theta \in S^{n-1}$.

Let $\mu \in \mathcal{P}_{[n]}$. For every $1 \leq k \leq n-1$ and $F \in G_{n,k}$, we define the F -marginal $\pi_F(\mu)$ of μ as follows: for every $A \in \mathcal{A}_F$,

$$(2.2) \quad \pi_F(\mu)(A) := \mu(P_F^{-1}(A)).$$

It is clear that $\pi_F(\mu) \in \mathcal{P}_{[\dim F]}$. Note that, by the definition, for every Borel measurable function $f : \mathbb{R}^n \rightarrow [0, \infty)$ we have

$$(2.3) \quad \int_F f(x) d\pi_F(\mu)(x) = \int_{\mathbb{R}^n} f(P_F(x)) d\mu(x).$$

The density of $\pi_F(\mu)$ is the function

$$(2.4) \quad \pi_F(f_\mu)(x) := f_{\pi_F(\mu)}(x) = \int_{x+F^\perp} f_\mu(y) dy.$$

Let $\mu_1 \in \mathcal{P}_{[n_1]}$ and $\mu_2 \in \mathcal{P}_{[n_2]}$. We will write $\mu_1 \otimes \mu_2$ for the measure in $\mathcal{P}_{[n_1+n_2]}$ which satisfies

$$(2.5) \quad (\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$$

for all $A_1 \in \mathcal{A}_{n_1}$ and $A_2 \in \mathcal{A}_{n_2}$. It is easily checked that $f_{\mu_1 \otimes \mu_2} = f_{\mu_1} f_{\mu_2}$.

2.3 Log-concave measures. We denote by $\mathcal{L}_{[n]}$ the class of all log-concave probability measures on \mathbb{R}^n . A measure μ on \mathbb{R}^n is called log-concave if for any $A, B \in \mathcal{A}_n$ and any $\lambda \in (0, 1)$,

$$(2.6) \quad \mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

A function $f : \mathbb{R}^n \rightarrow [0, \infty)$ is called log-concave if $\log f$ is concave.

It is known that if $\mu \in \mathcal{L}_{[n]}$ and $\mu(H) < 1$ for every hyperplane H , then $\mu \in \mathcal{P}_{[n]}$ and its density f_μ is log-concave (see [2]). As an application of the Prékopa-Leindler inequality ([10], [25], [26]) one can check that if f is log-concave then, for every $k \leq n-1$ and $F \in G_{n,k}$, $\pi_F(f)$ is also log-concave. As before, we write $\mathcal{CL}_{[n]}$ or $\mathcal{SL}_{[n]}$ for the centered or symmetric non degenerate $\mu \in \mathcal{L}_{[n]}$ respectively.

2.4 Convex bodies. A convex body in \mathbb{R}^n is a compact convex subset C of \mathbb{R}^n with non-empty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C has center of mass at the origin if $\int_C \langle x, \theta \rangle dx = 0$ for every $\theta \in S^{n-1}$.

The support function $h_C : \mathbb{R}^n \rightarrow \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$. The mean width of C is defined by

$$(2.7) \quad W(C) = \int_{S^{n-1}} h_C(\theta) \sigma(d\theta).$$

For each $-\infty < p < \infty$, $p \neq 0$, we define the p -mean width of C by

$$(2.8) \quad W_p(C) = \left(\int_{S^{n-1}} h_C^p(\theta) \sigma(d\theta) \right)^{1/p}.$$

The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C , the polar body C° of C is

$$(2.9) \quad C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in C\}.$$

Note that if K is a convex body in \mathbb{R}^n then the Brunn-Minkowski inequality implies that $\mathbf{1}_{\tilde{K}} \in \mathcal{L}_{[n]}$.

We will denote by $\mathcal{K}_{[n]}$ the class of convex bodies in \mathbb{R}^n and by $\tilde{\mathcal{K}}_{[n]}$ the subclass of bodies of volume 1. Also, $\mathcal{CK}_{[n]}$ is the class of convex bodies with center of mass at the origin and $\mathcal{SK}_{[n]}$ is the class of origin symmetric convex bodies in \mathbb{R}^n .

We refer to the books [28], [18] and [24] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

2.5 L_q -centroid bodies. Let $\mu \in \mathcal{P}_{[n]}$. For every $q \geq 1$ and $\theta \in S^{n-1}$ we define

$$(2.10) \quad h_{Z_q(\mu)}(\theta) := \left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^q f(x) dx \right)^{1/q},$$

where f is the density of μ . If $\mu \in \mathcal{L}_{[n]}$ then $h_{Z_q(\mu)}(\theta) < \infty$ for every $q \geq 1$ and every $\theta \in S^{n-1}$. We define the L_q -centroid body $Z_q(\mu)$ of μ to be the centrally symmetric convex set with support function $h_{Z_q(\mu)}$.

L_q -centroid bodies were introduced, with a different normalization, in [11] (see also [12] where an L_q affine isoperimetric inequality was proved). Here we follow the normalization (and notation) that appeared in [21]. The original definition concerned the class of measures $\mathbf{1}_K \in \mathcal{L}_{[n]}$ where K is a convex body of volume 1. In this case, we also write $Z_q(K)$ instead of $Z_q(\mathbf{1}_K)$.

If K is a compact set in \mathbb{R}^n and $|K| = 1$, it is easy to check that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \leq p \leq q \leq \infty$, where $Z_\infty(K) = \text{conv}\{K, -K\}$. Note that if $T \in SL_n$ then $Z_p(T(K)) = T(Z_p(K))$. Moreover, if K is convex body, as a consequence of the Brunn-Minkowski inequality (see, for example, [21]), one can check that

$$(2.11) \quad Z_q(K) \subseteq \bar{c}_0 q Z_2(K)$$

for every $q \geq 2$ and, more generally,

$$(2.12) \quad Z_q(K) \subseteq \bar{c}_0 \frac{q}{p} Z_p(K)$$

for all $1 \leq p < q$, where $\bar{c}_0 \geq 1$ is an absolute constant. Also, if K has its center of mass at the origin, then

$$(2.13) \quad Z_q(K) \supseteq \bar{c} K$$

for all $q \geq n$, where $\bar{c} > 0$ is an absolute constant. For a proof of this fact and additional information on L_q -centroid bodies, we refer to [20] and [22].

2.6 Isotropic probability measures. Let $\mu \in \mathcal{CP}_{[n]}$. We say that μ is isotropic if $Z_2(\mu) = B_2^n$. We write $\mathcal{I}_{[n]}$ and $\widetilde{\mathcal{IL}}_{[n]}$ for the classes of isotropic probability measures and isotropic log-concave probability measures on \mathbb{R}^n respectively.

We say that a convex body $K \in \widetilde{\mathcal{CK}}_{[n]}$ is isotropic if $Z_2(K)$ is a multiple of the Euclidean ball. We define the isotropic constant of K by

$$(2.14) \quad L_K := \left(\frac{|Z_2(K)|}{|B_2^n|} \right)^{1/n}.$$

So, K is isotropic if and only if $Z_2(K) = L_K B_2^n$. We write $\mathcal{IK}_{[n]}$ for the class of isotropic convex bodies in \mathbb{R}^n . Note that $K \in \mathcal{IK}_{[n]}$ if and only if $L_K^n \mathbf{1}_{\frac{K}{L_K}} \in \mathcal{IL}_{[n]}$. A convex body K is called *almost isotropic* if K has volume one and $\bar{K} \simeq T(K)$ where $T(K)$ is an isotropic linear transformation of K .

We refer to [16], [7] and [22] for additional information on isotropic convex bodies.

2.7 The bodies $K_p(\mu)$. A natural way to pass from log-concave measures to convex bodies was introduced by K. Ball in [1]. Here, we will give the definition in a somewhat more general setting: Let $\mu \in \mathcal{P}_{[n]}$. For every $p > 0$ we define a set $K_p(\mu)$ as follows:

$$(2.15) \quad K_p(\mu) := \left\{ x \in \mathbb{R}^n : p \int_0^\infty f_\mu(rx) r^{p-1} dr \geq f_\mu(0) \right\}.$$

It is clear that $K_p(\mu)$ is a star shaped body with gauge function

$$(2.16) \quad \|x\|_{K_p(\mu)} := \left(\frac{p}{f_\mu(0)} \int_0^\infty f_\mu(rx) r^{p-1} dr \right)^{-1/p}.$$

Let $1 \leq k < n$ and $F \in G_{n,k}$. For $\theta \in S_F$ we define

$$(2.17) \quad \|\theta\|_{B_{k+1}(\mu, F)} := \|\theta\|_{K_{k+1}(\pi_F(\mu))}.$$

In the following Proposition we give some basic properties of the star-shaped bodies $K_p(\mu)$. We refer to [1], [16], [22], [23] for the proofs and additional references.

Proposition 2.1. *Let $\mu \in \mathcal{P}_{[n]}$, $p > 0$, $1 \leq k < n$ and $F \in G_{n,k}$.*

- (i) *If $\mu \in \mathcal{L}_{[n]}$ then $K_p(\mu) \in \mathcal{K}_{[n]}$. Moreover, if $\mu \in \mathcal{SL}_{[n]}$ then $K_p(\mu) \in \mathcal{SK}_{[n]}$.*
- (ii) *If $\mu \in \mathcal{CL}_{[n]}$ then $K_{n+1}(\mu) \in \mathcal{CK}_{[n]}$. If $\mu \in \mathcal{SIL}_{[n]}$ then $\widetilde{K_{n+2}}(\mu) \in \widetilde{\mathcal{SK}}_{[n]}$.*

(iii) If $\mu \in \mathcal{IL}_{[n]}$ then $\widetilde{K_{n+1}}(\mu)$ is almost isotropic.

(iv) Let $1 \leq p \leq n$ and $\mu \in \mathcal{CL}_{[n]}$. Then, $f_\mu(0)^{\frac{1}{n}} Z_p(\mu) \simeq Z_p(\widetilde{K_{n+1}}(\mu))$.

(v) Let $1 \leq p \leq k < n$, $F \in G_{n,k}$, $\mu \in \mathcal{CL}_{[n]}$ and $K \in \mathcal{CL}_{[n]}$. Then,

$$(2.18) \quad f_{\pi_F(\mu)}(0)^{\frac{1}{k}} P_F(Z_p(\mu)) \simeq f_\mu(0)^{\frac{1}{n}} Z_p(\widetilde{B_{k+1}}(\mu, F))$$

and

$$(2.19) \quad |K \cap F^\perp|^{\frac{1}{k}} P_F(Z_p(K)) \simeq Z_p(\widetilde{B_{k+1}}(K, F)).$$

(vi) Let $1 \leq k < n$, $F \in G_{n,k}$ and $K \in \mathcal{IK}_{[n]}$. Then,

$$(2.20) \quad |K \cap F^\perp|^{\frac{1}{k}} \simeq \frac{L_{\widetilde{B_{k+1}}(K, F)}}{L_K}.$$

(vii) If $\mu \in \mathcal{IL}_{[n]}$, then

$$(2.21) \quad L_{K_{n+1}(\mu)} \simeq f_\mu(0)^{\frac{1}{n}}.$$

2.8 ψ_α -norm. Let $\mu \in \mathcal{P}_{[n]}$. Given $\alpha \geq 1$, the Orlicz norm $\|g\|_{\psi_\alpha}$ of a bounded measurable function $g: \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to μ is defined by

$$(2.22) \quad \|g\|_{\psi_\alpha} = \inf \left\{ t > 0 : \int_{\mathbb{R}^n} \exp \left(\left(\frac{|f(x)|}{t} \right)^\alpha \right) d\mu(x) \leq 2 \right\}.$$

It is not hard to check that

$$(2.23) \quad \|f\|_{\psi_\alpha} \simeq \sup \left\{ \frac{\|f\|_p}{p^{1/\alpha}} : p \geq \alpha \right\}.$$

Let $\theta \in S^{n-1}$. We say that μ satisfies a ψ_α -estimate with constant $\beta_{\alpha, \mu, \theta}$ in the direction of θ if

$$(2.24) \quad \|\langle \cdot, y \rangle\|_{\psi_\alpha} \leq \beta_{\alpha, \mu, \theta} \|\langle \cdot, y \rangle\|_2.$$

We say that μ is a ψ_α -measure with constant $\beta_{\alpha, \mu}$ where $\beta_{\alpha, \mu} := \sup_{\theta \in S^{n-1}} \beta_{\alpha, \mu, \theta}$, provided that this last quantity is finite.

Similarly, if $K \in \widetilde{\mathcal{K}}_{[n]}$ we define

$$(2.25) \quad \beta_{\alpha, K} := \sup_{\theta \in S^{n-1}} \sup_{p \geq \alpha} \frac{h_{Z_p(K)}(\theta)}{p^{1/\alpha} h_{Z_2(K)}(\theta)}.$$

Note that $\beta_{\alpha, \mu}$ is an affine invariant, since $\beta_{\alpha, \mu \circ T^{-1}} = \beta_{\alpha, \mu}$ for all $T \in SL_n$. Finally, we define

$$(2.26) \quad \mathcal{P}_{[n]}(\alpha, \beta) := \{\mu \in \mathcal{P}_{[n]} : \beta_{\alpha, \mu} \leq \beta\}$$

and

$$(2.27) \quad \mathcal{K}_{[n]}(\alpha, \beta) := \{K \in \tilde{\mathcal{K}}_{[n]} : \beta_{\alpha, K} \leq \beta\}.$$

2.9 The parameter $k_(C)$.* Let C be a symmetric convex body in \mathbb{R}^n . Define $k_*(C)$ as the largest positive integer $k \leq n$ for which

$$(2.28) \quad \mu_{n,k} \left(F \in G_{n,k} : \frac{1}{2}W(C)(B_2^n \cap F) \subseteq P_F(C) \subseteq 2W(C)(B_2^n \cap F) \right) \geq \frac{n}{n+k}.$$

Thus, $k_*(C)$ is the maximal dimension k such that a “random” k -dimensional projection of C is 4-Euclidean.

The parameter $k_*(C)$ is completely determined by the global parameters $W(C)$ and $R(C)$: There exist $c_1, c_2 > 0$ such that

$$(2.29) \quad c_1 n \frac{W(C)^2}{R(C)^2} \leq k_*(C) \leq c_2 n \frac{W(C)^2}{R(C)^2}$$

for every symmetric convex body C in \mathbb{R}^n . The lower bound appears in Milman’s proof of Dvoretzky’s theorem (see [13]) and the upper bound was proved in [19].

3 Negative moments of the Euclidean norm

Let $\mu \in \mathcal{P}_{[n]}$. If $-n < p \leq \infty$, $p \neq 0$, we define

$$(3.1) \quad I_p(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^p d\mu(x) \right)^{1/p}.$$

As usual, if K is a Borel subset of \mathbb{R}^n with Lebesgue measure equal to 1, we write $I_p(K) := I_p(\mathbf{1}_K)$.

Definition 3.1. Let $\mu \in \mathcal{P}_{[n]}$ and $\delta \geq 1$. We define

$$\begin{aligned} q_*(\mu) &:= \max\{k \leq n : k_*(Z_k(\mu)) \geq k\} \\ q_{-c}(\mu, \delta) &:= \max\{p \geq 1 : I_{-p}(\mu) \geq \frac{1}{\delta} I_2(\mu)\} \\ q_*(\mu, \delta) &:= \max\{k \leq n : k_*(Z_k(\mu)) \geq \frac{k}{\delta^2}\}. \end{aligned}$$

One of the main results of [23] asserts that the moments of the Euclidean norm on log-concave measures satisfy a strong reverse Hölder inequality up to the value q_* :

Theorem 3.2. *Let $\mu \in \mathcal{CL}_{[n]}$. Then for every $p \leq q_*(\mu)$,*

$$(3.2) \quad I_p(\mu) \leq CI_{-p}(\mu),$$

where $C > 0$ is an absolute constant.

It is clear from the statement that in order to apply Theorem 3.2 in a meaningful way one should have some non-trivial estimate for the parameter q_* . The next proposition (see [22, Proposition 3.10] or [23, Proposition 5.7]) gives a lower bound for q_* , with a dependence on the ψ_α constant, in the isotropic case.

Proposition 3.3. *Let $\mu \in \mathcal{I}_{[n]} \cap \mathcal{P}_{[n]}(\alpha, \beta)$. Then*

$$(3.3) \quad q_*(\mu) \geq c \frac{n^{\frac{\alpha}{2}}}{\beta^\alpha},$$

where $c > 0$ is an absolute constant.

Definition 3.4. Let $\mu \in \mathcal{P}_{[n]}$. We will say that μ is of *small diameter* (with constant $A > 0$) if for every $p \geq 2$ one has

$$(3.4) \quad I_p(\mu) \leq AI_2(\mu).$$

The definition that we give here is a direct generalization of the one given in [21] for the case of convex bodies.

Let $\mu \in \mathcal{P}_{[n]}$ and set $B := 4I_2(\mu)B_2^n$. Note that $\frac{3}{4} \leq \mu(B) \leq 1$. We define a new measure $\bar{\mu}$ on \mathcal{A}_n in the following way: for every $A \in \mathcal{A}_n$ we set

$$\bar{\mu}(A) := \frac{\mu(A \cap B)}{\mu(B)}.$$

Assume that, additionally, $\mu \in \mathcal{L}_{[n]}$. Then, it is not hard to check that

$$(3.5) \quad I_2(\mu) \simeq I_2(\bar{\mu}), \quad Z_2(\mu) \simeq Z_2(\bar{\mu}) \quad \text{and} \quad f_{\bar{\mu}}(0)^{\frac{1}{n}} \simeq f_\mu(0)^{\frac{1}{n}}.$$

Therefore, if $\mu \in \mathcal{L}_{[n]}$, we can always find a measure $\bar{\mu} \in \mathcal{L}_{[n]}$ which is of small diameter (with an absolute constant $C > 0$) and satisfies $f_{\bar{\mu}}(0)^{\frac{1}{n}} \simeq f_\mu(0)^{\frac{1}{n}}$.

Moreover, if μ is isotropic, then $\bar{\mu}$ is almost isotropic. As a consequence of [23, Theorem 5.6] we have the following:

Proposition 3.5. *Let $\mu \in \mathcal{L}$. Then,*

$$(3.6) \quad q_*(\bar{\mu}, \xi_1) \simeq q_{-c}(\bar{\mu}, \xi_2),$$

where $\xi_1, \xi_2 \geq 1$ are absolute constants.

4 Coherent classes of measures

Our starting point is a simple but crucial observation from the paper [5] of Bourgain, Klartag and Milman. First of all, one may observe that $L_n := \sup\{L_K : K \text{ is a convex body in } \mathbb{R}^n\}$ is, essentially, an increasing function of n : for every $k \leq n$, $L_k \leq CL_n$, where $C > 0$ is an absolute constant. So, using (2.20) we see

that if K_0 is an isotropic convex body in \mathbb{R}^n such that $L_{K_0} \simeq L_n$, then, for all $F \in G_{n,k}$,

$$(4.1) \quad |K_0 \cap F^\perp|^{1/k} \simeq \frac{L_{\widetilde{B_{k+1}(K_0, F)}}}{L_{K_0}} \leq C_1 \frac{L_k}{L_n} \leq C_2.$$

Building on the ideas of [5] one can use this property of a body K_0 with “extremal isotropic constant” to get upper bounds for the negative moments of the Euclidean norm on K_0 . Since we want to apply this argument in different situations, we will first introduce some terminology.

Definition 4.1. We define $\mathcal{P} := \cup_{i=1}^\infty \mathcal{P}_{[n]}$. Similarly, $\mathcal{IP} := \cup_{i=1}^\infty \mathcal{IP}_{[n]}$, etc.

Let \mathcal{U} be a subclass of \mathcal{P} . Set $\mathcal{U}_{[n]} = \mathcal{U} \cap \mathcal{P}_{[n]}$. We say that \mathcal{U} is *coherent* if it satisfies the following two conditions:

1. If $\mu \in \mathcal{U}_{[n]}$ then, for all $k \leq n$ and $F \in G_{n,k}$, $\pi_F(\mu) \in \mathcal{U}_{[\dim F]}$.
2. If $m \in \mathbb{N}$ and $\mu_i \in \mathcal{U}_{[n_i]}$, $i = 1, \dots, m$, then

$$\mu_1 \otimes \cdots \otimes \mu_m \in \mathcal{U}_{[n_1 + \cdots + n_m]}.$$

We also agree that the null class is coherent. Note that if \mathcal{U}_1 and \mathcal{U}_2 are coherent then $\mathcal{U}_1 \cap \mathcal{U}_2$ is also coherent.

The following proposition is a translation of well known results to this language.

Proposition 4.2. *The classes \mathcal{SP} , \mathcal{CP} , \mathcal{L} , \mathcal{I} are coherent.*

Note that the class $\mathcal{K} := \bigcup_{n=1}^\infty \{\mu \in \mathcal{P}_{[n]} : \mu = \mathbf{1}_{\widetilde{K}} ; K \in \mathcal{K}_{[n]}\}$ is *not* coherent.

Proposition 4.3. *Let \mathcal{U} be a coherent class of measures. If n is even then, for every $\mu \in \mathcal{U}_{[n]}$, $k = \frac{n}{2}$ and $F \in G_{n,k}$,*

$$(4.2) \quad f_{\pi_F(\mu)}(0)^{\frac{1}{k}} \leq \sup_{\mu \in \mathcal{U}_{[n]}} f_\mu(0)^{\frac{1}{n}}.$$

Moreover, if $\rho_n(\mathcal{U}) := \sup_{\mu \in \mathcal{U}_{[n]}} f_\mu(0)^{\frac{1}{n}}$, then

$$(4.3) \quad \rho_{n-1}(\mathcal{U}) \leq \rho_n(\mathcal{U}) \left(\frac{\rho_{n-1}(\mathcal{U})}{\rho_1(\mathcal{U})} \right)^{\frac{1}{n}}.$$

Proof. For the first assertion use the fact that $\pi_F(\mu) \otimes \pi_F(\mu) \in \mathcal{U}_{[n]}$ and

$$f_{(\pi_F(\mu) \otimes \pi_F(\mu))}(0) = [f_{\pi_F(\mu)}(0)]^2.$$

For the second assertion use the fact that if $\mu_1 \in \mathcal{U}_{[n-1]}$ and $\mu_2 \in \mathcal{U}_{[1]}$ then we have $\mu_1 \otimes \mu_2 \in \mathcal{U}_{[n]}$ and $f_{\mu_1 \otimes \mu_2}(0) = f_{\mu_1}(0)f_{\mu_2}(0)$. \square

In particular if a class satisfies

$$e^{-n} \leq \rho_n(\mathcal{U}) \leq e^n ,$$

it is enough to bound $\rho_n(\mathcal{U})$ for n even. Note that \mathcal{IL} is such a class.

Our goal in this section is to show that the class of measures that are ψ_α with constant less than β is coherent. To do that we will use a more convenient definition of “ ψ_α measures”.

Let $\mu \in \mathcal{CP}_{[n]}$. For every $\theta \in S^{n-1}$ and every $\lambda > 0$ we define

$$(4.4) \quad h_{\mu,\theta}(\lambda) := h(\lambda) = \log \left(\int_{\mathbb{R}^n} e^{\lambda \langle x, \theta \rangle} d\mu(x) \right).$$

Next, if $\alpha \in (1, \infty]$, we define

$$(4.5) \quad \tilde{\psi}_{\alpha,\mu}(\theta) := \sup_{\lambda > 0} \frac{1}{\lambda} h(\lambda)^{\frac{1}{\alpha_*}} = \sup_{\lambda > 0} \frac{1}{\lambda} \left(\log \int_{\mathbb{R}^n} e^{\lambda \langle x, \theta \rangle} d\mu(x) \right)^{\frac{1}{\alpha_*}},$$

where α_* is the conjugate exponent of α , i.e. $\frac{1}{\alpha} + \frac{1}{\alpha_*} = 1$.

Note that $\tilde{\psi}_{\alpha,\mu}(\theta) \in \mathbb{R}_+ \cup \{\infty\}$. Indeed, by Jensen’s inequality, $h(\lambda) \geq 0$, since $\mu \in \mathcal{CP}_{[n]}$. The function $h(\lambda)$ is strictly convex and, since $h(0) = 0$, the function $\lambda \mapsto \frac{h(\lambda)}{\lambda}$ is strictly increasing. In particular,

$$(4.6) \quad \tilde{\psi}_{\infty,\mu}(\theta) = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \int_{\mathbb{R}^n} e^{\lambda \langle x, \theta \rangle} d\mu(x).$$

Definition 4.4. Let μ be a probability measure on \mathbb{R}^n . For $\alpha \in (1, \infty]$ we define

$$(4.7) \quad \tilde{\beta}_{\mu,\alpha} := \sup_{\theta \in S^{n-1}} \frac{\tilde{\psi}_{\alpha,\mu}(\theta)}{h_{Z_2(\mu)}(\theta)}.$$

We also define

$$(4.8) \quad \mathcal{P}_\alpha(\beta) := \bigcup_{n=1}^{\infty} \{ \mu \in \mathcal{P}_{[n]} : \tilde{\beta}_{\mu,\alpha} \leq \beta \}$$

We will prove that, for every $\alpha \in (1, 2]$ and $\beta > 0$, the class $\mathcal{P}_\alpha(\beta)$ is coherent (see Proposition 4.11). Moreover, the next Proposition shows that for every $\mu \in \mathcal{CP}_{[n]}$ and every $\alpha > 1$ the quantities $\max\{\tilde{\psi}_{\alpha,\mu}(\theta), \tilde{\psi}_{\alpha,\mu}(-\theta)\}$ and $\|\langle \cdot, \theta \rangle\|_{\psi_\alpha}$ are equivalent up to an absolute constant.

Proposition 4.5. *Let $\mu \in \mathcal{CP}_{[n]}$. For every $\alpha \in (1, \infty)$ and every $\theta \in S^{n-1}$ we have that*

$$(4.9) \quad C_1 \max\{\tilde{\psi}_{\alpha,\mu}(\theta), \tilde{\psi}_{\alpha,\mu}(-\theta)\} \leq \|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leq C_2 \max\{\tilde{\psi}_{\alpha,\mu}(\theta), \tilde{\psi}_{\alpha,\mu}(-\theta)\},$$

where $C_1, C_2 > 0$ are absolute constants.

The proof will be based on the following two Lemmas:

Lemma 4.6. *If $p \in (0, 1]$ and $t > 0$, then*

$$(4.10) \quad (e^t - 1) \frac{1-t}{1-t^p} \leq \frac{t}{p} e^t.$$

Proof. We distinguish two cases:

Case 1. $t \in (0, 1)$: Let $F(p) = \frac{t}{p} e^t (1-t^p) - (e^t - 1)(1-t)$. It is enough to show that $F(p) \geq 0$. Note that $F(1) = (1-t)te^t - (e^t - 1)(1-t) = (1-t)(te^t - e^t + 1)$. Let $g(t) := te^t - e^t + 1$. Then $g(0) = 0$ and $g'(t) = te^t > 0$. This implies that $F(1) \geq 0$. So, the claim will follow if we show that F is a decreasing function of p on $(0, 1)$.

We compute

$$(4.11) \quad F'(p) = te^t \left(-\frac{1}{p^2} - \frac{t^p \log t}{p} + \frac{t^p}{p^2} \right) = \frac{te^t}{p^2} (t^p - pt^p \log t - 1).$$

Let $G(t) := t^p - pt^p \log t - 1$. Then, $G(1) = 0$ and

$$(4.12) \quad G'(t) = pt^{p-1} - p^2 t^{p-1} \log t - pt^{p-1} = -p^2 t^{p-1} \log t \geq 0.$$

So, G is increasing and $G(t) \leq 0$. This implies that F is decreasing and settles the first case.

Case 2. $t \geq 1$: Note that it is enough to show that

$$(4.13) \quad \frac{t-1}{t^p-1} \leq \frac{t}{p}.$$

We consider the function $H(t) := t^{p+1} - t - pt + p$. Note that

$$(4.14) \quad H'(t) = (p+1)t^p - (p+1) \geq 0$$

since $t \geq 1$. It follows that H is increasing. Since $H(1) = 0$, we have $H(t) \geq 0$ for all $t \geq 1$. This proves (4.13) and completes the proof. \square

Lemma 4.7. *If $r \geq 1$ and $t > 0$ then*

$$(4.15) \quad 1 + \sum_{k=1}^{\infty} \left(\frac{tr}{k} \right)^{k/r} \leq e^{5rt}.$$

Proof. Let $f(x) := \frac{t^x}{x^x}$, $x > 0$. Then f is increasing when $x \leq \frac{t}{e}$ and decreasing when $x \geq \frac{t}{e}$. Moreover, $\sup_{x>0} f(x) = e^{\frac{t}{e}}$.

Let $A := \{k \in \mathbb{N} : k \leq \frac{t \lfloor r+1 \rfloor}{e}\}$ and $A_v = \{k \in \mathbb{N} \setminus A : k \equiv v \pmod{\lfloor r+1 \rfloor}\}$. Note that A is possibly empty but, in any case, $|A| \leq \frac{t \lfloor r+1 \rfloor}{e}$. Note also that if $k \notin A$ then $\frac{k}{r} \geq \frac{k}{\lfloor r+1 \rfloor} > \frac{t}{e}$. So, in this case

$$(4.16) \quad \left(\frac{tr}{k} \right)^{k/r} \leq \left(\frac{t \lfloor r+1 \rfloor}{k} \right)^{k/\lfloor r+1 \rfloor}.$$

We write

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} \left(\frac{tr}{k}\right)^{k/r} &\leq 1 + \sum_{k \in A} \left(\frac{tr}{k}\right)^{k/r} + \sum_{k \in \mathbb{N} \setminus A} \left(\frac{tr}{k}\right)^{k/r} \\ &\leq 1 + \frac{t \lfloor r+1 \rfloor}{e} e^{\frac{t}{e}} + \sum_{k \in \mathbb{N} \setminus A} \left(\frac{t \lfloor r+1 \rfloor}{k}\right)^{k/\lfloor r+1 \rfloor}. \end{aligned}$$

Next, we write

$$\begin{aligned} \sum_{k \in \mathbb{N} \setminus A} \left(\frac{t \lfloor r+1 \rfloor}{k}\right)^{k/\lfloor r+1 \rfloor} &= \sum_{v=0}^{\lfloor r+1 \rfloor - 1} \sum_{m \in A_v} \left(\frac{t \lfloor r+1 \rfloor}{k}\right)^{k/\lfloor r+1 \rfloor} \\ &\leq \sum_{v=0}^{\lfloor r+1 \rfloor - 1} \sum_{m=1}^{\infty} \frac{t^{m + \frac{v}{\lfloor r+1 \rfloor}}}{\left(m + \frac{v}{\lfloor r+1 \rfloor}\right)^{m + \frac{v}{\lfloor r+1 \rfloor}}} \\ &\leq \sum_{v=0}^{\lfloor r+1 \rfloor - 1} t^{\frac{v}{\lfloor r+1 \rfloor}} \sum_{m=1}^{\infty} \frac{t^m}{m!} \\ &\leq (e^t - 1) \sum_{v=0}^{\lfloor r+1 \rfloor - 1} t^{\frac{v}{\lfloor r+1 \rfloor}} \\ &= (e^t - 1) \frac{1 - t}{1 - t^{\frac{1}{\lfloor r+1 \rfloor}}}. \end{aligned}$$

Combining the above, and using Lemma 4.6 with $p = 1/\lfloor r+1 \rfloor$, we get

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} \left(\frac{tr}{k}\right)^{k/r} &\leq e^t + t \lfloor r+1 \rfloor e^t + t \lfloor r+1 \rfloor e^t = (1 + 2t \lfloor r+1 \rfloor) e^t \\ &\leq e^{2t \lfloor r+1 \rfloor} e^t \leq e^{5rt}. \end{aligned}$$

This proves the Lemma. \square

Proof of Proposition 4.5. Let $\alpha \in (1, \infty)$ and let $\alpha_* \in (1, \infty)$ be the conjugate exponent of α . We set $\psi_{-1} := \tilde{\psi}_{\alpha, \mu}(-\theta)$, $\psi_1 := \tilde{\psi}_{\alpha, \mu}(\theta)$, $\psi_0 := \max\{\tilde{\psi}_{\alpha, \mu}(\theta), \tilde{\psi}_{\alpha, \mu}(-\theta)\}$ and $\psi_2 := \|\langle \cdot, \theta \rangle\|_{\psi_\alpha}$.

For every $\lambda > 0$,

$$(4.17) \quad \int_{\mathbb{R}^n} e^{\lambda \langle x, \theta \rangle} d\mu(x) \leq \exp(\lambda^{\alpha_*} \psi_1^{\alpha_*}).$$

So, by Markov's inequality we get that, for every $t > 0$,

$$(4.18) \quad \mu\{x : e^{\lambda \langle x, \theta \rangle} \geq e^{t^\alpha} e^{\lambda^{\alpha_*} \psi_1^{\alpha_*}}\} \leq e^{-t^\alpha}.$$

Equivalently,

$$(4.19) \quad \mu\left\{x : \langle x, \theta \rangle \geq \frac{t^\alpha}{\lambda} + \lambda^{\alpha_* - 1} \psi_1^{\alpha_*}\right\} \leq e^{-t^\alpha}.$$

Choosing $\lambda := \frac{t^{\alpha-1}}{\psi_1}$, we get

$$(4.20) \quad \mu\{x : \langle x, \theta \rangle \geq 2t\psi_1\} \leq e^{-t^\alpha}.$$

Similarly, for every $t > 0$ we have

$$(4.21) \quad \mu\{x : \langle x, -\theta \rangle \geq 2t\psi_{-1}\} \leq e^{-t^\alpha}.$$

Therefore,

$$\begin{aligned} \mu\{x : |\langle x, \theta \rangle| \geq 2t\psi_0\} &= \mu\{x : \langle x, \theta \rangle \geq 2t\psi_0\} + \mu\{x : \langle x, -\theta \rangle \geq 2t\psi_0\} \\ &\leq \mu\{x : \langle x, \theta \rangle \geq 2t\psi_1\} + \mu\{x : \langle x, -\theta \rangle \geq 2t\psi_{-1}\} \\ &\leq 2e^{-t^\alpha}. \end{aligned}$$

The last inequality implies that $\psi_2 \leq C_1\psi_0$.

In the opposite direction, using Lemma 4.7 we write

$$\begin{aligned} \int_{\mathbb{R}^n} e^{\lambda\langle x, \theta \rangle} d\mu(x) &\leq \int_{\mathbb{R}^n} e^{\lambda|\langle x, \theta \rangle|} d\mu(x) \leq 1 + \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \frac{\lambda^k |\langle x, \theta \rangle|^k}{k!} d\mu(x) \\ &\leq 1 + \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \frac{(e\lambda)^k |\langle x, \theta \rangle|^k}{k^k} d\mu(x) \leq 1 + \sum_{k=1}^{\infty} \frac{(e\lambda)^k (k)^{\frac{k}{\alpha}} \psi_2^k}{k^k} \\ &= 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{e\lambda}{\alpha^{1/\alpha_*}}\right)^k \psi_2^k}{\left(\frac{k}{\alpha_*}\right)^{\frac{k}{\alpha_*}}} = 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{(e\lambda\psi_2)^{\alpha_*}}{\alpha_*}\right)^{\frac{k}{\alpha_*}}}{\left(\frac{k}{\alpha_*}\right)^{\frac{k}{\alpha_*}}} \\ &\leq e^{5\alpha_* \frac{\lambda^{\alpha_*} (e\psi_2)^{\alpha_*}}{\alpha_*}} \leq e^{\lambda^{\alpha_*} (5e\psi_2)^{\alpha_*}}. \end{aligned}$$

It follows that

$$(4.22) \quad \psi_1 := \sup_{\lambda > 0} \frac{1}{\lambda} \left(\log \int_{\mathbb{R}^n} e^{\lambda\langle x, \theta \rangle} d\mu(x) \right)^{\frac{1}{\alpha_*}} \leq 5e\psi_2.$$

In a similar way we check that $\psi_{-1} \leq 5e\psi_2$. This completes the proof. \square

Corollary 4.8. *For every $\alpha \in (1, 2]$,*

$$(4.23) \quad \mathcal{CP}_\alpha(\beta) \subseteq \mathcal{CP}(\alpha, c_1\beta)$$

and

$$(4.24) \quad \mathcal{CP}(\alpha, c_2\beta) \subseteq \mathcal{CP}_\alpha(\beta),$$

where $c_1, c_2 > 0$ are universal constants.

Proof. Indeed, if $\mu \in \mathcal{CP}_\alpha(\beta)$ then Proposition 4.5 implies that

$$(4.25) \quad \sup_{\theta \in S^{n-1}} \frac{h_{\psi_\alpha(\mu)}(\theta)}{h_{Z_2(\mu)}(\theta)} \leq c_1 \sup_{\theta \in S^{n-1}} \frac{\tilde{\psi}_{\alpha,\mu}(\theta)}{h_{Z_2(\mu)}(\theta)} \leq c_1 \beta$$

which means that $\mu \in \mathcal{CP}(\alpha, c_1\beta)$ (recall (2.26)). The second inclusion is proved in a similar way. \square

Next, we prove that the class $\mathcal{P}_\alpha(\beta)$ is coherent. The behavior of $\tilde{\psi}_{\alpha,\mu}$ for products of measures is described by the following:

Proposition 4.9. *Let k be a positive integer and let $\mu_i \in \mathcal{CP}_{[n_i]}$ and $\theta_i \in S^{n_i-1}$, $i = 1, \dots, k$. If $\tilde{\psi}_{\alpha,\mu_i}(\theta_i) < \infty$ for all $i \leq k$ and some $\alpha \in (1, \infty)$, then*

$$(4.26) \quad \tilde{\psi}_{\alpha,\mu}((\theta_1, \dots, \theta_k)) \leq \left(\sum_{i=1}^k \tilde{\psi}_{\alpha,\mu_i}^{\alpha_*}(\theta_i) \right)^{\frac{1}{\alpha_*}},$$

where $\mu = \mu_1 \otimes \dots \otimes \mu_k$. In the case $\alpha = \infty$,

$$(4.27) \quad \tilde{\psi}_{\infty,\mu}((\theta_1, \dots, \theta_k)) = \sum_{i=1}^k \tilde{\psi}_{\infty,\mu_i}(\theta_i).$$

Proof. For every $\lambda > 0$ we can write

$$\frac{1}{\lambda^{\alpha_*}} \log \left(\int_{\mathbb{R}^{n_1}} \dots \int_{\mathbb{R}^{n_k}} e^{\lambda \sum_{i=1}^k \langle x_i, \theta_i \rangle} d\mu_k(x_k) \dots d\mu_1(x_1) \right)$$

as follows:

$$\begin{aligned} \frac{1}{\lambda^{\alpha_*}} \log \left(\prod_{i=1}^k \int_{\mathbb{R}^{n_i}} e^{\lambda \langle x_i, \theta_i \rangle} d\mu_i(x_i) \right) &= \frac{1}{\lambda^{\alpha_*}} \sum_{i=1}^k \log \int_{\mathbb{R}^{n_i}} e^{\lambda \langle x_i, \theta_i \rangle} d\mu_i(x_i) \\ &\leq \frac{1}{\lambda^{\alpha_*}} \sum_{i=1}^k \lambda^{\alpha_*} \tilde{\psi}_{\alpha,\mu_i}^{\alpha_*}(\theta_i) \\ &\leq \sum_{i=1}^k \tilde{\psi}_{\alpha,\mu_i}^{\alpha_*}(\theta_i). \end{aligned}$$

Taking the supremum with respect to $\lambda > 0$ we get the result. The case $\alpha = \infty$ can be treated in a similar way, taking into account (4.6). \square

The behavior of marginals is described by the following:

Proposition 4.10. *Let $\mu \in \mathcal{CP}_{[n]}$. Let $F \in G_{n,k}$ and $\theta \in S_F$. If $\alpha \in (1, \infty)$, then*

$$(4.28) \quad \tilde{\psi}_{\alpha,\pi_F(\mu)}(\theta) \leq \tilde{\psi}_{\alpha,\mu}(\theta).$$

If $\alpha = \infty$, then

$$(4.29) \quad \tilde{\psi}_{\infty,\pi_F(\mu)}(\theta) = \tilde{\psi}_{\infty,\mu}(\theta).$$

Proof. Note that, for every $\lambda > 0$,

$$(4.30) \quad \int_{\mathbb{R}^n} e^{\lambda\langle x, \theta \rangle} d\mu(x) = \int_F e^{\lambda\langle x, \theta \rangle} d\pi_F(\mu)(x)$$

It follows that

$$(4.31) \quad \frac{1}{\lambda^{\alpha_*}} \log \int_F e^{\lambda\langle x, \theta \rangle} d\pi_F(\mu)(x) = \frac{1}{\lambda^{\alpha_*}} \log \int_{\mathbb{R}^n} e^{\lambda\langle x, \theta \rangle} d\mu(x) \leq \tilde{\psi}_{\alpha_*, \mu}^{\alpha_*}(\theta).$$

Taking the supremum with respect to $\lambda > 0$ we get the result. The case $\alpha = \infty$ can be treated in a similar way. \square

Proposition 4.11. *Let $\alpha \in (1, 2]$ and let $\beta > 0$. Then the class $\mathcal{P}_\alpha(\beta)$ is coherent.*

Proof. Let $\mu \in (\mathcal{P}_\alpha(\beta))_{[n]}$. Fix $1 \leq k < n$ and $F \in G_{n,k}$. Then, using (4.28) and the fact that $h_{Z_2(\pi_F(\mu))}(\theta) = h_{Z_2(\mu)}(\theta)$ for $\theta \in S_F$, we see that

$$(4.32) \quad \tilde{\beta}_{\pi_F(\mu), \alpha} = \sup_{\theta \in S_F} \frac{\tilde{\psi}_{\alpha, \pi_F(\mu)}(\theta)}{h_{Z_2(\pi_F(\mu))}(\theta)} \leq \sup_{\theta \in S_F} \frac{\tilde{\psi}_{\alpha, \mu}(\theta)}{h_{Z_2(\mu)}(\theta)} \leq \tilde{\beta}_{\mu, \alpha}.$$

So, $\pi_F(\mu) \in \mathcal{P}_\alpha(\beta)$.

Next, let $\mu_i \in (\mathcal{P}_\alpha(\beta))_{[n_i]}$, $i := 1, \dots, k$ and set $N := n_1 + \dots + n_k$. Since

$$h_{Z_2(\mu_1 \otimes \dots \otimes \mu_k)}(\theta_1, \dots, \theta_k) = \left(\sum_{i=1}^k h_{Z_2(\mu_i)}^2(\theta_i) \right)^{\frac{1}{2}},$$

$$\begin{aligned} \tilde{\beta}_{\mu_1 \otimes \dots \otimes \mu_k, \alpha} &= \sup_{(\theta_1, \dots, \theta_k) \in S^{N-1}} \frac{\tilde{\psi}_{\alpha, \mu_1 \otimes \dots \otimes \mu_k}(\theta_1, \dots, \theta_k)}{h_{Z_2(\mu_1 \otimes \dots \otimes \mu_k)}(\theta_1, \dots, \theta_k)} \\ &\leq \sup_{(\theta_1, \dots, \theta_k) \in S^{N-1}} \frac{\left(\sum_{i=1}^k \tilde{\psi}_{\alpha, \mu_i}^{\alpha_*}(\theta_i) \right)^{\frac{1}{\alpha_*}}}{\left(\sum_{i=1}^k h_{Z_2(\mu_i)}^2(\theta_i) \right)^{\frac{1}{2}}} \\ &\leq \beta \sup_{(\theta_1, \dots, \theta_k) \in S^{N-1}} \frac{\left(\sum_{i=1}^k h_{Z_2(\mu_i)}^{\alpha_*}(\theta_i) \right)^{\frac{1}{\alpha_*}}}{\left(\sum_{i=1}^k h_{Z_2(\mu_i)}^2(\theta_i) \right)^{\frac{1}{2}}} \\ &\leq \beta \end{aligned}$$

since $\alpha_* \in [2, \infty)$, and $\|x\|_{\ell_{\alpha_*}^k} \leq \|x\|_{\ell_2^k}$. So, $\mu_1 \otimes \dots \otimes \mu_k \in \mathcal{P}_\alpha(\beta)$. \square

5 M -positions and extremal bodies

All the results in this section are stated for the case where the dimension is even. Proposition 4.3 shows that this is sufficient for our purposes. However, with minor changes in the proofs, all the results remain valid in the case where the dimension is odd.

Our main goal in this section is to prove the following:

Proposition 5.1. *Let $\mathcal{U} \subseteq \mathcal{IL}$ be a coherent class of probability measures, let $n \geq 2$ even, $\alpha \in (1, 2)$ and $t \geq \left(\frac{C_0}{2-\alpha}\right)^{\frac{1}{\alpha}}$. Then, there exists $\mu_1 \in \mathcal{U}_{[n]}$ such that*

$$(5.1) \quad f_{\mu_1}(0)^{\frac{1}{n}} \geq C_1 \sup_{\nu \in \mathcal{U}_{[n]}} f_{\nu}(0)^{\frac{1}{n}}$$

and

$$(5.2) \quad I_{-c_2 \frac{n}{(2-\alpha)t^\alpha}}(\mu_1) \leq C_3 t \sqrt{n} f_{\mu_1}(0)^{-\frac{1}{n}},$$

where $C_0, C_1, C_3 > 0$ and $c_2 \geq 2$ are absolute constants.

Moreover, if $\mathcal{U} = \mathcal{IL}$, μ_1 can be chosen to be of small diameter (with an absolute constant $C_4 > 0$).

Recall that if K and C are convex bodies in \mathbb{R}^n , then the covering number of K with respect to C is the minimum number of translates of C whose union covers K :

$$(5.3) \quad N(K, C) := \min \left\{ k \in \mathbb{N} : \exists z_1, \dots, z_k \in \mathbb{R}^n : K \subset \bigcup_{i=1}^k (z_i + C) \right\}.$$

Let K be a convex body of volume 1 in \mathbb{R}^n . Milman (see [14], [15] and also [16] for the not necessarily symmetric case) proved that there exists an ellipsoid \mathcal{E} with $|\mathcal{E}| = 1$, such that

$$(5.4) \quad \log N(K, \mathcal{E}) \leq \kappa n,$$

where $\kappa > 0$ is an absolute constant. We will use the existence of α -regular M -ellipsoids for symmetric convex bodies. More precisely, we need the following theorem of Pisier (see [24]; the result is stated and proved in the case of symmetric convex bodies but it can be easily extended to the non-symmetric case):

Theorem 5.2. *Let K be a convex body of volume 1 in \mathbb{R}^n with center of mass at the origin. For every $\alpha \in (0, 2)$ there exists an ellipsoid \mathcal{E} with $|\mathcal{E}| = 1$ such that, for every $t \geq 1$,*

$$(5.5) \quad \log N(K, t\mathcal{E}) \leq \frac{\kappa(\alpha)}{t^\alpha} n,$$

where $\kappa(\alpha) > 0$ is a constant depending only on α . One can take $\kappa(\alpha) \leq \frac{\kappa}{2-\alpha}$, where $\kappa > 0$ is an absolute constant.

We will also need the following facts about ellipsoids:

Lemma 5.3. *Let \mathcal{E} be an ellipsoid in \mathbb{R}^n . Assume that there exists a diagonal matrix T with entries $\lambda_1 \geq \dots \geq \lambda_n > 0$ such that $\mathcal{E} = T(B_2^n)$. Then,*

$$(5.6) \quad \max_{F \in G_{n,k}} |\mathcal{E} \cap F| = \max_{F \in G_{n,k}} |P_F(\mathcal{E})| = \omega_k \prod_{i=1}^k \lambda_i$$

and

$$(5.7) \quad \min_{F \in G_{n,k}} |\mathcal{E} \cap F| = \min_{F \in G_{n,k}} |P_F(\mathcal{E})| = \omega_k \prod_{i=n-k+1}^n \lambda_i$$

for all $1 \leq k \leq n-1$.

Proof. A proof of the equality $\min_{F \in G_{n,k}} |\mathcal{E} \cap F| = \omega_k \prod_{i=n-k+1}^n \lambda_i$ is outlined in [9, Lemma 4.1]. Let $F_s(k) = \text{span}\{e_{n-k+1}, \dots, e_n\}$. Then, for every $F \in G_{n,k}$ we have

$$(5.8) \quad |P_{F_s(k)}(\mathcal{E})| = |\mathcal{E} \cap F_s(k)| \leq |\mathcal{E} \cap F| \leq |P_F(\mathcal{E})|.$$

This shows that

$$(5.9) \quad \min_{F \in G_{n,k}} |P_F(\mathcal{E})| = |P_{F_s(k)}(\mathcal{E})| = \omega_k \prod_{i=n-k+1}^n \lambda_i$$

and completes the proof of (5.7).

Observe that $\mathcal{E}^\circ = T^{-1}(B_2^n)$ is also an ellipsoid; since the diagonal entries of T^{-1} are $\lambda_n^{-1} \geq \dots \geq \lambda_1^{-1} > 0$, the same reasoning shows that

$$(5.10) \quad \min_{F \in G_{n,k}} |\mathcal{E}^\circ \cap F| = \min_{F \in G_{n,k}} |P_F(\mathcal{E}^\circ)| = \omega_k \left(\prod_{i=1}^k \lambda_i \right)^{-1}.$$

Since $P_F(\mathcal{E})$ is an ellipsoid in F and $\mathcal{E}^\circ \cap F$ is its polar in F , by the affine invariance of the product of volumes of a body and its polar, we get $|P_F(\mathcal{E})| \cdot |\mathcal{E}^\circ \cap F| = |B_2^n \cap F|^2 = \omega_k^2$ for every $F \in G_{n,k}$. This observation and (5.10) prove (5.6). \square

Lemma 5.4. *Let n be even and let \mathcal{E} be an ellipsoid in \mathbb{R}^n . Assume that there exists a diagonal matrix T with entries $\lambda_1 \geq \dots \geq \lambda_n > 0$ such that $\mathcal{E} = T(B_2^n)$. Then, there exists $F \in G_{n,n/2}$ such that $P_F(\mathcal{E}) = \lambda_{n/2}(B_2^n \cap F)$.*

Proof. The proof can be found in [30, pp. 125-6], but we sketch it for the reader's convenience. We may assume that $\lambda_1 > \dots > \lambda_n > 0$. Write $n = 2s$. Then, $\mathcal{E}^\circ \cap e_n^\perp = \left\{ x \in \mathbb{R}^{2s-1} : \sum_{i=1}^{2s-1} \lambda_i^2 x_i^2 \leq 1 \right\}$ (the reason for this step is that the argument in [30, pp. 125-6] works in odd dimensions). Since $\lambda_i > \lambda_s > \lambda_{2s-i}$ for every $i \leq s-1$, we can define $b_1, \dots, b_{s-1} > 0$ by the equations

$$(5.11) \quad \lambda_i^2 b_i^2 + \lambda_{2s-i}^2 = \lambda_s^2 (b_i^2 + 1).$$

Consider the subspace $F = \text{span}\{v_1, \dots, v_s\} \in G_{2s,s}$, where $v_s = e_s$ and

$$(5.12) \quad v_i = \frac{b_i e_i + e_{2s-i}}{\sqrt{b_i^2 + 1}}, \quad i = 1, \dots, s-1.$$

It is easy to check that $\{v_1, \dots, v_s\}$ is an orthonormal basis for F and, using (5.11) and (5.12), we see that, for every $x \in F$,

$$(5.13) \quad \lambda_s^2 \|x\|_2^2 = \lambda_s \sum_{i=1}^s \langle x, v_i \rangle^2 = \sum_{i=1}^{2s-1} \lambda_i^2 \langle x, e_i \rangle^2 = \|x\|_{\mathcal{E}}^2.$$

This proves that $\mathcal{E}^\circ \cap F = \lambda_s^{-1}(B_2^n \cap F)$ and, by duality, $P_F(\mathcal{E}) = \lambda_s(B_2^n \cap F) = \lambda_{n/2}(B_2^n \cap F)$. \square

Proposition 5.5. *Let $K \in \widetilde{\mathcal{IK}}_{[n]}$. Let $1 \leq k \leq n-1$ and set*

$$(5.14) \quad \gamma := \max_{F \in G_{n,k}} |K \cap F^\perp|^{\frac{1}{k}}.$$

Then,

$$(5.15) \quad \min_{H \in G_{n,n-k}} |K \cap H^\perp|^{\frac{1}{n-k}} \geq \gamma \left(\frac{\eta}{\gamma} \right)^{\frac{n}{n-k}},$$

where $0 < \eta < 1$ is an absolute constant.

Proof. Fix $\alpha = 1$ and consider an α -regular M -ellipsoid \mathcal{E} for K given by Theorem 5.2. By the invariance of the isotropic position under orthogonal transformations, we may assume that there exists a diagonal matrix T with entries $\lambda_1 \geq \dots \geq \lambda_n > 0$ such that $\mathcal{E} = T(B_2^n)$. Recall that $|\mathcal{E}| = 1$.

Let $F \in G_{n,k}$, $1 \leq k \leq n-1$. Since projecting a covering creates a covering of the projection, we have

$$(5.16) \quad \frac{|P_F(K)|}{|P_F(\mathcal{E})|} \leq N(K, \mathcal{E}) \leq e^{\kappa n}.$$

We will use the Rogers-Shephard inequality (see [27]) for K and \mathcal{E} : since $|K| = 1$, we know that

$$(5.17) \quad c_1 \leq (|K \cap F^\perp| |P_F(K)|)^{\frac{1}{k}} \leq \binom{n}{k}^{\frac{1}{k}} \leq \frac{en}{k},$$

where $c_1 > 0$ is a universal constant (see [29] or [17] for the left hand side inequality).

From (5.17) and the definition of γ in (5.14), we see that

$$(5.18) \quad |P_F(K)|^{\frac{1}{k}} \geq \frac{c_1}{\gamma}.$$

Using (5.16) we get

$$(5.19) \quad \frac{c_1}{\gamma} \leq e^{\frac{\kappa n}{k}} |P_F(\mathcal{E})|^{\frac{1}{k}}.$$

In other words,

$$(5.20) \quad \min_{F \in G_{n,k}} |P_F(\mathcal{E})|^{\frac{1}{k}} \geq \frac{c_1}{\gamma} e^{-\frac{\kappa n}{k}}.$$

We can now apply the upper bound from (5.17) to get

$$(5.21) \quad \frac{c_1}{\gamma} |\mathcal{E} \cap F^\perp|^{\frac{1}{k}} \leq e^{\frac{\kappa n}{k}} (|P_F(\mathcal{E})| |\mathcal{E} \cap F^\perp|)^{\frac{1}{k}} \leq e^{\frac{\kappa n}{k}} \frac{en}{k} \leq e^{\frac{\kappa_1 n}{k}}.$$

It follows that

$$(5.22) \quad \max_{H \in G_{n,n-k}} |\mathcal{E} \cap H| \leq \frac{e^{\kappa_1 n} \gamma^k}{c_1^k}.$$

Lemma 5.3 implies that

$$(5.23) \quad \max_{H \in G_{n,n-k}} |P_H(\mathcal{E})| \leq \frac{e^{\kappa_1 n} \gamma^k}{c_1^k},$$

and hence,

$$(5.24) \quad |P_H(K)| \leq e^{\kappa n} |P_H(\mathcal{E})| \leq \frac{e^{\kappa_2 n} \gamma^k}{c_1^k}$$

for every $H \in G_{n,n-k}$, where we have used again (5.16). Applying (5.17) once again, we have

$$(5.25) \quad c_1 \leq (|K \cap H^\perp| |P_H(K)|)^{\frac{1}{n-k}} \leq |K \cap H^\perp|^{\frac{1}{n-k}} e^{\frac{\kappa_2 n}{n-k}} \left(\frac{\gamma}{c_1} \right)^{\frac{k}{n-k}}.$$

This proves that

$$(5.26) \quad \min_{H \in G_{n,n-k}} |K \cap H^\perp|^{\frac{1}{n-k}} \geq \gamma \left(\frac{\eta}{\gamma} \right)^{\frac{n}{n-k}}$$

with $\eta = c_1 e^{-\kappa_2}$, as claimed. \square

Lemma 5.6. *Let $K \in \widetilde{CK}_{[n]}$. Assume that, for some $s > 0$,*

$$(5.27) \quad r_s := \log N(K, sB_2^n) < n.$$

Then,

$$(5.28) \quad I_{-r_s}(K) \leq 3es.$$

Proof. Let $z_0 \in \mathbb{R}^n$ such that $|K \cap (-z_0 + sB_2^n)| \geq |K \cap (z + sB_2^n)|$ for every $z \in \mathbb{R}^n$. It follows that

$$(5.29) \quad |(K + z_0) \cap sB_2^n| \cdot N(K, sB_2^n) \geq |K| = 1.$$

Let $q := r_s < n$. Then, using Markov's inequality, the definition of $I_{-q}(K + z_0)$ and (5.27), we get

$$(5.30) \quad |(K + z_0) \cap 3^{-1}I_{-q}(K + z_0)B_2^n| \leq 3^{-q} < e^{-q} = e^{-r_s} \leq \frac{1}{N(K, sB_2^n)}.$$

From (5.29) we obtain

$$(5.31) \quad |(K + z_0) \cap 3^{-1}I_{-q}(K + z_0)\widetilde{B}_2^n| < |(K + z_0) \cap sB_2^n|,$$

and this implies

$$(5.32) \quad 3^{-1}I_{-q}(K + z_0) \leq s.$$

Since K has center of mass at the origin, as an application of Fradelizi's theorem (see [6]), we have that $I_{-k}(K + z) \geq \frac{1}{e}I_{-k}(K)$ for any $1 \leq k < n$ and $z \in \mathbb{R}^n$ (a proof appears in [23, Proposition 4.6]). This proves the Lemma. \square

Theorem 5.7. *Let n be even and let $K \in \widetilde{\mathcal{IK}}_{[n]}$. Set*

$$(5.33) \quad \gamma := \max_{F \in G_{n, \frac{n}{2}}} |K \cap F^\perp|^{\frac{2}{n}}.$$

Then, there exists $K_1 \in \widetilde{\mathcal{IK}}_{[n]}$ such that:

- (i) $\frac{\eta_1}{\gamma}L_K \leq L_{K_1} \leq \eta_2\gamma L_K$, where $\eta_1, \eta_2 > 0$ are absolute constants.
- (ii) If $\alpha \in (1, 2)$ one has that for every $t \geq C_1\gamma^2$

$$\log N(K_1, t\sqrt{n}B_2^n) \leq C_2\gamma^2 \frac{\kappa(\alpha)n}{t^\alpha},$$

where $\kappa(\alpha) \leq \frac{\kappa}{2-\alpha}$ and $C_1, C_2 > 0$ are absolute constants.

(iii) *If K is a body of small diameter (with some constant $A > 1$) then K_1 is also a body of small diameter (with constant $C_3\gamma^2A > 1$, where C_3 is an absolute constant).*

Proof. Let \mathcal{E} be an α -regular M -ellipsoid for K given by Theorem 5.2. As in the proof of Proposition 5.5, we assume that $\mathcal{E} = T(B_2^n)$ for some diagonal matrix T with entries $\lambda_1 \geq \dots \geq \lambda_n > 0$. From (5.20) and Lemma 5.3 we have

$$(5.34) \quad \omega_{\frac{n}{2}}(\lambda_{\frac{n}{2}})^{\frac{n}{2}} \geq \omega_{\frac{n}{2}} \prod_{i=\frac{n}{2}+1}^n \lambda_i = \min_{F \in G_{n, \frac{n}{2}}} |P_F \mathcal{E}| \geq e^{-\kappa n} \left(\frac{c_1}{\gamma} \right)^{\frac{n}{2}},$$

and hence (recall that $\omega_k^{1/k} \simeq 1/\sqrt{k}$),

$$(5.35) \quad \lambda_{\frac{n}{2}} \geq \frac{c_2\sqrt{n}}{\gamma}.$$

Similarly, (5.22) and Lemma 5.3 imply that

$$(5.36) \quad \omega_{\frac{n}{2}} (\lambda_{\frac{n}{2}})^{\frac{n}{2}} \leq \omega_{\frac{n}{2}} \prod_{i=1}^{\frac{n}{2}} \lambda_i = \max_{H \in G_{n, \frac{n}{2}}} |\mathcal{E} \cap H| \leq e^{\kappa_1 n} \left(\frac{\gamma}{c_1} \right)^{\frac{n}{2}},$$

and hence,

$$(5.37) \quad \lambda_{\frac{n}{2}} \leq c_3 \gamma \sqrt{n}.$$

Then, by Lemma 5.4 we can find $F_0 \in G_{n, \frac{n}{2}}$ such that

$$(5.38) \quad \frac{c_2 \sqrt{n}}{\gamma} (B_2^n \cap F_0) \subseteq P_{F_0}(\mathcal{E}) \subseteq c_3 \gamma \sqrt{n} (B_2^n \cap F_0).$$

Let $K_0 := \widetilde{B_{\frac{n}{2}+1}^n}(K, F_0)$ and $K_1 := T(K_0 \times K_0) \in \mathbb{R}^n$, where $T \in SL_n$ is such that K_1 is isotropic. Note that $K_0 \times K_0$ has volume 1, center of mass at the origin and is almost isotropic. In other words T is almost an isometry. We will show that K_1 satisfies (i), (ii) and (iii).

(i) From Proposition 2.1(vi) we know that

$$(5.39) \quad \bar{c}_2 L_K |K \cap F_0^\perp|^{\frac{2}{n}} \leq L_{K_0} \leq \bar{c}_1 L_K |K \cap F_0^\perp|^{\frac{2}{n}},$$

where $\bar{c}_1, \bar{c}_2 > 0$ are absolute constants. Then, Proposition 5.5 shows that

$$(5.40) \quad \frac{\eta_1}{\gamma} L_K \leq L_{K_0} \leq \eta_2 \gamma L_K,$$

where $\eta_1 = \eta^2 \bar{c}_2$, $\eta_2 = \bar{c}_1$. Note that $L_{K_1} = L_{K_0}$. This completes the proof of (i).

(ii) From Proposition 2.1(v) and from the fact that $\bar{c} \operatorname{conv}\{C, -C\} \subseteq Z_{\frac{n}{2}}(C) \subseteq \operatorname{conv}\{C, -C\}$ for all C in $\widetilde{CK}_{[\frac{n}{2}]}$, we get

$$\begin{aligned} \operatorname{conv}\{K_0, -K_0\} &\subseteq \frac{1}{\bar{c}} Z_{\frac{n}{2}}(\widetilde{B_{\frac{n}{2}+2}^n}(K, F_0)) \\ &\subseteq \frac{1}{\bar{c} \bar{c}_3} |K \cap F_0^\perp|^{\frac{2}{n}} P_{F_0}(Z_{\frac{n}{2}}(K)) \\ &\subseteq \frac{1}{\bar{c} \bar{c}_3} \gamma P_{F_0}(\operatorname{conv}\{K, -K\}) \end{aligned}$$

and, similarly,

$$\begin{aligned} \operatorname{conv}\{K_0, -K_0\} &\supseteq Z_{\frac{n}{2}}(\widetilde{B_{\frac{n}{2}+2}^n}(K, F_0)) \\ &\supseteq \frac{1}{\bar{c}_4} |K \cap F_0^\perp|^{\frac{2}{n}} P_{F_0}(Z_{\frac{n}{2}}(K)) \\ &\supseteq \frac{\eta^2}{\bar{c}_4} \frac{\bar{c}}{2\bar{c}_0} \frac{1}{\gamma} P_{F_0}(\operatorname{conv}\{K, -K\}), \end{aligned}$$

where we have used the fact that $Z_{\frac{n}{2}}(K) \supseteq \frac{1}{2\bar{c}_0}Z_n(K) \supseteq \frac{\bar{c}}{2\bar{c}_0}\text{conv}\{K, -K\}$. In other words,

$$(5.41) \quad \frac{\bar{c}_5}{\gamma}P_{F_0}(\text{conv}\{K, -K\}) \subseteq \text{conv}\{K_0, -K_0\} \subseteq \bar{c}_6\gamma P_{F_0}(\text{conv}\{K, -K\}),$$

where $\bar{c}_5, \bar{c}_6 > 0$ are absolute constants.

For $s > 0$ we have

$$\begin{aligned} N(K_1, s\sqrt{n}B_2^n) &= N(T(K_0 \times K_0), s\sqrt{n}B_2^n) \\ &\leq N(K_0 \times K_0, cs\sqrt{n}B_2^n) \\ &\leq N(K_0 \times K_0, \sqrt{2}cs\sqrt{n}(B_2^n \cap F_0 \times B_2^n \cap F_0)) \\ &\leq N(K_0, c's\sqrt{n}B_2^n \cap F_0)^2, \end{aligned}$$

where we have used the fact that T is almost an isometry, and hence, $T(K_0 \times K_0) \subseteq \frac{1}{c}(K_0 \times K_0)$. Moreover, we have used the fact that if K, C are convex bodies, then

$$(5.42) \quad N(K \times K, C \times C) \leq N(K, C)^2$$

and $B_2^k \times B_2^k \supseteq \frac{1}{\sqrt{2}}B_2^{2k}$.

Recall that c_2 and c_3 are the constants in (5.38). For every $r > 0$,

$$\begin{aligned} N(K_0, c_3r\gamma\sqrt{n}(B_2^n \cap F_0)) &\leq N(\text{conv}\{K_0, -K_0\}, c_3r\gamma\sqrt{n}(B_2^n \cap F_0)) \\ &\leq N(\text{conv}\{K_0, -K_0\}, rP_{F_0}(\mathcal{E})) \\ &\leq N(\bar{c}_6\gamma P_{F_0}(\text{conv}\{K, -K\}), rP_{F_0}(\mathcal{E})) \\ &\leq N(\bar{c}_6\gamma \text{conv}\{K, -K\}, r\mathcal{E}) \\ &\leq N\left(K - K, \frac{r}{\bar{c}_6\gamma}\mathcal{E}\right) \\ &\leq N\left(K, \frac{r}{2\bar{c}_6\gamma}\mathcal{E}\right)^2. \end{aligned}$$

So, we can write

$$(5.43) \quad N(K_1, t\sqrt{n}B_2^n) \leq N\left(K, \frac{t}{\bar{c}_7\gamma^2}\mathcal{E}\right)^4$$

for every $t > 0$, where $\bar{c}_7 = \sqrt{2}c_2\bar{c}_6$. Since \mathcal{E} is a α -regular ellipsoid for K , for every $t \geq \bar{c}_7\gamma^2$ we have

$$(5.44) \quad \log N(K_1, t\sqrt{n}B_2^n) \leq 4 \log N\left(K, \frac{t}{\bar{c}_7\gamma^2}\mathcal{E}\right) \leq \frac{4\bar{c}_7\kappa(\alpha)\gamma^2n}{t^\alpha}.$$

This completes the proof of (ii).

(iii) We have that $R(K_0) \leq c\gamma A\sqrt{n}L_K$. Indeed, by Proposition 2.1,

$$\begin{aligned}
R(K_0) &= R\left(\widetilde{B_{\frac{n}{2}+1}}(K, F_0)\right) \\
&\leq cR\left(Z_{\frac{n}{2}+1}\left(\widetilde{B_{\frac{n}{2}+1}}(K, F_0)\right)\right) \\
&\leq c'|K \cap F_0^\perp|^{\frac{2}{n}} R(P_{F_0}Z_{\frac{n}{2}+1}(K)) \\
&\leq c'\gamma R(\text{conv}\{K, -K\}) \\
&\leq 2c'\gamma R(K) \leq c\gamma A\sqrt{n}L_K.
\end{aligned}$$

Also,

$$(5.45) \quad R(K_1) = R(K_0 \times K_0) = \sqrt{2}R(K_0).$$

To see this, write

$$(5.46) \quad R^2(K_0 \times K_0) = \max_{(x,y) \in K_0 \times K_0} \|x\|_2^2 + \|y\|_2^2 = 2R^2(K_0).$$

So, using (i) we get that

$$(5.47) \quad R(K_1) \leq \sqrt{2}R(K_0) \leq c\sqrt{2}\gamma A\sqrt{n}L_K \leq C_3\gamma^2 A\sqrt{n}L_{K_1}.$$

This completes the proof. \square

Lemma 5.8. *Let $\mu \in \mathcal{IL}_{[n]}$. Fix $1 \leq k < n - 1$ and $F \in G_{n,k}$. Then,*

$$(5.48) \quad |\widetilde{K_{n+1}}(\mu) \cap F^\perp|^{\frac{1}{k}} \simeq \frac{f_{\pi_F(\mu)}(0)^{\frac{1}{k}}}{f_\mu(0)^{\frac{1}{n}}},$$

$$(5.49) \quad L_{B_{k+1}(\mu, F)} \simeq f_{\pi_F(\mu)}(0)^{\frac{1}{k}} \simeq L_{B_{k+1}(\widetilde{K_{n+1}}(\mu), F)},$$

and

$$(5.50) \quad f_\mu(0)^{\frac{1}{n}} \widetilde{B_{k+1}}(\mu, F) \simeq \widetilde{B_{k+1}}(\widetilde{K_{n+1}}(\mu), F).$$

Proof. We will make use of the following facts (see Proposition 4.2 and Theorem 4.4 in [23]): If $\mu \in \mathcal{IL}_{[n]}$, then

$$(5.51) \quad f_{\pi_F(\mu)}(0)^{\frac{1}{k}} |P_F Z_k(\mu)|^{\frac{1}{k}} \simeq 1,$$

and if $K \in \widetilde{\mathcal{CK}}_{[n]}$ then

$$(5.52) \quad |K \cap F^\perp|^{\frac{1}{k}} |P_F Z_k(K)|^{\frac{1}{k}} \simeq 1.$$

Then, taking into account Proposition 2.1(iv), we get

$$(5.53) \quad |\widetilde{K_{n+1}}(\mu) \cap F^\perp|^{\frac{1}{k}} \simeq |P_F Z_k(\widetilde{K_{n+1}}(\mu))|^{-\frac{1}{k}} \simeq f_\mu(0)^{-\frac{1}{n}} |P_F Z_k(\mu)|^{-\frac{1}{k}} \simeq \frac{f_{\pi_F(\mu)}(0)^{\frac{1}{k}}}{f_\mu(0)^{\frac{1}{n}}}.$$

This proves (5.48).

(ii) Using Proposition 2.1(v) and (iv), we have that

$$\begin{aligned}
Z_2 \left(\widetilde{B}_{k+1}(\widetilde{K}_{n+1}(\mu), F) \right) &\simeq |\widetilde{K}_{n+1}(\mu) \cap F^\perp|^{\frac{1}{k}} P_F \left(Z_2 \left(\widetilde{K}_{n+1}(\mu) \right) \right) \\
&\simeq \frac{f_{\pi_F(\mu)}(0)^{\frac{1}{k}}}{f_\mu(0)^{\frac{1}{n}}} f_\mu(0)^{\frac{1}{n}} P_F(Z_2(\mu)) \\
&= f_{\pi_F(\mu)}(0)^{\frac{1}{k}} P_F(Z_2(\mu)) \\
&= f_{\pi_F(\mu)}(0)^{\frac{1}{k}} B_F,
\end{aligned}$$

because $Z_2(\mu) = B_2^n$. Taking volumes we see that

$$(5.54) \quad L_{B_{k+1}(\widetilde{K}_{n+1}(\mu), F)} \simeq f_{\pi_F(\mu)}(0)^{\frac{1}{k}}$$

and we conclude by Proposition 2.1(vii) and (2.17).

(iii) By Proposition 2.1(v),

$$(5.55) \quad \widetilde{B}_{k+1}(\mu, F) \simeq Z_k \left(\widetilde{B}_{k+1}(\mu, F) \right) \simeq \frac{\pi_F(\mu)(0)^{\frac{1}{k}}}{f_\mu(0)^{\frac{1}{n}}} P_F Z_k(\mu)$$

and, by Proposition 2.1(v) and then (iv),

$$\begin{aligned}
\widetilde{B}_{k+1}(\widetilde{K}_{n+1}(\mu), F) &\simeq Z_k \left(\widetilde{B}_{k+1}(\widetilde{K}_{n+1}(\mu), F) \right) \\
&\simeq |\widetilde{K}_{n+1}(\mu) \cap F^\perp|^{\frac{1}{k}} P_F \left(Z_k \left(\widetilde{K}_{n+1}(\mu) \right) \right) \\
&\simeq \frac{\pi_F(\mu)(0)^{\frac{1}{k}}}{f_\mu(0)^{\frac{1}{n}}} f_\mu(0)^{\frac{1}{n}} P_F(Z_k(\mu)) \\
&= \pi_F(\mu)(0)^{\frac{1}{k}} P_F(Z_k(\mu)).
\end{aligned}$$

We have thus shown that

$$(5.56) \quad \widetilde{B}_{k+1}(\widetilde{K}_{n+1}(\mu), F) \simeq \pi_F(\mu)(0)^{\frac{1}{k}} P_F(Z_k(\mu)).$$

Combining (5.55) and (5.56) we see that

$$(5.57) \quad f_\mu(0)^{\frac{1}{n}} \widetilde{B}_{k+1}(\mu, F) \simeq \widetilde{B}_{k+1}(\widetilde{K}_{n+1}(\mu), F).$$

This completes the proof. \square

Proof of Proposition 5.1. (i) Let $\nu \in \mathcal{U}_{[n]}$ such that $\sup_{\mu \in \mathcal{U}_{[n]}} f_\mu(0)^{\frac{1}{n}} = f_\nu(0)^{\frac{1}{n}}$. Let

$$(5.58) \quad K := T \left(\widetilde{K}_{n+1}(\nu) \right),$$

where $T \in SL_n$ is such that $K \in \widetilde{\mathcal{IK}}_{[n]}$. Note that, from Proposition 2.1, T is almost an isometry and $L_K \simeq f_\nu(0)^{\frac{1}{n}}$.

If $\mathcal{U} = \mathcal{I}\mathcal{L}$ we take $K := T\left(\widetilde{K_{n+1}(\bar{\nu})}\right)$. By Proposition 2.1 and (2.14) we have that $L_K \simeq f_\nu(0)^{\frac{1}{n}}$. The proof of the first two assertions is identical in both cases. We write μ for either ν or $\bar{\nu}$.

(i) Let $F_0 \in G_{n, \frac{n}{2}}$, $K_0 \in \widetilde{\mathcal{I}\mathcal{K}}_{[\frac{n}{2}]}$ and $K_1 \in \widetilde{\mathcal{I}\mathcal{K}}_{[n]}$ as in the proof of Theorem 5.7. Let $\mu_1 := \pi_{F_0}(\mu) \otimes \pi_{F_0}(\mu)$. Assume that the two copies of $\pi_{F_0}(\mu)$ live on F and F^\perp respectively, where $F \in G_{2n, n}$. Since $\mu \in \mathcal{U}$ and \mathcal{U} is coherent, we have $\mu_1 \in \mathcal{U}$.

Moreover, using again Proposition 2.1, we have that

$$\begin{aligned} f_{\mu_1}(0)^{\frac{1}{n}} &= f_{\pi_{F_0}(\mu)}(0)^{\frac{2}{n}} \simeq L_{\widetilde{K_{\frac{n}{2}+1}(\pi_{F_0}(\mu))}} \\ &= L_{\widetilde{B_{\frac{n}{2}+1}(\mu, F_0)}} \simeq L_{\widetilde{B_{\frac{n}{2}+1}(K_{n+1}(\mu), F_0)}} \\ &\simeq L_{B_{\frac{n}{2}+1}(K, F_0)} = L_{K_0} = L_{K_1} \\ &\simeq f_\mu(0)^{\frac{1}{n}}. \end{aligned}$$

This settles the first assertion of the Proposition.

(ii) Since \mathcal{U} is coherent, for every $F \in G_{n, \frac{n}{2}}$ we have

$$(5.59) \quad f_{\pi_F(\mu)}(0)^{\frac{2}{n}} \leq f_\mu(0)^{\frac{1}{n}}.$$

Set $\gamma := \max_{F \in G_{n, \frac{n}{2}}} |K \cap F^\perp|^{\frac{2}{n}}$. Then,

$$(5.60) \quad \gamma \simeq \frac{L_{\widetilde{B_{\frac{n}{2}+1}(K_{n+1}(\mu), F)}}}{L_K} \simeq \frac{f_{\pi_F(\mu)}(0)^{\frac{2}{n}}}{f_\mu(0)^{\frac{1}{n}}} \leq C,$$

where we have used again Lemma 5.8. So, by Theorem 5.7 we have that

$$(5.61) \quad \log N(K, t\sqrt{n}B_2^n) \leq \frac{Cn}{t^\alpha(2-\alpha)}.$$

Note that, for every $p > 0$ and every pair of probability measures ν_1, ν_2 living in F, F^\perp respectively, we have $P_F Z_p(\nu_1 \otimes \nu_2) = Z_p(\nu_1)$ and $P_{F^\perp} Z_p(\nu_1 \otimes \nu_2) = Z_p(\nu_2)$. Indeed, if $\theta \in S_F$, we have that

$$\begin{aligned} h_{Z_p(\nu_1 \otimes \nu_2)}^p(\theta) &= \int_F \int_{F^\perp} |\langle x + y, \theta \rangle|^p d\nu_2(y) d\nu_1(x) \\ &= \int_F |\langle x, \theta \rangle|^p d\nu_1(x) = h_{Z_p(\nu_1)}^p(\theta). \end{aligned}$$

Note that for every convex body K and $F \in G_{n, k}$ one has

$$(5.62) \quad K \subseteq P_F(K) \otimes P_{F^\perp}(K).$$

So, we have that

$$\begin{aligned}
\widetilde{K_{n+1}}(\mu_1) &\subseteq P_F \left(\widetilde{K_{n+1}}(\mu_1) \right) \times P_{F^\perp} \left(\widetilde{K_{n+1}}(\mu_1) \right) \\
&\simeq P_F \left(Z_{\frac{n}{2}} \left(\widetilde{K_{n+1}}(\mu_1) \right) \right) \times P_{F^\perp} \left(Z_{\frac{n}{2}} \left(\widetilde{K_{n+1}}(\mu_1) \right) \right) \\
&\simeq f_\mu(0)^{\frac{1}{n}} P_F \left(Z_{\frac{n}{2}} (\pi_{F_0}(\mu) \otimes \pi_{F_0}(\mu)) \right) \\
&\quad \times f_\mu(0)^{\frac{1}{n}} P_{F^\perp} \left(Z_{\frac{n}{2}} (\pi_{F_0}(\mu) \otimes \pi_{F_0}(\mu)) \right) \\
&\simeq f_\mu(0)^{\frac{1}{n}} Z_{\frac{n}{2}} (\pi_{F_0}(\mu)) \times f_\mu(0)^{\frac{1}{n}} Z_{\frac{n}{2}} (\pi_{F_0}(\mu)) \\
&\simeq f_\mu(0)^{\frac{1}{n}} \widetilde{B_{\frac{n}{2}+1}}(\mu, F_0) \times f_\mu(0)^{\frac{1}{n}} \widetilde{B_{\frac{n}{2}+1}}(\mu, F_0) \\
&\simeq \widetilde{B_{\frac{n}{2}+1}} \left(\widetilde{K_{n+1}}(\mu), F_0 \right) \times \widetilde{B_{\frac{n}{2}+1}} \left(\widetilde{K_{n+1}}(\mu), F_0 \right) \\
&\simeq \widetilde{B_{\frac{n}{2}+1}}(K, F_0) \times \widetilde{B_{\frac{n}{2}+1}}(K, F_0) \\
&= K_0 \times K_0 = K_1.
\end{aligned}$$

Therefore,

$$(5.63) \quad R(\widetilde{K_{n+1}}(\mu_1)) \leq cR(K_1)$$

and

$$(5.64) \quad \log N \left(\widetilde{K_{n+1}}(\mu_1), t\sqrt{n}B_2^n \right) \leq \log N \left(K, ct\sqrt{n}B_2^n \right) \leq \frac{Cn}{t^\alpha(2-\alpha)}.$$

We have assumed that $t^\alpha(2-a) \geq C$, and hence, by Lemma 5.6 we have

$$(5.65) \quad I_{-p}(\widetilde{K_{n+1}}(\mu)) \leq 3et\sqrt{n},$$

where $p = \frac{Cn}{t^\alpha(2-a)} < n$. Note that if $\mu \in \mathcal{CL}$ then for every $1 \leq p \leq n-1$ one has (see Proposition 3.4 in [23])

$$(5.66) \quad I_{-p}(\mu) f_\mu(0)^{\frac{1}{n}} \simeq I_{-p}(\widetilde{K_{n+1}}(\mu)).$$

It follows that

$$(5.67) \quad I_{-\frac{Cn}{t^\alpha(2-a)}}(\mu_1) \leq C't\sqrt{n}f_{\mu_1}(0)^{-\frac{1}{n}},$$

and the proof of the second assertion is complete.

For the rest of the proof we set $\mu = \bar{\nu}$. In this case, K is a body of small diameter. Indeed, for $p \geq 2$, by Proposition 2.1(iv) we have

$$(5.68) \quad I_p(K) \simeq I_p(\widetilde{K_{n+1}}(\bar{\nu})) \simeq I_p(\mu) f_{\bar{\nu}}(0)^{\frac{1}{n}} \simeq \sqrt{n} f_{\bar{\nu}}(0)^{\frac{1}{n}} \simeq I_2(\widetilde{K_{n+1}}(\bar{\nu})) \simeq I_2(K).$$

From Theorem 5.7 we have that K_1 is a body of small diameter, and this implies that $\frac{R(K_1)}{I_2(K_1)} \simeq 1$. Also, by the first assertion we have that

$$(5.69) \quad L_{K_1} \simeq f_{\bar{\nu}}(0)^{\frac{1}{n}} \simeq f_{\mu_1}(0)^{\frac{1}{n}} \simeq L_K.$$

Then, from (5.63) we see that for $p \geq 2$,

$$(5.70) \quad \frac{I_p(\mu_1)}{I_2(\mu_1)} \simeq \frac{I_p(\widetilde{K_{n+1}}(\mu_1))}{\sqrt{n}f_{\mu_1}(0)^{\frac{1}{n}}} \leq c \frac{R(\widetilde{K_{n+1}}(\mu_1))}{\sqrt{n}L_{K_1}} \simeq \frac{R(K_1)}{I_2(K_1)} \simeq 1.$$

So, μ_1 is a measure of small diameter. The proof is complete. \square

6 Proof of the main result

We are now ready to state and prove the main result of the paper:

Theorem 6.1. *Let \mathcal{U} be a coherent subclass of \mathcal{IL} and let $n \geq 2$ and $\delta \geq 1$. Then,*

$$(6.1) \quad \sup_{\mu \in \mathcal{U}_{[n]}} f_{\mu}(0)^{\frac{1}{n}} \leq C\delta \sup_{\mu \in \mathcal{U}_{[n]}} \sqrt{\frac{n}{q_{-c}(\mu, \delta)} \log \left(\frac{en}{q_{-c}(\mu, \delta)} \right)},$$

where $C > 0$ is an absolute constant. Moreover if $\mathcal{U} = \mathcal{IL}$ then the supremum on right hand side can be taken over all $\bar{v} \in \mathcal{IL}$.

Proof. By Proposition 4.3 we can assume that n is even. Let $q := \inf_{\mu \in \mathcal{U}_{[n]}} q_{-c}(\mu, \delta)$. Let $\alpha := 2 - \frac{1}{\log(e\frac{2n}{q})}$ and $t = C_1 \sqrt{\frac{n}{q} \log \frac{en}{q}}$, where the absolute constant $C_1 > 0$ can be chosen large enough to ensure that $t^\alpha(2 - \alpha) \geq C_0$, where $C_0 > 0$ is the constant that appears in Proposition 5.1. We have

$$(6.2) \quad t^\alpha(2 - \alpha) \simeq \frac{n}{q} \log \frac{en}{q} \frac{1}{\log \frac{en}{q}} = \frac{n}{q},$$

and hence,

$$(6.3) \quad \frac{n}{t^\alpha(2 - \alpha)} \simeq q.$$

By Proposition 5.1 there exists a measure $\mu_1 \in \mathcal{U}_{[n]}$ such that $f_{\mu_1}(0)^{\frac{1}{n}} \simeq \sup_{\mu \in \mathcal{U}} f_{\mu}(0)^{\frac{1}{n}}$ and

$$(6.4) \quad I_{-q}(\mu_1) = I_{-\frac{en}{t^\alpha(2-\alpha)}}(\mu_1) \leq C't\sqrt{n}f_{\mu_1}(0)^{-\frac{1}{n}} \leq C''\sqrt{\frac{n}{q} \log \frac{en}{q}}\sqrt{n}f_{\mu_1}(0)^{-\frac{1}{n}}.$$

On the other hand, by the definition of q , we have

$$(6.5) \quad \frac{\sqrt{n}}{\delta} = \frac{1}{\delta}I_2(\mu_1) \leq I_{-q_{-c}(\mu_1, \delta)}(\mu_1) \leq I_{-q}(\mu_1).$$

Combining the above we get the result. \square

Remark. Observe that for the choice $\delta = \sup_{\mu \in \mathcal{U}_{[n]}} f_{\mu}(0)^{\frac{1}{n}}$ we have

$$\inf_{\mu \in \mathcal{U}_{[n]}} q_{-c}(\mu, \delta) \simeq n$$

(see Proposition 4.8 in [23]). This shows that the preceding result is sharp (up to a universal constant).

Theorem 3.2 shows that there exists an absolute constant $\xi > 0$ such that $q_{-c}(\mu, \xi) \geq q_*(\mu)$ for every $\mu \in \mathcal{IL}$. So we get the following:

Corollary 6.2. *Let \mathcal{U} be a coherent subclass of \mathcal{IL} . Then for any $n \geq 1$,*

$$(6.6) \quad \sup_{\mu \in \mathcal{U}_{[n]}} f_\mu(0)^{\frac{1}{n}} \leq C \sup_{\mu \in \mathcal{U}_{[n]}} \sqrt{\frac{n}{q_*(\mu)} \log \left(\frac{en}{q_*(\mu)} \right)},$$

where $C > 0$ is an absolute constant.

Corollary 6.3. *Let $\alpha \in (1, 2]$, let $\beta > 0$ and $\mu \in (\mathcal{P}_\alpha(\beta) \cap \mathcal{IL})_{[n]}$. Then,*

$$(6.7) \quad f_\mu(0)^{\frac{1}{n}} \leq C \sqrt{n^{\frac{2-\alpha}{2}} \beta^\alpha} \sqrt{\log \left(n^{\frac{2-\alpha}{2}} \beta^\alpha \right)},$$

where $C > 0$ is an absolute constant.

Proof. Since $\mu \in \mathcal{CP}_\alpha(\beta)$, by Corollary 4.8 we have that $\mu \in \mathcal{CP}(\alpha, c_1\beta)$. Then, Proposition 3.3 shows that $q_*(\mu) \geq c \frac{n^{\frac{\alpha}{2}}}{b^\alpha}$.

Therefore, the result follows from Corollary 6.2. \square

Theorem 6.4. *Let $\alpha \in (1, 2]$ and $b_\alpha > 0$. Let μ be an isotropic ψ_α log-concave measure with constant β_α in \mathbb{R}^n . Then,*

$$(6.8) \quad f_\mu(0)^{\frac{1}{n}} \leq C \sqrt{n^{\frac{2-\alpha}{2}} \beta_\alpha^\alpha} \sqrt{\log \left(n^{\frac{2-\alpha}{2}} \beta_\alpha^\alpha \right)},$$

where $C > 0$ is an absolute constant. In particular, if μ is ψ_2 with constant $\beta_2 > 0$, then

$$(6.9) \quad f_\mu(0)^{\frac{1}{n}} \leq C \beta_2 \sqrt{\log \beta_2}.$$

Moreover, for every isotropic log-concave measure μ ,

$$(6.10) \quad f_\mu(0)^{\frac{1}{n}} \leq C n^{\frac{1}{4}} \sqrt{\log n}.$$

Proof. Recall that, from Corollary 4.8, if $\mu \in \mathcal{CP}(\alpha, \beta_\alpha)$ then $\mu \in \mathcal{CP}_\alpha(c_1\beta_\alpha)$. Then (6.8) follows from Corollary 6.3. In the special case $\alpha = 2$ or α tending to 1, one gets (6.9) and (6.10) respectively (recall that log-concave measures are uniformly ψ_1). \square

Remark. In the proof of Corollary 6.2 we have used the fact that $q_*(\mu) \leq q_{-c}(\mu)$. One may check that in general this is not sharp (for example one may check that for $f_\mu := \mathbf{1}_{\widehat{B}_1^n}$ one has $q_*(\mu) \ll q_{-c}(\mu, \xi)$ for $\xi \simeq 1$). As Proposition 3.5 shows, this is not the case for measures of small diameter.

We conclude with the following:

Theorem 6.5. *The following statements are equivalent:*

(a) *There exists $C_1 > 0$ such that*

$$\sup_n \sup_{\mu \in \mathcal{IL}_{[n]}} f_\mu(0)^{\frac{1}{n}} \leq C_1.$$

(b) *There exist $C_2, \xi_1 > 0$ such that*

$$\sup_n \sup_{\mu \in \mathcal{IL}_{[n]}} \frac{n}{q_{-c}(\mu, \xi_1)} \leq C_2.$$

(c) *There exists $C_3, \xi_2 > 0$ such that*

$$\sup_n \sup_{\mu \in \mathcal{IL}_{[n]}} \frac{n}{q_*(\bar{\mu}, \xi_2)} \leq C_3.$$

Proof. The claim that (a) implies (b) is an immediate consequence of the remark after Theorem 6.1. The implication from (b) to (c) follows immediately from Proposition 3.4. The claim that (c) implies (a) follows from Theorem 6.1. \square

We close by noting that there is a strong connection between the existence of supergaussian directions and small ball probability estimates, and hence, in view of Theorem 6.5, with the hyperplane conjecture as well. This connection will appear elsewhere.

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