# An inequality for moments of log-concave functions on Gaussian random vectors.

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**Abstract**

We prove a sharp moment inequality for a log-concave or a log-convex function, on Gaussian random vectors. As an application we take a stability result for the classical logarithmic Sobolev inequality of L. Gross in the case where the function is log-concave.

## 1 Introduction and main results

A non-negative function $f : \mathbb{R}^k \to [0, +\infty)$ is called *log-concave on its support*, if and only if

$$f\big((1-\lambda)x + \lambda y\big) \geq f(x)^{(1-\lambda)} f(y)^{\lambda}.$$

for every $\lambda \in [0, 1]$ and $x, y \in \mathrm{supp}(f)$. Respectively, is called log-*convex on its support*, if nd only if

$$f\big((1-\lambda)x + \lambda y\big) \leq f(x)^{(1-\lambda)} f(y)^{\lambda}.$$

for every $\lambda \in [0, 1]$ and $x, y \in \mathrm{supp}(f)$. The aim of this note is to present a sharp inequality for Gaussian moments of a log-concave or a log-convex function, stated below as Theorem 1.1.

We work on $\mathbb{R}^k$, equipped with the standard scalar product $\langle \cdot, \cdot \rangle$. We denote by $|\cdot|$, the corresponding Euclidean norm and the absolute value of a real number. We additionally use the notation $X \sim N(\xi, T)$, if $X$ is a Gaussian random vector in $\mathbb{R}^k$, with expectation $\xi \in \mathbb{R}^k$ and covariance the $k \times k$ positive semi-definite matrix $T$. We say that $X$ is centered, whenever $\mathbb{E}X = 0$, and that $X$ is a standard Gaussian random vector if it is centered with covariance matrix the identity in $\mathbb{R}^k$, where in that case $\gamma_k$ stands for its distribution law. Finally, $\mathcal{L}^{p,s}(\gamma_k)$ stand for the class of all functions $f \in L^p(\gamma_k)$ whose partial derivatives up to order s, are also in $L^p(\gamma_k)$.

**Theorem 1.1.** *Let $k \in \mathbb{N}$, $f : \mathbb{R}^k \to [0, +\infty)$ be a* log-*concave, $g : \mathbb{R}^k \to [0, +\infty)$ be a* log-*convex function, and $X$ be Gaussian random vector in $\mathbb{R}^k$. Then,* *(i) for every $r \in [0,1]$*

$$\mathbb{E} f\big(\sqrt{r}X\big) \geq \left(\mathbb{E} f(X)^r\right)^{\frac{1}{r}} \quad \text{and} \quad \mathbb{E} g\big(\sqrt{r}X\big) \leq \left(\mathbb{E} g(X)^r\right)^{\frac{1}{r}}, \tag{1.1}$$

*(ii) for every $q \in [1,+\infty)$*

$$\mathbb{E} f\big(\sqrt{q}X\big) \leq \left(\mathbb{E} f(X)^q\right)^{\frac{1}{q}} \quad \text{and} \quad \mathbb{E} g\big(\sqrt{q}X\big) \geq \left(\mathbb{E} g(X)^q\right)^{\frac{1}{q}}. \tag{1.2}$$

*In any case, equality holds if $r = 1 = q$ or if $f(x) = g(x) = e^{-\langle \mathrm{a}, x\rangle + c}$, where $\mathrm{a} \in \mathbb{R}^k$ and $c \in \mathbb{R}$.*

In section 2 we prove theorem 1.1. In the main step of the proof, which is summarized in proposition 2.9, we combine techniques from [7] along with Barthe's inequality [2].

In section 3, we prove a stability type result for the logarithmic Sobolev inequality. Let $X$ be a random vector in $\mathbb{R}^k$. Define the entropy of a function $f \in L(X)$, with respect to $X$ to be the quantity

$$\mathrm{Ent}_X(f) := \mathbb{E}|f(X)| \log |f(X)| - \mathbb{E}|f(X)| \, \log \mathbb{E}|f(X)|,$$

provided that the expectations make sense. The classical Logarithmic Sobolev inequality, proved by L. Gross in [10], states that if $X \sim N(0, I_n)$, then

$$\mathrm{Ent}_X(|f|^2) \leq 2\,\mathbb{E}|\nabla f(X)|^2 \tag{1.3}$$

for every function $f \in L^2(\gamma_k)$. Of course we may state this for $f \geq 0$ without loss of generality. Moreover, Carlen proved in [6], that equality holds if and only if $f$ is an exponential function. For more details about the logarithmic Sobolev inequality we refer the reader to [4], [13], [17], [18] and to the references therein.

Theorem 1.1, after an application of the Gaussian integration by parts formula (see lemma 3.2), leads us to the following sharp, quantitative stability result for Gross' inequality, when the function is log concave.

**Theorem 1.2.** *Let $X$ be a standard Gaussian random vector in $\mathbb{R}^k$ and $f = e^{-v} \in \mathcal{L}^{2,1}(\gamma_k)$, where $v : \mathbb{R}^k \to \mathbb{R}$ is a convex function (on its support). Then*

$$2\,\mathbb{E}|\nabla f(X)|^2 - \mathbb{E} f(X)^2 \Delta v(X) \leq \mathrm{Ent}_X(f^2) \leq 2\,\mathbb{E}|\nabla f(X)|^2 \tag{1.4}$$

Theorem 1.2 is actually a quantitative stability result, asserts that, as long as Gross' right-hand side bound $2\,\mathbb{E}|\nabla f(X)|^2$ is finite, then the closer a log-concave function is to be a maximizer (exponential function), the sharper this bound become.

Moreover, Theorem 1.2 retrieves Carlen's result, for log-concave functions. Namely we have the following corollary

**Corollary 1.3.** *Let $f = e^{-v} \in \mathcal{L}^{2,1}(\gamma_k)$, where $v : \mathbb{R}^k \to \mathbb{R}$ is a convex function (on its support) such that*

$$\mathrm{Ent}_X(|f|^2) = 2\,\mathbb{E}|\nabla f(X)|^2.$$

*Then $f$ is an exponential function (a.e)*

*Proof.* Since we have equality everywhere in (1.4) we get that $\mathbb{E}f(X)^2\Delta v(X) = 0$. Thus $\Delta v = 0$ a.e. and this means that there exist $\mathrm{a} \in \mathbb{R}^k$ and $c \in \mathbb{R}$ such that $v(x) = \langle x, \mathrm{a}\rangle + c$ for almost every $x \in \mathbb{R}^k$. $\square$

For more stability results on the logarithmic-Sobolev inequalities we refer to the papers [11], [9], [8] and the references therein.

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## 2 Proof of the main result

The first main tool in the proof theorem 1.1 is the following inequality for Gaussian random vectors, proved in [7]. Recall that for two $N \times N$ matrices $A$ and $B$, we say that $A \leq B$ if and only if $B - A$ is positive semi-definite.

**Theorem 2.1.** *Let $m, n_1, \ldots, n_m \in \mathbb{N}$ and set $N = \sum_{i=1}^m n_i$. For every $1 \leq i \leq m$, let $X_i$ be a Gaussian random vector in $\mathbb{R}^{n_i}$, such that $\mathbf{X} := (X_1, \ldots, X_m)$, is a Gaussian random vector in $\mathbb{R}^N$ with covariance the $N \times N$ matrix $T = (T_{ij})_{1 \leq i,j \leq m}$, where $T_{ij}$ is the covariance matrix between $X_i$ and $X_j$ for $1 \leq i, j \leq m$. Let $P$ be the block diagonal matrix,*

$$P = \mathrm{diag}(p_1 T_{11}, \ldots, p_m T_{mm}).$$

*Then for any set of nonnegative measurable functions $f_i$ on $\mathbb{R}^{n_i}$, $i = 1, \ldots, m$,*

(*i*) *If $T \leq P$, then*

$$\mathbb{E}\prod_{i=1}^m f_i(X_i) \leq \prod_{i=1}^m \Big(\mathbb{E}f_i(X_i)^{p_i}\Big)^{\frac{1}{p_i}}. \tag{2.1}$$

(*ii*) *If $T \geq P$, then*

$$\mathbb{E}\prod_{i=1}^m f_i(X_i) \geq \prod_{i=1}^m \Big(\mathbb{E}f_i(X_i)^{p_i}\Big)^{\frac{1}{p_i}}. \tag{2.2}$$

Theorem 2.1 generalizes many fundamental results in analysis, such as Hölder inequality and its reverse, Sharp Young inequality and its reverse (see [3] and [5]), and Nelson's Gaussian Hypercontractivity and its reverse (see [15] and [14]). Actually, the first part of theorem 2.1 is a reforlmulation of the famous Bascamp-Lieb inequality, first prooved in [5] (see also [12] for the fully generalized version), while the second part provides us with its generalized reverse form.

The second main tool in our proof, is the other famous reverse form of the Brascamp-Lieb inequality proved by F. Barthe [2], that generalizes the Prékopa-Leindler inequality. Next we state the Geometric form of Barthe's theorem, first put forward by k.Ball [1]:

**Theorem 2.2.** *Let $n, m, n_1, \ldots, n_m \in \mathbb{N}$. For every $i = 1, \ldots, m$ let $U_i$ be a $n_i \times n$ matrix with $U_i U_i^* = I_{n_i}$ and $c_1, \ldots, c_m$ be positive numbers such that*

$$\sum_{i=1}^{m} c_i \, U_i^* U_i = I_n$$

*Let $h : \mathbb{R}^n \to [0, +\infty)$ and $f_i : \mathbb{R}^{n_i} \to [0, +\infty)$, $i = 1, \ldots, m$ measurable functions such that*

$$h\left(\sum_{i=1}^{N} c_i U_i^* \xi_i\right) \geq \prod_{i=1}^{m} f_i(\xi_i)^{c_i} \quad \forall \, \xi_i \in \mathbb{R}^{n_i} \tag{2.3}$$

*then*

$$\int_{\mathbb{R}^n} h(x) \, d\gamma_n(x) \geq \prod_{i=1}^{m} \left(\int_{\mathbb{R}^{n_i}} f_i(x) \, d\gamma_{n_i}(x)\right)^{c_i} \tag{2.4}$$

### 2.1 Decomposing the identity

We are going to apply theorem 2.1 in the special case where the covariance matrix is of the form $T = \big([T_{ij}]\big)_{i,j \leq n}$ $kn \times kn$, with $T_{ii} = I_k$ and $T_{ij} = tI_k$ if $i \neq j$, $t \in [-\frac{1}{n-1}, 1]$. Equivalently, in this case $X_1, \cdots, X_n$ are standard Gaussian random vectors in $\mathbb{R}^k$, such that

$$\mathbb{E}(X_i X_j^*) = \begin{cases} I_k \, , & i = j \\ tI_k \, , & i \neq j \end{cases} \tag{2.5}$$

For any $t \in [0, 1]$, a natural way to construct such random vectors is to consider $n$ independent copies $Z_1, \ldots, Z_n$, of a $Z \sim N(0, I_k)$ and set

$$X_i := \sqrt{t}\, Z + \sqrt{1-t}\, Z_i \, , \qquad i = 1, \ldots, n.$$

It's then easy to check that condition (2.5) holds true for these vectors. However, we are going to construct such vectors using a more geometric language. We first make this construction the "$k = 1$" case of the theorems, and then we pass it for any $k \in \mathbb{N}$, using a tensorization argument. We begin with the definition of the SR-simplex.

**Definition 2.3.** *We say that* $S = \mathrm{conv}\{\mathrm{v}_1, \ldots, \mathrm{v}_n\} \subseteq \mathbb{R}^{n-1}$ *is the spherico-regular simplex (in short SR-simplex) if* $\mathrm{v}_1, \ldots, \mathrm{v}_n$ *are unit vectors in* $\mathbb{R}^{n-1}$ *enjoying the properties*

*(SR1)* $\langle \mathrm{v}_i, \mathrm{v}_j \rangle = -\frac{1}{n-1}$, *for any* $i \neq j$

*(SR2)* $\sum_{i=1}^n \mathrm{v}_i = 0$.

Using the vertices of the SR-simplex in $\mathbb{R}^{n-1}$, one can create $n$ vectors in $\mathbb{R}^n$ with the same angle between them. This is done in next lemma, which is a special case of a more general fact, observed in [7, sec. 3.1]

**Lemma 2.4.** *Let* $n \geq 2$ *and* $\mathrm{v}_1, \ldots, \mathrm{v}_n$ *be the vertices of any RS-Simplex in* $\mathbb{R}^{n-1}$. *For every* $t \in [-\frac{1}{n-1}\,, 1]$, *let* $u_1, \ldots, u_n$ *in* $\mathbb{R}^n$ *be the unit vectors in* $\mathbb{R}^n$ *with*

$$u_i = u_i(t) = \sqrt{\frac{t(n-1)+1}{n}}\ \ e_n\ +\ \sqrt{\frac{n-1}{n}(1-t)}\ \ \mathrm{v}_i. \tag{2.6}$$

*Then we have that*

$$\langle u_i, u_j \rangle = t\ , \qquad \forall\, i \neq j. \tag{2.7}$$

*Moreover, using those vectors we can decompose the identity in* $\mathbb{R}^n$*:*

*(i) If* $t \in [0,1]$, *then*

$$\frac{1}{t(n-1)+1} \sum_{i=1}^n u_i u_i^* + \frac{nt}{t(n-1)+1} \sum_{j=1}^{n-1} e_j e_j^* = I_n. \tag{2.8}$$

*(ii) If* $t \in [-\frac{1}{n-1}, 0]$, *then*

$$\frac{1}{1-t} \sum_{i=1}^n u_i u_i^* + \frac{-nt}{1-t}\, e_n e_n^* = I_n. \tag{2.9}$$

*Proof.* A direct computation shows that (2.7), (2.8) and (2.9) holds true. $\square$

**Remark 2.5.** *If* $Z \sim N(0, I_n)$, *then* $X_i := \langle u_i, Z \rangle$, $i = 1, \ldots, n$, *are standard Gaussian random variables, satisfying the condition* (2.5) *in the* 1*-dimensional case.*

In order to make the same construction in the general $k$-dimensional case, we use a more or less standard tensorization argument. We start with the definition of the *tensor product* between two matrices.

**Definition 2.6.** *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{k\times \ell}$. Then the tensor product of $A$ and $B$ is the matrix*

$$A = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{km\times \ell n}.$$

Every vector $a \in \mathbb{R}^n$ is considered to be a column $n \times 1$ matrix, and with this notation in mind, we state some basic properties for the tensor product.

**Lemma 2.7.** *1. Let $a = (a_1, \ldots, a_m)^* \in \mathbb{R}^m$ and $b = (b_1, \ldots, b_n)^* \in \mathbb{R}^n$. Then*

$$a \otimes b^* = ab^* = \begin{bmatrix} a_1b_1 & \cdots & a_1b_n \\ \vdots & \ddots & \vdots \\ a_mb_1 & \cdots & a_mb_n \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

*As linear transformation: $a \otimes b^* = ab^* : \mathbb{R}^n \mapsto \mathbb{R}^m$ with*

$$(a \otimes b^*)(x) = (ab^*)(x) = \langle x, b\rangle\, a,$$

*for every $x \in \mathbb{R}^n$.*

*2. Let $A_i \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{k\times \ell}$. Then $(\sum_i A_i) \otimes B = \sum_i A_i \otimes B$*

*3. Let $A_1 \in \mathbb{R}^{m\times n}$, $B_1 \in \mathbb{R}^{k\times \ell}$, and $A_2 \in \mathbb{R}^{n\times r}$, $B_2 \in \mathbb{R}^{\ell\times s}$.Then*

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2 \in \mathbb{R}$$

*4. For all $A$ and $B$,*

$$(A \otimes B)^* = A^* \otimes B^*$$

Consider now the matrices

$$\begin{aligned} U_i &:= u_i^* \otimes I_k = \begin{bmatrix} [u_{i1}I_k] & \cdots & [u_{in}I_k] \end{bmatrix} \quad (k \times kn)\,, \qquad i = 1, \ldots, n \quad (2.10) \\ E_j &:= e_j^* \otimes I_k = \begin{bmatrix} [e_{j1}I_k] & \cdots & [e_{jn}I_k] \end{bmatrix} \quad (k \times kn)\,, \qquad j = 1, \ldots, n. \quad (2.11) \end{aligned}$$

Then,

$$U_i^*U_i = (u_i^* \otimes I_k)^*(u_i^* \otimes I_k) = u_iu_i^* \otimes I_k, \quad kn \times kn$$

and

$$E_j^*E_j = (e_j^* \otimes I_k)^*(e_j^* \otimes I_k) = e_je_j^* \otimes I_k, \quad kn \times kn$$

and thus, by taking the tensor product with $I_k$, in both sides of (2.8), we have that

$$\frac{1}{p}\sum_{i=1}^{n} U_i^* U_i + \frac{nt}{p}\sum_{j=1}^{n-1} E_j^* E_j = I_{kn}, \tag{2.12}$$

for every $t \in [0,1]$, where $p := (n-1)t + 1$.

With the help of these matrices we are ready now to construct the general situation, describing in (2.5). We summarize in next lemma.

**Lemma 2.8.** *Let $Z_1, \dots, Z_n$ be iid $N(0, I_k)$, $\mathbf{Z} = (Z_1, \dots, Z_n) \sim N(0, I_{kn})$, end for every $i = 1, \dots, n$ consider the random vectors*

$$X_i := U_i \mathbf{Z} = \sum_{a=1}^{n} u_{ia} Z_a \tag{2.13}$$

*Then $X_i \sim N(0, I_k)$ for every $i = 1, \dots, n$, while for $i \neq j$*

$$\mathbb{E}\big[X_i \otimes X_j^*\big] = \big[\mathbb{E} X_{ir} X_{j\ell}\big]_{r,\ell \le k} = \big[t\delta_{r\ell}\big]_{r,\ell \le k} = tI_k \tag{2.14}$$

*Proof.* Clearly, $\mathbb{E} X_i = 0$, for every $i, j = 1, \dots, n$, and since

$$\mathbb{E}\big[Z_a \otimes Z_b^*\big] = \big[\mathbb{E} Z_{ar} Z_{b\ell}\big]_{r,\ell \le k} = \delta_{\alpha\beta} I_k$$

we have that

$$\begin{aligned}
\mathbb{E} X_{ir} X_{j\ell} &= \mathbb{E}\left(\sum_{a=1}^{n} u_{ia} Z_{ar}\right)\left(\sum_{b=1}^{n} u_{jb} Z_{b\ell}\right) \\
&= \sum_{a=1}^{n}\sum_{b=1}^{n} u_{ia} u_{jb}\, \mathbb{E} Z_{ar} Z_{b\ell} \\
&= \sum_{a=1}^{n} u_{ia} u_{ja}\, \mathbb{E} Z_{ar} Z_{a\ell} \\
&= \sum_{a=1}^{n} u_{ia} u_{ja}\, \delta_{r\ell} \\
&= \langle u_i, u_j \rangle\, \delta_{r\ell}.
\end{aligned}$$

and from (2.7) the proof is complete. ∎

## 2.2 Proof of theorem 1.1

Next proposition, that has a separate interest by its own, gives the first step for the proof of our main result, theorem 1.1.

**Proposition 2.9.** *Let $t \in [0,1]$, $n \in \mathbb{N}$, $p = t(n-1)+1$, $X$ be a standard Gaussian random vector in $\mathbb{R}^k$, $k \in \mathbb{N}$ and $X_1, \cdots, X_n$ be copies of $X$ such that*

$$\mathbb{E}(X_i X_j^*) = \left(\mathbb{E}X_{ir}X_{j\ell}\right)_{r,\ell\leq k} = tI_k, \quad i \neq j.$$

*Then, for any* log-*concave (on its support) function $f : \mathbb{R}^k \to [0,+\infty)$, we have that*

$$\mathbb{E}\left(\prod_{i=1}^n f(X_i)\right)^{\frac{1}{n}} \leq \left(\mathbb{E}f(X)^{\frac{p}{n}}\right)^{\frac{n}{p}} \leq \mathbb{E}f\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \tag{2.15}$$

*Note that, since $f$ is* log-*concave we always have that $\left(\prod_{i=1}^n f(X_i)\right)^{\frac{1}{n}} \leq f\left(\frac{1}{n}\sum_{i=1}^n X_i\right)$, while equality is achieved if $f(x) = e^{\langle \mathrm{a}, x\rangle + c}$, $\mathrm{a} \in \mathbb{R}^k$ and $c \in \mathbb{R}$.*

*Proof.* The left-hand side inequality in (2.15), follows after the application of theorem 2.1 in the special case describing in lemma 2.8. Note that the assumption that $f$ is log-concave is not needed here. This inequality holds for any measurable function $f$. To make this more precise, the following simple remark is helpful.

**Remark 2.10.** *Let $t \in [-\frac{1}{n-1}, 1]$ and $X_1, \ldots, X_n$ be standard Gaussian random vectors in $\mathbb{R}^k$ satisfying the condition (2.14) of lemma 2.8. Thus $\mathbf{X} := (X_1, \ldots, X_n)$, is a centered Gaussian vector in $\mathbb{R}^{kn}$ with covariance matrix $T = [T_{i,j}]_{i,j\leq n}$, with block entries the $k \times k$ matrices $T_{ii} = I_k$ for every $i = 1, \ldots, n$, and $T_{ij} = tI_k$, for $i \neq j$. If we set*

$$p := (n-1)t + 1 \quad \text{and} \quad q := 1 - t,$$

*then it's not hard to check that, for $t \leq 0$ $q$ is the biggest and $p$ is the smallest singular value of $T$. On the other hand, if $t \geq 0$ then, $p$ is the biggest singular value of $T$ and $q$ is the smallest one. Thus we have that*

*(i) if $t \geq 0$ then*

$$qI_{kn} \leq T \leq pI_{kn}$$

*(ii) if $t \leq 0$ then*

$$pI_{kn} \leq T \leq qI_{kn}$$

Thus, in the above situation, theorem 2.1 reads as follows:

**Theorem 2.11.** *Let $k, n \in \mathbb{N}$, $t \in [-\frac{1}{n-1}, 1]$ and let $X_1, \ldots, X_n$ be standard Gaussian random vectors in $\mathbb{R}^k$, with $\mathbb{E}[X_i \otimes X_j^*] = tI_k$, for all $i \neq j$. Setting $p := (n-1)t + 1$ and $q := 1 - t$, we have that for every set of measurable functions $f_i : \mathbb{R}^k \to [0, +\infty)$, $i = 1, \ldots, n$,*

*(i) if $t \in [0, 1]$, then*

$$\prod_{i=1}^{n} \left(\mathbb{E} f_i(X_i)^q\right)^{1/q} \leq \mathbb{E} \prod_{i=1}^{n} f_i(X_i) \leq \prod_{i=1}^{n} \left(\mathbb{E} f_i(X_i)^p\right)^{1/p}, \tag{2.16}$$

*(ii) if $t \in [-\frac{1}{n-1}, 0]$, then*

$$\prod_{i=1}^{n} \left(\mathbb{E} f_i(X_i)^p\right)^{1/p} \leq \mathbb{E} \prod_{i=1}^{n} f_i(X_i) \leq \prod_{i=1}^{n} \left(\mathbb{E} f_i(X_i)^q\right)^{1/q} \tag{2.17}$$

Now, the left-hand side inequality of (2.15), follows immediately from (2.16), by taking $f_i = f$ for every $i = 1, \ldots, n$.

In order to prove the right-hand side inequality of (2.15), we apply Barthe's theorem, using the decomposition of the identity (2.12). To do so we first state, in the following lemma, some technical details we are going to need.

**Lemma 2.12.** *Let $U_i$ and $E_i$, $i = 1, \ldots, n$ the matrices defined in* (2.10) *and* (2.11), *and set $p = (n-1)t + 1$, $q = 1 - t$. Then*

$$\begin{aligned} U_i^* &= \sqrt{\frac{p}{n}}\, e_n \otimes I_k + \sqrt{\frac{n-1}{n} q}\, \mathrm{v}_i \otimes I_k \ \in \mathbb{R}^{kn \times k}. \\ U_i U_j^* &= \langle u_i, u_j \rangle I_k \\ U_i E_j^* &= \sqrt{\frac{n-1}{n} q}\, \langle \mathrm{v}_i, e_j \rangle I_k \end{aligned}$$

*for every $i \leq n$ and $j \leq n - 1$.*

*Proof.* The first and the second can be verified after some obvious and trivial computations. For the third one, we have

$$\begin{aligned}
U_i E_j^* &= (u_i^* \otimes I_k)(e_j^* \otimes I_k)^* \\
&= \left( \sqrt{\frac{p}{n}}\, e_n^* \otimes I_k + \sqrt{\frac{n-1}{n} q}\, \mathrm{v}_i^* \otimes I_k \right) (e_j \otimes I_k) \\
&= \sqrt{\frac{p}{n}}\, (e_n^* \otimes I_k)(e_j \otimes I_k) + \sqrt{\frac{n-1}{n} q}\, (\mathrm{v}_i^* \otimes I_k)(e_j \otimes I_k) \\
&= \sqrt{\frac{p}{n}}\, e_n^* e_j \otimes I_k + \sqrt{\frac{n-1}{n} q}\, \mathrm{v}_i^* e_j \otimes I_k \\
&= \sqrt{\frac{p}{n}}\, \langle e_n, e_j \rangle I_k + \sqrt{\frac{n-1}{n} q}\, \langle \mathrm{v}_i, e_j \rangle I_k \\
&= \mathbb{O} + \sqrt{\frac{n-1}{n} q}\, \langle \mathrm{v}_i, e_j \rangle I_k .
\end{aligned}$$

$\square$

To this end, we will apply Barthe's theorem 2.2, using the decomposition of the identity appearing in (2.12). More precisely, we choose the parameters: $n \leftrightarrow kn$, $m := 2n-1$, $n_i := k$ for all $i = 1, \dots, 2n-1$, and

$$c_i := \begin{cases} \frac{1}{p} & , \quad i = 1, \dots, n \\ \frac{nt}{p} & , \quad i = n+1, \dots, 2n-1 \end{cases}$$

and we apply theorem 2.2 to the functions,

$$\tilde{f}_i(x) := \begin{cases} f(x)^{\frac{p}{n}} & , \quad i = 1, \dots, n \\ 1 & , \quad i = n+1, \dots, 2n-1 \end{cases} , \quad x \in \mathbb{R}^k$$

and

$$h(x) := f\left( \frac{1}{n} \sum_{i=1}^{n} U_i x \right), \quad x \in \mathbb{R}^{kn}.$$

Note then that under lemma 2.12, we have that for every $\xi_j \in \mathbb{R}^k$, $j = 1, \dots, n$,

$$\begin{aligned}
& h\left(\sum_{j=1}^{n} \frac{1}{p} U_j^* \xi_j + \sum_{a=1}^{n-1} \frac{nt}{p} E_a^* \xi_{n+a}\right) \\
= \ & f\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{p} U_i U_j^* \xi_j + \frac{1}{n}\sum_{i=1}^{n}\sum_{a=1}^{n-1} \frac{nt}{p} U_i E_a^* \xi_{n+a}\right) \\
= \ & f\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{p} U_i U_j^* \xi_j + \frac{1}{n}\sum_{i=1}^{n}\sum_{a=1}^{n-1} \frac{nt}{p} \sqrt{\frac{n-1}{n}} q \langle \mathrm{v}_i, e_a\rangle \xi_{n+a}\right) \\
= \ & f\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{p} U_i U_j^* \xi_j\right) \qquad \Big(\text{since } \sum \mathrm{v}_i = 0\Big) \\
= \ & f\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{p} \langle u_i, u_j\rangle \xi_j\right) \\
= \ & f\left(\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{1}{p}\xi_i + \sum_{j\neq i} \frac{t}{p}\xi_j\Big)\right) \\
= \ & f\left(\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{1}{p} + (n-1)\frac{t}{p}\Big)\xi_i\right) \\
= \ & f\left(\frac{1}{n}\sum_{i=1}^{n}\xi_i\right) \geq \prod_{i=1}^{n} f(\xi_i)^{\frac{1}{n}} = \prod_{i=1}^{n}\Big(f(\xi_i)^{\frac{p}{n}}\Big)^{\frac{1}{p}} = \prod_{i=1}^{n} \tilde{f}(\xi_i)^{c_i}
\end{aligned}$$

Thus, theorem 2.2 gives that

$$\mathbb{E}f\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \mathbb{E}f\left(\frac{1}{n}\sum_{i=1}^{n} U_i Z\right) \geq \prod_{i=1}^{n}\Big(\mathbb{E}f(X_i)^{\frac{p}{n}}\Big)^{\frac{1}{p}} = \Big(\mathbb{E}f(X)^{\frac{p}{n}}\Big)^{\frac{n}{p}} \tag{2.18}$$

and the proof is complete ☐

*Proof of theorem 1.1.* Suppose first that $X \sim N(0, I_k)$. Then, under the notation of lemma 2.8 we have that

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} U_i\mathbf{Z} &= \frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{p}{n}}\,(e_n^*\otimes I_k)\mathbf{Z} \;+\; \frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{n-1}{n}q}\,(\mathrm{v}_i^*\otimes I_k)\mathbf{Z} \\
&= \sqrt{\frac{p}{n}}\,(e_n^*\otimes I_k)\mathbf{Z} \;+\; \frac{1}{n}\sqrt{\frac{n-1}{n}q}\,\left(\sum_{i=1}^{n}\mathrm{v}_i^*\right)\otimes I_k\,\mathbf{Z} \\
&= \sqrt{\frac{p}{n}}\,E_n\mathbf{Z} \;+\; \frac{1}{n}\sqrt{\frac{n-1}{n}q}\,\left(\sum_{i=1}^{n}\mathrm{v}_i\right)^*\otimes I_k\,\mathbf{Z} \\
&= \sqrt{\frac{p}{n}}\,Z_n.
\end{aligned}$$

Thus, the right hand side of (2.15) can be written as

$$\mathbb{E}f\left(\sqrt{\frac{p}{n}}\,X\right) \geq \left(f(X)^{\frac{p}{n}}\right)^{\frac{n}{p}}. \tag{2.19}$$

where $p=(n-1)t+1$, $n\in\mathbb{N}$, and $t\in[0,1]$.

Consequently, if $f:\mathbb{R}^k\to[0,+\infty)$ is a log-concave function and $r\in(0,1]$, then there exist, $t\in[0,1]$ and $n\in\mathbb{N}$, such that $r=\frac{p}{n}=\frac{(n-1)t+1}{n}$, and so by (2.19) we get that

$$\mathbb{E}f\big(\sqrt{r}X\big) \geq \left(\mathbb{E}f(X)^r\right)^{\frac{1}{r}} \tag{2.20}$$

for every $r\in(0,1]$. We deal independently with the case where $r=0$. Since $f$ is *log*-concave, there exists a convex function $v:\mathbb{R}^k\mapsto\mathbb{R}$, such that $f=e^{-v}$. Then for $r=0$, inequality (1.1) is equivalent to Jensen's inequality

$$v(0)=v(\mathbb{E}X)\leq\mathbb{E}v(X), \tag{2.21}$$

and the proof of (1.1) is now complete.

For every $q\geq 1$ consider $r=\frac{1}{q}\in(0,1]$. Let $F(x)=f(x/\sqrt{r})^{1/r}$ which is also log-concave and so (2.20) for $F$ and $r$ implies

$$\mathbb{E}f(X)^q \geq \big(\mathbb{E}f(\sqrt{q}X)\big)^q, \tag{2.22}$$

and (1.2) follows.

Assume now that $g:\mathbb{R}^n\to[0,+\infty)$ is log-convex and $r\in(0,1]$. By the log-convexity of $g$ and theorem 2.11(i), we have that

$$\mathbb{E}g\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) \leq \mathbb{E}\prod_{i=1}^{n}g(X_i)^{\frac{1}{n}} \leq \left(\mathbb{E}g(Z)^{\frac{p}{n}}\right)^{\frac{n}{p}}. \tag{2.23}$$

As we have seen at the beginning of the proof, we have that $\frac{1}{n}\sum_{i=1}^n X_i \sim \sqrt{\frac{p}{n}}X$. So, using (2.23) for $t\in[0,1]$ and $n\in\mathbb{N}$ such that $\frac{p}{n}=\frac{(n-1)t+1}{n}=r$, we derive that

$$\mathbb{E}g\left(\sqrt{r}Z\right)\le\left(\mathbb{E}g(Z)^r\right)^{\frac{1}{r}}.$$

for every $r\in(0,1]$. The rest of the proof for a log-convex function $g$ is identical to the log-concave one.

Finally for the equality case, a straightforward computation shows that for $f(x)=e^{\langle \mathrm{a},x\rangle+c}$, we have that

$$\mathbb{E}f(\sqrt{q}X)=C\exp\left(\frac{q}{2}|\mathrm{a}|^2\right)=\left(\mathbb{E}f(X)^q\right)^{\frac{1}{q}}.$$

for every $q\ge 0$.

At the end, suppose that $X$ is a general Gaussian random vector in $\mathbb{R}^k$ with expectation $\xi\in\mathbb{R}^k$ and covariance matrix $T=UU^*$ where $U\in\mathbb{R}^{k\times k}$. Note, that if $f$ is a log-concave (or log-convex) and positive function on $\mathbb{R}^k$, then so is $F(x):=f(Ux-\xi)$. Moreover, if $Z\sim N(0,I_k)$ then $UZ-\xi\overset{d}{=}X\sim N(0,T)$. Thus, we get the general theorem by applying the previous case with function $F$.

□

# 3 Entropy Inequalities and Stability in Log-Sobolev

**Proposition 3.1.** *Let $X$ be a Gaussian random vector in $\mathbb{R}^k$, and $f:\mathbb{R}^k\to[0,+\infty)$. Then,*

*(i) if $f$ is log-concave, then*

$$\mathrm{Ent}_X(f)\ge\frac{1}{2}\mathbb{E}\langle X,\nabla f(X)\rangle \tag{3.1}$$

*(ii) if $f$ is log-convex, then*

$$\mathrm{Ent}_X(f)\le\frac{1}{2}\mathbb{E}\langle X,\nabla f(X)\rangle \tag{3.2}$$

*In any case, one has equality when $f(x)=\exp\left(\langle \mathrm{a},x\rangle+c\right)$, $\mathrm{a}\in\mathbb{R}^k$, $c\in\mathbb{R}$.*

*Proof.* Let $M(q):=\left(\mathbb{E}f(X)^q\right)^{\frac{1}{q}}$ and $H(q):=\mathbb{E}f(\sqrt{q}X)$. Then we have that

$$M(1)=\mathbb{E}f(X)=H(1)\quad\text{and}\quad M'(1)=\mathrm{Ent}_X(f)\,,\quad H'(1)=\frac{1}{2}\mathbb{E}\langle X,\nabla f(X)\rangle.$$

Thus, Theorem 1.1 immediately implies the desired result. □

Gaussian random vectors have a special property known as the *Gaussian Integration by Parts* formula, which we state in the next lemma (see [16, Appendix 4] for a simple proof).

**Lemma 3.2.** *Let $X, Y_1, \ldots, Y_n$ be centered jointly Gaussian random variables, and $F$ be a real valued function on $\mathbb{R}^n$, that satisfy the growth condition*

$$\lim_{|x|\to\infty} |F(x)| \exp\left(-a|x|^2\right) = 0 \qquad \forall\, a > 0. \tag{3.3}$$

*Then*

$$\mathbb{E} X F(Y_1, \ldots, Y_n) = \sum_{i=1}^{n} \mathbb{E} X Y_i \, \mathbb{E} \partial_i F(Y_1, \ldots, Y_n). \tag{3.4}$$

Involving the Gaussian Integration by Parts formula, we can further elaborate proposition 3.1 in order to prove theorem 1.2.

More precisely, let $\mathcal{G}_k$, be the class all the functions in $\mathbb{R}^k$, such that their first derivatives satisfy the growth condition (3.3). Then for any $f \in \mathcal{G}_k$, lemma 3.2 implies that

$$\mathbb{E}\langle X, \nabla f(X)\rangle = \sum_{i=1}^{k} \mathbb{E} X_i \partial_i f(X) = \sum_{i=1}^{k}\sum_{j=1}^{k} \mathbb{E} X_i X_j \, \mathbb{E} \partial_{ij} f(X) \; = \mathbb{E}\, \mathrm{tr}\big(T H_f(X)\big). \tag{3.5}$$

where, $H_f(x)$ stands for the Hessian matrix of $f$ at $x \in \mathbb{R}^k$. In the special case where $X \sim N(0, I_k)$, we have proved the following

**Corollary 3.3.** *Let $k \in \mathbb{N}$, and $X$ be a standard Gaussian vector in $\mathbb{R}^k$. Then*

*(i) for every log-concave function $f \in \mathcal{G}_k$ we have that*

$$\mathrm{Ent}_X(f) \geq \frac{1}{2} \mathbb{E} \Delta f(X), \tag{3.6}$$

*(ii) for every log-convex function $f \in \mathcal{G}_k$ we have that*

$$\mathrm{Ent}_X(f) \leq \frac{1}{2} \mathbb{E} \Delta f(X). \tag{3.7}$$

*Proof of Theorem 1.2.* Let $f \in \mathcal{L}^{2,1}(\gamma_k)$, and without loss of of generality we may also assume that $\mathbb{E} f^2(X) = 1$. Suppose first that $f$ has a bounded support. Then $f^2 \in \mathcal{G}_k$, and so Corollary 3.3, after an application of the chain rule $\frac{1}{2}\Delta f^2 = |\nabla f|^2 + f\Delta f$, gives that

$$\mathbb{E}|\nabla f(X)|^2 + \mathbb{E} f(X)\Delta f(X) \leq \mathrm{Ent}_X(f^2) \leq 2\, \mathbb{E}|\nabla f(X)|^2 \tag{3.8}$$

Finally, for $f = e^{-v}$, where $v : supp(f) \to \mathbb{R}$ is a convex function, and by another application of the chain rule:

$$f\Delta f = f^2|\nabla v|^2 - f^2\Delta v = |\nabla f|^2 - f^2\Delta v,$$

we get that

$$\mathbb{E}f(X)\Delta f(X) = \mathbb{E}|\nabla f(X)|^2 - \mathbb{E}f(X)^2\Delta v(X). \tag{3.9}$$

Equation (3.8) combined with (3.9), proves theorem 1.2 in this case.

In order to drop the assumption of the bounded support, we proceed with a standard approximation argument. We consider the functions $f_n := f\,\mathbf{1}_{nB_2^k}$, where $\mathbf{1}_{nB_2^k}$ is the indicator function of the Euclidean Ball in $\mathbb{R}^k$ with radius $n \in \mathbb{N}$. Then, every $f_n$ has bounded support and we also have that $0 \leq f_n \nearrow f$, $0 \leq |\nabla f_n|^2 \nearrow |\nabla f|^2$, and $0 \leq f_n^2\Delta v_n \nearrow f^2\Delta v$. Thus by the monotone convergence theorem

$$\mathbb{E}|\nabla f_n(X)|^2 \longrightarrow \mathbb{E}|\nabla f(X)|^2 < \infty \quad \text{and} \quad \mathbb{E}f_n(X)^2\Delta v_n(X) \longrightarrow \mathbb{E}f(X)^2\Delta v(X) \tag{3.10}$$

Moreover, $f_n^2\log f_n^2 \to f^2\log f^2$ and $|f_n^2\log f_n^2| \leq |f^2\log f^2|$, for every $n \in \mathbb{N}$ (where we have taken that $0\log 0 = 0$). By Gross' inequality $|f^2\log f^2| \in L^1(\gamma_k)$, and so after applying the Lebesgue's dominated convergence theorem we also get that

$$\mathrm{Ent}_X(f_n^2) \longrightarrow \mathrm{Ent}_X(f^2). \tag{3.11}$$

Since equation (1.4) holds true for every $f_n$, we pass to the limit using (3.10) and (3.11), and we get that (1.4) is also true for $f$. Theorem is now proved. $\square$

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