Gaussian Methods in Linear Dvoretzky Theory

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Mathematical Analysis in honor of S. Argyros

Modes of convergence

Let $(X, \|\cdot\|)$ be a normed space and let $(x_n) \subset X$.

- Absolute Convergence (AC): $\sum \|x_n\| < \infty$.
- Unconditional Convergence (UC): $\forall\, \varepsilon_n = \pm 1 \implies \sum \varepsilon_n x_n$ converges.
- (AC) $\Rightarrow$ (UC). [$m < n$, $\left\|\sum_{k=m+1}^{n} \varepsilon_k x_k\right\| \leq \sum_{k=m+1}^{n} \|x_k\|$.]
- (AC) $\Leftarrow$ (UC) ?
  - Yes, if $\dim X < \infty$.
    [On $\mathbb{R}$ choose $\varepsilon_i = \mathrm{sgn}(x_i)$.]
  - No, if X is *infinite Hilbert space*.
    [Let (x_n) be o.s. Pythagoras's thm $\left\|\sum_{i=m+1}^{n} \varepsilon_i x_i\right\|^2 = \sum_{i=m+1}^{n} \|x_i\|^2$.

Question (Banach 1932): *Is it true that if (AC) $\iff$ (UC) on X, then* $\dim X < \infty$?

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- Early (negative) results by Orlicz, Macphail, . . .
- Dvoretzky, Rogers (1950): (AC) $\Longleftrightarrow$ (UC) on X iff $\dim X < \infty$. Why? All high-dimensional normed spaces contain relatively large Besselian systems. They proved the following *local* phenomenon:

Lemma (Dvoretzky, Rogers 1950)

Let $(X, \|\cdot\|)$ be a normed space and let $F \subset X$ with $\dim F = n$. Then, there exist $m \geq \sqrt{n}$ and $x_1, \ldots, x_m \in F$ with $\|x_j\| = 1$ such that

$$\left\| \sum_{j=1}^{m} \alpha_j x_j \right\| \leq 2 \sqrt{\sum_{j=1}^{m} \alpha_j^2}, \quad \forall (\alpha_j) \subset \mathbb{R}.$$

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A question of Grothendieck

In fact, Dvoretzky and Rogers proved the following:

$$\frac{1}{\sqrt{3}} \max_{j \leq m} |\alpha_j| \leq \left\| \sum_{j=1}^{m} \alpha_j x_j \right\| \leq 2 \sqrt{\sum_{j=1}^{m} \alpha_j^2}, \quad \forall (\alpha_j) \subset \mathbb{R}.$$

Question (Grothendieck, 1950): Is it possible to have a two-sided ℓ_2 estimate and $m = m(n) \to \infty$ as $n \to \infty$?
Let $k = k(n, \varepsilon)$ be the largest k for which *any* n-dimensional space $(X, \|\cdot\|)$ admits vectors $x_1, \ldots, x_k$ such that

$$(1-\varepsilon)\left(\sum_j \alpha_j^2\right)^{1/2} \leq \left\| \sum_j \alpha_j x_j \right\| \leq (1+\varepsilon)\left(\sum_j \alpha_j^2\right)^{1/2}, \quad \forall (\alpha_j) \subset \mathbb{R}.$$

Determine the asymptotic behavior of $k(n, \varepsilon)$.

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Dvoretzky's theorem

Alternatively, for any normed space $(X, \|\cdot\|)$ and for any $\varepsilon > 0$ define

$$k(X,\varepsilon) := \sup\left\{k \in \mathbb{N} \mid \ell_2^k \overset{1+\varepsilon}{\hookrightarrow} X\right\}.$$

Then, we have

$$k(n,\varepsilon) := \inf\left\{k(X,\varepsilon) \mid \dim X = n\right\}.$$

Theorem (Dvoretzky, 1960)

For all $n \in \mathbb{N}$ and for any $\varepsilon \in (0,1)$ one has $k(n,\varepsilon) \geq c\varepsilon \frac{\sqrt{\log n}}{\log\log n}$.

Fact. $k(\ell_\infty^n, \varepsilon) \asymp \frac{1}{\log(1/\varepsilon)} \log n$. Thus, $k(n,\varepsilon) \leq \frac{C}{\log(1/\varepsilon)} \log n$.

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V. Milman's approach

1. Probabilistic method (sieving): Introduce a probability space with rdm objects of interest, i.e., rdm operators

$$G = (g_{ij}) : \mathbb{R}^k \to X \equiv (\mathbb{R}^n, \|\cdot\|), \quad g_{ij} \sim N(0,1).$$

2. Concentration of measure: for arbitrary (but fixed) $\theta \in S^{k-1}$ note that $G\theta \overset{d}{=} Z \sim N(0, I_n)$. Hence,

$$\mathbb{P}\Big(\underbrace{\big|\|G\theta\| - \mathbb{E}\|Z\|\big| > \varepsilon \mathbb{E}\|Z\|}_{B_\theta}\Big) \le 2e^{-c\varepsilon^2 k(X)}, \quad k(X) := \frac{(\mathbb{E}\|Z\|)^2}{\mathrm{Lip}^2(\|\cdot\|)}.$$

3. "Discretize" the sphere S^{k-1} using a ε-net $\mathcal{N}$ with $\mathrm{card}(\mathcal{N}) \le (3/\varepsilon)^k$; apply the previous estimate to obtain

$$\mathbb{P}\Big(\bigcup_{\theta \in \mathcal{N}} B_\theta\Big) \le 2\mathrm{card}(\mathcal{N}) e^{-c\varepsilon^2 k(X)} \ll 1,$$

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Theorem (V. Milman, 1971)

Let $X = (\mathbb{R}^n, \|\cdot\|)$. *Then,* $k(X,\varepsilon) \geq c(\varepsilon)k(X)$, *where* $c(\varepsilon) \gtrsim \frac{\varepsilon^2}{|\log \varepsilon|}$, $k(X) := (\mathbb{E}\|Z\|)^2/\mathrm{Lip}^2(\|\cdot\|)$, *and* $Z \sim N(0, I_n)$.

- Choosing the linear structure appropriately first (before we apply the aforementioned random procedure) to optimize the parameters, we may achieve $k(X) \geq ck(\ell_\infty^{\lfloor n/2 \rfloor}) \asymp \log n$.
- (V. Milman '71). The $\log n$ in $k(n, \varepsilon)$ is optimal.
- Milman's approach yields the existence of "many" (w.r.t. to the Haar measure on Grassmannian) almost Euclidean subspaces (in some canonical position of the ambient space).
- Introduces the *randomized* Dvoretzky number $k_r(X,\varepsilon)$. Clearly, $k(X,\varepsilon) \geq k_r(X,\varepsilon)$.

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Theorem (Paouris, V., '18)

For any norm $\|\cdot\|$ on $\mathbb{R}^n$, there exists $T \in GL(n)$ such that

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- *Probabilistic component*: Quantifying the superconcentration via Talagrand's $L_1 - L_2$ bound.

Theorem (Talagrand 1994; Cordero-Erausquin, Ledoux 2013)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a sufficiently smooth . Then,

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Theorem (Alon, Milman, 1983; Talagrand 1995)

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be in John's position. Then, either $k(X) > cn^{1/3}$ or there exists $F \leq X$ with $\dim F > cn^{1/2}$ and $\mathrm{unc} F \ll \sqrt{\dim F}$.

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Summary of the talk

- Dvoretzky's theorem: Every normed space $X = (\mathbb{R}^n, \|\cdot\|)$ contains almost isometrically copies of ℓ_2^k for $k = k(n,\varepsilon) \to \infty$ as $n \to \infty$.
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  - Concentration of measure phenomenon
  - $k_r(n,\varepsilon) \gtrsim c\varepsilon^2 \log n$; (optimal in n).
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