7nth Math@NTUA Summer School in honour of Spyros Argyros. 27/6/2024.

• Metric Embeddings and Invariants I | Αλέξανδρος Εσκενάζης
(Μετρικές Εμφυτεύσεις και Αναλλοίωτες)

Notation: Denote by $L_p := \{f : (0,1) \to \mathbb{R} \,/\, \|f\|_p = \left(\int_{(0,1)} |f|^p\right)^{1/p} < +\infty\}$

$\ell_p := \{(\alpha_n) / \sum |\alpha_n|^p < +\infty\}$, $\ell_p^n := (\mathbb{R}^n, \|\cdot\|_p)$.

Definition: A linear operator $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a D-isomorphic embedding iff $\exists\, \sigma > 0$:

$$\sigma \cdot \|x\|_X \le \|Tx\|_Y \le \sigma \cdot D \|x\|_X \quad \forall x \in X.$$

Denote $C_Y(X) =$ least such $D = \inf\{\|T\| \cdot \|T^{-1}\| / T : X \to Y\}$

→ Banach's problem (1932): For which $p, q \in [1, +\infty)$ is L_p isomorphic to a closed subspace of L_q?

Thm 1: (Banach '32, Paley '36) $L_p \not\hookrightarrow L_q$ unless $p = 2$ or $1 \le q < p < 2$.

Thm 2: (Kadec '58) $L_p \hookrightarrow L_q$ if $p = 2$ or $1 \le q < p < 2$.

Proof of Kadec for $p=2$: Let $g_1, g_2, \dots, g_n, \dots$ a sequence of i.i.d Gaussian r.v in a probability space $(\Omega, \mathbb{P})$. Consider: $T : \ell_2 \to L_q(\Omega, \mathbb{P})$ given by

$$T\alpha = \sum_{i \ge 1} \alpha_i g_i \quad , \quad \alpha = (\alpha_i)_{i \ge 1} \in \ell_2$$

• Claim: $\lambda, \mu \in \mathbb{R}$: g_1, g_2 independent $N(0,1)$:

$\lambda \cdot g_1 + \lambda \cdot g_2 \overset{(d)}{=} \sqrt{\lambda^2+\mu^2} \cdot g$ (*).

Therefore $T\alpha \overset{(d)}{=} \|\alpha\|_2 \cdot g$ (1). $\Rightarrow T\alpha \in L_q(\Omega, \mathbb{P})$ $\forall q < +\infty$.

(1) $\Rightarrow \|T\alpha\|_{L_q(\Omega,\mathbb{P})} = \|g\|_q \cdot \|\alpha\|_2 \Rightarrow$

$\Rightarrow$ T is a rescaled isometry $\ell_2 \hookrightarrow L_q$ ∎.

Question: Given $p > 0$, $\exists X_1, X_2$ r.v (i.i.d):

$\lambda \cdot X_1 + \mu \cdot X_2 \overset{(d)}{=} (|\lambda|^p + |\mu|^p)^{1/p} \cdot X_1$?

Equivalently want $\mathbb{E}[e^{itX_1}] = e^{-c \cdot |t|^p}$

Thm (Lévy, 1951): Such r.v exist with $c = 1$ iff $0 < p \le 2$. Moreover:

$\exists \delta_p > 0 : \lim_{t \to +\infty} t \cdot \mathbb{P}[|X_1| \ge t] = \delta_p$

(standard symmetric p-stable r.v.)

Therefore:

$$\mathbb{E}[|X_1|^p] = \int_0^{+\infty} \mathbb{E}[q \cdot t^{q-1} \cdot \mathbb{1}_{\{t \le |X_1|\}}] dt =$$

$$= q \cdot \int_0^{+\infty} t^{q-1} \cdot \underbrace{\mathbb{P}[|X_1| \ge t]}_{\sim \frac{1}{t^p}} dt \rightsquigarrow \text{converges iff } q < p.$$

• Prove that $\ell_p \hookrightarrow L_q$ if $1 \le q < p < 2$.

Let $X_1, X_2, \dots$ sequence i.i.d standard p-stable r.v in $(\Omega, \mathbb{P})$.

Let $T : \ell_p \to L_q(\Omega, \mathbb{P})$ given by

$T\alpha = \sum_{i \ge 1} a_i X_i \overset{(d)}{=} \|\alpha\|_{\ell_p} \cdot X_1 \in L_q(\Omega, \mathbb{P})$ for $q < p$

This is a rescaled isometry ∎

Rmk: Due to classical facts, to prove $L_p \hookrightarrow L_q$ it suffices to know that L_p is finitely representable in ℓ_p.

$\rightsquigarrow$ Question: Given $p \neq q$, estimate $c_q(\ell_p^n)$.

Thm (...   ): We have:

$$c_q^{lin}(\ell_p^n) \underset{p,q}{\asymp} \begin{cases} 1, & 1 \le q \le p \le 2 \quad \text{(Kadec)} \\ n^{\frac{1}{p}-\frac{1}{q}}, & 1 \le p \le q \le 2 \\ n^{\frac{1}{p}-\frac{1}{2}}, & 1 \le p \le 2 \le q \end{cases} \left(\begin{matrix}\text{type}\\ \text{range}\end{matrix}\right)$$

$$\begin{cases} n^{\frac{1}{q}-\frac{1}{p}}, & 2 \le q \le p \\ n^{\frac{1}{2}-\frac{1}{p}}, & q \le 2 \le p \end{cases} \left(\begin{matrix}\text{cotype}\\ \text{range}\end{matrix}\right)$$

$$n^{\frac{(q-p)(p-2)}{p^2(q-2)}}, \quad 2 < p < q \quad (X_p \text{ range})$$

"proportional"

Exercise: Prove the upper bounds.

§ Smoothness and Convexity:

Definition: (Ball-Carlen-Lieb, '93): A Banach space X is said to be p-uniformly smooth iff $\exists S > 0$:

$$(\forall x,y \in X)\left(\frac{\|x\|^p + \|y\|^p}{2} \le \left\|\frac{x+y}{2}\right\|^p + S^p \cdot \left\|\frac{x-y}{2}\right\|^p\right.$$

$(1 < p < 2)$

Similarly, X is q-uniformly convex ($2 \le q < +\infty$) iff: $\exists K > 0$:

$$(\forall x, y \in X)\left(\left\|\frac{x+y}{2}\right\|^q + \frac{1}{K^q}\cdot\left\|\frac{x-y}{2}\right\|^q \le \frac{\|x\|^q + \|y\|^q}{2}\right)$$

· The best constants are $S_p(X), K_q(X)$.

Ex: (Lindenstrauss' duality formula):

If $\frac{1}{p} + \frac{1}{q} = 1$, $S_p(X) = K_q(X^*)$ and vice versa.

Thm 1: (Clarckson's inequality): For $1 < p \le 2$, $S_p(L_p) = 1$. For $2 \le q < +\infty$, $K_q(L_q) = 1$.

Thm 2: For $1 < p \le 2$, $K_2(L_p) \le \frac{1}{\sqrt{p-1}}$. For $2 \le q < +\infty$, $S_2(L_q) \le \sqrt{q-1}$

Proof of Thm 1: For $q \ge 2$, it suffices to prove the inequality for $a, b \in \mathbb{R}$. Then,

$$LHS^{1/q} := \left(\left|\frac{a+b}{2}\right|^q + \left|\frac{a-b}{2}\right|^q\right)^{1/q} \underset{\|\cdot\|_{\ell_q} \le \|\cdot\|_{\ell_2}}{\le} \left(\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2\right)^{1/2}$$

$$= \left(\frac{a^2+b^2}{2}\right)^{1/2} \le \left(\frac{|a|^q + |b|^q}{2}\right)^{1/q} =: RHS^{1/q}$$

$\|\cdot\|_{L_2(\mathbb{P})} \le \|\cdot\|_{L_q(\mathbb{P})}$.

· Metric Embeddings II. [After Break]

Proposition 1 (Bonami): For $a, b \in \mathbb{R}$,

$$\left(a^2+(p-1)\cdot b^2\right)^{1/2} \leq \left(\frac{|a+b|^p+|a-b|^p}{2}\right)^{1/p} \quad (\text{Ex.})$$

Proposition 2 (Hannersineq): If $f, g \in L_p$, $1 < p \leq 2$, then

$$(\|f\|_p - \|g\|_p)^p + (\|f\|_p + \|g\|_p)^p \leq \|f+g\|_p^p + \|f-g\|_p^p.$$

Lemma: For $r \in [0,1]$, let $\alpha(r) := (1+r)^{p-1} + (1-r)^{p-1}$

and $\beta(r) := \dfrac{(1+r)^{p-1} - (1-r)^{p-1}}{r^{p-1}}$

Then, $\forall A, B \in \mathbb{R}$

$$\max_{r \in [0,1]} \{\alpha(r)\cdot|A|^p + \beta(r)\cdot|B|^p\} \leq |A+B|^p + |A-B|^p \quad (\text{Ex.})$$

Proof of Hannersineq: $\|f\|_p \geq \|g\|_p$ and let:

$A := |f(x)|$, $B := |g(x)|$

$r = \dfrac{\|g\|_p}{\|f\|_p} \in [0,1]$

Applying Lemma:

$$|f(x)+g(x)|^p + |f(x)-g(x)|^p \geq \left\{\left(1+\frac{\|g\|_p}{\|f\|_p}\right)^{p-1} + \left(1-\frac{\|g\|_p}{\|f\|_p}\right)^{p-1}\right\}\cdot$$

$$\cdot |f(x)|^p + \frac{\left(1+\frac{\|g\|_p}{\|f\|_p}\right)^{p-1} - \left(1-\frac{\|g\|_p}{\|f\|_p}\right)^{p-1}}{\left(\|g\|_p/\|f\|_p\right)^{p-1}} \cdot |g(x)|^p$$

$$\|f+g\|_p^p+\|f-g\|_p^p \geq \{(\|f\|_p+\|g\|_p)^{p-1}+(\|f\|_p-\|g\|_p)^{p-1}\}\cdot\|f\|_p$$
$$+\{(\|f\|_p+\|g\|_p)^{p-1}-(\|f\|_p-\|g\|_p)^{p-1}\}\cdot\|g\|_p =$$
$$=(\|f\|_p+\|g\|_p)^p+(\|f\|_p-\|g\|_p)^p \qquad \blacksquare$$

Proof of Thm 2:

Let $x,y\in L_p$, $1<p\leq 2$ and consider $f=\frac{x+y}{2}$, $g=\frac{x-y}{2}$. We'll show:

$$(\|f\|_p^2+(p-1)\cdot\|g\|_p^2)\leq\left(\frac{\|f+g\|_p^2+\|f-g\|_p^2}{2}\right)^{1/2}$$

Then:

$$LHS\leq\left(\frac{(\|f\|_p+\|g\|_p)^p+(\|f\|_p-\|g\|_p)^p}{2}\right)^{1/p} \overset{\text{Hannersiney}}{\leq}$$
$$\leq\left(\frac{\|f+g\|_p^p+\|f-g\|_p^p}{2}\right)^{1/p} \underset{\|\cdot\|_{L_p(\mathbb{P})}\leq\|\cdot\|_{L_2(\mathbb{P})}}{\leq} \left(\frac{\|f-g\|_p+\|f+g\|_p}{2}\right)^{1/2}$$

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§ Martingales in Banach Spaces:

Definition: Let X be a Banach Space. A sequence $\{M_k:\{-1,1\}^k\to X\}_{k=0}^{n}$ is a X-valued martingale iff $(\forall k=1,\dots,n \wedge \forall(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{k-1})\in\{-1,1\}^{k-1})$.

$$(M_{k-1}(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{k-1})=\frac{M_k(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{k-1},1)+M_k(\varepsilon_1,\dots,\varepsilon_{k-1},-1)}{2}$$

If $x_1, x_2, \dots, x_n \in X$, $M_0 := 0$, $M_k = \sum_{i=1}^{k} \varepsilon_i x_i$ is a martingale.

Rmk (Ex.): If $X = \mathbb{R}$ and $\{M_k\}_{k=0}^{n}$ is a martingale then:
$$\mathbb{E}_\varepsilon\left[|M_n(\varepsilon) - M_0|^2\right] = \sum_{k=1}^{n} \mathbb{E}_\varepsilon\left[|M_k(\varepsilon) - M_{k-1}(\varepsilon)|^2\right]$$

Definition: A Banach space X has martingale type p, if-f $\exists T>0$:

$\forall X$-valued martingale $\{M_k\}_{k=0}^{n}$, we have
$$\mathbb{E}\left[\|M_n - M_0\|_X^p\right] \leq T^p \cdot \sum_{k=1}^{n} \mathbb{E}\left[\|M_k - M_{k-1}\|_X^p\right]$$

Similarly, X has martingale cotype q if-f $\exists C>0$: $\forall$ X-valued $\{M_k\}_{k=0}^{n}$
$$\mathbb{E}\left[\|M_n - M_0\|_X^q\right] \geq \frac{1}{C^q} \cdot \sum_{k=1}^{n} \mathbb{E}\left[\|M_k - M_{k-1}\|_X^q\right];$$

Thm (Pisier): If a space X is p-smooth (resp. q-convex) then it has martingale type p (resp. cotype q) with constant $S_p(X)$ (resp. $K_q(X)$).

Proof of type: Recall that $(\forall x, y \in X)$:
$$\frac{\|x\|^p + \|y\|^p}{2} \leq \left\|\frac{x+y}{2}\right\|^p + S_p(X)^p \cdot \left\|\frac{x-y}{2}\right\|^p$$

For X-valued $\{M_k\}_{k=0}^{n}$ with $M_0 := 0$, we have. $\rightarrow$

$$\mathbb{E}[\|\mathcal{U}_n\|^p] = \mathbb{E}_{\vec{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_n)}\left[\frac{\|\mathcal{U}_n(\vec{\varepsilon},1)\|^p+\|\mathcal{U}_n(\vec{\varepsilon},-1)\|^p}{2}\right]$$

$$\leq \mathbb{E}_{\vec{\varepsilon}}\left[\left\|\frac{\mathcal{U}_n(\vec{\varepsilon},1)+\mathcal{U}_n(\vec{\varepsilon},-1)}{2}\right\|^p + S^p\cdot\left\|\frac{\mathcal{U}_n(\vec{\varepsilon},1)-\mathcal{U}_n(\vec{\varepsilon},-1)}{2}\right\|^p\right]$$

$$= \mathbb{E}[\|\mathcal{U}_{n-1}\|^p] + S_p(X)^p\cdot\mathbb{E}[\|\mathcal{U}_n-\mathcal{U}_{n-1}\|^p]$$

tensorize:

$$\leq \underbrace{\cancel{\mathbb{E}[\|\mathcal{U}_0\|^p]}}_{\to 0} + S_p(X)^p\cdot\sum_{k=1}^{n}\mathbb{E}[\|\mathcal{U}_k-\mathcal{U}_{k-1}\|^p]$$

∎

- Corollary 1: If $1<q\leq 2$, $\mathbb{E}[\|\mathcal{U}_n-\mathcal{U}_0\|_{L_q}^q]\leq$

$$\leq\sum_{k=1}^{n}\mathbb{E}[\|\mathcal{U}_k-\mathcal{U}_{k-1}\|_{L_q}^q]$$

- $\mathbb{E}[\|\mathcal{U}_n-\mathcal{U}_0\|_{L_q}^2]\geq(q-1)\cdot\sum_{k=1}^{n}\mathbb{E}[\|\mathcal{U}_k-\mathcal{U}_{k-1}\|_{L_q}^2]$

- If $2\leq q<+\infty$, $\mathbb{E}[\|\mathcal{U}_n-\mathcal{U}_0\|_{L_q}^q]\geq\sum_{k=1}^{n}\mathbb{E}[\|\mathcal{U}_k-\mathcal{U}_{k-1}\|_{L_q}^q]$

and

$$\mathbb{E}[\|\mathcal{U}_n-\mathcal{U}_0\|_{L_q}^2]\leq(q-1)\cdot\sum_{k=1}^{n}\mathbb{E}[\|\mathcal{U}_k-\mathcal{U}_{k-1}\|_{L_q}^2]$$

Corollary 2: Let $x_1,x_2,\dots,x_n\in L_q$. If $q\leq 2$

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|_{L_q}^q\right]\leq\sum_{i=1}^{n}\|x_i\|_{L_q}^q,$$

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|_{L_q}^2\right]\geq(q-1)\cdot\sum_{i=1}^{n}\|x_i\|_{L_q}^2 \quad \text{(Rademacher cotype 2)}$$

- If $q \geq 2$:

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^{n} \varepsilon_i \cdot x_i\Big\|_{L_q}^{q}\Big] \geq \sum_{i=1}^{n} \|x_i\|_q^q \quad \text{(Rademacher cotype } q\text{)}.$$

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^{n} \varepsilon_i \cdot x_i\Big\|_{L_q}^{2}\Big] \leq (q-1) \cdot \sum_{i=1}^{n} \|x_i\|_{L_q}^2 \quad \text{(R-type 2)}$$

Recall:

$$c_q^{lin}(\ell_p^n) \underset{p,q}{\asymp} \begin{cases} n^{\frac{1}{p}-\frac{1}{q}}, & 1 \leq p \leq q \leq 2 \\ n^{\frac{1}{p}-\frac{1}{2}}, & 1 \leq p \leq 2 \leq q \end{cases} \left(\text{type range}\right)$$
$$\begin{cases} n^{\frac{1}{q}-\frac{1}{p}}, & 2 \leq q \leq p \\ n^{\frac{1}{2}-\frac{1}{p}}, & q \leq 2 \leq p. \end{cases} \left(\text{cotype range}\right)$$

Proof of $1 \leq p \leq 2 \leq q$: Let $T: \ell_p^n \to L_q$. Take: $x_i = Te_i \in L_q$. Then type 2 in $L_q \Rightarrow$

$$(**)\quad \mathbb{E}\Big[\Big\|\sum_{i=1}^{n} \varepsilon_i \cdot Te_i\Big\|_{L_q}^{2}\Big] \leq (q-1) \cdot \sum_{i=1}^{n} \|Te_i\|_{L_q}^2 \leq$$

$$\leq (q-1) \cdot n \cdot \|T\|^2, \quad (**): \frac{n^{2/p}}{\|T^{-1}\|^2} \leq \dots$$

$$\Rightarrow \|T\|^2 \cdot \|T^{-1}\|^2 \geq \frac{n^{\frac{2}{p}-1}}{q-1} \Rightarrow c_p^{lin}(\ell_p^n) \geq \frac{n^{\frac{1}{p}-\frac{1}{2}}}{\sqrt{q-1}} \quad \blacksquare$$

Fact: L_1 has R-type 2 with constant $\frac{1}{\sqrt{2}}$, but not mart. cotype 2

Q: How well can the finite metric space $(\{1,2,\dots,m\}^n, \|\cdot\|_p)$ be embedded in L_q?

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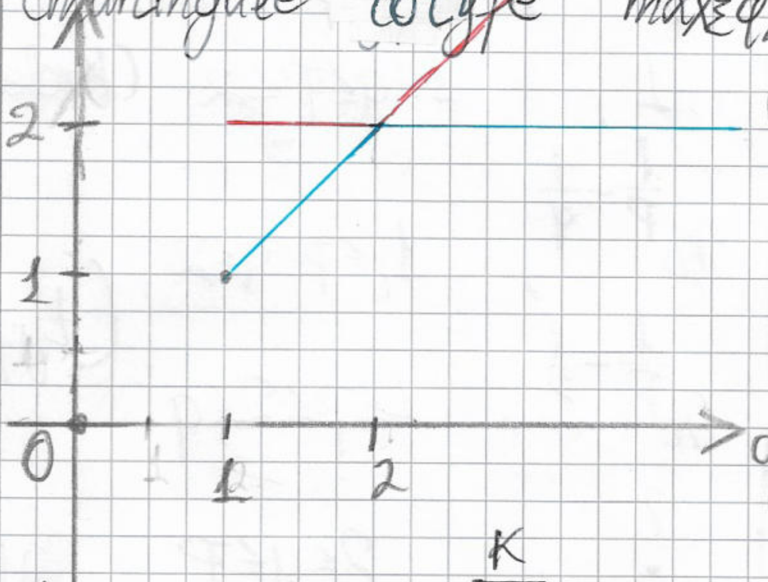
Yesterday: If $\{u_k\}_{k=0}^{n}$ is an L_q-valued martingale then

$$\mathbb{E}\left[\|u_n - u_0\|_{L_q}^{\min\{q,2\}}\right] \le \max\{q-1,1\} \cdot \sum_{k=1}^{n} \mathbb{E}\|u_k - u_{k-1}\|^{\min\{q,2\}}$$

(martingale type $\min\{q,2\}$).

$$\mathbb{E}\left[\|u_n - u_0\|_{L_q}^{\max\{q,2\}}\right] \ge \min\{q-1,1\} \cdot \sum_{k=1}^{n} \mathbb{E}\|u_k - u_{k-1}\|_{L_q}^{\max\{q,2\}}$$

(martingale cotype $\max\{q,2\}$).



(type: is a parameter which gets better as q increases)

(cotype: on the other hand, gets better as q is lower).

Applied to $u_k = \sum_{j=1}^{k} \varepsilon_j x_j \rightarrow$ Rademacher type/cotype

$\Rightarrow$ bounds for linear distortion $C_q^{lin}(\ell_p^n)$

• Question: How well can $(\{1,2,\dots,m\}^n, \|\cdot\|_p)$ be embedded into L_q?

Notation: Given metric spaces $(\mathcal{M}, d_{\mathcal{M}})$, (N, d_N)
$f: \mathcal{M} \to N$ is a D-isomorphic embedding iff
$\exists \sigma > 0$:

$$(\forall x, y \in \mathcal{M})(\sigma \cdot d_{\mathcal{M}}(x,y) \le d_N(f(x), f(y)) \le \sigma \cdot D \cdot d_{\mathcal{M}}(x,y))$$

Denote $c_N(\mathcal{M}) :=$ the least such $D =$
$= \inf\{ \|f\|_{Lip} \cdot \|f^{-1}\|_{Lip} \mid f: \mathcal{M} \to N\}$.

Question: Quantify $c_q([m]_p^n)$

Thm:

$$c_q([m]_p^n) \underset{p,q}{\asymp} \begin{cases} 1, & 1 \le q \le p \le 2 \text{ (Kadec)}. \\ n^{\frac{1}{p} - \frac{1}{q}}, & 1 \le p < q \le 2 \\ n^{\frac{1}{p} - \frac{1}{2}}, & 1 \le p \le 2 < q \end{cases} \text{(type)}$$
$$\begin{cases} ?, & 2 \le q \le p \\ \min\{ n^{\frac{1}{2} - \frac{1}{p}}, m^{1 - \frac{2}{p}}\}, & 1 \le q < 2 \le p \end{cases} \text{(cotype)}$$
$$\min\{ n^{\frac{(q-p)(p-2)}{p^2(q-2)}}, m^{1 - \frac{2}{q}}\}, \quad 2 \le p < q \quad (X_p)$$

For $(*)$ $2 \le q \le p$,

$$\min\{ n^{\frac{1}{q} - \frac{1}{p}}, m^{1 - \frac{q}{p}}\} \underset{p,q}{\lesssim} c_q([m]_p^n) \underset{p,q}{\lesssim} \min\{ n^{\frac{1}{q} - \frac{1}{p}}, m^{1 - \frac{2}{p}}\}.$$

Goal: Prove the lower bounds.

§ Metric Type

Definition (Enflo, '69) A metric space $(\mathcal{U}, d)$ has Enflo type p iff $\exists T>0 : \forall n \in \mathbb{N}$ $f:\{-1,1\}^n \to \mathcal{U}$, we have:

$$\mathbb{E}_\varepsilon[d(f(\varepsilon), f(-\varepsilon))^p] \le T^p \sum_{k=1}^{n} \mathbb{E}_\varepsilon[d(f(\varepsilon), f(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k-1}, -\varepsilon_k, \dots, \varepsilon_n)$$

(we are flipping one point).

Rmk: If $\mathcal{U} = X$ is a normed space,

$$f(\varepsilon) = \sum_{i=1}^{n} \varepsilon_i \cdot x_i,$$

$$LHS := 2^p \cdot \mathbb{E}\left[\left\|\sum_{i=1}^{n} \varepsilon_i \cdot x_i\right\|^p\right], \quad RHS := 2^p \cdot \sum_{i=1}^{n} \|x_i\|^p.$$

Enflo type $\Rightarrow$ Rademacher type.

Theorem (Oleszkiewicz, '96):

Martingale type $\Rightarrow$ Enflo type

Proof: Fix $n \in \mathbb{N}$, $f:\{-1,1\}^n \to X$, where X has martingale type p with const T

$\to$ Def: $\mathcal{U}_k : \{-1,1\}^k \to X$, given by

$$\mathcal{U}_k(\varepsilon) = \mathbb{E}_{\delta_{k+1}, \dots, \delta_n}[f(\varepsilon_1, \dots, \varepsilon_k, \delta_{k+1}, \dots, \delta_n)]$$

Then: $\mathbb{E}\|\mathcal{U}_n - \mathcal{U}_0\|_X^p = \mathbb{E}\|f - \mathbb{E}f\|_X^p \le T^p \sum_{k=1}^{n} \mathbb{E}\|\mathcal{U}_k - \mathcal{U}_{k-1}\|_X^p$

but $\mathcal{U}_k(\varepsilon) - \mathcal{U}_{k-1}(\varepsilon) = \mathbb{E}_{\delta_{k+1}, \dots, \delta_n}[f(\varepsilon,$

$\longrightarrow$

but

$$u_k(\varepsilon)-u_{k-1}(\varepsilon)=\mathbb{E}_{\delta_{k+1},\dots,\delta_n}\Big[f(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k,\delta_{k+1},\dots,\delta_n)-$$

$$-\mathbb{E}_{\delta_k}\Big[f(\varepsilon_1,\dots,\varepsilon_{k-1},\delta_k,\dots,\delta_n)\Big]$$

$$\frac{f(\varepsilon_1,\dots,\varepsilon_k,\delta_{k+1},\dots,\delta_n)+f(\varepsilon_1,\dots,\varepsilon_{k-1},-\varepsilon_k,\delta_{k+1},\dots,\delta_n)}{2}$$

$$=\frac{1}{2}\cdot\mathbb{E}_{\delta_k,\dots,\delta_n}\Big[f(\varepsilon_1,\dots,\varepsilon_k,\delta_{k+1},\dots,\delta_n)-f(\varepsilon_1,\dots,\varepsilon_{k-1},-\varepsilon_k,$$

$$\delta_{k+1},\dots,\delta_n)\Big]$$

$$\Rightarrow \mathbb{E}\Big[\|u_k-u_{k-1}\|^p\Big]\underset{\text{Jensen}}{\leq}\frac{1}{2^p}\cdot\mathbb{E}\|f(\varepsilon)-f(\varepsilon_1,\dots,\varepsilon_{k-1},-\varepsilon_k,\varepsilon_{k+1},$$

$$\dots,\varepsilon_n)\|_X^p$$

So:

$$\mathbb{E}\Big[\|f(\varepsilon)-f(-\varepsilon)\|^p\Big]\underset{\text{tr. ineq}}{\leq}2^p\cdot\mathbb{E}\Big[\|f-\mathbb{E}f\|_X^p\Big]\leq$$

$$\leq T^p\cdot\sum_{k=1}^n\mathbb{E}\Big[\|f(\varepsilon)-f(\varepsilon_1,\dots,\varepsilon_{k-1},-\varepsilon_k,\dots,\varepsilon_n)\|_X^p\Big] \quad \blacksquare$$

Rmk: Ivanisvili – von Handel – Volberg (2020) showed.
Rademacher type $\Rightarrow$ Enflo type.

Proof of distortion: Take $1\leq p\leq q\leq 2$.

Then:

$$c_q([m]_p^n)\geq c_q([2]^n)=c_q(\{-1,1\}^n).$$

Take $f:(\{-1,1\}^n,\|\cdot\|_p)\to L_q$. Then by type q in L_q

→

$$\frac{1}{\|f^{-1}\|_{Lip}^{q}}\cdot n^{q/p} \le \mathbb{E}\|f(\varepsilon)-f(-\varepsilon)\|_{L_q}^{q} \le \sum_{k}\mathbb{E}[d(f(\varepsilon), f(\varepsilon_{k-1}, -\varepsilon_k, \dots, \varepsilon_n$$

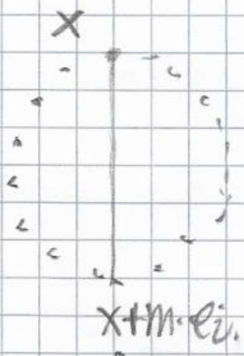
$$\le n\cdot\|f\|_{Lip}^{q} \Rightarrow \|f\|_{Lip}\cdot\|f^{-1}\|_{Lip} \ge n^{1/p-1/q}. \quad \blacksquare$$

Metric World: The duality disappears even on the definition

§ Metric Cotype

• Definition (Mendel–Naor, 2008): A metric space (M,d) has metric cotype q iff
$\exists c>0 : (\forall n\in\mathbb{N})(\exists m=m(n,M)\in\mathbb{N})$:
any $f:\mathbb{Z}_{2m}^{n}\to M$ satisfies:

$$\sum_{i=1}^{n}\mathbb{E}_{x\sim\mathbb{Z}_{2m}^{n}}[d(f(x), f(x+m\cdot e_i))^{q}] \le c^{q}m^{q}\,\mathbb{E}_{x}\,\mathbb{E}_{\varepsilon\sim\{-1,1\}^{n}}[d(f(x+\varepsilon), f(x$$



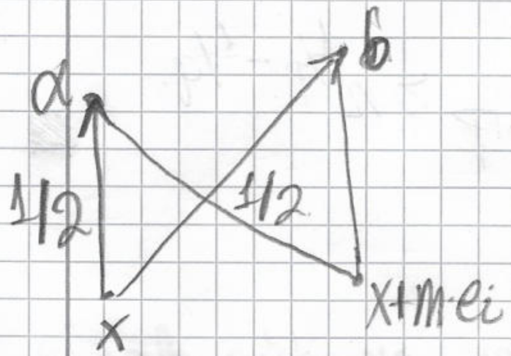
Remark: If (M,d) has a metric cotype of q,
$|M|\ge 2 \Rightarrow m(n,M)\gtrsim n^{1/q}$

Proof: Take $f:\mathbb{Z}_{2m}^{n}\to M$ a random function s.t. $(\forall x\in\mathbb{Z}_{2m}^{n})$

$$f(x)=\begin{cases} a, & \text{w.p. } \frac{1}{2} \\ b, & \text{w.p. } \frac{1}{2}\end{cases} \quad \text{(independent over}$$

$\mathbb{E}_f[LHS] \sim n \cdot d(a,b)^q$, $\mathbb{E}_f[RHS] \sim c^q \cdot m^q \cdot d(a,b)^q \Rightarrow$

$\Rightarrow m^q \gtrsim n$ $\blacksquare$



Thm (Giladi-Mendel-Naor, '11)

Rademacher cotype $q \Rightarrow$ metric cotype q with $m \simeq n^{1+\frac{1}{q}}$

Open ?? $m \asymp n^{1/q}$?? under Rademacher cotype q.

Thm (E.-Mendel-Naor, '19): martingale cotype q with $m \asymp n^{1/q}$

Metric Embeddings IV [After Break]

Proof (of the last Theorem):

For $h:\{-1,1\}^n \to X$, let

$(E_i h)(\varepsilon) = \mathbb{E}_{\delta_{i+1},\dots,\delta_n}[h(\varepsilon_1,\dots,\varepsilon_i,\delta_{i+1},\dots,\delta_n)]$: $\{E_i h\}_{i=0}^n$ is a martingale.

Let $f:\mathbb{Z}_{4m}^n \to X$. Given $x \in \mathbb{Z}_{4m}^n$, let

$f_x:\{-1,1\}^n \to X$, by $f_x(\varepsilon)=f(x+\varepsilon)$.

Then $LHS = \sum_{i=1}^n \mathbb{E}_{x,\varepsilon}\|f(x+\varepsilon+2me_i)-f(x+\varepsilon)\|^q =$

$= \sum_{i=1}^n \mathbb{E}\|f_{x+2me_i}(\varepsilon)-f_x(\varepsilon)\|^q \le$

$\le \sum_{i=1}^n \mathbb{E}(\|f_{x+2me_i}(\varepsilon)-E_i f_{x+2me_i}(\varepsilon)\| + \|E_i f_{x+2me_i}(\varepsilon) -$

$- E_i f_x(\varepsilon)\| + \|E_i f_x(\varepsilon)-f_x(\varepsilon)\|)^q \lesssim$

$\lesssim_q \sum_{i=1}^n \mathbb{E}[\underbrace{\|E_i f_x(\varepsilon)-f_x(\varepsilon)\|^q}_{①\ \text{approx.}} + \underbrace{\|E_i f_{x+2me_i}(\varepsilon)-E_i f_x(\varepsilon)\|^q}_{②\ \text{smoothing}}]$

Bounding:

$(1) \lesssim_q \sum_{i=1}^n \{\underbrace{\mathbb{E}\|E_i f_x(\varepsilon)-f(x)\|^q}_{\le \mathbb{E}\|f_x(\varepsilon)-f(x)\|^q} + \mathbb{E}\|f_x(\varepsilon)-f(x)\|^q\} \le$

$E_i f_x$ martingale

$\lesssim n \cdot \mathbb{E}\|f(x+\varepsilon)-f(x)\|^q$

For term ②: apply triangle inequality:

$\forall \varepsilon \in \{-1,1\}^n$

$$\mathbb{E}_x \| E_i f_{x+2me_i}(\varepsilon) - E_i f_x(\varepsilon)\|^q \overset{\text{Hölder}}{\le}$$

$$\le m^{q-1} \cdot \sum_{k=1}^{m} \mathbb{E}_x \| E_i f_{x+2k\varepsilon_i e_i}(\varepsilon) - E_i f_{x+2(k-1)\varepsilon_i e_i}(\varepsilon) \|^q$$

Telescopic Sum.

$(y = x - 2(k-1)\varepsilon_i e_i - e_i)$

$$= m^q \cdot \mathbb{E}_x \| E_i f_{x-2\varepsilon_i e_i}(\varepsilon) - E_i f_x(\varepsilon)\|^q$$

translation inv.

- Key Claim:

$$E_i f_{x-2\varepsilon_i e_i}(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_i) = \mathbb{E}_{\delta_{i+1},\dots,\delta_n} f(\underbrace{x - 2\varepsilon_i e_i + (\varepsilon_1,\dots,\varepsilon_i,\delta_{i+1},\dots,\delta_n)}_{=\,x+(\varepsilon_1,\dots,\varepsilon_{i-1},-\varepsilon_i,\delta_{i+1},\dots,\delta_n)})$$

$$= \mathbb{E}_i f_x(\varepsilon_1,\dots,\varepsilon_{i-1},-\varepsilon_i)$$

$E_i f_x(\varepsilon)$

$E_{i-1} f_x(\varepsilon)$

$E_i f_x(\varepsilon_1,\dots,-\varepsilon_i)$

So: $\mathbb{E}_x \| E_i f_{x-2\varepsilon_i e_i}(\varepsilon) - E_i f_x(\varepsilon)\|^q =$

$$= 2^q \cdot \mathbb{E}_x \| E_i f_x(\varepsilon) - E_{i-1} f_x(\varepsilon)\|^q$$

So: $② \le (2m)^q \cdot \sum_{i=1}^{n} \mathbb{E}_{x,\varepsilon} \| E_i f_x(\varepsilon) - E_{i-1} f_x(\varepsilon)\|^q \le$

$$\le C^q (2m)^q \cdot \mathbb{E}_{x,\varepsilon} \| f_x(\varepsilon) - E_\delta f_x(\delta)\|^q \lesssim_q$$

$$\lesssim_q C^q m^q \cdot \mathbb{E} \| f(x+\varepsilon) - f(x)\|^q$$

■

§ Coarse Embeddings.

Def (Gromov): We say that $(\mathcal{U}, d_{\mathcal{U}})$ embeds coarsely into (N, d_N) iff

$\exists\ w, \Omega : [0,+\infty) \to [0,+\infty)$ and $f : \mathcal{U} \to N$:

$$w(d_{\mathcal{U}}(x,y)) \leq d_N(f(x), f(y)) \leq \Omega(d_{\mathcal{U}}(x,y)) \quad \forall x,y \in \mathcal{U}.$$

and $\lim_{t\to+\infty} w(t) = \infty$.

Cor: If $2<q<p$, L_p does not embed coarsely in L_q.

Proof: Suppose $F: L_p \to L_q$ coarse embedding.
Identify: $\mathbb{Z}_{2m}^n$ with $s[2m]^n \subseteq \ell_p$.
consider $f = F|_{\mathbb{Z}_{2m}^n} : \mathbb{Z}_{2m}^n \to L_q$. Then for $m \simeq n^{1/q}$

$$\sum_{i=1}^n \mathbb{E}_{x,\varepsilon}[\|f(x+me_i) - f(x)\|_{L_q}^q] \lesssim m^q \cdot \mathbb{E}_{x,\varepsilon}\|f(x+\varepsilon) - f(x)\|_{L_q}^q$$

$$\leq m^q \cdot \Omega(sn^{1/p})^q$$

$$\geq n \cdot w(sm) = \sum_{i=1}^n w(sn)^q$$

choose $m = n^{1/q}$, $s = n^{-1/p}$. $\quad n w(n^{\frac{1}{q}-\frac{1}{p}})^q \lesssim n\Omega(1)^q$

$\downarrow$
$+\infty$

contradiction ■

Parenthesis: $\forall p<q \quad L_p \xhookrightarrow{\text{coarsely}} L_q$

$1 \leq q \leq p \leq 2 \quad L_p \hookrightarrow L_q$

§ Metric Kwapien thm's.

· Thm (Kwapien '72) If a normed space has Rademacher type 2 and Rademacher cotype 2 $\Rightarrow$ it is isomorphic to Hilbert space

<u>Question</u>: ∃? Kwapien thm for metric spaces?

<u>Thm</u> (Alexandrov, '96): If a geodesic metric space embeds isometrically in a space of ≥ 0 sectional/Alexandrov curv. and a space of ≤ 0 Alexandrov/Sectional curv. $\Rightarrow$
$\Rightarrow$ M isometric to a convex subset of a Hilbert Space.

<u>Thm</u> (E.-Mendel-Naor, '24) ∃ a metric space which embeds bi-Lipschitzly into a space of
$\begin{cases} \geq 0 \text{ curvature} \rightsquigarrow \text{all Known notions of metric type 2.} \\ \text{and} \\ \leq 0 \text{ curvature} \rightsquigarrow \text{all Known notions of (sharp) metric cotype.} \end{cases}$

but does not embed coarsely in Hilbert Space.