Some inequalities about mixed volumes

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Abstract

We prove inequalities about the quermassintegrals $V_k(K)$ of a convex body K in $\mathbb{R}^n$ (here, $V_k(K)$ is the mixed volume $V((K,k),(B_n,n-k))$ where B_n is the Euclidean unit ball).

(i) The inequality

$$\frac{V_k(K+L)}{V_{k-1}(K+L)} \geq \frac{V_k(K)}{V_{k-1}(K)} + \frac{V_k(L)}{V_{k-1}(L)}$$

is true for every pair of convex bodies K and L in $\mathbb{R}^n$ if and only if $k=2$ or $k=1$.

ii) Let $0 \leq k \leq p \leq n$. Then, for every p-dimensional subspace E of $\mathbb{R}^n$,

$$\frac{V_{n-k}(K)}{|K|} \geq \frac{1}{\binom{n-p+k}{n-p}} \frac{V_{p-k}(P_E K)}{|P_E K|}.$$

where $P_E K$ denotes the orthogonal projection of K onto E. The proof is based on a sharp upper estimate for the volume ratio $|K|/|L|$ in terms of $V_{n-k}(K)/V_{n-k}(L)$, whenever L and K are two convex bodies in $\mathbb{R}^n$ such that $K \subseteq L$.

1 Introduction

Let $\mathcal{K}_n$ denote the class of all non-empty compact convex subsets of $\mathbb{R}^n$. If $K \in \mathcal{K}_n$ has non-empty interior, we will say that K is a convex body. We denote by $|K|$ the volume of a convex body K in $\mathbb{R}^n$. Generally there is no ambiguity, but when A is a convex body in a p-dimensional subspace of $\mathbb{R}^n$, $1 \leq p \leq n-1$, then $|A|$ means its p-dimensional volume.

In this paper we prove some inequalities about mixed volumes of convex bodies. Mixed volumes are introduced by a classical theorem of Minkowski which describes the way volume behaves with respect to the operations of addition and multiplication of convex bodies by nonnegative reals: If $K_1, \ldots, K_m \in \mathcal{K}_n$, $m \in \mathbb{N}$, then the volume of $t_1K_1 + \ldots + t_mK_m$ is a homogeneous polynomial of degree n in $t_i \geq 0$ (see [BZ], [Sc]). That is,

$$\left|t_1K_1 + \ldots + t_mK_m\right| = \sum_{1 \leq i_1, \ldots, i_n \leq m} V(K_{i_1}, \ldots, K_{i_n}) t_{i_1} \ldots t_{i_n},$$

where the coefficients $V(K_{i_1}, \ldots, K_{i_n})$ are chosen to be invariant under permutations of their arguments. The coefficient $V(K_{i_1}, \ldots, K_{i_n})$ is called the mixed volume of the n-tuple $(K_{i_1}, \ldots, K_{i_n})$.

Steiner's formula is a special case of Minkowski's theorem. Let B_n denote the Euclidean unit ball in $\mathbb{R}^n$. Then, the volume of $K + tB_n$, $t > 0$, can be expanded as a polynomial in t:

$$|K + tB_n| = \sum_{k=0}^{n} \binom{n}{k} V_{n-k}(K) t^k,$$

where $V_{n-k}(K) := V((K, n-k), (B_n, k))$ is the k-th quermassintegral of K.

The Aleksandrov-Fenchel inequality states that if $K, L, K_3, \ldots, K_n \in \mathcal{K}_n$, then

$$V(K, L, K_3, \ldots, K_n)^2 \geq V(K, K, K_3, \ldots, K_n) V(L, L, K_3, \ldots, K_n).$$

In particular, this implies that the sequence $(V_0(K), \ldots, V_n(K))$ is log-concave. A consequence of the Aleksandrov-Fenchel inequality is the Brunn-Minkowski inequality as well as the following generalization for the quermassintegrals:

$$V_k(K + L)^{1/k} \geq V_k(K)^{1/k} + V_k(L)^{1/k}, \tag{1}$$

for every $k = 1, \ldots, n$.

There is a close relationship between inequalities about quermassintegrals of convex bodies and inequalities about symmetric functions of positive reals or determinants of symmetric matrices. For example, an inequality of Bergstrom asserts that if A and B are symmetric positive definite matrices and A_i, B_i denote the submatrices obtained by deleting the i-th row and column, then

$$\frac{\det(A + B)}{\det(A_i + B_i)} \geq \frac{\det(A)}{\det(A_i)} + \frac{\det(B)}{\det(B_i)}.$$

Milman asked if there is a version of Bergstrom's inequality in the theory of mixed volumes. This question can be formulated as follows: For which values of k is it true that

$$\frac{V_k(K + L)}{V_{k-1}(K + L)} \geq \frac{V_k(K)}{V_{k-1}(K)} + \frac{V_k(L)}{V_{k-1}(L)} \tag{2}$$

for every pair of convex bodies K and L in $\mathbb{R}^n$? If true for all $k = 1, \ldots, n$, this would formally imply (1).

The same question (case $k=n$) was asked by Dembo, Cover and Thomas in [DCT], where the inequality

$$\frac{|K+L|}{|\partial(K+L)|} \geq \frac{|K|}{|\partial K|} + \frac{|L|}{|\partial L|}$$

is proposed as the dual of the Fisher information inequality

$$J(X+Y)^{-1} \geq J(X)^{-1} + J(Y)^{-1}.$$

Here, $|\partial A|$ denotes the surface area of A, while $J(X)$ is the Fisher information of the random vector X in $\mathbb{R}^n$. It is worth mentioning that (2) holds true for every k when $L = rB_n$ (this is a simple consequence of the Aleksandrov-Fenchel inequality; see [GHP]). As we shall see in Section 2, the answer to the above question is negative in general. In fact, the only values of k for which (2) is always true are $k=2$ and $k=1$.

Theorem 1.1 *Let $1 \leq k \leq n$. Then, the inequality*

$$\frac{V_k(K+L)}{V_{k-1}(K+L)} \geq \frac{V_k(K)}{V_{k-1}(K)} + \frac{V_k(L)}{V_{k-1}(L)}$$

is true for every pair of convex bodies K and L in $\mathbb{R}^n$ if and only if $k=2$ or $k=1$.

It is an interesting question to describe the class $\mathcal{L}$ of compact convex subsets L of $\mathbb{R}^n$ for which (2) holds true for every convex body K. In particular, if line segments belonged to this class, then taking $k=n$ and $L=[-\theta,\theta]$ where $\theta \in S^{n-1}$ we would have

$$\frac{|\partial(P_{\theta^\perp}K)|}{|P_{\theta^\perp}K|} \leq \frac{|\partial K|}{|K|} \tag{3}$$

for every convex body K in $\mathbb{R}^n$, where $P_{\theta^\perp}$ denotes the orthogonal projection onto $\theta^\perp$. In [GHP] it was conjectured that this inequality is correct. Moreover, it was proved that

$$\frac{|\partial(P_E K)|}{|P_E K|} \leq \frac{2(n-1)}{n}\frac{|\partial K|}{|K|}, \tag{4}$$

for every convex body K and every $(n-1)$-dimensional subspace E of $\mathbb{R}^n$.

In Section 4 we show that (3) is not true; this gives a negative answer to the question of Dembo, Cover and Thomas (alternative to the one in Theorem 1.1). In fact, the constant in (4) is optimal. Moreover, we present a generalization of this last inequality to subspaces of arbitrary dimension and quermassintegrals of any order.

Theorem 1.2 *Let K be a convex body in $\mathbb{R}^n$ and let $0 \leq k \leq p \leq n$. Then for every p-dimensional subspace E of $\mathbb{R}^n$, if $P_E K$ denotes the orthogonal projection of K onto E, we have*

$$\frac{V_{n-k}(K)}{|K|} \geq \frac{1}{\binom{n-p+k}{n-p}}\frac{V_{p-k}(P_E K)}{|P_E K|} = \frac{1}{\prod_{i=1}^{k}\left(1+\frac{n-p}{i}\right)}\frac{V_{p-k}(P_E K)}{|P_E K|}.$$

We show by examples that the constants in Theorem 1.2 are sharp, although there are no cases of equality. The proof of Theorem 1.2 is based on an inequality which estimates the volume ratio $|K|/|L|$ in terms of $V_{n-k}(K)/V_{n-k}(L)$, whenever L and K are two convex bodies in $\mathbb{R}^n$ such that $K \subseteq L$.

Theorem 1.3 *Let L and K be two convex bodies in $\mathbb{R}^n$ such that $K \subseteq L$. Then, for $1 \leq k < n$ we have*

$$\frac{|K|}{|L|} \leq \alpha_{n,k}\left(\frac{V_{n-k}(K)}{V_{n-k}(L)}\right),$$

where $\alpha_{n,k} : [0,1] \mapsto [0,1]$ is defined by

$$\alpha_{n,k}(t) = \binom{n}{k} \int_0^t \left(1 - s^{\frac{1}{n-k}}\right)^k ds.$$

The proof of Theorem 1.3, as well as examples showing that it is optimal, will be given in Section 3.

Basic references on classical convexity and the theory of mixed volumes are the books [BZ] and [Sc]. The reader may wish to consult [BB] for numerical and matrix inequalities related to the questions discussed in this work.

2 Mixed volumes of Minkowski sums

In this section we prove Theorem 1.1. Our first lemma is a consequence of the Aleksandrov-Fenchel inequality. Inequalities of this type may be found in [Sc, Section 6.4], but we reproduce a proof for completeness (the argument below is due to R. Schneider).

Lemma 2.1 *Let $\mathcal{C} = (K_3, \ldots, K_n)$ be an $(n-2)$-tuple of $K_j \in \mathcal{K}_n$. If $A, B \in \mathcal{K}_n$, we denote $V(A, B, \mathcal{C})$ by $V(A, B)$. Then, for all $A, B, C \in \mathcal{K}_n$ we have*

$$\begin{aligned}(V(B,A)V(C,A) - V(B,C)V(A,A))^2 \quad &\leq \quad [V(B,A)^2 - V(A,A)V(B,B)] \\ &\quad \times [V(C,A)^2 - V(A,A)V(C,C)]. \qquad \square\end{aligned}$$

Proof: By the Aleksandrov-Fenchel inequality, for all $t, s \geq 0$ we have

$$V(B + tA, C + sA)^2 - V(B + tA, B + tA)V(C + sA, C + sA) \geq 0$$

and

$$V(sB + tC, A)^2 - V(sB + tC, sB + tC)V(A, A) \geq 0.$$

Using the linearity of mixed volumes, from the first inequality we arrive at

$$\begin{aligned} 0 \quad \leq \quad & g(t,s) + t^2\left(V(C,A)^2 - V(A,A)V(C,C)\right) \\ & + s^2\left(V(B,A)^2 - V(A,A)V(B,B)\right) \\ & + 2ts\left(V(B,C)V(A,A) - V(B,A)V(C,A)\right), \end{aligned}$$

where g is a linear function of t and s. It follows that the quadratic term is non-negative and hence, either $V(B,C)V(A,A) > V(B,A)V(C,A)$ or its discriminant

$$\begin{aligned}(V(B,A)V(C,A) - V(B,C)V(A,A))^2 \quad - \quad & [V(B,A)^2 - V(A,A)V(B,B)] \\ & \times[V(C,A)^2 - V(A,A)V(C,C)]\end{aligned}$$

is non-positive.

Working in the same way with the second inequality, we arrive at

$$\begin{aligned}0 \quad \leq \quad & t^2(V(C,A)^2 - V(A,A)V(C,C)) + s^2(V(B,A)^2 - V(A,A)V(B,B)) \\ & +2ts(V(B,A)V(C,A) - V(B,C)V(A,A)).\end{aligned}$$

This shows that if $V(B,C)V(A,A) > V(B,A)V(C,A)$ then the discriminant of this second quadratic form (which is the same as before) is non-positive. Thus, the lemma is proved. □

From the previous lemma, we deduce the following inequality.

Proposition 2.1 *Let $\mathcal{C} = (K_3, \ldots, K_n)$ be an $(n-2)$-tuple of $K_j \in \mathcal{K}_n$. In the notation of Lemma 2.1, for all $A, B, C \in \mathcal{K}_n$ we have*

$$\frac{V(B+C,B+C)}{V(B+C,A)} \geq \frac{V(B,B)}{V(B,A)} + \frac{V(C,C)}{V(C,A)}.$$

Proof: From Lemma 2.1 and the arithmetic-geometric means inequality we get

$$\begin{aligned}& V(B,A)V(C,A) - V(B,C)V(A,A) \\ & \quad \leq \quad \big(V(B,A)^2 - V(A,A)V(B,B)\big)^{1/2}\big(V(C,A)^2 - V(A,A)V(C,C)\big)^{1/2} \\ & \quad \leq \quad \frac{1}{2} \times \frac{V(C,A)}{V(B,A)}\big(V(B,A)^2 - V(A,A)V(B,B)\big) \\ & \qquad\quad +\frac{1}{2} \times \frac{V(B,A)}{V(C,A)}\big(V(C,A)^2 - V(A,A)V(C,C)\big).\end{aligned}$$

Thus

$$2V(B,C) \geq \frac{V(C,A)}{V(B,A)} \times V(B,B) + \frac{V(B,A)}{V(C,A)} \times V(C,C) \ .$$

From this and the linearity of mixed volumes, we have

$$\begin{aligned}V(B+C,B+C) \quad & = \quad V(B,B) + 2V(B,C) + V(C,C) \\ & \geq \quad V(B,B)\left(1 + \frac{V(C,A)}{V(B,A)}\right) + V(C,C)\left(1 + \frac{V(B,A)}{V(C,A)}\right) \ ,\end{aligned}$$

which is the inequality

$$\frac{V(B+C,B+C)}{V(B+C,A)} \geq \frac{V(B,B)}{V(B,A)} + \frac{V(C,C)}{V(C,A)}. \quad \square$$

Setting $B = K$, $C = L$ and $A = K_3 = \ldots = K_n = B_n$, we immediately get the following.

Proposition 2.2 *Let L and K be two convex bodies in $\mathbb{R}^n$. Then,*

$$\frac{V_2(K+L)}{V_1(K+L)} \geq \frac{V_2(K)}{V_1(K)} + \frac{V_2(L)}{V_1(L)}. \qquad \square$$

We will next show that if $3 \leq k \leq n$, then the inequality

$$(*) \qquad \frac{V_k(K+L)}{V_{k-1}(K+L)} \geq \frac{V_k(K)}{V_{k-1}(K)} + \frac{V_k(L)}{V_{k-1}(L)}$$

is not true for all pairs of convex bodies L and K in $\mathbb{R}^n$.

Our proof will make use of tangential bodies. Let $0 \leq p \leq n-1$. If K and M are convex bodies in $\mathbb{R}^n$ and $M \subseteq K$, then K is called a p-tangential body of M if every $(n-p-1)$-extreme support plane of K is a support plane of M (we refer to [Sc, Section 2.2] for more details). It is easily seen that if $p < q \leq n-1$ then every p-tangential body of M is a q-tangential body of M. Tangential bodies of balls are closely related to the question of equality in the Aleksandrov-Fenchel inequalities for the quermassintegrals of convex bodies. A result which we will need is the following.

Fact. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$ and let $1 \leq k \leq n-1$. Then, we have*

$$V_k(K)^2 = V_{k+1}(K)V_{k-1}(K)$$

if and only if K is a $(k-1)$-tangential body of a ball.

We will also use the observation that for every $0 \leq p < n-2$ there exist centrally symmetric $(p+1)$-tangential bodies of a ball which are not p-tangential bodies of a ball.

Lemma 2.2 *Let $1 \leq k \leq n$. Assume that $(*)$ holds true for all convex bodies K and L in $\mathbb{R}^n$. Then, the function*

$$g(t) = \frac{V_k(K+tL)}{V_{k-1}(K+tL)}$$

is concave on $[0, +\infty)$ for every K and L. In particular, if $3 \leq k \leq n$, for every convex body K in $\mathbb{R}^n$ we have

$$\begin{aligned} &kV_{k-2}(K)\big(V_{k-1}^2(K) - V_k(K)V_{k-2}(K)\big) \\ &\quad \geq \ (k-2)V_k(K)\big(V_{k-2}^2(K) - V_{k-1}(K)V_{k-3}(K)\big). \end{aligned}$$

Proof: We check that

$$\begin{aligned} g\left(\frac{t+s}{2}\right) &= \frac{V_k((K+tL)/2 + (K+sL)/2)}{V_{k-1}((K+tL)/2 + (K+sL)/2)} \\ &\geq \frac{V_k((K+tL)/2)}{V_{k-1}((K+tL)/2)} + \frac{V_k((K+sL)/2)}{V_{k-1}((K+sL)/2)} \\ &= \frac{1}{2}\frac{V_k(K+tL)}{V_{k-1}(K+tL)} + \frac{1}{2}\frac{V_k(K+sL)}{V_{k-1}(K+sL)} \\ &= \frac{g(t)+g(s)}{2}. \end{aligned}$$

For the second assertion, let K be a convex body in $\mathbb{R}^n$, $n \geq 3$. For every $k \leq n$ we set $f_k(t) = V_k(K + tB_n)$. Then,

$$f_k(t+\varepsilon) = f_k(t) + \varepsilon k f_{k-1}(t) + O(\varepsilon^2),$$

and therefore

$$f_k'(t) = k f_{k-1}(t).$$

The derivative of the function

$$g_k(t) = \frac{f_k(t)}{f_{k-1}(t)} = \frac{V_k(K+tB_n)}{V_{k-1}(K+tB_n)}$$

is thus given by

$$g_k'(t) = k - (k-1)\frac{f_k(t) f_{k-2}(t)}{f_{k-1}^2(t)},$$

By the first part of the lemma, g_k is concave. This implies that $f_k f_{k-2}/f_{k-1}^2$ is an increasing function, and differentiating again we see that

$$k f_{k-1}^2 f_{k-2} + (k-2) f_k f_{k-1} f_{k-3} - 2(k-1) f_k f_{k-2}^2 \geq 0$$

on $(0, +\infty)$. This can be equivalently written in the form

$$k f_{k-2}(f_{k-1}^2 - f_k f_{k-2}) \geq (k-2) f_k (f_{k-2}^2 - f_{k-1} f_{k-3}).$$

Letting $t \to 0^+$, we conclude the lemma. □

Proposition 2.3 *Let $3 \leq k \leq n$. There exist convex bodies K and L in $\mathbb{R}^n$ for which $(*)$ is not true.*

Proof: Assume the contrary and let K be a centrally symmetric $(k-2)$-tangential body of a ball. Then, $V_{k-1}^2(K) - V_k(K)V_{k-2}(K) = 0$ and Lemma 2.2 implies that $V_{k-2}^2(K) - V_{k-1}(K)V_{k-3}(K) = 0$. This shows that K is a $(k-3)$-tangential body of a ball.

On the other hand, for every $0 \leq p < n-2$ there exist $(p+1)$-tangential bodies of a ball which are not p-tangential bodies of a ball. One can easily construct such an example by taking the convex hull of the ball and $2(p+1)$ suitably chosen points outside the ball. This leads to a contradiction. □

Finally, we observe that when $k=1$, then $(*)$ reduces to the inequality $V_1(A+B) \geq V_1(A) + V_1(B)$, which holds as an equality for every pair of convex bodies; mean width is linear with respect to Minkowski addition. This remark and Propositions 2.2 and 2.3 prove the following.

Theorem 2.1 *Let $1 \leq k \leq n$. Then, the inequality*

$$\frac{V_k(K+L)}{V_{k-1}(K+L)} \geq \frac{V_k(K)}{V_{k-1}(K)} + \frac{V_k(L)}{V_{k-1}(L)}$$

is true for every pair of convex bodies K and L in $\mathbb{R}^n$ if and only if $k=2$ or $k=1$.

Remark.
An interesting special case is when $n = 3$ and $k = 2$. If A and w denote surface area and mean width respectively, we obtain the inequality

$$\frac{A(K+L)}{w(K+L)} \geq \frac{A(K)}{w(K)} + \frac{A(L)}{w(L)}$$

for all convex bodies K and L in $\mathbb{R}^3$.

3 Comparison of the mixed volumes of K and L when $K \subseteq L$

Recall that, for every $1 \leq k < n$, the function $\alpha_{n,k} : [0,1] \mapsto [0,1]$ is defined by

$$\alpha_{n,k}(t) = \binom{n}{k} \int_0^t \left(1 - s^{\frac{1}{n-k}}\right)^k ds.$$

Proposition 3.1 *Let L and K be two convex bodies in $\mathbb{R}^n$ such that $K \subseteq L$. Let $1 \leq k < n$ and write P for the orthogonal projection onto an $(n-k)$-dimensional subspace E of $\mathbb{R}^n$. Then,*

$$\frac{|K|}{|L|} \leq \alpha_{n,k}\Big(\frac{|PK|}{|PL|}\Big).$$

Proof: Let F be the orthogonal subspace of E in $\mathbb{R}^n$. For notational convenience we may assume that $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k = E \times F$. Let L_1 and K_1 be the Schwarz symmetrals of L and K with respect to E. It is clear that $|L_1| = |L|$, $|K_1| = |K|$, $PL_1 = PL$ and $PK_1 = PK$. Moreover, there is a non-negative concave function $f : PL \to \mathbb{R}$ such that

$$L_1 = \{(x,y) : \ x \in PL, \ y \in f(x)\tilde{B_k}\}$$

where $\tilde{B_k}$ denotes the Euclidean ball of volume 1 in $\mathbb{R}^k$. Therefore, if we define

$$K' = \{(x,y) : \ x \in PK, \ y \in f(x)\tilde{B_k}\},$$

then it is clear that $K_1 \subseteq K'$ and thus

$$|K| = |K_1| \leq |K'|.$$

For $u \in [0, \max f]$, we define $\phi(u) = |\{f \geq u\}|$. Then, $\phi : [0, \max f] \mapsto [0, |PL|]$ is non-increasing and

(1) either there exists $t \in [0, \max f]$ such that $\phi(t) = |PK|$,
(2) or $\phi(u) > |PK|$ for all $u \in [0, \max f]$.

In case (1), by the definition of t we have $|PK\cap\{f<t\}| = |(PL\setminus PK)\cap\{f\ge t\}|$. Therefore,

$$\begin{aligned}
|K|\le|K'| &= \int_{PK} f^k(x)dx \\
&= \int_{PK\cap\{f\ge t\}} f^k(x)dx + \int_{PK\cap\{f<t\}} f^k(x)dx \\
&\le \int_{PK\cap\{f\ge t\}} f^k(x)dx + t^k|PK\cap\{f<t\}| \\
&= \int_{PK\cap\{f\ge t\}} f^k(x)dx + t^k|(PL\setminus PK)\cap\{f\ge t\}| \\
&\le \int_{PK\cap\{f\ge t\}} f^k(x)dx + \int_{(PL\setminus PK)\cap\{f\ge t\}} f^k(x)dx \\
&= \int_{\{f\ge t\}} f^k(x)dx.
\end{aligned}$$

It follows that if $K'' = \{(x,y):\ f(x)\ge t,\ y\in f(x)\tilde{B_k}\}$, then

$$|K'|\le|K''|.$$

Now,

$$|L| = \int_{PL} f^k(x)dx = k\int_{PL}\int_0^{f(x)} u^{k-1}dudx = k\int_0^{\max f}\phi(u)u^{k-1}du$$

and

$$\begin{aligned}
|K''| &= k\phi(t)\int_0^t u^{k-1}du + k\int_t^{\max f}\phi(u)u^{k-1}du \\
&= t^k\phi(t) + k\int_t^{\max f}\phi(u)u^{k-1}du.
\end{aligned}$$

By the Brunn-Minkowski inequality, $\phi^{\frac{1}{n-k}}(u) = |\{f\ge u\}|^{\frac{1}{n-k}}$ is concave and non-increasing on $[0,\max f]$. We set $\lambda = \left(\frac{|PK|}{|PL|}\right)^{\frac{1}{n-k}}$ and for every $u\ge 0$ we define

$$\psi(u) = \phi(0)^{\frac{1}{n-k}}\max\big(1-(1-\lambda)u/t, 0\big).$$

Then, ψ is affine on $[0,\frac{t}{1-\lambda}]$ and satisfies

$$\psi(0) = \phi(0)^{\frac{1}{n-k}} = |PL|^{\frac{1}{n-k}}$$

and

$$\psi(t) = \lambda\phi(0)^{\frac{1}{n-k}} = |PK|^{\frac{1}{n-k}} = \phi(t)^{\frac{1}{n-k}}.$$

Moreover, it is easy to see that $\psi^{n-k} \leq \phi$ on $[0,t]$, $\psi^{n-k} \geq \phi$ on $[t, \max f]$ and $\psi \geq 0$ on $[t, \frac{t}{1-\lambda}]$.
Thus, we get

$$\begin{aligned}
\frac{|L|}{|K|} \geq \frac{|L|}{|K''|} &= \frac{k\int_0^{\max f} u^{k-1}\phi(u)du}{t^k\phi(t) + k\int_t^{\max f} u^{k-1}\phi(u)du} \\
&= \frac{k\int_0^t u^{k-1}\phi(u)du + k\int_t^{\max f} u^{k-1}\phi(u)du}{t^k\phi(t) + k\int_t^{\max f} u^{k-1}\phi(u)du} \\
&\geq \frac{k\int_0^t u^{k-1}\psi^{n-k}(u)du + k\int_t^{\max f} u^{k-1}\phi(u)du}{t^k\phi(t) + k\int_t^{\max f} u^{k-1}\phi(u)du} \\
&= 1 + \frac{k\int_0^t u^{k-1}\psi^{n-k}(u)du - t^k\phi(t)}{t^k\phi(t) + k\int_t^{\max f} u^{k-1}\phi(u)du}.
\end{aligned}$$

Now, since ψ is non increasing, we have

$$k\int_0^t u^{k-1}\psi^{n-k}(u)du - t^k\phi(t) = k\int_0^t u^{k-1}\psi^{n-k}(u)du - t^k\psi^{n-k}(t) \geq 0$$

and since $\phi \leq \psi^{n-k}$ on $[t, \max f]$ we get

$$\begin{aligned}
\frac{|L|}{|K''|} &\geq 1 + \frac{k\int_0^t u^{k-1}\psi^{n-k}(u)du - t^k\psi^{n-k}(t)}{t^k\psi^{n-k}(t) + k\int_t^{+\infty} u^{k-1}\psi^{n-k}(u)du} \\
&= \frac{k\int_0^{+\infty} u^{k-1}\psi^{n-k}(u)du}{t^k\psi^{n-k}(t) + k\int_t^{+\infty} u^{k-1}\psi^{n-k}(u)du}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{|L|}{|K|} \geq \frac{|L|}{|K''|} &\geq \frac{k\int_0^{\frac{t}{1-\lambda}} \left(1 - \frac{(1-\lambda)u}{t}\right)^{n-k} u^{k-1}du}{t^k\lambda^{n-k} + k\int_t^{\frac{t}{1-\lambda}} \left(1 - \frac{(1-\lambda)u}{t}\right)^{n-k} u^{k-1}du} \\
&= \frac{1}{\binom{n}{k}\left((1-\lambda)^k\lambda^{n-k} + k\int_{1-\lambda}^1 (1-v)^{n-k}v^{k-1}dv\right)} \\
&= \frac{1}{\alpha_{n,k}(\lambda^{n-k})}
\end{aligned}$$

In case (2), we have $|\{f = \max f\}| > |PK|$. We consider a convex body A in $\mathbb{R}^k$ such that

$$A \subset \{f = \max f\} \text{ and } |A| = |PK|,$$

and we define

$$K'' = \{(x,y) : \ x \in A, \ y \in f(x)\tilde{B}_k\}.$$

Then $|K'| \leq |K''|$ and we may apply the same method as in case (1). $\square$

Theorem 3.1 *Let L and K be two convex bodies in $\mathbb{R}^n$ such that $K \subseteq L$. Then, for $1 \le k < n$ we have*

$$\frac{|K|}{|L|} \le \alpha_{n,k}\Big(\frac{V_{n-k}(K)}{V_{n-k}(L)}\Big).$$

In particular, for every $1 \le k \le n$, the following inequality holds :

$$\frac{V_{n-k}(K)}{|K|} \ge \frac{1}{\binom{n}{k}} \frac{V_{n-k}(L)}{|L|}$$

Proof: Since $\alpha_{n,k} : [0,1] \mapsto [0,1]$ is increasing, $\beta_{n,k} = \alpha_{n,k}^{-1}$ is increasing. Proposition 3.1 shows that for every orthogonal projection P onto an $(n-k)$-dimensional subspace of $\mathbb{R}^n$, we have

$$\alpha_{n,k}\Big(\frac{|PK|}{|PL|}\Big) \ge \frac{|K|}{|L|}.$$

It follows that

$$\frac{|PK|}{|PL|} \ge \beta_{n,k}\Big(\frac{|K|}{|L|}\Big),$$

that is

$$|PK| \ge \beta_{n,k}\Big(\frac{|K|}{|L|}\Big)\ |PL|.$$

Integrating over the Grassman manifold $G_{n,n-k}$ of all $(n-k)$-dimensional subspaces we get

$$V_{n-k}(K) \ge \beta_{n,k}\Big(\frac{|K|}{|L|}\Big) V_{n-k}(L).$$

Since $\alpha_{n,k} = \beta_{n,k}^{-1}$ we return to the desired result. On observing that actually $\alpha_{n,k}(t) \le \binom{n}{k} t$ for every $t \in [0,1]$, we obtain the second inequality. □

Cases of equality: We will show by examples that the estimates in Proposition 3.1 and Theorem 3.1 are optimal.

1. *The inequality of Proposition 3.1 is sharp.*

Fix $\lambda = \Big(\frac{|PK|}{|PL|}\Big)^{\frac{1}{n-k}} \in (0,1)$. The proof shows that there is equality in Proposition 3.1 if and only if $\psi = \phi^{\frac{1}{n-k}}$ and $K' = K_1$. This is satisfied if and only if

(1) The Schwarz symmetral L_1 of L with respect to E is the convex hull of PL and $x_0 + aB$ for some $x_0 \in PL$ and some $a > 0$, where B denotes the Euclidean unit ball of $E^\perp$

(2) The Schwarz symmetral K_1 of K with respect to E is

$$K_1 = \{x \in L : Px \in x_0 + \lambda(PL - x_0)\}.$$

For example, these conditions are satisfied in the following situation: Let G be some $(n-k)$-dimensional subspace of $\mathbb{R}^n$ such that $\mathbb{R}^n = E^\perp \oplus G$ and let Q be the projection from $\mathbb{R}^n$ onto G parallel to $E^\perp$. Let D and C be two convex bodies in $E^\perp$ and G respectively. If $L = \text{conv}(D \cup C)$ and $K = \{x \in L : Qx \in \lambda C\}$, then (1) and (2) hold true. But this may happen in many other cases. In fact, there does not seem to exist a complete characterization of those convex bodies L which satisfy the following: for some k-dimensional subspace E of $\mathbb{R}^n$ and for some $x_0 \in PL$,

$$\binom{n}{k}|L| = |P_E L| \cdot |(E^\perp + x_0) \cap L|,$$

where P_E denotes the orthogonal projection onto E.

2. *The inequalities of Theorem 3.1 are sharp.*

We write $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ and denote the orthogonal projection onto $\mathbb{R}^{n-k}$ by P_{n-k} and the Euclidean unit ball of $\mathbb{R}^k$ by B_k. Given $b > 0$, let

$$L_b = \text{conv}\big((B_{n-k} \times \{0\}) \cup (\{0\} \times bB_k)\big).$$

For a fixed $r \in (0,1)$, let

$$K_b = \{x \in L_b : P_{n-k}x \in rB_{n-k}\}.$$

Then, for every $b > 0$

$$\frac{|L_b|}{|K_b|} = \frac{1}{\alpha_{n,k}(r^{n-k})}$$

and by Lemma 3.1 below,

$$\lim_{b \to 0} \frac{V_{n-k}(K_b)}{V_{n-k}(L_b)} = r^{n-k}.$$

This shows that Theorem 3.1 is optimal.

Lemma 3.1 *Let M be a convex body in $\mathbb{R}^n$ and let P_E be the orthogonal projection onto an $(n-k)$-dimensional subspace E. Let $b > 0$ and*

$$M_b = \{x + by : x \in E, y \in E^\perp, (x,y) \in M\}.$$

Then

$$\lim_{b \to 0} V_{n-k}(M_b) = c_{n,k}|P_E M|,$$

where $c_{n,k}$ depends only on n and k.

Proof: This follows from the continuity of mixed volumes and the fact that $M_b \to P_E M$ in the sense of Hausdorff. □

Remarks.

1. When $k = 1$, Theorem 3.1 shows that for any two convex bodies K, L in $\mathbb{R}^n$ such that $K \subseteq L$,

$$\frac{|\partial K|}{|K|} \geq \frac{1}{n}\frac{|\partial L|}{|L|}.$$

This was proved by Wills [W] as a consequence of the following fact: If M is a convex body in $\mathbb{R}^n$ and $r > 0$ is the inradius of M, then

$$\frac{|\partial M|}{n|M|} \leq \frac{1}{r} \leq \frac{|\partial M|}{|M|}.$$

2. When $k = n-1$, Proposition 3.1 shows that for $K \subseteq L$,

$$1 - \frac{w(K,\theta)}{w(L,\theta)} \leq \left(1 - \frac{|K|}{|L|}\right)^{1/n}$$

for every $\theta \in S^{n-1}$, where $w(M,\theta)$ denotes the width of M in the direction of θ. If we assume that K and L are centrally symmetric, this implies that for $K \subseteq L$,

$$\left(1 - \left(1 - \frac{|K|}{|L|}\right)^{1/n}\right) L \subseteq K,$$

a fact which appears already in [GMP].

3. The results of Theorem 3.1 can be extended to mixed volumes with zonoids, instead of quermassintegrals.

4 Comparison of the mixed volumes of a convex body and the mixed volumes of its projections

We begin with two simple lemmas.

Lemma 4.1 *Let K be a convex body in $\mathbb{R}^n$ and, for $1 \leq p \leq n-1$, let E be a p-dimensional subspace of $\mathbb{R}^n$ and $F = E^\perp$. Let P_F be the orthogonal projection onto F, and for every $y \in P_F K$ write*

$$K^y = \{x \in E : x + y \in K\}.$$

Then, for every $0 \leq k \leq p$ the following holds:

$$\binom{n}{k} V_{n-k}(K) \geq \binom{n}{k} V((K, n-k), (B_p(E), k)) = \binom{p}{k} \int_{P_F K} V_{p-k}(K^y) dy.$$

Proof: Let B_n be the Euclidean unit ball in $\mathbb{R}^n$ and $B_p(E) = B_n \cap E$ be the Euclidean unit ball of E. Since mixed volumes are increasing, we have

$$V_{n-k}(K) = V((K, n-k), (B_n, k)) \geq V((K, n-k), (B_p(E), k)).$$

Now, it easy to see that for $t \geq 0$

$$K + tB_p(E) = \{x' + y : y \in P_F K, x' \in K^y + tB_p(E)\}$$

and hence,

$$|K+tB_p(E)| = \int_{P_F K} |K^y + tB_p(E)| dy$$

For every $y \in P_F K$ we write

$$\begin{aligned}|K^y + tB_p(E)| &= \sum_{k=0}^{p} \binom{p}{k} V((K^y, p-k), (B_p(E), k)) t^k \\ &= \sum_{k=0}^{p} \binom{p}{k} V_{p-k}(K^y) t^k.\end{aligned}$$

Integrating, we get

$$|K+tB_p(E)| = \sum_{k=0}^{p} \binom{p}{k} \Big(\int_{P_F K} V_{p-k}(K^y) dy \Big) t^k,$$

and thus, for every integer k with $0 \le k \le p$ we have

$$\binom{n}{k} V_{n-k}(K) \ge \binom{n}{k} V((K, n-k), (B_p(E), k)) = \binom{p}{k} \int_{P_F K} V_{p-k}(K^y) dy. \qquad \square$$

The next lemma is in the spirit of Berwald's inequality [Be].

Lemma 4.2 *Let h be a concave function on $[0,1]$ such that $h(0) = 0$. Let C be a convex body in $\mathbb{R}^q$ and $\phi : C \to \mathbb{R}$ a non-negative function such that $\phi^{1/r}$ is concave on C for some $r > 0$ and $\sup_C \phi = 1$. Then,*

$$\int_C h(\phi(y)) dy \le \frac{\int_0^1 \left(1 - t^{1/r}\right)^q h'(t) dt}{\int_0^1 \left(1 - t^{1/r}\right)^q dt} \int_C \phi(y) dy$$

Proof: We write

$$\int_C h(\phi(y)) dy = \int_C \Big(\int_0^{\phi(y)} h'(t) dt \Big) dy = \int_0^1 |\{y \in C; \phi(y) \ge t\}| h'(t) dt.$$

Let $\psi(t) = |\{y \in C; \phi(y) \ge t\}|$. Then, ψ in non-increasing on $[0,1]$, and since $\phi^{1/r}$ is concave, using the Brunn-Minkowski inequality in $\mathbb{R}^q$, we get that the function $g : [0,1] \mapsto \mathbb{R}$ defined by $g(t) = \psi^{1/q}(t^r)$ is concave and non-increasing on $[0,1]$. Let

$$\gamma = \frac{\int_0^1 \psi(t) dt}{\int_0^1 \left(1 - t^{1/r}\right)^q dt}$$

and define ψ_1 on $[0,1]$ by

$$\psi_1(t) = \gamma \left(1 - t^{1/r}\right)^q.$$

Then,

$$\int_0^1 \psi(t)dt = \int_0^1 \psi_1(t)dt = \gamma \int_0^1 \left(1 - t^{1/r}\right)^q dt.$$

The function

$$F(s) = \int_0^s \left(\psi_1(t) - \psi(t)\right)\, dt$$

satisfies

$$F(0) = F(1) = 0$$

and

$$F'(s) = \psi_1(s) - \psi(s).$$

Claim: $F(s) \geq 0$ for every $s \geq 0$.

Proof of the claim: For $s \geq 0$, we define $g_1(s) = \psi_1^{1/q}(s^r)$. It is clear that g_1 is affine. We have seen that g is is concave and non-increasing, and this implies that $g_1 - g$ changes sign not more than once on $[0, 1]$. Since $0 = g_1(1) \leq g(1)$ and

$$r \int_0^1 \left(g_1^q(u) - g^q(u)\right) u^{r-1} du = \int_0^1 \left(\psi_1(t) - \psi(t)\right) dt = 0,$$

it is easy to see that there exists $u_0 \in [0,1]$ such that $g_1 \geq g$ on $[0, u_0]$ and $g_1 \leq g$ on $[u_0, 1]$. It follows that $\psi_1 \geq \psi$ on $[0, u_0]$ and $\psi_1 \leq \psi$ on $[u_0, 1]$. Since $F' = \psi_1 - \psi$ and $F(0) = F(1) = 0$, we conclude the proof of the claim. □

We now go back to the proof of the lemma: clearly,

$$\frac{\int_0^1 \left(1 - t^{1/r}\right)^q h'(t)dt}{\int_0^1 \left(1 - t^{1/r}\right)^q dt} \int_C \phi(y)dy - \int_C h(\phi(y))dy = \int \left(\psi_1(s) - \psi(s)\right) h'(s)ds$$

$$= \int_0^1 F'(s)h'(s)ds.$$

Since h is concave, h' is non-increasing. Since $F \geq 0$ and $F(0) = F(1) = 0$, we conclude using the second mean value theorem. □

Theorem 4.1 *Let K be a convex body in $\mathbb{R}^n$ and let $0 \leq k \leq p \leq n$. Then for every p-dimensional subspace E of $\mathbb{R}^n$, we have*

$$\frac{V_{n-k}(K)}{|K|} \geq \frac{1}{\binom{n-p+k}{n-p}} \frac{V_{p-k}(P_E K)}{|P_E K|}$$

where $P_E K$ denotes the orthogonal projection of K onto E.

Proof: Since quermassintegrals decrease by Schwarz symmetrization, we may replace K by its Schwarz symmetral with respect to E. In the notation of Lemma 4.1 we have

$$\binom{n}{k} V_{n-k}(K) \geq \binom{p}{k} \int_{P_F K} V_{p-k}(K^y)dy.$$

We define $\phi : P_F K \to \mathbb{R}$ by

$$\phi(y) = \frac{V_{p-k}(K^y)}{V_{p-k}(P_E K)}.$$

Then, $\sup_{P_F K} \phi = 1$ and

$$\binom{n}{k} V_{n-k}(K) \geq \binom{p}{k} V_{p-k}(P_E K) \int_{P_F K} \phi(y) dy.$$

By the Aleksandrov-Fenchel inequality, $\phi^{1/(p-k)}$ is concave on $P_F K$. We apply Lemma 4.2 with

$$C = P_F K \subset \mathbb{R}^{n-p}, \; q = n-p \; , r = p-k$$

and

$$h(t) = \alpha_{p,k}(t) = \binom{p}{k} \int_0^t \left(1 - s^{\frac{1}{p-k}}\right)^k ds$$

to get

$$\int_{P_F K} \alpha_{p,k}(\phi(y)) dy \leq \frac{\int_0^1 \left(1 - t^{1/r}\right)^q \alpha'_{p,k}(t) dt}{\int_0^1 \left(1 - t^{1/r}\right)^q dt} \int_{P_F K} \phi(y) dy.$$

After some computations, this inequality takes the form

$$\int_{P_F K} \phi(y) dy \geq \frac{\binom{n}{p-k}}{\binom{n-k}{p-k}\binom{p}{k}} \int_{P_F K} \alpha_{p,k}(\phi(y)) dy.$$

If follows that

$$\begin{aligned} V_{n-k}(K) &\geq \frac{\binom{p}{k}}{\binom{n}{k}} V_{p-k}(P_E K) \int_{P_F K} \phi(y) dy \\ &\geq \frac{\binom{p}{k}}{\binom{n}{k}} \frac{\binom{n}{p-k}}{\binom{n-k}{p-k}\binom{p}{k}} V_{p-k}(P_E K) \int_{P_F K} \alpha_{p,k}(\phi(y)) dy. \end{aligned}$$

For every $y \in P_F K$, the convex bodies K^y and $P_E K$ in $E (= \mathbb{R}^p)$ clearly satisfy

$$K^y \subset P_E K.$$

Applying Theorem 3.1, we get

$$\alpha_{p,k}\big(\phi(y)\big) = \alpha_{p,k}\Big(\frac{V_{p-k}(K^y)}{V_{p-k}(P_E K)}\Big) \geq \frac{|K^y|}{|P_E K|},$$

and hence,

$$\int_{P_F K} \alpha_{p,k}\big(\phi(y)\big) dy \geq \frac{1}{|P_E K|} \int_{P_F K} |K^y| dy = \frac{|K|}{|P_E K|}.$$

It follows that

$$V_{n-k}(K) \geq \frac{V_{p-k}(P_E K)}{\binom{n-p+k}{k}} \frac{|K|}{|P_E K|}.$$

The case $k = p$ follows form the fact that if $F = E^{\perp}$, then

$$\binom{n}{k} \frac{V_{n-k}(K)}{|K|} \geq \frac{|P_F K|}{|K|} |B_p|$$

by Lemma 4.1, and the observation that $|K| \leq |P_E K| \times |P_F K|$ and $|B_p| = V_0(P_E K)$. Since the cases $k = 0$ and $p = n$ are trivial, the proof is complete. □

Case of equality in Theorem 4.1.

The constants are sharp but there is no case of equality in the inequalities of this theorem.

To see this, we need to go back to the case of equality in Proposition 3.1 and Theorem 3.1, and see what happens in the lemmas 4.1 and 4.2 which were used in the proof of Theorem 4.1.

(i) If $\mathbb{R}^p = \mathbb{R}^k \times \mathbb{R}^{p-k}$, let $C \subset \mathbb{R}^k$ and $D \subset \mathbb{R}^{p-k}$ be two convex bodies with $0 \in C \cap D$. Denote the orthogonal projections from $\mathbb{R}^p$ onto $\mathbb{R}^k$ and $\mathbb{R}^{p-k}$ by P_k and P_{p-k}. For every $b > 0$, we define the convex body $L(b)$ in $\mathbb{R}^p$ to be the convex hull of $bC := bC \times 0$ and $D := 0 \times D$. Then,

$$L(b) = \{\lambda x + (1-\lambda) y : x \in bC, y \in D, 0 \leq \lambda \leq 1\}$$

For every $t \in [0,1]$, we also define

$$K_t(b) = \{z \in L(b); P_{p-k} z \in tD\}.$$

It is clear that $|P_{p-k}(K_t(b))| = t^{p-k}|D| = t^{p-k}|P_{p-k}(L(b))|$ for every $b > 0$. By the equality case in Proposition 3.1,

$$\frac{|K_t(b)|}{|L(b)|} = \alpha_{p,k}(t^{p-k}).$$

Now, when $b \to 0$, it is easy to see that

$$\frac{V_{p-k}\big(K_t(b)\big)}{V_{p-k}(L(b))} \to \frac{|P_{p-k}\big(K_t(b)\big)|}{|P_{p-k}\big(L(b)\big)|} = t^{p-k}$$

uniformly in $t \in [0,1]$. It follows that when $b \to 0$,

$$\frac{|L(b)|}{V_{p-k}\big(L(b)\big)} \frac{\int_0^1 t^{n-p-1} V_{p-k}\big(K_{1-t}(b)\big) dt}{\int_0^1 t^{n-p-1} |K_{1-t}(b)| dt} = \frac{\displaystyle\int_0^1 t^{n-p-1} \frac{V_{p-k}\big(K_{1-t}(b)\big)}{V_{p-k}\big(L(b)\big)} \, dt}{\int_0^1 t^{n-p-1} a_{p,k}\big((1-t)^{p-k}\big) dt}$$

$$\to \frac{\int_0^1 t^{n-p-1}(1-t)^{p-k}dt}{\int_0^1 t^{n-p-1}a_{p,k}\big((1-t)^{p-k}\big)dt}.$$

This means that

$$\frac{\int_0^1 t^{n-p-1}V_{p-k}\big(K_{1-t}(b)\big)dt}{\int_0^1 t^{n-p-1}|K_{1-t}(b)|dt} \sim_{b\to 0} \frac{V_{p-k}\big(L(b)\big)}{|L(b)|}\frac{\int_0^1 t^{n-p-1}(1-t)^{p-k}dt}{\int_0^1 t^{n-p-1}a_{p,k}\big((1-t)^{p-k}\big)dt}$$

and a short calculation gives

$$\frac{\int_0^1 t^{n-p-1}V_{p-k}\big(K_{1-t}(b)\big)dt}{\int_0^1 t^{n-p-1}|K_{1-t}(b)|dt} \sim_{b\to 0} \frac{\binom{n}{n-p+k}}{\binom{p}{k}\binom{n-k}{p-k}}\frac{V_{p-k}\big(L(b)\big)}{|L(b)|}.$$

So given $\varepsilon > 0$, we may choose b_0 small enough so that

$$\frac{\int_0^1 t^{n-p-1}V_{p-k}\big(K_{1-t}(b_0)\big)dt}{\int_0^1 t^{n-p-1}|K_{1-t}(b_0)|dt} \leq \sqrt{1+\varepsilon}\ \frac{\binom{n}{n-p+k}}{\binom{p}{k}\binom{n-k}{p-k}}\frac{V_{p-k}\big(L(b_0)\big)}{|L(b_0)|}.$$

(ii) Let now A be a convex body in $\mathbb{R}^{n-p}$ containing 0 in its interior, and consider the Minkowski functional

$$||w||_A = \inf\{\mu \geq 0 : w \in \mu A\}, \quad w \in \mathbb{R}^{n-p}.$$

Following the notation of **(i)**, we define a body M in $\mathbb{R}^n = \mathbb{R}^{n-p} \times \mathbb{R}^p$ by

$$M = \{w + z :\ w \in A, z \in K_{1-\|w\|_A}(b_0)\}.$$

Then M is convex. Indeed, we have

$$M = \mathrm{conv}(A \times P_{p-k}(L(b_0)), 0 \times L(b_0))$$

Now, for $a > 0$, we define a new convex body $M(a)$ in $\mathbb{R}^n$ by

$$M(a) = \{aw + z : w \in \mathbb{R}^{n-p}, z \in \mathbb{R}^p, w + z \in M\}.$$

Let $T_a : \mathbb{R}^n \mapsto \mathbb{R}^n$ be the linear mapping defined by $T_a(w) = aw$ if $w \in \mathbb{R}^{n-p} \times 0$ and $T_a z = z$ for $z \in 0 \times \mathbb{R}^p$. Then $M(a) = T_a(M)$ so that

$$\begin{aligned} \frac{V_{n-k}\big(M(a)\big)}{|M(a)|} &= \frac{V\big((T_a(M), n-k), (B_n, k)\big)}{|T_a(M)|} \\ &= \frac{V\big((M, n-k), (T_a^{-1}(B_n), k)\big)}{|M|}. \end{aligned}$$

If B_p denotes the Euclidean ball of $\mathbb{R}^p$,

$$T_a^{-1}(B_n) \to B_p$$

in the sense of Hausdorff as $a \to +\infty$. From the continuity of mixed volumes, we get

$$\binom{n}{k} \frac{V_{n-k}\big(M(a)\big)}{|M(a)|} \to \binom{n}{k} \frac{V\big((M, n-k), (B_p, k)\big)}{|M|}$$

From Lemma 4.1 and the definition of M we actually get

$$\begin{aligned} \binom{n}{k} \frac{V_{n-k}\big(M(a)\big)}{M(a)} &\to \binom{p}{k} \frac{\int_A V_{p-k}\big(K_{1-\|w\|_A}(b_0)\big)\, dw}{\int_A |K_{1-\|w\|_A}(b_0)|\; dw} \\ &= \binom{p}{k} \frac{\int_0^1 V_{p-k}\big(K_{1-t}(b_0)\big) t^{n-p-1} dt}{\int_0^1 |K_{1-t}(b_0)| t^{n-p-1} dt}. \end{aligned}$$

Now if P_p is the canonical projection from $\mathbb{R}^n = \mathbb{R}^{n-p} \times \mathbb{R}^p$ to $\mathbb{R}^p$, we have

$$P_p\big(M(a)\big) = L(b_0),$$

so that

$$\frac{V_{p-k}\Big(P_p\big(M(a)\big)\Big)}{|P_p\big(M(a)\big)|} = \frac{V_{p-k}\big(L(b_0)\big)}{|L(b_0)|}.$$

For a big enough, the estimate that we got in **(i)** shows that

$$\begin{aligned} \frac{V_{n-k}\big(M(a)\big)}{|M(a)|} &\le \sqrt{1+\varepsilon} \frac{\binom{p}{k}}{\binom{n}{k}} \; \frac{\int_0^1 V_{p-k}\big(K_{1-t}(b_0)\big) t^{n-p-1} dt}{\int_0^1 |K_{1-t}(b_0)| t^{n-p-1} dt} \\ &\le \frac{\binom{p}{k}}{\binom{n}{k}} \frac{\binom{n}{n-p+k}}{\binom{p}{k}\binom{n-k}{p-k}} (1+\varepsilon) \frac{V_{p-k}\Big(P_p\big(M(a)\big)\Big)}{|P_p\big(M(a)\big)|} \\ &= \frac{(1+\varepsilon)}{\binom{n+k-p}{k}} \frac{V_{p-k}\Big(P_p\big(M(a)\big)\Big)}{|P_p\big(M(a)\big)|}. \end{aligned}$$

(iii) The above discussion indicates that if the estimate of Theorem 4.1 is sharp, then the limiting bodies are degenerated in two different directions.

Remarks.

1. When $p = n-1$, Theorem 4.1 gives

$$\frac{V_{n-k}(K)}{|K|} \ge \frac{1}{k+1} \frac{V_{n-1-k}(P_H K)}{|P_H K|}$$

for every convex body K in $\mathbb{R}^n$ and every $(n-1)$-dimensional subspace H. For $k = 1$ this was proved in [GHP]. From this and Steiner's formula we have

$$\frac{|K + tB| - |tB|}{|K|} = \sum_{k=0}^{n-1} \binom{n}{k} \frac{V_{n-k}(K)}{|K|} \; t^k$$

$$
\begin{aligned}
&\geq \sum_{k=0}^{n-1} \binom{n}{k} \frac{V_{n-1-k}(P_H K)}{|P_H K|} \frac{t^k}{k+1} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \frac{V_{n-1-k}(P_H K)}{|P_H K|} \int_0^t s^k ds \ .
\end{aligned}
$$

Using $\binom{n}{k} \geq \binom{n-1}{k}$ we get

$$
\frac{|K+tB|}{|K|} \geq \frac{|K+tB|-|tB|}{|K|} \geq \int_0^t \frac{|P_H(K+sB)|}{|P_H K|} \ ds \ .
$$

2. Let E_{n-i}, $2 \leq i \leq n-1$, be a decreasing sequence of $(n-i)$-dimensional subspaces of $\mathbb{R}^n$. Write P_{n-i} for the orthogonal projection onto E_{n-i}. The results from [GHP] show that

$$
\frac{V_{n-1}(K)}{|K|} \geq \frac{1}{2} \frac{V_{n-2}(P_{n-1}K)}{|P_{n-1}K|}
$$

$$
\frac{V_{n-2}(P_{n-1}K)}{|P_{n-1}K|} \geq \frac{1}{2} \frac{V_{n-3}(P_{n-2}K)}{|P_{n-2}K|}
$$

$$
\cdots
$$

Therefore, for all $1 \leq q \leq n-2$, we have

$$
\frac{V_{n-1}(K)}{|K|} \geq \frac{1}{2^q} \frac{V_{n-1-q}(P_{n-q}K)}{|P_{n-q}K|}.
$$

Applying Theorem 4.1 directly, we get

$$
\frac{V_{n-1}(K)}{|K|} \geq \frac{1}{q+1} \frac{V_{n-1-q}(P_{n-q}K)}{|P_{n-q}K|},
$$

which is a better estimate.

3. It might be possible to generalize Theorem 4.1 as follows: Let $0 \leq l \leq k \leq p$ and let P_E be the orthogonal projection onto a p-dimensional subspace E of $\mathbb{R}^n$. Then,

$$
\frac{V_{n-k}(K)}{V_{n-l}(K)} \geq \frac{1}{\binom{n-p+k-l}{n-p}} \frac{V_{p-k}(P_E K)}{V_{n-l}(P_E K)}.
$$

Theorem 4.1 corresponds to the case $l=0$, while the case $p=n-1$ and $l=k-1$ was established in [GHP].

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