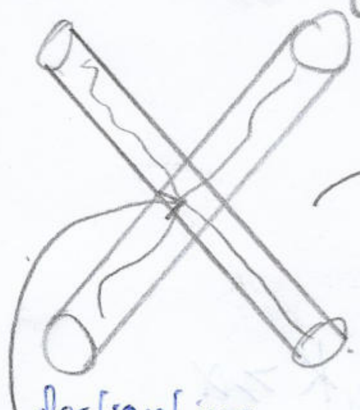
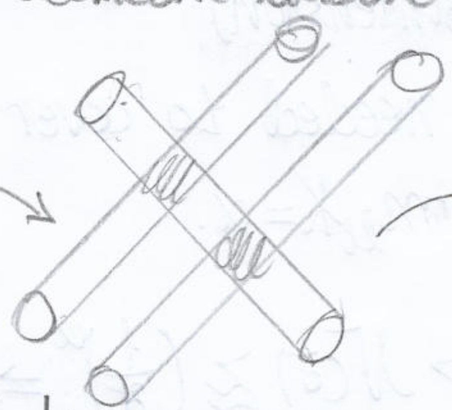
Μαρίνα Ηλιοπούλου | Marina Iliopoulou. 1/7/2024.

• Η πολυωνυμική Μέθοδος | The polynomial Method

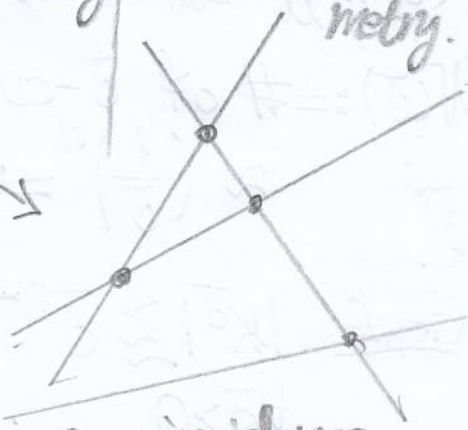
Harmonic Analysis. | Geometric Measure Theory | Incidence Geometry.



destructive interference.



Kakeya Conjecture
tubes rarely meet.



incidence geom. lines rarely meet. (2008)

Notation: $\cdot\; A_m \lesssim B_m \Leftrightarrow A_m \leq C B_m$ → depends only on dimension.

$\cdot\; A_m \gtrsim B_m \Leftrightarrow B_m \lesssim A_m$.

$\cdot\; A_m \sim B_m \Leftrightarrow cB_m \leq A_m \leq C_1 B_m \;\forall m$.

(Ex: $\binom{n}{2} = \frac{n(n-1)}{2} \sim n^2$, $10 \sim 1$) (We don't care about multiplicative constants, only for powers)

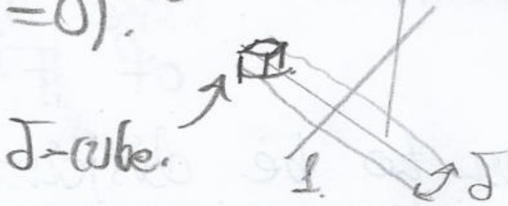
$\cdot\; A(\delta) \lessapprox B(\delta)$: $A(\delta) \leq C_1 \cdot (\log \frac{1}{\delta})^{c'} \cdot B(\delta)$

(Ex: For us, $\log\frac{1}{\delta} \approx 1$, $\log R \approx 1$).

Definition: A Kakeya set in $\mathbb{R}^n$ is a compact set containing a unit line segment in each direction.

S^{n-1}

(A Kakeya set can have Leb. measure $=0$).



Kakeya set conjecture: $\dim_H K = n$, $\dim_M K = n$.
($\dim_H$: Hausdorff dimension).
($\dim_M$: Minkowski dimension)

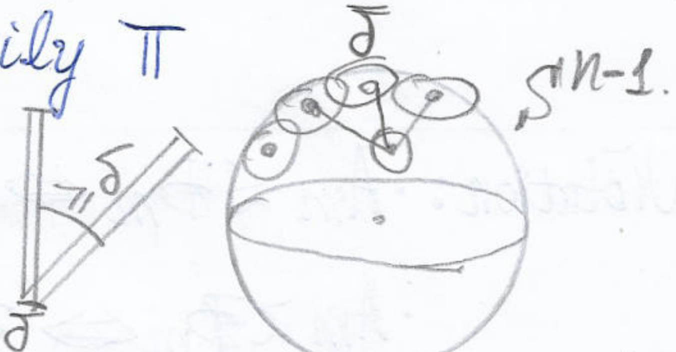
$N(\delta) := \#$ of δ-cubes needed to cover K^δ
$\approx \left(\frac{1}{\delta}\right)^\alpha \Rightarrow \dim_M K = \alpha.$

• If $|K^\delta| \gtrapprox \delta^{n-\alpha}$ $\forall \delta > 0$. $\Rightarrow N(\delta) \gtrapprox \left(\frac{1}{\delta}\right)^\alpha \Rightarrow \dim_M K \geq \alpha.$

Goal: $|K^\delta| \gtrapprox 1$, $\forall \delta > 0$. [True in two dimensions]

Equiv: For every maximal family $\mathbb{T}$ of δ-seperated δ-tubes,

$$\Big|\bigcup_{T \in \mathbb{T}} T\Big| \gtrapprox 1. \left(= \sum_{T \in \mathbb{T}} |T|\right)^{(*)}.$$

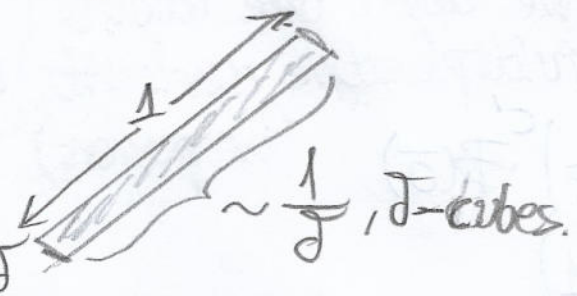


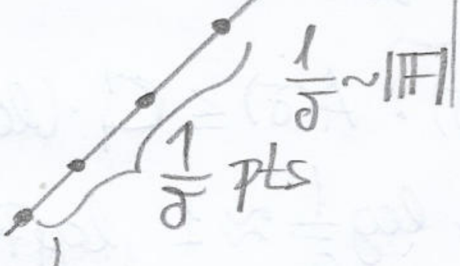
{Thinking of simplyfying the problem...}

Looking for a discrete setting where: $\mathbb{F}^n$ (finite field)

$\# \mathbb{T} \sim \frac{1}{\delta^{n-1}}$

$|T| \sim \delta^{n-1}$.





$N(\delta)$
$\searrow$ $\# Q \approx \frac{1}{\delta^n}$

K: 1 line in each direction
$\# K \gtrsim |\mathbb{F}|^n$
(the union covers a proportion of $\mathbb{F}^n$).

(*): δ-tubes seem to be disjoint.

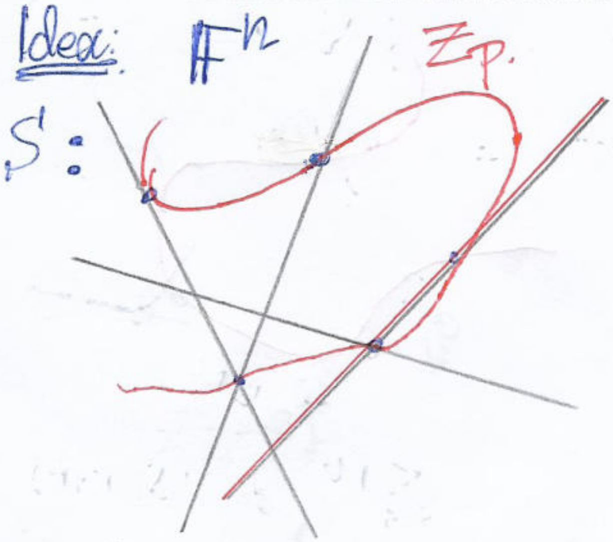
** In a finite field we define:

$l_{\vec{a},\vec{b}} := \{\vec{a} + t\cdot\vec{b} \,/\, t\in\mathbb{F}\}$ {direction in a finite field}

↓direction of line

**

<u>Idea:</u>



① Find a low degree polynomial $p\in\mathbb{F}[x_1,x_2,\dots,x_n]$ vanishing on S' (set of points).

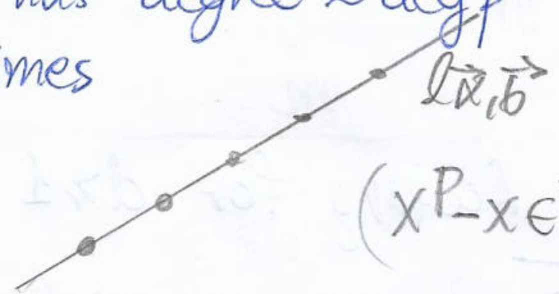
② See how this vanishing spreads out to the lines.

Lemma: If $0\neq p\in\mathbb{F}[x_1,\dots,x_n]$ vanishes on some line $l=l_{\vec{a},\vec{b}}$ $>\deg p$ times, $p|_{l_{\vec{a},\vec{b}}}=0\in\mathbb{F}[t]$

If $p|_{l_{\vec{a},\vec{b}}}(t)=p(\vec{a}+t\cdot\vec{b})$ has degree $\leq \deg p$

so if it vanishes $>\deg p$ times

$\Rightarrow p|_{l_{\vec{a},\vec{b}}}=0.$



$(x^p - x\in\mathbb{Z}_p[x])$

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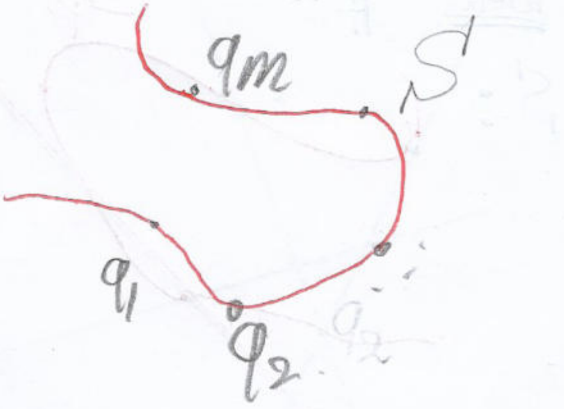
(3) $\forall l$ ↗ $l\subseteq\mathbb{Z}_p$ | structure

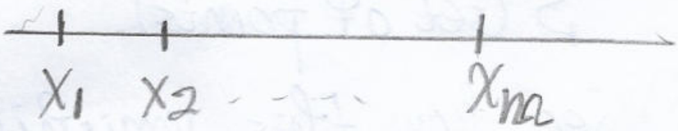
↘ $l\cap S'$ is small | control $\# S'$.

How is this all connected to the Kakeya Problem?

Dvir's vanishing Lemma: Let $S \subseteq \mathbb{F}^n$ be a set of m points. Then, $\exists\, 0 \neq p \in \mathbb{F}[x_1, \dots, x_n]$, $\deg(p) \lesssim m^{1/n}$ vanishing on S

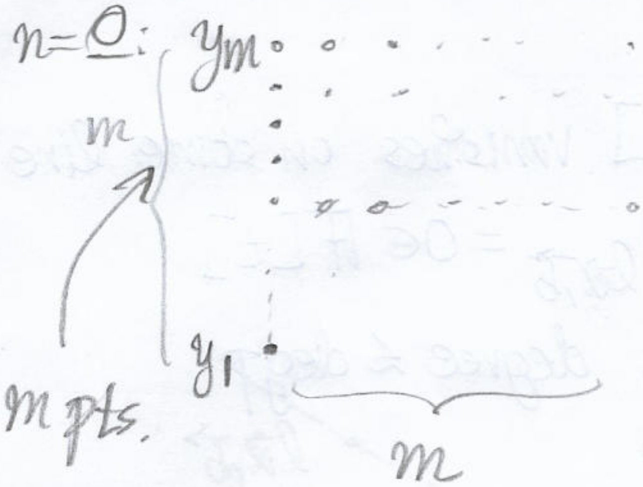


$n=1$:



$(x-x_1)\cdot(x-x_2) \cdots (x-x_m)$

$n=2$: $(y-y_1)(y-y_2)\cdots(y-y_m)$.



Proof: Looking for $d \geq 1$ and $p(x_1, x_2, \dots, x_n) =$

$$= \sum_{\alpha_1 + \dots + \alpha_n \leq d} \underbrace{c_{\alpha_1, \alpha_2, \dots, \alpha_n}}_{\text{unknowns}} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

s.t. $\left.\begin{cases} p(q_1) = 0 \\ p(q_2) = 0 \\ \vdots \\ p(q_m) = 0. \end{cases}\right\} \rightsquigarrow$ m linear equations with $\binom{d+n}{n} \sim d^n$ unknowns

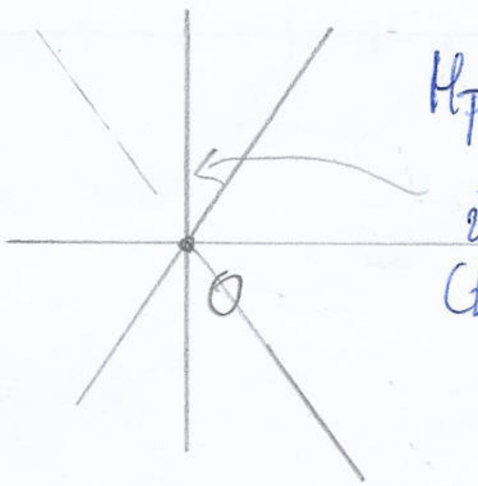
We are ok as long as $d^n \gtrsim m \Leftrightarrow d \gtrsim m^{1/n}$

Pick $\boxed{d \sim m^{1/n}}$

Thm (Dvir, 2008): Let $K \subseteq \mathbb{F}^n$ be a Kakeya set ($\mathbb{F}$: finite field). Then $|K| \gtrsim |\mathbb{F}|^n$

<u>Proof:</u>

P vanishing on K.



H_p vanishes on $\mathbb{F}^n$ impossible (too low degree)

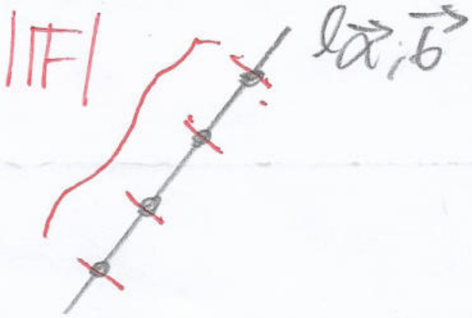
Suppose $|K| < C_n \cdot |\mathbb{F}|^n$

$$\exists\, 0 \neq p \in \mathbb{F}[x_1, \ldots, x_n],\ \deg(p) \leq A_n \cdot |K|^{1/n} < \overset{\lneq 1}{A_n \cdot C_n^{1/n}} \cdot |\mathbb{F}| \lneq |\mathbb{F}|$$

vanishing on K.

$\forall \vec{b} \in \mathbb{F}^n \setminus \{0\},\ \exists\, l_{\vec{a},\vec{b}} \subseteq K$

$\Rightarrow P$ vanishes $|\mathbb{F}| > \deg(p)$ times on $l_{\vec{a},\vec{b}} \Rightarrow P_{|l_{\vec{a},\vec{b}}} = 0 \in \mathbb{F}[t]$.

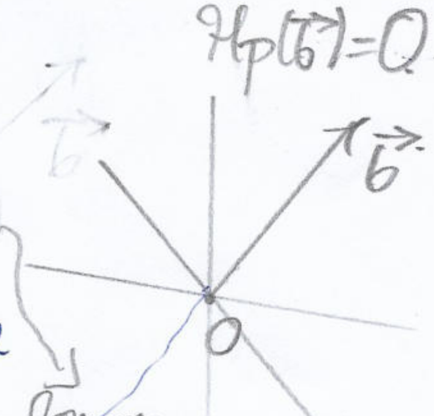


$H_p(x_1, \ldots, x_n) :=$ homogeneous part of p, of degree $= \deg(p)$.

⚠ $H_p(\vec{b}) = 0$

$$P_{|l_{\vec{a},\vec{b}}}(t) = P(\vec{a} + t\cdot\vec{b}) = t^{\deg(p)} \cdot \overset{\to 0}{H_p(\vec{b})} + \underbrace{\ldots}_{\text{lower degree terms.}}$$

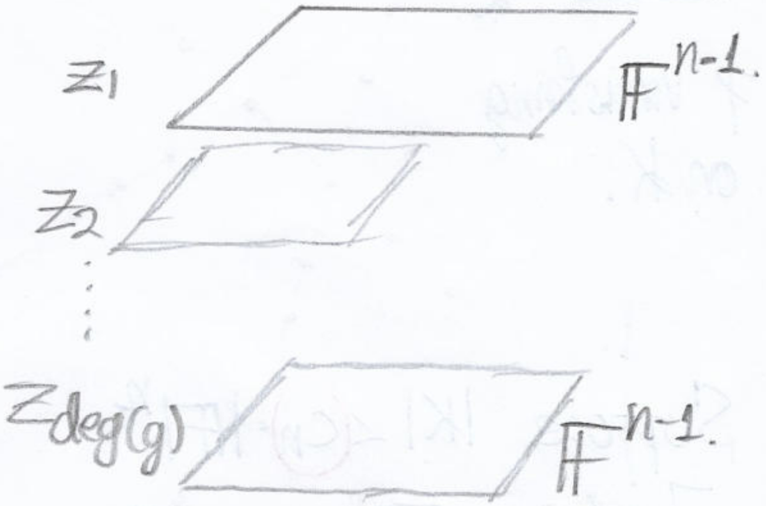
$$p(x_1, \ldots, x_n) = \sum_{d_1 + \ldots + d_n \leq \deg p} c_{d_1, d_2, \ldots, d_n} \cdot x_1^{d_1} \cdots x_n^{d_n}$$



Schwartz-Zippel lemma: Let $0\neq g\in \mathbb{F}[x_1,\dots,x_n]$
Then, $\#Z_g \le \deg(g)\cdot|\mathbb{F}|^{n-1}$

$\mathbb{F}^n$

$n=1$: 

n: 

$g(x)=(x_n-z_1)\cdots(x_n-z_{\deg(g)})$

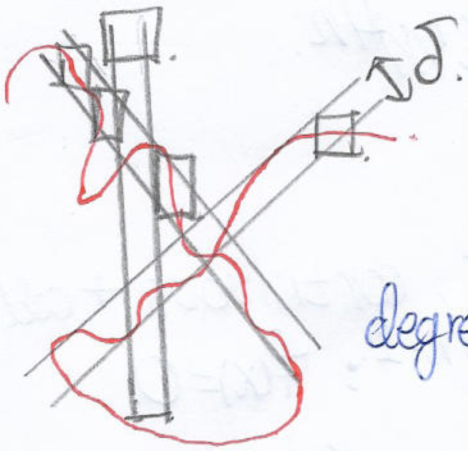
By S-Z lemma: $\#Z_{H_p} \le \deg H_p\cdot|\mathbb{F}|^{n-1} \lneq |\mathbb{F}|^n$

$\deg H_p = \deg(p) \lneq |\mathbb{F}|$.

End of Part I.

• The Polynomial Method. [Part II]

Adapt to the Kakeya conj. ($|\bigcup_{T\in\mathbb{T}} T| \gtrsim 1$):



<u>Goal</u>: $\#Q \approx \left(\frac{1}{\delta}\right)^n$

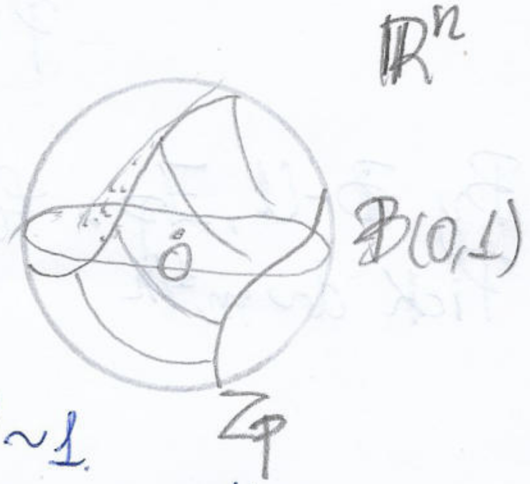
① Find a polynomial p of low degree $\lesssim \#Q^{1/n}$ s.t. Z_p bisects each δ-cube in Q.

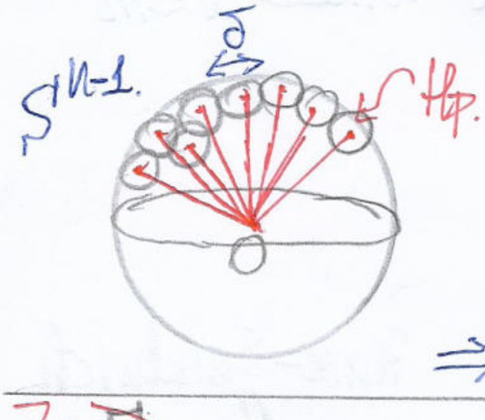
② Show: H_p vanishes on dir(T) $\forall T\in\mathbb{T}$

③ Show this is impossible.

Wongkew's theorem:

$|N_\delta(Z_p)\cap B(0,1)| \lesssim \deg(p)\cdot\delta.$

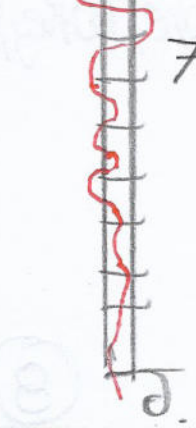




$|N_\delta(Z_{H_p})\cap B(0,1)| \sim 1$

$\lesssim \deg(p)\cdot\delta \lesssim \#Q^{1/n}\cdot\delta.$

$\Rightarrow \#Q \gtrsim \left(\frac{1}{\delta}\right)^n$



$\not\Rightarrow \exists \ell \subseteq T: p|_\ell = 0.$

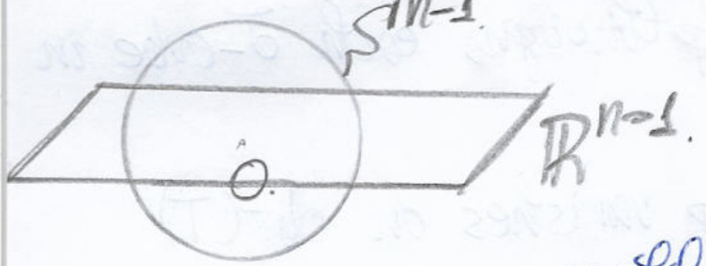
Proposition: Let $S_1, S_2, \dots, S_m$ be sets $\subseteq \mathbb{R}^n$ of finite, positive volume [figure]

Then, $\exists\, 0 \neq p \in \mathbb{R}[x_1, \dots, x_n]$, $\deg(p) \lesssim m^{1/n}$

s.t.: $\forall i$, $|S_i \cap \{p>0\}| = |S_i \cap \{p<0\}|$

Borsuk-Ulam: Let $f: S^{n-1} \to \mathbb{R}^{n-1}$, continuous + odd

Then, $\exists x \in S^{n-1}$: $f(x) = 0$.



p of $\deg \leq d \rightsquigarrow \binom{d+n}{n} \sim d^n$

coeff. $\in \mathbb{R}^{d^n}$

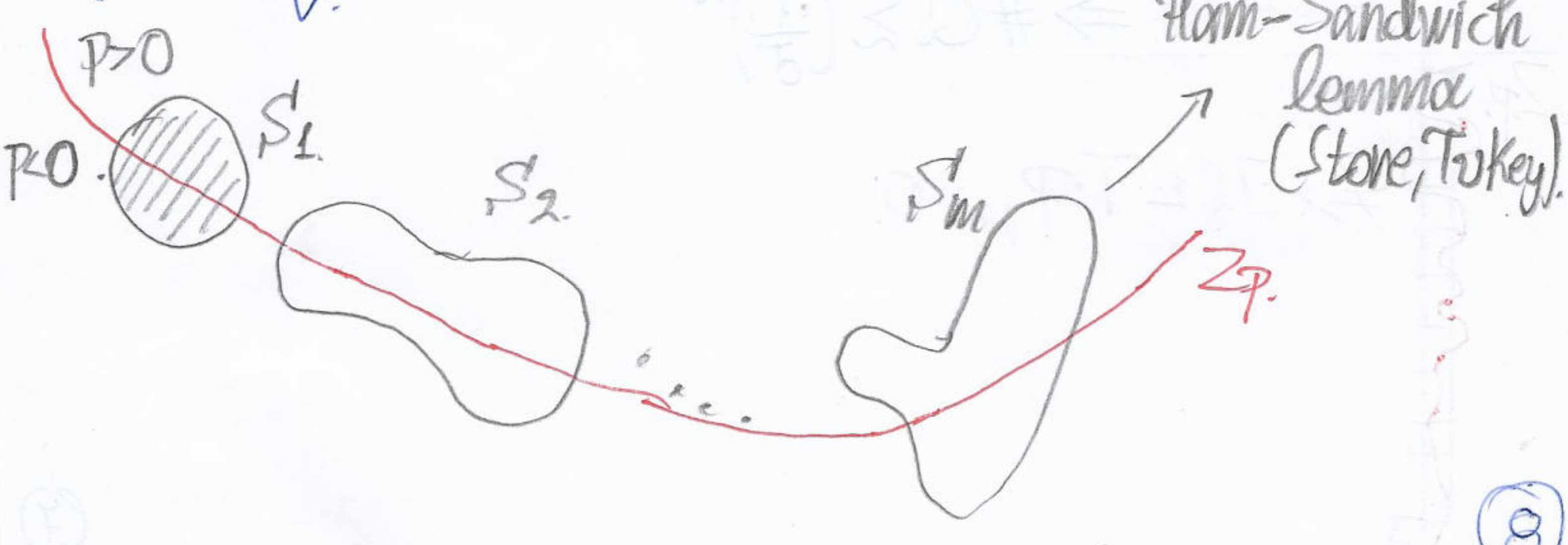
$\to f: S^{d^n - 1} \to \mathbb{R}^m$

$p \longmapsto (|S_1 \cap \{p>0\}| - |S_1 \cap \{p<0\}|, \dots, |S_m \cap \{p>0\}| - |S_m \cap \{p<0\}|)$.

By B-U, $\exists p$: $f(p) = 0$ as long as $d^n \gtrsim m \Leftrightarrow d \gtrsim m^{1/n}$

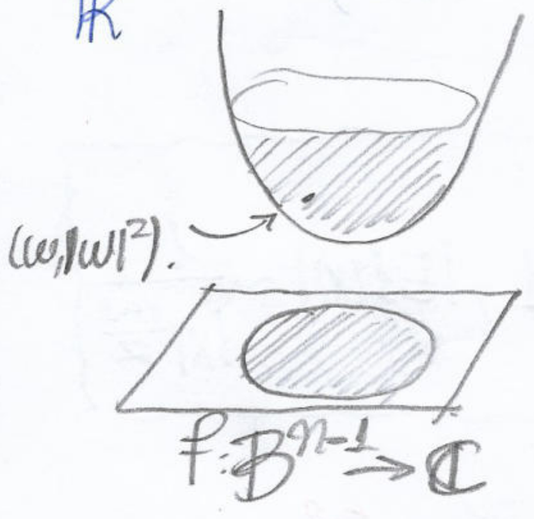
Pick $d \sim m^{1/n}$

[figure]:



Harmonic Analysis Approach.

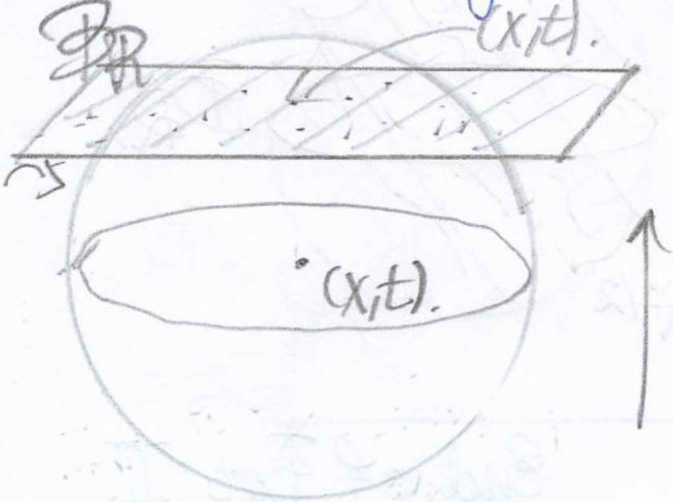
Restriction Conjecture: → Understanding the Fourier transform of functions defined on curves.



E: extension operator. $\cdot(x,t)$ $\mathbb{R}^n$

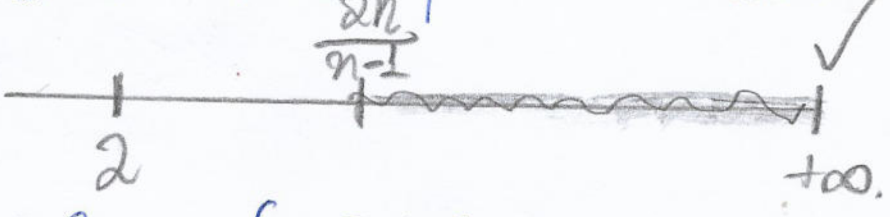
$$Ef(x,t)=\int_{\mathbb{B}^{n-1}} e^{-2\pi i\langle(x,t),(\omega,|\omega|^2)\rangle} f(\omega)\,d\omega.$$

We'll be focusing on:



$\|Ef\|_{L^p(\mathbb{B}_R)}$
$//$

Q: For what p do we have $\|Ef\|_p \lesssim \|f\|_p$?



$|Ef(x,t)| \leq \|f\|_1 \lesssim \|f\|_\infty$

$$Ef(x,t)=\int_{\mathbb{B}^{n-1}} e^{-2\pi i\langle x,\omega\rangle}\cdot\left[e^{-2\pi i t|\omega|^2} f(\omega)\right]d\omega=\hat{f}_t(x),\ \forall x\in\mathbb{R}^{n-1}$$

($x,\omega \in \mathbb{R}^{n-1}$.)

$$\Rightarrow \|Ef(\cdot,t)\|_2=\|\hat{f}_t\|_2 \underset{\text{(Plancherel)}}{=} \|f_t\|_2=\|f\|_2 \Rightarrow$$

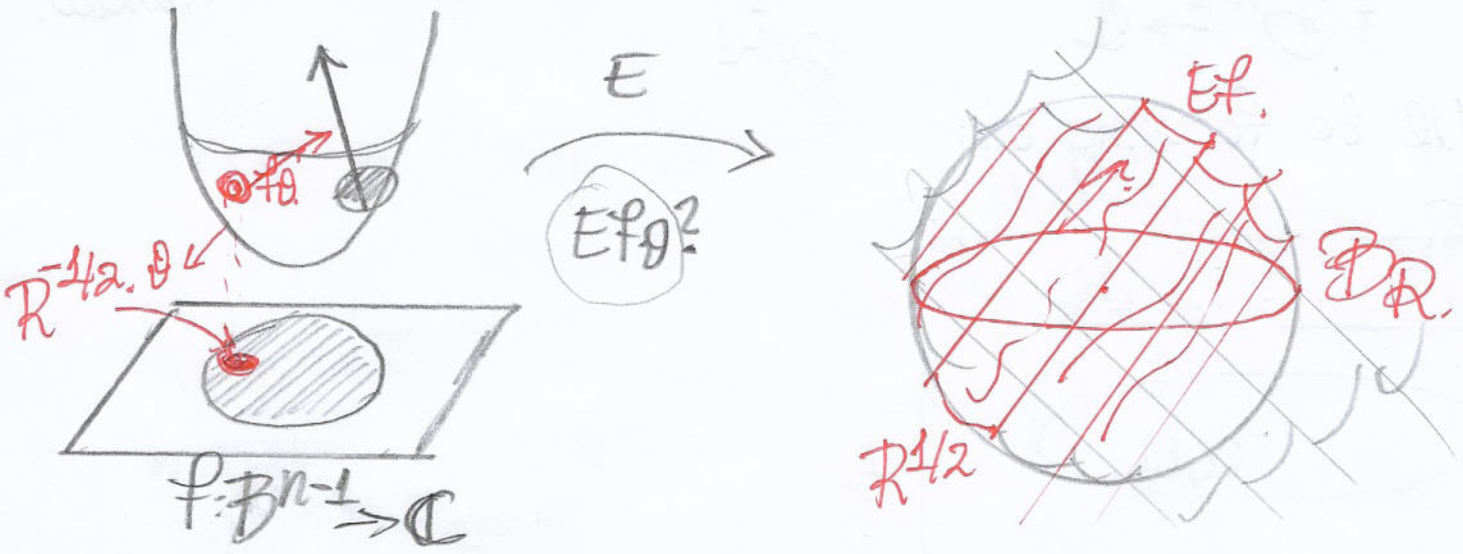
$$\Rightarrow \|Ef\|_{L^2(\mathbb{R}^{n-1}\times(-R,R))} \sim R^{1/2}\cdot\|f\|_2.$$

Restriction Conjecture: Let Σ be a hyperplane in $\mathbb{R}^n$ of non-vanishing Gaussian curvature. Then,

$$\|Ef\|_{L^p(B_R)} \lessapprox \|f\|_p, \quad \forall R>1$$

$\forall\, p \geq \frac{2n}{n-1}.$

$\Sigma =$  $, f=1., |E1(x)| \sim \frac{1}{|x|^{\frac{n-1}{2}}}$



End of Part II.

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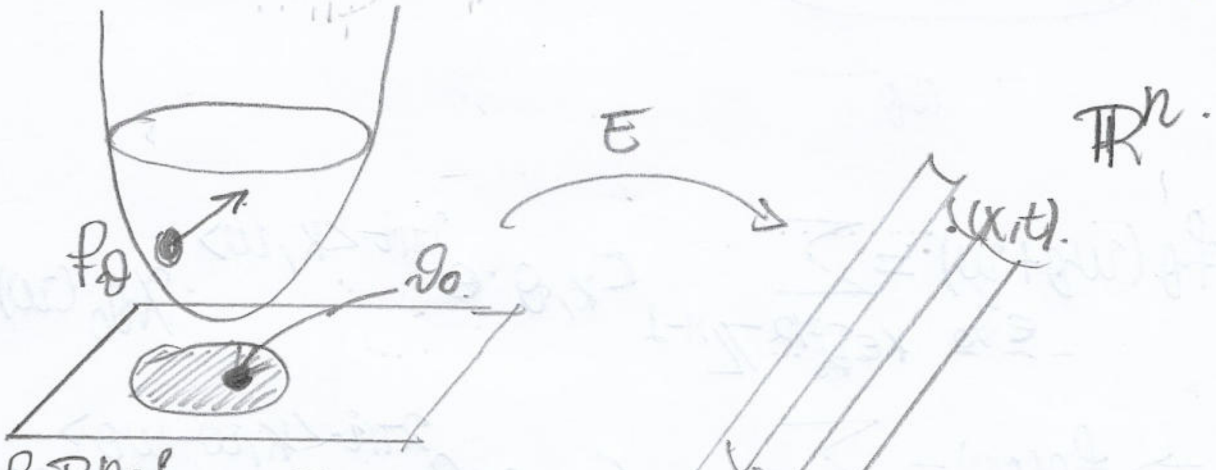
Η πολυωνυμική μέθοδος | The polynomial Method (Parts III & IV)

Remember:



$f: B^{n-1} \to \mathbb{C}$

$$Ef(x,t) = \int_{B^{n-1}} e^{-2\pi i\cdot\langle (x,t),(\omega,|\omega|^2)\rangle} f(\omega)\,d\omega.$$

Ef_θ.

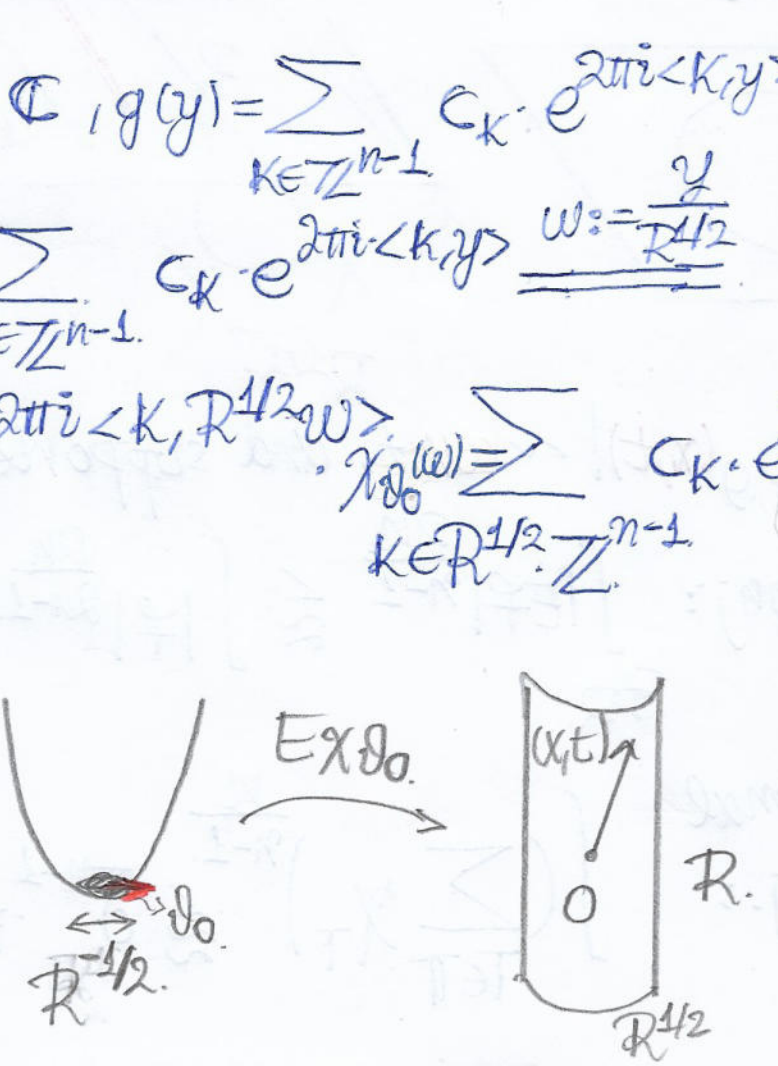
$$\forall g:[0,1]^{n-1}\to\mathbb{C},\ g(y)=\sum_{k\in\mathbb{Z}^{n-1}} c_k\cdot e^{2\pi i\langle k,y\rangle}$$

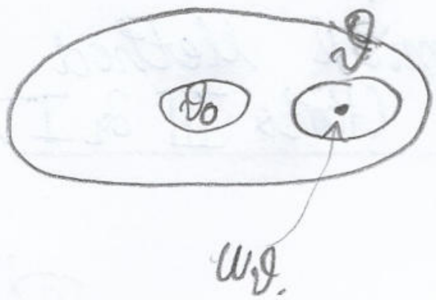
$$\cdot\ f_{\theta_0}\left(\frac{y}{R^{1/2}}\right) = \sum_{k\in\mathbb{Z}^{n-1}} c_k\cdot e^{2\pi i\cdot\langle k,y\rangle} \overset{\omega := \frac{y}{R^{1/2}}}{=\!=\!=}$$

$$= \sum_{k\in\mathbb{Z}^{n-1}} c_k\cdot e^{2\pi i\langle k, R^{1/2}\omega\rangle}\cdot\chi_{\theta_0}(\omega) = \sum_{k\in R^{1/2}\cdot\mathbb{Z}^{n-1}} c_k\cdot e^{2\pi i\cdot\langle k,\omega\rangle}\cdot\chi_{\theta_0}(\omega)$$

$(\forall y\in[0,1]^{n-1})$.

Motivation:

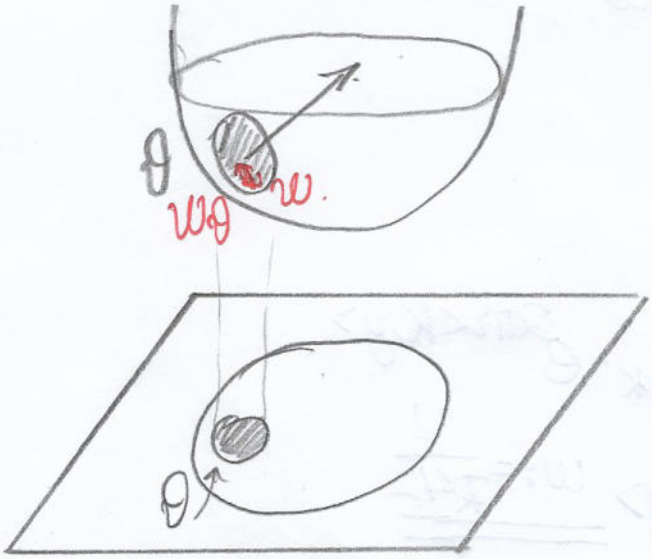


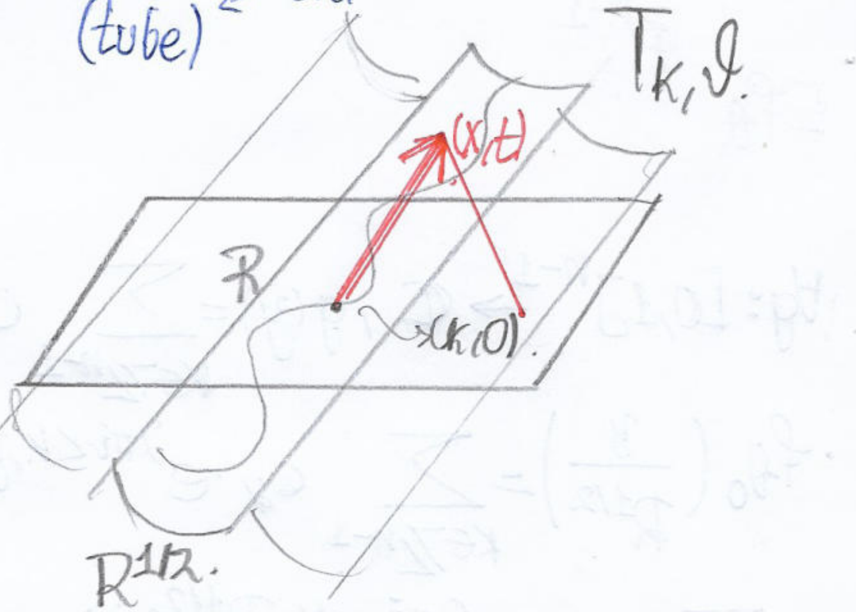


$$f_\vartheta(\underbrace{w_\vartheta + w}_{\in \vartheta_0}) = \sum_{k \in R^{1/2}\cdot\mathbb{Z}^{n-1}} c_{k,\vartheta}\cdot e^{2\pi i\cdot\langle k, w\rangle}\cdot \chi_{\vartheta_0}(w) \Rightarrow$$

$$\Rightarrow f_\vartheta(w) = \sum_{k \in R^{1/2}\cdot\mathbb{Z}^{n-1}} \underbrace{c_{k,\vartheta}\cdot e^{2\pi i\cdot\langle k, w - w_\theta\rangle}\cdot \chi_\vartheta(w)}_{=: f_{T_{k,\vartheta}}}$$

(tube)





Claim: $|Ef_{T_{k,\vartheta}}(x,t)| \sim$ const. and supported in $T_{k,\vartheta}$.

Restriction Conj: $\int_{B_R} |Ef|^{\frac{2n}{n-1}} \lessapprox \int |f|^{\frac{2n}{2n-1}}$ $\forall f: B^{n-1} \to \mathbb{C}$

$\Downarrow$

Kakeya maximal operator conj.: $\int \Big(\sum_{T\in\mathbb{T}} \chi_T\Big)^{\frac{n}{n-1}} \lessapprox \delta^{n-1}\cdot\#\mathbb{T}$, for (*)

$(\longrightarrow)$

any δ-seperated family $\mathbb{T}$ of δ-tubes.

$\rightsquigarrow$ Kakeya max. $\Rightarrow$ Kakeya set conj: let $K \subseteq \mathbb{R}^n$ Kakeya set Goal: $|K^\delta| \gtrapprox 1$.

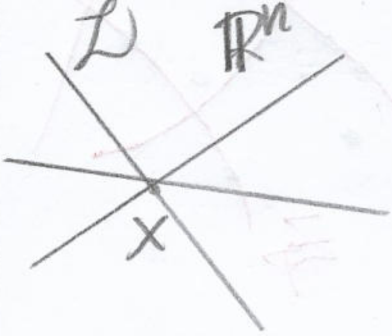
(*): $\delta^{n-1} \cdot \#\mathbb{T} = \sum_T |T| = \int (\sum \chi_T)^1$, so on avg, $\forall x \in \mathbb{R}^n$, $\#\{\text{tubes in } \mathbb{T} \text{ through } x\} \approx 1$.

Take a maximal δ-seperated family $\mathbb{T}$ of δ-tubes in K^δ.

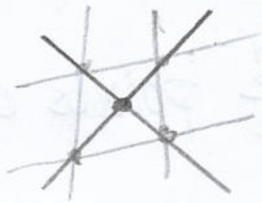
$$\underbrace{\#\mathbb{T}}_{\sim\delta^{-(n-1)}} \cdot \delta^{n-1} = \int_{K^\delta} \sum_{T\in\mathbb{T}} \chi_T \le \underbrace{\left[\int \left(\sum \chi_T\right)^{\frac{n}{n-1}}\right]^{\frac{n-1}{n}}}_{\lessapprox (\delta^{n-1}\cdot\#\mathbb{T})^{\frac{n-1}{n}} \ (\sim 1)} \cdot |K^\delta|^{1/n} \lessapprox$$

$|K^\delta|^{1/n} \Rightarrow |K^\delta| \gtrapprox 1.$

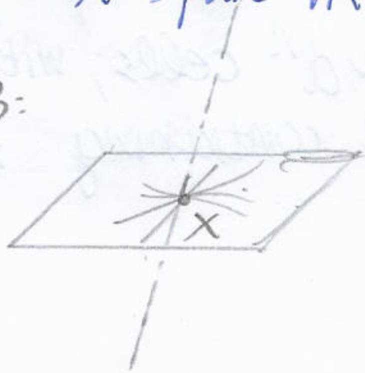
• Joints ploblem: Let $\mathcal{L}$ be a family of L lines in $\mathbb{R}^n$. A joint formed by $\mathcal{L}$ is a point $x \in \mathbb{R}^n$: the lines in $\mathcal{L}$ through x span $\mathbb{R}^n$.



, $n=2$:



, $n=3$:



$\#J \lessapprox \binom{L}{2} \sim L^2$.

Conj: $\#J \lessapprox L^{\frac{n}{n-1}}$ (2008: Quildrdn / Kaplan-Sharir-Shustin)

Q: Show: $\sum_{x\in J} \#\{\text{lines in } \mathcal{L} \text{ through } x\}^{n/n-1} \underset{(n=3)}{\lessapprox} (\log L) \cdot L^{\frac{n}{n-1}}$?

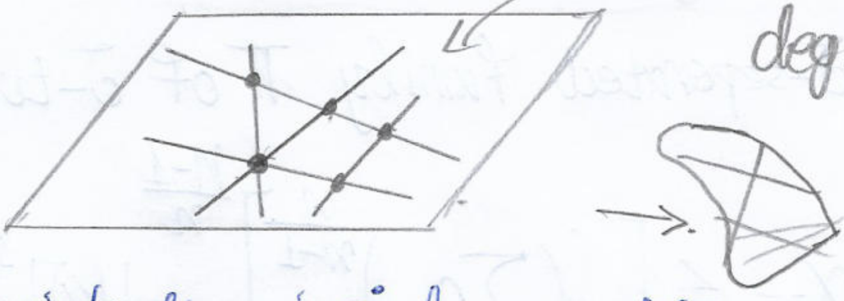
$\forall\, \kappa$ dyadic, $J_\kappa := \{x \in J \,/\, x$ lies in $\sim \kappa$ lines in $\mathcal{L}\}$

<u>Goal:</u> $|J_\kappa| \leq C_\varepsilon \cdot \left(\frac{L}{\kappa}\right)^{\frac{n}{n-1}+\varepsilon}$, $\forall \varepsilon > 0$.

Polynomial Partitioning. (Guth-Katz, 2010).

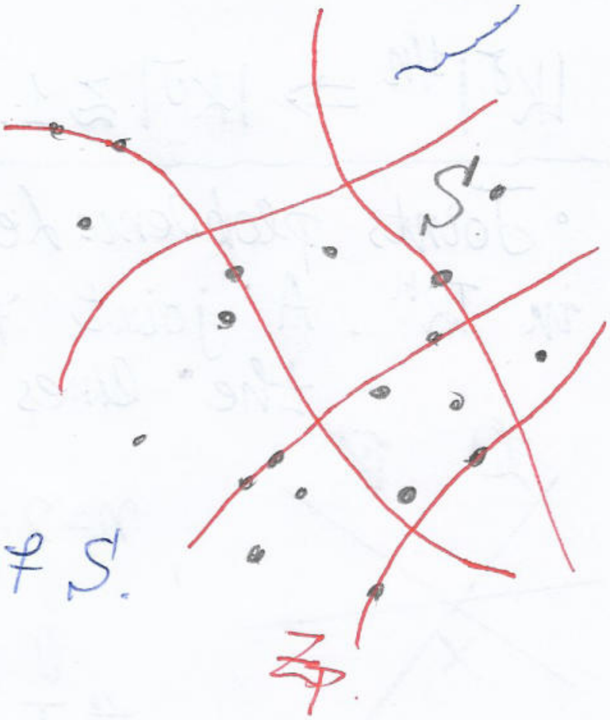
Parenthesis: Why does the polynomial method help?

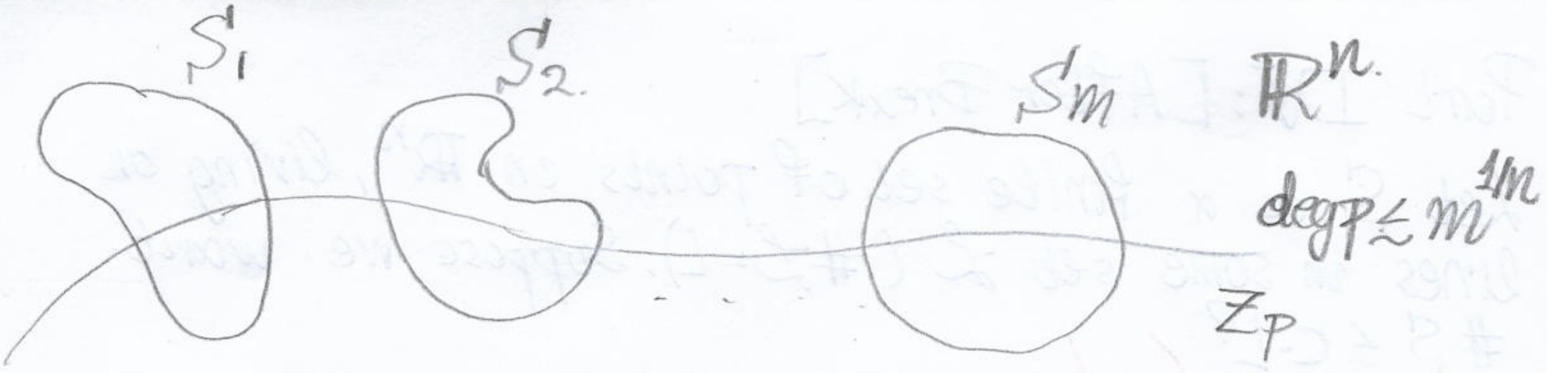
zero set of polynomial of degree $=1$.



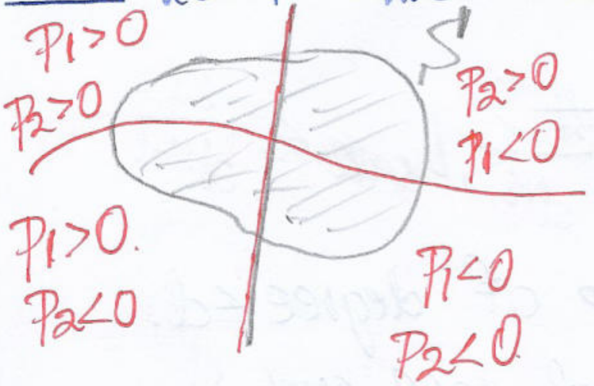
In point-line incidence problems, the "enemies" are the situations where the lines lie on zero sets of <u>low</u> degree polynomial.

Let S be a finite set of points in $\mathbb{R}^n$, $d \geq 1$. The $\exists\, 0 \neq p \in \mathbb{R}[x_1, x_2, \dots, x_n]$ $\deg p \leq d$ s.t. Z_p splits $\mathbb{R}^n$ in $\sim d^n$ cells, with each cell containing $\lesssim \frac{\# S}{d^n}$ points of S.





<u>Proof</u>: Let S have finite, positive volume.



$\exists P_1$, $\deg P_1 \lesssim 1^{1/n}$, Z_{P_1} splits S in 2 cells, each cell contains $\frac{|S|}{2}$ mass of S.

- $\deg(P_2) \lesssim 2^{1/n} \to 2^2$ cells, each with $\frac{|S|}{2^2}$ of the mass

$\vdots$ (Do this again and again)

- $\exists P_J$, $\deg P_J \lesssim (2^{J-1})^{1/n}$
  $\rightsquigarrow 2^J$ cells, each with $\frac{|S|}{2^J}$ of the mass.

and we stop when:

$$1^{1/n} + 2^{1/n} + \dots + (2^{J-1})^{1/n} = d.$$

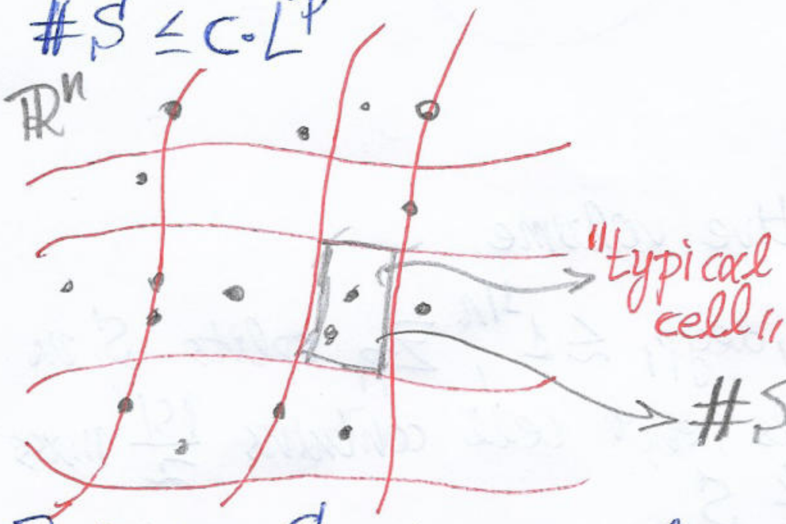
$$\Rightarrow (2^J)^{1/n} \sim d \Rightarrow 2^J \sim d^n$$

[Break].

Part IV: [After Break].

Let S be a finite set of points in $\mathbb{R}^n$, living on lines in some set $\mathcal{L}$ ($\#\mathcal{L}=L$). Suppose we want.

$\#S \leq C\cdot L^P$



$\#S_{cell} \sim \frac{\#S}{d^n}$, $L_{cell} \lesssim \frac{L}{d^{n-1}}$.

Partition S using a polynomial p of degree $\leq d$.
Z_P splits $\mathbb{R}^n$ in $\sim d^n$ cells, each cell contains $\lesssim \frac{\#S}{d^n}$ pts of S.

- <u>Cellural case:</u> $>50\%$ of our pts in S lie inside the cells.
- Algebraic case: $>50\%$ " " " lie on Z_p

Cellural case $\Rightarrow$ 99% of the cells contain $\sim \frac{\#S}{d^n}$ pts of S each.

- Any line l can enter $\leq d+1$ cells.

(if l enter $> d+1$ cells $\Rightarrow$ it would hit Z_p $>d$ times $\Rightarrow l \subseteq Z_p \Rightarrow l$ enters 0 cells

So. $$\sum_{cells} L_{cell} = \sum_{l\in\mathcal{L}} \underbrace{\#\{\text{cells } l \text{ enters}\}}_{\lesssim d} \lesssim L\cdot d.$$

$$\Rightarrow \frac{1}{d^n}\sum_{cells} L_{cell} = \frac{1}{d^n}\sum_{l\in\mathcal{L}} \#\{\text{cells } l \text{ enters}\} \lesssim \frac{L\cdot d}{d^n} = \frac{L}{d^{n-1}}$$

L_{cell} := # lines in $\mathcal{L}$ that enter the cell.

We can prove: > 2% of the cells satisfy: $L_{cell} \le \frac{L}{d^{n-1}}$

By induction on $\#\mathcal{L}$:

$$\#S'_{cell} \le C\cdot L^p_{cell} \Rightarrow \frac{\#S'}{d^n} \lesssim C\cdot \frac{L^p}{d^{(n-1)p}} \Rightarrow$$

$$\Rightarrow \#S' \le C\cdot A\cdot L^p\cdot \frac{1}{d^{(n-1)p-n}}$$

{* When $(n-1)p-n>0$, then the "bad" constant A can be absorbed}.

⇝ Want: $(n-1)\cdot p-n>0 \Leftrightarrow p>\frac{n}{n-1}$.

$\mathcal{L}$: set of L lines in $\mathbb{R}^n$, $n=3$.

Fix k, $J_k := \{x\in J \,/\, x$ lies in $\sim k$ lines in $\mathcal{L}\}$

joints.

Show: $\#J_k \le C_\varepsilon\cdot\left(\frac{L}{k}\right)^{\frac{n}{n-1}+\varepsilon}$.

Lemma (Quilodrán): Let J be a set of joints formed by $\mathcal{L}$. Suppose $p\in\mathbb{R}[x_1,\dots,x_n]$ vanishes on J.

Then: $\#J \le \deg(p)\cdot L$.

Proof: Step 1: $\exists \ell\in\mathcal{L}$: $\#(J\cap\ell)\le\deg(p)$.

Step 2: Throw away $\le\deg(p)$ joints

$L\left\{\begin{array}{l} +\le\deg(p) \text{ joints} \\ \vdots \\ \le\deg(p) \text{ joints.}\end{array}\right.$

$\Rightarrow$

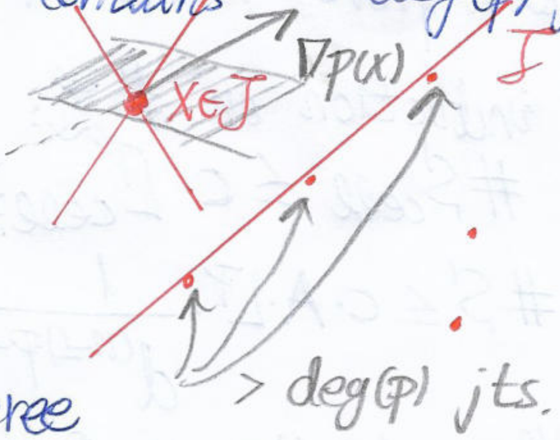
$\Rightarrow \le L\cdot\deg(p)$ joints.

Step 1: Suppose: each $l \in \mathcal{L}$ contains $> \deg(p)$ jts.

$\Rightarrow$ each $l \in \mathcal{L}$ lies in Z_p.

$\Rightarrow (\forall x \in J)(\nexists$ a tangent space to Z_p at x. $\Rightarrow$

$\Rightarrow \forall x \in J$, $\underbrace{\nabla p(x)}_{\text{polys of lower degree vanishing on } J} = 0$.



- Contradiction: if we worked with p of lowest degree vanishing on J.

$(\deg p \lesssim \# J^{1/n} \Rightarrow \# J \lesssim L^{\frac{n}{n-1}})$.

- Partition J_K with a polynomial p, $\deg(p) \le d$.

$$\underbrace{\# J_{K,cell}}^{\frac{\# J_K}{d^n}} \le C_\varepsilon \cdot \left(\frac{L_{cell}}{K}\right)^{\frac{n}{n-1}+\varepsilon}$$

$$\lesssim C_\varepsilon \cdot \left(\frac{L}{K}\right)^{\frac{n}{n-1}+\varepsilon} \cdot \left(\frac{L}{d^{n-1}}\right)^{\frac{n}{n-1}+\varepsilon}$$

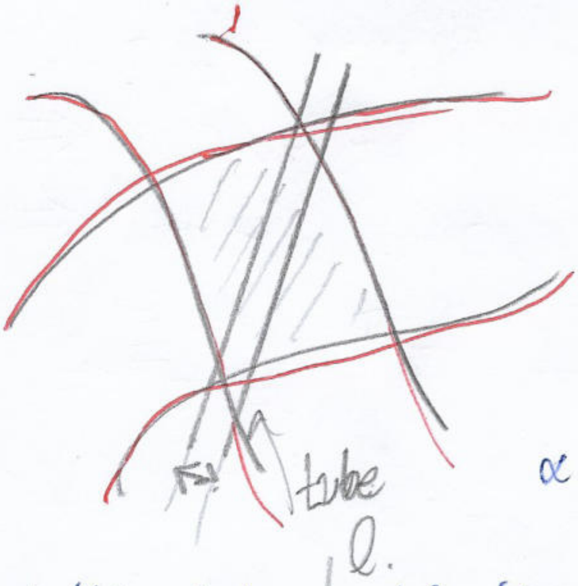
$$\Rightarrow \# J_K \le A \frac{d^n}{d^{n+\varepsilon(n-1)}} \cdot C_\varepsilon \left(\frac{L}{K}\right)^{\frac{n}{n-1}+\varepsilon}$$

(***)

- Cellular case: $\Leftrightarrow$ 99% of the cells contain $\frac{\# J_K}{d^n}$ pts of J_K each.

(***): $\Rightarrow \# J_K \le C_\varepsilon \cdot \left(\frac{L}{K}\right)^{\frac{n}{n-1}+\varepsilon}$, if d is large enough.

- Algebraic case: $> 50\%$ pts in J_K lie on Z_p and $\deg(p) = \overset{\text{constant}}{C_\varepsilon} \Rightarrow \# J_K \lesssim \deg(p) \cdot L \lesssim C_\varepsilon \cdot L \ll C_\varepsilon \cdot L^{\frac{n}{n-1}+\varepsilon}$



$$\int |Ef|^{\frac{2n}{n-1}}$$

the same $\int_{cell} |Ef|^{\frac{2n}{n-1}}$ $\forall$ cell.

• Each l enters $\leq d+1$ cells, but a tube can enter more...

* We take a thickening of the zero set...

$\Rightarrow$ each tube enters $\leq d+1$ cells shrunken cells.

The End!