M-estimates for isotropic convex bodies and their L_q-centroid bodies

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Abstract

Let K be a centrally-symmetric convex body in $\mathbb{R}^n$ and let $\|\cdot\|$ be its induced norm on $\mathbb{R}^n$. We show that if $K \supseteq rB_2^n$ then:

$$\sqrt{n}M(K) \leqslant C \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \min\left(\frac{1}{r}, \frac{n}{k}\log\left(e+\frac{n}{k}\right)\frac{1}{v_k^-(K)}\right).$$

where $M(K) = \int_{S^{n-1}} \|x\|\, d\sigma(x)$ is the mean-norm, $C > 0$ is a universal constant, and $v_k^-(K)$ denotes the minimal volume-radius of a k-dimensional orthogonal projection of K. We apply this result to the study of the mean-norm of an isotropic convex body K in $\mathbb{R}^n$ and its L_q-centroid bodies. In particular, we show that if K has isotropic constant L_K then:

$$M(K) \leqslant \frac{C\log^{2/5}(e+n)}{\sqrt[10]{n}L_K}.$$

1 Introduction

Let K be a centrally-symmetric convex compact set with non-empty interior ("body") in Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. We write $\|\cdot\|$ for the norm induced on $\mathbb{R}^n$ by K and h_K for the support function of K; this is precisely the dual norm $\|\cdot\|^*$. The parameters:

$$M(K) = \int_{S^{n-1}} \|x\|\, d\sigma(x) \quad \text{and} \quad M^*(K) = \int_{S^{n-1}} h_K(x)\, d\sigma(x), \tag{1.1}$$

where σ denotes the rotationally invariant probability measure on the unit Euclidean sphere S^{n-1}, play a central role in the asymptotic theory of finite dimensional normed spaces.

Let $\mathrm{vrad}(K) := (|K|/|B_2^n|)^{1/n}$ denote the volume-radius of K, where $|A|$ denotes Lebesgue measure in the linear hull of A and B_2^n denotes the unit Euclidean ball. It is easy to check that:

$$M(K)^{-1} \leqslant \mathrm{vrad}(K) \leqslant M^*(K) = M(K^\circ), \tag{1.2}$$

where $K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\}$ is the polar body to K, i.e. the unit-ball of the dual norm $\|\cdot\|^*$. Indeed, the left-hand side is a simple consequence of Jensen's inequality after we express the volume of K as an integral in polar coordinates, while the right-hand side is the classical Urysohn inequality. In particular, one always has $M(K)M^*(K) \geqslant 1$.

In the other direction, it is known from results of Figiel–Tomczak-Jaegermann [11], Lewis [18] and Pisier's estimate [30] on the norm of the Rademacher projection, that for any centrally-symmetric convex body K, there exists $T \in GL(n)$ such that:

$$M(TK)M^*(TK) \leqslant C \log n, \tag{1.3}$$

where $C > 0$ is a universal constant. Throughout this note, unless otherwise stated, all constants $c, c', C, \dots$ denote universal numeric constants, independent of any other parameter, whose value may change from one occurrence to the next. We write $A \simeq B$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 A \leqslant B \leqslant c_2 A$.

The role of the linear map T in (1.3) is to put the body in a good "position", since without it $M(K)M^*(K)$ can be arbitrarily large. The purpose of this note is to obtain good upper bounds on the parameter $M(K)$, when K is already assumed to be in a good position - the isotropic position. A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, its barycenter is at the origin, and there exists a constant $L_K > 0$ such that:

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2, \text{ for all } \theta \in S^{n-1}. \tag{1.4}$$

It is not hard to check that every convex body K has an isotropic affine image which is uniquely determined up to orthogonal transformations [24]. Consequently, the isotropic constant L_K is an affine invariant of K. A central question in asymptotic convex geometry going back to Bourgain [5] asks if there exists an absolute constant $C > 0$ such that $L_K \leqslant C$ for every (isotropic) convex body K in $\mathbb{R}^n$ and every $n \geqslant 1$. Bourgain [6] proved that $L_K \leqslant C\sqrt[4]{n}\log n$ for every centrally-symmetric convex body K in $\mathbb{R}^n$. The currently best-known general estimate, $L_K \leqslant C\sqrt[4]{n}$, is due to Klartag [14] (see also the work of Klartag and E. Milman [16] and a further refinement of their approach by Vritsiou [33]).

It is known that if K is a centrally-symmetric isotropic convex body in $\mathbb{R}^n$ then $K \supseteq L_K B_2^n$, and hence trivially $M(K) \leqslant 1/L_K$. It seems that, until recently, the problem of bounding $M(K)$ in isotropic position had not been studied and there were no other estimates besides the trivial one. The example of the normalized ℓ_∞^n ball shows that the best one could hope is $M(K) \leqslant C\sqrt{\log n}/\sqrt{n}$. Note that obtaining a bound of the form $M(K) \leqslant n^{-\delta}L_K^{-1}$ immediately provides a non-trivial upper bound on L_K, since $M(K) \geqslant \mathrm{vrad}(K)^{-1} \simeq 1/\sqrt{n}$, and hence $L_K \leqslant c^{-1}n^{\frac{1}{2}-\delta}$. The current best-known upper bound on L_K suggests that $M(K) \leqslant C(n^{1/4}L_K)^{-1}$ might be a plausible goal.

Paouris and Valettas (unpublished) proved that for every isotropic centrally-symmetric convex body K in $\mathbb{R}^n$ one has:

$$M(K) \leqslant \frac{C\sqrt[3]{\log(e+n)}}{\sqrt[12]{n}L_K}. \tag{1.5}$$

Subsequently, this was extended by Giannopoulos, Stavrakakis, Tsolomitis and Vritsiou in [12] to the case of the L_q-centroid bodies $Z_q(\mu)$ of an isotropic log-concave probability measure μ on $\mathbb{R}^n$ (see Section 5 for the necessary definitions). The approach of [12] was based on a number of observations regarding the local structure of $Z_q(\mu)$; more precisely, lower bounds for the in-radius of their proportional projections and estimates for their dual covering numbers (we briefly sketch an improved version of this approach in Section 7).

In this work we present a different method, applicable to general centrally-symmetric convex bodies, which yields better quantitative estimates. As always, our starting point is Dudley's entropy estimate (see e.g. [32, Theorem 5.5]):

$$\sqrt{n}M^*(K) \leqslant C \sum_{k \geqslant 1} \frac{1}{\sqrt{k}} e_k(K, B_2^n), \tag{1.6}$$

where $e_k(K, B_2^n)$ are the entropy numbers of K. Recall that the covering number $N(K, L)$ is defined to be the minimal number of translates of L whose union covers K, and that $e_k(K, L) := \inf \left\{ t > 0 \, : \, N(K, tL) \leqslant 2^k \right\}$.

Our results depend on the following natural volumetric parameters associated with K for each $k = 1, \dots, n$:

$$w_k(K) := \sup \left\{ \mathrm{vrad}(K \cap E) : E \in G_{n,k} \right\} \ , \ v_k^-(K) := \inf \left\{ \mathrm{vrad}(P_E(K)) : E \in G_{n,k} \right\},$$

where $G_{n,k}$ denotes the Grassmann manifold of all k-dimensional linear subspaces of $\mathbb{R}^n$, and P_E denotes orthogonal projection onto $E \in G_{n,k}$. Note that by the Blaschke–Sanataló inequality and its reverse form due to Bourgain and V. Milman (see Section 2), it is immediate to verify that $w_k(K^\circ) \simeq \frac{1}{v_k^-(K)}$.

Theorem 1.1. *For every centrally-symmetric convex body K in $\mathbb{R}^n$ and $k \geqslant 1$:*

$$e_k(K, B_2^n) \leqslant C \frac{n}{k} \log\left(e + \frac{n}{k}\right) \sup_{1 \leqslant m \leqslant \min(k,n)} \left\{ 2^{-\frac{k}{3m}} w_m(K) \right\}.$$

By invoking Carl's theorem (see Section 2), a slightly weaker version of Theorem 1.1 may be deduced from the following stronger statement:

Theorem 1.2. *Let K be a centrally-symmetric convex body in $\mathbb{R}^n$. Then for any $k = 1, \dots, \lfloor n/2 \rfloor$ there exists $F \in G_{n,n-2k}$ so that:*

$$K \cap F \subseteq C \frac{n}{k} \log\left(e + \frac{n}{k}\right) w_k(K) B_2^n \cap F, \tag{1.7}$$

and dually, there exists $F \in G_{n,n-2k}$ so that:

$$P_F(K) \supseteq \frac{1}{C \frac{n}{k} \log(e + \frac{n}{k})} v_k^-(K) P_F(B_2^n). \tag{1.8}$$

A weaker version of Theorem 1.2, with the parameters $w_k(K)$, $v_k^-(K)$ above replaced by:

$$v_k(K) := \sup \left\{ \mathrm{vrad}(P_E(K)) : E \in G_{n,k} \right\} \ , \ w_k^-(K) := \inf \left\{ \mathrm{vrad}(K \cap E) : E \in G_{n,k} \right\},$$

respectively, was obtained by V. Milman and G. Pisier in [25] (see Theorem 4.1). Our improved version is crucial for properly exploiting the corresponding properties of isotropic convex bodies.

By (essentially) inserting the estimates of Theorem 1.1 into (1.6) (with K replaced by K°), we obtain that if K is a centrally-symmetric convex body in $\mathbb{R}^n$ with $K \supseteq rB_2^n$ then:

$$\sqrt{n}M(K) \leqslant C \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \min\left(\frac{1}{r}, \frac{n}{k} \log\left(e + \frac{n}{k}\right) \frac{1}{v_k^-(K)} \right). \tag{1.9}$$

In the case of the centroid bodies $Z_q(\mu)$ of an isotropic log-concave probability measure μ on $\mathbb{R}^n$, one can obtain precise information on the growth of the parameters $v_k^-(Z_q(\mu))$. We recall the relevant definitions in Section 5, and use (1.9) to deduce in Section 6 that:

$$2 \leqslant q \leqslant q_0 := (n \log n)^{2/5} \quad \Longrightarrow \quad M(Z_q(\mu)) \leqslant C \frac{\sqrt{\log q}}{\sqrt[4]{q}}. \tag{1.10}$$

In particular, since $Z_n(\mu) \supseteq Z_{q_0}(\mu)$ and $M(K) \simeq M(Z_n(\lambda_{K/L_K}))/L_K$, where λ_A denotes the uniform probability measure on A, we immediately obtain:

Theorem 1.3. *If K is a centrally-symmetric isotropic convex body in $\mathbb{R}^n$ then:*

$$M(K) \leqslant \frac{C \log^{2/5}(e+n)}{\sqrt[10]{n} L_K}. \tag{1.11}$$

It is clear that (1.11) is not optimal. Note that if (1.10) were to remain valid until $q_0 = n$, we would obtain the bound $M(K) \leqslant C \frac{\sqrt{\log(e+n)}}{n^{1/4} L_K}$, which as previously explained would in turn imply that $L_K \leqslant C \sqrt{\log(e+n)}\, n^{1/4}$, in consistency with the best-known upper bound on the isotropic constant. We believe that it is an interesting question to extend the range where (1.10) remains valid. In Section 6, we obtain such an extension when μ is in addition assumed to be Ψ_α (see Section 6 for definitions).

Our entire method is based on Pisier's regular versions of V. Milman's M-ellipsoids associated to a given centrally-symmetric convex body K, comparing between volumes of sections and projections of K and those of its associated regular ellipsoids. This expands on an approach already employed in [32, 7, 17, 15, 12].

We conclude the introduction by remarking that the dual question of providing an upper bound for the mean-width $M^*(K)$ of an isotropic convex body K has attracted more attention in recent years. Until recently, the best known estimate was $M^*(K) \leqslant C n^{3/4} L_K$, where $C > 0$ is an absolute constant (see [9, Chapter 9] for a number of proofs of this inequality). The second named author has recently obtained in [21] an essentially optimal answer to this question - for every isotropic convex body K in $\mathbb{R}^n$ one has $M^*(K) \leqslant C\sqrt{n} \log^2 n\, L_K$.

2 Preliminaries and notation from the local theory

Let us introduce some further notation. Given $F \in G_{n,k}$, we denote $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$. A centrally-symmetric convex body K in $\mathbb{R}^n$ is a compact convex set with non-empty interior so that $K = -K$. The norm induced by K on $\mathbb{R}^n$ is given by $\|x\|_K = \min\{t \geqslant 0 : x \in tK\}$. The support function of K is defined by $h_K(y) := \|y\|_K^* = \max\{\langle y, x\rangle : x \in K\}$, with K° denoting the unit-ball of the dual-norm. By the Blaschke–Santaló inequality (the right-hand side below) and its reverse form due to Bourgain and V. Milman [8] (the left-hand side), it is known that:

$$0 < c \leqslant \mathrm{vrad}(K)\mathrm{vrad}(K^\circ) \leqslant 1. \tag{2.1}$$

Recall that the k-th entropy number is defined as

$$e_k(K, L) := \inf\left\{t > 0 \ : \ N(K, tL) \leqslant 2^k\right\}.$$

A deep and very useful fact about entropy numbers is the Artstein–Milman–Szarek duality of entropy theorem [1], which states that:

$$e_k(B_2^n, K) \leqslant C e_{ck}(K^\circ, B_2^n) \tag{2.2}$$

for every centrally-symmetric convex body K and $k \geqslant 1$.

In what follows, a crucial role is played by G. Pisier's regular version of V. Milman's M-ellipsoids. It was shown by Pisier (see [31] or [32, Chapter 7]) that for any centrally-symmetric convex body K in $\mathbb{R}^n$ and $\alpha \in (0,2)$, there exists an ellipsoid $\mathcal{E} = \mathcal{E}_{K,\alpha}$ so that:

$$\max\{e_k(K,\mathcal{E}), e_k(K^\circ,\mathcal{E}^\circ), e_k(\mathcal{E},K), e_k(\mathcal{E}^\circ,K^\circ)\} \leqslant P_\alpha \left(\frac{n}{k}\right)^{1/\alpha}, \tag{2.3}$$

where $P_\alpha \leqslant C\left(\frac{\alpha}{2-\alpha}\right)^{1/2}$ is a positive constant depending only on α.

Given a pair of centrally-symmetric convex bodies K, L in $\mathbb{R}^n$, the Gelfand numbers $c_k(K,L)$ are defined as:

$$c_k(K,L) := \begin{cases} \inf\{\mathrm{diam}_{L\cap F}(K\cap F) : F \in G_{n,n-k}\} & k = 0,\dots,n-1 \\ 0 & \text{otherwise} \end{cases},$$

where $\mathrm{diam}_A(B) := \inf\{R > 0 : B \subseteq RA\}$. We denote $c_k(K) = c_k(K, B_2^n)$ and $e_k(K) = e_k(K, B_2^n)$.

Carl's theorem [10] relates any reasonable Lorentz norm of the sequence of entropy numbers $\{e_m(K,L)\}$ with that of the Gelfand numbers $\{c_m(K,L)\}$. In particular, for any $\alpha > 0$, there exist constants $C_\alpha, C'_\alpha > 0$ such that for any $k \geqslant 1$:

$$\sup_{m=1,\dots,k} m^\alpha e_m(K,L) \leqslant C_\alpha \sup_{m=1,\dots,k} m^\alpha c_m(K,L), \tag{2.4}$$

and:

$$\sum_{m=1}^k m^{-1+\alpha} e_m(K,L) \leqslant C'_\alpha \sum_{m=1}^k m^{-1+\alpha} c_m(K,L). \tag{2.5}$$

In fact, Pisier deduces the covering estimates of (2.3) from an application of Carl's theorem, after establishing the following estimates:

$$\max\{c_k(K,\mathcal{E}), c_k(K^\circ,\mathcal{E}^\circ)\} \leqslant P_\alpha \left(\frac{n}{k}\right)^{1/\alpha} \quad \text{for all } k \in \{1,\dots,n\}. \tag{2.6}$$

Our estimates depend on a number of volumetric parameters of K, already defined in the Introduction, which we now recall:

$$w_k(K) := \sup\{\mathrm{vrad}(K\cap E) : E \in G_{n,k}\}, v_k(K) := \sup\{\mathrm{vrad}(P_E(K)) : E \in G_{n,k}\},$$

and

$$w_k^-(K) := \inf\{\mathrm{vrad}(K\cap E) : E \in G_{n,k}\}, v_k^-(K) := \inf\{\mathrm{vrad}(P_E(K)) : E \in G_{n,k}\}.$$

Note that $0 < c \leqslant w_k^-(K) v_k(K^\circ), v_k^-(K) w_k(K^\circ) \leqslant 1$ by (2.1). Also observe that $k \mapsto v_k(K)$ is non-increasing by the Alexandrov inequalities and Kubota's formula, and that $k \mapsto w_k^-(K)$ is non-decreasing by polar-integration and Jensen's inequality.

We refer to the books [26] and [32] for additional basic facts from the local theory of normed spaces.

3 New covering estimates

The main result of this section provides a general upper bound for the entropy numbers $e_k(K, B_2^n)$.

Theorem 3.1. *Let K be a centrally-symmetric convex body in $\mathbb{R}^n$, and let $k \geqslant 1$. Then:*

$$e_k(K, B_2^n) \leqslant C\frac{n}{k}\log\left(e + \frac{n}{k}\right) \sup_{1 \leqslant m \leqslant \min(k,n)} \left\{2^{-\frac{k}{3m}} w_m(K)\right\}.$$

We combine this fact with Dudley's entropy estimate

$$\sqrt{n}M^*(K) \leqslant C \sum_{k \geqslant 1} \frac{1}{\sqrt{k}} e_k(K, B_2^n). \tag{3.1}$$

(see [32, Theorem 5.5] for this formulation). As an immediate consequence, we obtain:

Corollary 3.2. *Let K be a centrally-symmetric convex body in $\mathbb{R}^n$ with $K \subseteq RB_2^n$. Then:*

$$\sqrt{n}M^*(K) \leqslant C \sum_{k \geqslant 1} \frac{1}{\sqrt{k}} \min\left\{R, \frac{n}{k}\log\left(e + \frac{n}{k}\right) \sup_{1 \leqslant m \leqslant \min(k,n)} \left\{2^{-\frac{k}{3m}} w_m(K)\right\}\right\}.$$

Dually, let K be a centrally-symmetric convex body in $\mathbb{R}^n$ with $K \supseteq rB_2^n$. Then:

$$\sqrt{n}M(K) \leqslant C \sum_{k \geqslant 1} \frac{1}{\sqrt{k}} \min\left\{\frac{1}{r}, \frac{n}{k}\log\left(e + \frac{n}{k}\right) \sup_{1 \leqslant m \leqslant \min(k,n)} \left\{2^{-\frac{k}{3m}} \frac{1}{v_m^-(K)}\right\}\right\}$$

Proof. The first claim follows by a direct application of (3.1) if we estimate $e_k(K, B_2^n)$ using Theorem 3.1 and the observation that $e_k(K, B_2^n) \leqslant R$ for all $k \geqslant 1$ (recall that $K \subseteq RB_2^n$). Then, the second claim follows by duality since $w_m(K^\circ) \simeq \frac{1}{v_m^-(K)}$. □

We will see in the next section that the supremum over m above is unnecessary and that one may always use $m = k$, only summing over $k = 1, \ldots, n$. But we proceed with the proof of Theorem 3.1, as it is a simpler approach.

Proof of Theorem 3.1. Assume without loss of generality that k is divisible by 3, and use the estimate:

$$e_k(K, B_2^n) \leqslant e_{k/3}(K, \mathcal{E}) e_{2k/3}(\mathcal{E}, B_2^n),$$

where $\mathcal{E} = \mathcal{E}_{K,\alpha_k}$ is Pisier's α_k-regular M-ellipsoid associated to K, with $\alpha_k \in [1, 2)$ to be determined. The first term is controlled directly by Pisier's regular covering estimate (2.3). For the second term we use the following simple fact about covering numbers of ellipsoids (see e.g. [32, Remark 5.15]):

$$e_j(\mathcal{E}, B_2^n) \simeq \sup_{1 \leqslant m \leqslant n} 2^{-j/m} w_m(\mathcal{E}) \simeq \sup_{1 \leqslant m \leqslant \min(j,n)} 2^{-j/m} w_m(\mathcal{E});$$

the latter equivalence follows since $w_m(\mathcal{E})$ is the geometric average of the m largest principal radii of $\mathcal{E}$, and so $m \mapsto w_m(\mathcal{E})$ is non-increasing. Now recall that

$$w_m(\mathcal{E}) \simeq 1/v_m^-(\mathcal{E}^\circ). \tag{3.2}$$

To estimate $v_m^-(\mathcal{E}^\circ)$, we use a trivial volumetric bound: for any $E \in G_{n,m}$,

$$\begin{aligned}\frac{\mathrm{vrad}(P_E(K^\circ))}{\mathrm{vrad}(P_E(\mathcal{E}^\circ))e_s(K^\circ,\mathcal{E}^\circ)} &\leqslant N(P_E(K^\circ), e_s(K^\circ,\mathcal{E}^\circ)P_E(\mathcal{E}^\circ))^{1/m} \\ &\leqslant N(K^\circ, e_s(K^\circ, E^\circ)\mathcal{E}^\circ)^{1/m} \leqslant 2^{s/m},\end{aligned}$$

for $s \geqslant 1$ to be determined. Consequently:

$$v_m^-(\mathcal{E}^\circ) \geqslant \frac{1}{2^{s/m} e_s(K^\circ,\mathcal{E}^\circ)} v_m^-(K^\circ),$$

and plugging this back into (3.2), we deduce:

$$w_m(\mathcal{E}) \leqslant C 2^{s/m} e_s(K^\circ,\mathcal{E}^\circ) w_m(K),$$

and hence:

$$e_{2k/3}(\mathcal{E}, B_2^n) \leqslant C \sup_{1 \leqslant m \leqslant \min(k,n)} 2^{\frac{s-2k/3}{m}} e_s(K^\circ,\mathcal{E}^\circ) w_m(K).$$

Setting $s = k/3$, we conclude that:

$$e_{2k/3}(\mathcal{E}, B_2^n) \leqslant C e_{k/3}(K^\circ,\mathcal{E}^\circ) \sup_{1 \leqslant m \leqslant \min(k,n)} 2^{-\frac{k}{3m}} w_m(K).$$

Combining everything, we obtain:

$$\begin{aligned} e_k(K,\mathcal{E}) &\leqslant C e_{k/3}(K,\mathcal{E}) e_{k/3}(K^\circ,\mathcal{E}^\circ) \sup_{1 \leqslant m \leqslant \min(k,n)} 2^{-\frac{k}{3m}} w_m(K) \\ &\leqslant \frac{C'}{2-\alpha_k}\left(\frac{n}{k}\right)^{\frac{2}{\alpha_k}} \sup_{1 \leqslant m \leqslant \min(k,n)} 2^{-\frac{k}{3m}} w_m(K).\end{aligned}$$

Setting $\alpha_k = 2 - \frac{1}{\log(e+n/k)}$, the assertion follows. $\square$

Remark 3.3. Theorem 3.1 implies the following dual covering estimate:

$$e_k(B_2^n, K) \leqslant C\frac{n}{k}\log\left(e+\frac{n}{k}\right) \sup_{1 \leqslant m \leqslant \min(k,n)} \left\{2^{-\frac{k}{3m}} \frac{1}{v_m^-(K)}\right\}. \tag{3.3}$$

Indeed, this is immediate from the duality of entropy theorem (2.2) and the fact that $w_m(K^\circ) \simeq \frac{1}{v_m^-(K)}$. Alternatively, one may simply repeat the proof of Theorem 3.1 with the roles of K and B_2^n exchanged.

4 New diameter estimates

This section may be read independently of the rest of this work, and contains a refinement of the following result of V. Milman and G. Pisier from [25], as exposed in [32, Lemma 9.2]:

Theorem 4.1 (Milman–Pisier)**.** *Let K be a centrally-symmetric convex body in $\mathbb{R}^n$. Then, for any $k = 1, \ldots, n/2$:*

$$c_{2k}(K) \leqslant C\frac{n}{k}\log\left(e+\frac{n}{k}\right) v_k(K).$$

In other words, there exists $F \in G_{n,n-2k}$ so that:

$$K \cap F \subseteq C\frac{n}{k}\log\Big(e+\frac{n}{k}\Big)v_k(K)B_F, \tag{4.1}$$

and dually, there exists $F \in G_{n,n-2k}$ so that:

$$P_F(K) \supseteq \frac{1}{C\frac{n}{k}\log(e+\frac{n}{k})}w_k^-(K)B_F. \tag{4.2}$$

Our version refines these estimates by replacing $v_k(K)$ and $w_k^-(K)$ above by the stronger $w_k(K)$ and $v_k^-(K)$ parameters, respectively; this refinement is crucial for our application in this paper.

Theorem 4.2. *Let K be a centrally-symmetric convex body in $\mathbb{R}^n$. Then for any $k = 1,\dots,n/2$:*

$$c_{2k}(K) \leqslant C\frac{n}{k}\log\Big(e+\frac{n}{k}\Big)w_k(K).$$

In other words, there exists $F \in G_{n,n-2k}$ so that:

$$K \cap F \subseteq C\frac{n}{k}\log\Big(e+\frac{n}{k}\Big)w_k(K)B_F, \tag{4.3}$$

and dually, there exists $F \in G_{n,n-2k}$ so that:

$$P_F(K) \supseteq \frac{1}{C\frac{n}{k}\log(e+\frac{n}{k})}v_k^-(K)B_F. \tag{4.4}$$

Our refinement will come from exploiting the full strength of Pisier's result on the existence of regular M-ellipsoids. In contrast, the Milman–Pisier result is based on V. Milman's quotient-of-subspace theorem, from which it seems harder to obtain enough regularity to deduce our proposed refinement.

Proof of Theorem 4.2. Given $k = 1,\dots,n/2$, let $\mathcal{E} = \mathcal{E}_{K,\alpha_k}$ denote Pisier's α_k-regular M-ellipsoid, for some $\alpha_k \in [1,2)$ to be determined. By the second estimate in (2.6), we know that there exists $E \in G_{n,n-k}$ so that:

$$P_E(K) \supseteq \frac{1}{P_{\alpha_k}}\left(\frac{k}{n}\right)^{1/\alpha_k} P_E(\mathcal{E}).$$

For the ellipsoid $\mathcal{E}' := P_E(\mathcal{E}) \subseteq E$, we may always find a linear subspace $F \subseteq E$ of codimension m in E so that:

$$P_F(\mathcal{E}') \supseteq \inf_{H\in G_m(E)}\sup_{H'\subseteq H}\{\mathrm{vrad}(P_{H'}(\mathcal{E}'))\}\,B_F,$$

where $G_m(E)$ is the Grassmannian of all m-dimensional linear subspaces of E. Indeed, this is immediate by choosing H to be the subspace spanned by the m shortest axes of $\mathcal{E}'$, and setting F to be its orthogonal complement. Consequently, there exists a subspace $F \in G_{n,n-(k+m)}$ so that:

$$P_F(K) \supseteq \frac{1}{P_{\alpha_k}}\left(\frac{k}{n}\right)^{1/\alpha_k}\inf_{H\in G_{n,m}}\sup_{H'\subseteq H}\{\mathrm{vrad}(P_{H'}(\mathcal{E}))\}\,B_F. \tag{4.5}$$

We now deviate from the proof of our refined version, to show how one may recover the Milman–Pisier estimate. Assume for simplicity that $k < n/3$. By the first estimate in (2.6), we know that there exists $J \in G_{n,n-k}$ so that:

$$K \cap J \subseteq P_{\alpha_k} \left(\frac{n}{k}\right)^{1/\alpha_k} \mathcal{E} \cap J.$$

Given $H \in G_{n,m}$ and denoting $H' := H \cap J \in G_{m'}(H)$ with $m' \in [m-k, m]$, it follows that:

$$P_{H'}(\mathcal{E}) \supseteq \mathcal{E} \cap H' \supseteq \frac{1}{P_{\alpha_k}} \left(\frac{k}{n}\right)^{1/\alpha_k} K \cap H'.$$

Setting $m = 2k$, it follows from (4.5) that there exists $F \in G_{n,n-3k}$ so that:

$$P_F(K) \supseteq \frac{1}{P_{\alpha_k}^2} \left(\frac{k}{n}\right)^{2/\alpha_k} \inf \{\mathrm{vrad}(K \cap H') \,:\, H' \in G_{n,m'}\,,\ m' \in [k, 2k]\}\, B_F.$$

Noting that the sequence $m' \mapsto w^-_{m'}(K)$ is non-decreasing, and setting $\alpha_k = 2 - \frac{1}{\log(e+n/k)}$, we have found $F \in G_{n,n-3k}$ such that

$$P_F(K) \supseteq \frac{c}{\frac{n}{k}\log(e+\frac{n}{k})} w^-_k(K),$$

as asserted in (4.2) (with perhaps an immaterial constant 3 instead of 2). The assertion of (4.1) follows by duality.

To obtain our refinement, we will use instead of the first estimate in (2.6), the covering estimate (2.3) (which Pisier obtains from (2.6) by an application of Carl's theorem, requiring the entire sequence of c_k estimates, not just the one for our specific k). Setting $m = k$, we use a trivial volumetric estimate to control $\mathrm{vrad}(P_H(\mathcal{E}))$, exactly as in the proof of Theorem 3.1: for any $H \in G_{n,k}$,

$$\frac{\mathrm{vrad}(P_H(K))}{\mathrm{vrad}(P_H(\mathcal{E}))e_k(K,\mathcal{E})} \leqslant N(P_H(K), e_k(K,\mathcal{E})P_H(\mathcal{E}))^{1/k} \leqslant N(K, e_k(K,E)\mathcal{E})^{1/k} \leqslant 2.$$

Together with (2.3), we obtain:

$$\mathrm{vrad}(P_H(\mathcal{E})) \geqslant \frac{1}{2e_k(K,\mathcal{E})}\mathrm{vrad}(P_H(K)) \geqslant \frac{1}{2P_{\alpha_k}} \left(\frac{k}{n}\right)^{1/\alpha_k} \mathrm{vrad}(P_H(K)).$$

Plugging this into (4.5) and setting as usual $\alpha_k = 2 - \frac{1}{\log(e+n/k)}$, the asserted estimate (4.4) follows. The other estimate (4.3) follows by duality. $\square$

As immediate corollaries, we have:

Corollary 4.3. *For every centrally-symmetric convex body K in $\mathbb{R}^n$, $k = 1, \ldots, n$ and $\alpha > 0$:*

$$e_k(K, B_2^n) \leqslant C_\alpha \sup_{m=1,\ldots,k} \left(\frac{m}{k}\right)^\alpha \frac{n}{m} \log\left(e + \frac{n}{m}\right) w_m(K),$$

where $C_\alpha > 0$ is a constant depending only on α.

Proof. This is immediate from Theorem 4.2 and Carl's theorem (2.4). Note that $k \mapsto c_k(K, B_2^n)$ is non-increasing, and so there is no difference whether we take the supremum on the right-hand-side just on the even integers. $\square$

Corollary 4.4. *For every centrally-symmetric convex body K in $\mathbb{R}^n$ so that $K \subseteq RB_2^n$, we have:*

$$\sqrt{n}M^*(K) \leqslant C \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \min\left(R, \frac{n}{k} \log\left(e + \frac{n}{k}\right) w_k(K)\right).$$

Dually, for every centrally-symmetric convex body K in $\mathbb{R}^n$ so that $K \supseteq rB_2^n$, we have:

$$\sqrt{n}M(K) \leqslant C \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \min\left(\frac{1}{r}, \frac{n}{k} \log\left(e + \frac{n}{k}\right) \frac{1}{v_k^-(K)}\right).$$

Proof. Let us verify the first claim, the second follows by duality. Indeed, this is immediate from Dudley's entropy estimate (1.6) coupled with Carl's theorem (2.5):

$$\sqrt{n}M^*(K) \leqslant C \sum_{k=1}^{n} \frac{1}{\sqrt{k}} e_k(K) \leqslant C' \sum_{k=1}^{n} \frac{1}{\sqrt{k}} c_k(K).$$

Obviously $c_k(K) \leqslant R$ for all k, and so the assertion follows from the estimates of Theorem 4.2. $\square$

Both Corollaries should be compared with the results of the previous section.

Remark 4.5. It may be insightful to compare Theorem 4.2 to some other known estimates on diameters of k-codimensional sections, besides the Milman–Pisier Theorem 4.1. One sharp estimate is the Pajor–Tomczak-Jaegermann refinement [27] of V. Milman's low-M^* estimate [22]:

$$c_k(L) \leq C\sqrt{\frac{n}{k}} M^*(L), \tag{4.6}$$

for any origin-symmetric convex L and $k = 1, \ldots, n$. However, for our application, we cannot use this to control $c_k(K^\circ)$ since we do not a-priori know $M^*(K^\circ) = M(K)$. A type of dual low-M estimate was observed by Klartag [13]:

$$c_k(L) \leq C^{\frac{n}{k}} \mathrm{vrad}(L)^{\frac{n}{k}} M(L)^{\frac{n-k}{k}}.$$

Since $M(K^\circ) = M^*(K)$ is now well understood for an isotropic origin-symmetric convex body [21], this would give good estimates for low-dimensional sections (large codimension k), but unfortunately this is not enough for controlling $M(K)$. Klartag obtains the latter estimate from the following one, which is more in the spirit of the estimates we obtain in this work:

$$c_k(L) \leq C^{\frac{n}{k}} \frac{\mathrm{vrad}(L)^{\frac{n}{k}}}{w_{n-k}(L)^{\frac{n-k}{k}}}.$$

Again, this seems too rough for controlling the diameter of high-dimensional sections.

5 Preliminaries from asymptotic convex geometry

An absolutely continuous Borel probability measure μ on $\mathbb{R}^n$ is called log-concave if its density f_μ is of the form $\exp(-\varphi)$ with $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex. Note that the uniform probability measure on K, denoted λ_K, is log-concave for any convex body K.

The barycenter of μ is denoted by $\mathrm{bar}(\mu) := \int_{\mathbb{R}^n} x d\mu(x)$. The isotropic constant of μ, denoted L_μ, is the following affine invariant quantity:

$$L_\mu := (\sup_{x \in \mathbb{R}^n} f_\mu(x))^{\frac{1}{n}} \det \mathrm{Cov}(\mu)^{\frac{1}{2n}}, \tag{5.1}$$

where $\mathrm{Cov}(\mu) := \int x \otimes x d\mu(x) - \int x d\mu(x) \otimes \int x d\mu(x)$ denotes the covariance matrix of μ. We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if $\mathrm{bar}(\mu) = 0$ and $\mathrm{Cov}(\mu)$ is the identity matrix. Note that a convex body K of volume 1 is isotropic if and only if the log-concave probability measure λ_{K/L_K} is isotropic, and that L_{λ_K} indeed coincides with L_K. It was shown by K. Ball [2, 3] that given $n \geqslant 1$:

$$\sup_\mu L_\mu \leqslant C \sup_K L_K,$$

where the suprema are taken over all log-concave probability measures μ and convex bodies K in $\mathbb{R}^n$, respectively (see e.g. [14] for the non-even case). Klartag's bound on the isotropic constant [14] thus reads $L_\mu \leqslant C n^{1/4}$ for all log-concave probability measures μ on $\mathbb{R}^n$.

Given $E \in G_{n,k}$, we denote by $\pi_E \mu := \mu \circ P_E^{-1}$ the push-forward of μ via P_E. Obviously, if μ is centered or isotropic then so is $\pi_E \mu$, and by the Prékopa–Leindler theorem, the same also holds for log-concavity.

Given a log-concave probability measure μ on $\mathbb{R}^n$ and $q \geqslant 1$, the L_q-centroid body of μ, denoted $Z_q(\mu)$, is the centrally-symmetric convex body with support function:

$$h_{Z_q(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^q d\mu(x) \right)^{1/q}. \tag{5.2}$$

Observe that μ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$. By Jensen's inequality $Z_1(\mu) \subseteq Z_p(\mu) \subseteq Z_q(\mu)$ for all $1 \leqslant p \leqslant q < \infty$. Conversely, it follows from work of Berwald [4] or by employing Borell's lemma (see [26, Appendix III]), that:

$$1 \leqslant p \leqslant q \implies Z_q(\mu) \subseteq C \frac{q}{p} Z_p(\mu).$$

When $\mu = \lambda_K$ is the uniform probability measure on a centrally-symmetric convex body K in $\mathbb{R}^n$, it is easy to check (e.g. [9]) using the Brunn–Minkowski inequality that:

$$cK \subseteq Z_n(\lambda_K) \subseteq K.$$

Let μ denote an isotropic log-concave probability measure μ on $\mathbb{R}^n$. It was shown by Paouris [28] that

$$1 \leqslant q \leqslant \sqrt{n} \implies M^*\big(Z_q(\mu)\big) \simeq \sqrt{q}, \tag{5.3}$$

and that:

$$1 \leqslant q \leqslant n \implies \mathrm{vrad}(Z_q(\mu)) \leqslant C\sqrt{q}. \tag{5.4}$$

Conversely, it was shown by Klartag and E. Milman in [16] that:

$$1 \leqslant q \leqslant \sqrt{n} \quad \Longrightarrow \quad \mathrm{vrad}(Z_q(\mu)) \geqslant c_1\sqrt{q}. \tag{5.5}$$

This determines the volume radius of $Z_q(\mu)$ for all $1 \leqslant q \leqslant \sqrt{n}$. For larger values of q one can still use the lower bound:

$$1 \leqslant q \leqslant n \quad \Longrightarrow \quad \mathrm{vrad}(Z_q(\mu)) \geqslant c_2\sqrt{q}\, L_\mu^{-1}, \tag{5.6}$$

obtained by Lutwak, Yang and Zhang [20] via symmetrization.

We refer to the book [9] for further information on isotropic convex bodies and log-concave measures.

6 M-estimates for isotropic convex bodies and their L_q-centroid bodies

Let μ denote an isotropic log-concave probability measure on $\mathbb{R}^n$, and fix $H \in G_{n,k}$. A very useful observation is that:

$$P_H\big(Z_q(\mu)\big) = Z_q\big(\pi_H(\mu)\big).$$

It follows from (5.5) that:

$$1 \leqslant q \leqslant \sqrt{k} \quad \Longrightarrow \quad \mathrm{vrad}(P_H(Z_q(\mu))) \geqslant c\sqrt{q}. \tag{6.1}$$

Furthermore, using (5.6), we see that:

$$q \geq \sqrt{k} \quad \Longrightarrow \quad \mathrm{vrad}(P_H(Z_q(\mu))) \geqslant c' \max\left(\sqrt[4]{k}, \frac{\sqrt{\min(q,k)}}{L_{\pi_H\mu}}\right). \tag{6.2}$$

Unfortunately, we can only say in general that $\sup\{L_{\pi_H\mu} : H \in G_{n,k}\} \leqslant C\sqrt[4]{k}$, and so the estimate (6.2) is not very useful, unless we have some additional information on μ. Recalling the definition of $v_k^-(Z_q(\mu))$, we summarize this (somewhat sloppily) in:

Lemma 6.1. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $q \geqslant 1$ and $k = 1, \ldots, n$ we have:*

$$v_k^-(Z_q(\mu)) \geqslant c\sqrt{\min(q, \sqrt{k})}.$$

Assuming that $\sup\{L_{\pi_H\mu} : H \in G_{n,k}\} \leqslant A_k$ *we have:*

$$v_k^-(Z_q(\mu)) \geqslant \frac{c'}{A_k}\sqrt{\min(q,k)}.$$

6.1 Estimates for $Z_q(\mu)$

Plugging these lower bounds for $v_k^-(Z_q(\mu))$ into either Theorem 3.1 or Corollary 4.3 coupled with Remark 3.3, we immediately obtain estimates on the entropy numbers $e_k(B_2^n, Z_q(\mu))$. Similar estimates on the maximal (with respect to $F \in G_{n,n-k}$) in-radius of $P_F(Z_q(\mu))$ are obtained by invoking Theorem 4.2.

Theorem 6.2. *Given $q \geqslant 2$ and an integer $k = 1, \dots, n$, denote:*

$$R_{k,q} := \min\left\{1, C\frac{1}{\min(\sqrt{q}, \sqrt[4]{k})}\frac{n}{k}\log\left(e+\frac{n}{k}\right)\right\}.$$

Then, for any isotropic log-concave probability measure μ on $\mathbb{R}^n$:

$$e_k(B_2^n, Z_q(\mu)) \leqslant R_{k,q},$$

and there exists $F \in G_{n,n-k}$ so that:

$$P_F(Z_q(\mu)) \supseteq \frac{1}{R_{k,q}} B_F.$$

Proof. From (3.3) and Lemma 6.1 we have

$$e_k(B_2^n, Z_q(\mu)) \leqslant C\frac{n}{k}\log\left(e+\frac{n}{k}\right)\sup_{1\leqslant m\leqslant \min(k,n)}\left\{2^{-\frac{k}{3m}}\frac{1}{\min(\sqrt{q}, \sqrt[4]{m})}\right\}.$$

Then, it suffices to observe that

$$\begin{aligned}\sup_{1\leqslant m\leqslant \min(k,n)}\left\{2^{-\frac{k}{3m}}\frac{1}{\min(\sqrt{q}, \sqrt[4]{m})}\right\} &\simeq \sup_{1\leqslant m\leqslant \min(k,n)}\left\{2^{-\frac{k}{3m}}\left(\frac{1}{\sqrt{q}}+\frac{1}{\sqrt[4]{m}}\right)\right\}\\ &\leqslant C\left(\frac{1}{\sqrt{q}}+\frac{1}{\sqrt[4]{k}}\right) \simeq \frac{1}{\min(\sqrt{q}, \sqrt[4]{k})},\end{aligned}$$

because $2^{-\frac{k}{3m}}/\sqrt{q} \leqslant 1/\sqrt{q}$ for all $1 \leqslant m \leqslant k$, and the function $m \mapsto 2^{\frac{k}{3m}}\sqrt[4]{m}$ attains its minimum at $m = k$, which shows that $\sup_{1\leqslant m\leqslant\min(k,n)}(2^{-\frac{k}{3m}}/\sqrt[4]{m}) \leqslant 1/\sqrt[4]{k}$. We also use the fact that in a certain range of values for $q \geqslant 2$ and $k \geqslant 1$, we might as well use the trivial estimates:

$$e_k(B_2^n, Z_q(\mu)) \leqslant 1\ ,\ \ P_F(Z_q(\mu)) \supseteq B_F, \tag{6.3}$$

which hold since $Z_q(\mu) \supseteq Z_2(\mu) = B_2^n$. □

An elementary computation based on Corollary 4.4 then yields a non-trivial estimate for $M(Z_q(\mu))$. It is interesting to note that without using the trivial information that $Z_q(\mu) \supseteq B_2^n$ (or equivalently, the trivial estimates in (6.3)), Corollary 4.4 would not yield anything meaningful.

Theorem 6.3. *For any isotropic log-concave probability measure μ on $\mathbb{R}^n$:*

$$2 \leqslant q \leqslant q_0 := (n\log(e+n))^{2/5} \quad\Longrightarrow\quad M(Z_q(\mu)) \leqslant C\frac{\sqrt{\log q}}{\sqrt[4]{q}}.$$

Proof. We use the estimate

$$\sqrt{n}M(Z_q(\mu)) \leqslant C\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\min\left\{1, C\frac{1}{\min(\sqrt{q}, \sqrt[4]{k})}\frac{n}{k}\log\left(e+\frac{n}{k}\right)\right\},$$

which follows from Corollary 4.4 combined with Theorem 6.2. We set $k(n,q) = n\log q/\sqrt{q}$. Note that if $k \geqslant k(n,q)$ then $k \geqslant cq^2$. Therefore, we may write

$$\begin{aligned}\sqrt{n}M(Z_q(\mu)) &\leqslant C\sum_{k=1}^{k(n,q)}\frac{1}{\sqrt{k}} + \frac{Cn}{\sqrt{q}}\sum_{k=k(n,q)}^{n}\frac{1}{k^{3/2}}\log\left(e+\frac{n}{k}\right)\\ &\leqslant C_1\sqrt{k(n,q)} + C_2\frac{n\log q}{\sqrt{qk(n,q)}} \leqslant C_3\frac{\sqrt{n\log q}}{\sqrt[4]{q}}.\end{aligned}$$

The result follows. $\square$

For larger values of q, we obtain no additional information beyond the trivial monotonicity:

$$q_0 \leqslant q \quad\Longrightarrow\quad M(Z_q(\mu)) \leqslant M(Z_{q_0}(\mu)) \leqslant C\frac{\log^{2/5}(e+n)}{n^{1/10}}.$$

If K is an isotropic centrally-symmetric convex body in $\mathbb{R}^n$, using that λ_{K/L_K} is isotropic log-concave and that $Z_n(\lambda_{K/L_K})$ is isomorphic to K/L_K, one immediately translates the above results to corresponding estimates for K.

Theorem 6.4. *Given $k = 1,\ldots,n$, set:*

$$R_k := \min\left\{1, C\frac{1}{\sqrt[4]{k}}\frac{n}{k}\log\left(e+\frac{n}{k}\right)\right\}.$$

Then, for any isotropic centrally-symmetric convex body K in $\mathbb{R}^n$:

$$e_k(B_2^n, K) \leqslant \frac{R_k}{L_K},$$

and there exists $F \in G_{n,n-k}$ so that:

$$P_F(K) \supseteq \frac{L_K}{R_k}B_F.$$

Moreover:

$$M(K) \leqslant \frac{C}{L_K}\frac{\log^{2/5}(e+n)}{n^{1/10}}.$$

6.2 Assuming that the isotropic constant is bounded

It is interesting to perform the same calculations under the assumption that $L_\mu \leqslant C$ for any log-concave probability measure μ (regardless of dimension). In that case:

$$v_k^-(Z_q(\mu)) \geqslant c\sqrt{\min(q,k)}.$$

This would yield the following conditional result:

Theorem 6.5. *Given $q \geqslant 2$ and an integer $k = 1,\ldots,n$, denote:*

$$R_{k,q} := \min\left\{1, C\frac{1}{\sqrt{\min(q,k)}}\frac{n}{k}\log\left(e+\frac{n}{k}\right)\right\}.$$

Assuming that $L_\mu \leqslant C$ for any log-concave probability measure (regardless of dimension), then for any isotropic log-concave probability measure μ on $\mathbb{R}^n$:

$$e_k(B_2^n, Z_q(\mu)) \leqslant R_{k,q},$$

and there exists $F \in G_{n,n-k}$ so that:

$$P_F(Z_q(\mu)) \supseteq \frac{1}{R_{k,q}} B_F.$$

Furthermore:

$$M(Z_q(\mu)) \leqslant C \frac{\sqrt{\log q}}{\sqrt[4]{q}} \quad \text{for all } \; 2 \leqslant q \leqslant (n \log n)^{2/3}.$$

Consequently, for every isotropic convex body K in $\mathbb{R}^n$ one would have:

$$M(K) \leqslant C \frac{\log^{1/3}(e+n)}{n^{1/6}}.$$

6.3 ψ_α–measures

Finally, rather than assuming that L_μ is always bounded, we repeat the calculations for a log-concave measure μ which is assumed to be ψ_α-regular. Recall that μ is called ψ_α with constant b_α ($\alpha \in [1,2]$) if:

$$Z_q(\mu) \subseteq b_\alpha q^{1/\alpha} Z_2(\mu) \quad \text{for all } q \geqslant 2.$$

Note that this property is inherited by all marginals of μ, and that any log-concave measure is ψ_1 with $b_1 = C$ a universal constant.

It was shown by Klartag and E. Milman [16] that when μ is a ψ_α log-concave probability measure on $\mathbb{R}^n$ with constant b_α, then:

$$1 \leqslant q \leqslant C \frac{n^{\frac{\alpha}{2}}}{b_\alpha^\alpha} \quad \Longrightarrow \quad \mathrm{vrad}(Z_q(\mu)) \geqslant c\sqrt{q},$$

and:

$$L_\mu \leqslant C \sqrt{b_\alpha^\alpha n^{1-\alpha/2}}.$$

This implies that for such a measure, for any $H \in G_{n,k}$:

$$1 \leqslant q \leqslant C \frac{k^{\frac{\alpha}{2}}}{b_\alpha^\alpha} \quad \Longrightarrow \quad \mathrm{vrad}(P_H(Z_q(\mu))) \geqslant c\sqrt{q}.$$

By (5.6), we know that:

$$q \geq q_0 := C \frac{k^{\frac{\alpha}{2}}}{b_\alpha^\alpha} \quad \Longrightarrow \quad \mathrm{vrad}(P_H(Z_q(\mu))) \geqslant c' \max\left(\sqrt{q_0}, \frac{\sqrt{\min(q,k)}}{L_{\pi_H \mu}}\right). \tag{6.4}$$

Unfortunately, since we only know that:

$$L_{\pi_H \mu} \leqslant C \sqrt{b_\alpha^\alpha k^{1-\alpha/2}},$$

we again see that the maximum in (6.4) is always attained by the $\sqrt{q_0}$ term. Summarizing, we have:

Lemma 6.6. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ which is ψ_α with constant b_α for some $\alpha \in [1,2]$. Then for any $q \geqslant 1$ and $k = 1, \ldots, n$ we have:*

$$v_k^-(Z_q(\mu)) \geqslant c\sqrt{\min\left(q, \frac{k^{\alpha/2}}{b_\alpha^\alpha}\right)}.$$

Plugging this estimate into the general results of Sections 3 and 4, we obtain:

Theorem 6.7. *Let μ denote an isotropic log-concave probability measure on $\mathbb{R}^n$ which is ψ_α with constant b_α for some $\alpha \in [1,2]$. Given $q \geqslant 2$ and an integer $k = 1, \ldots, n$, denote:*

$$R_{k,q} := \min\left\{1, C\frac{1}{\sqrt{\min\left(q, \frac{k^{\alpha/2}}{b_\alpha^\alpha}\right)}}\frac{n}{k}\log\left(e + \frac{n}{k}\right)\right\}.$$

Then:

$$e_k(B_2^n, Z_q(\mu)) \leqslant R_{k,q},$$

and there exists $F \in G_{n,n-k}$ so that:

$$P_F(Z_q(\mu)) \supseteq \frac{1}{R_{k,q}}B_F.$$

Furthermore:

$$M(Z_q(\mu)) \leqslant C\frac{\sqrt{\log q}}{\sqrt[4]{q}} \quad \text{for all } 2 \leqslant q \leqslant c\frac{(n\log(e+n))^{\frac{2\alpha}{\alpha+4}}}{b_\alpha^{\frac{4\alpha}{\alpha+4}}}.$$

Consequently, for every isotropic convex body K in $\mathbb{R}^n$ so that λ_K is ψ_α with constant b_α, one has:

$$M(K) \leqslant \frac{C}{L_K}b_\alpha^{\frac{\alpha}{\alpha+4}}\frac{\log^{\frac{2}{\alpha+4}}(e+n)}{n^{\frac{\alpha}{2(\alpha+4)}}}.$$

Remark 6.8. Better estimates for the entropy-numbers $e_k(B_2^n, Z_q(\mu))$ and Gelfand numbers $c_k(Z_q(\mu)^\circ)$ may be obtained for various ranges of k by employing the alternative known estimates mentioned in Remark 4.5. However, these do not result in improved estimates on $M(Z_q(\mu))$, which was our ultimate goal. We therefore leave these improved estimates on the entropy and Gelfand numbers to the interested reader. We only remark that even the classical low-M^* estimate (4.6) coupled with our estimate on $M(Z_q(\mu))$ yield improved estimates for e_k and c_k in a certain range - a type of "bootstrap" phenomenon.

7 Concluding remarks

In this section we briefly describe an improved and simplified version of the arguments from [12] and compare the resulting improved estimates to the ones from the previous section. Following the general approach we employ in this work, the arguments are presented for general centrally–symmetric convex bodies, and this in fact further simplifies the exposition of [12].

We mainly concentrate on presenting an alternative proof of the following slightly weaker variant of Theorem 6.2:

Theorem 7.1. *Let K be a centrally-symmetric convex body in $\mathbb{R}^n$. For any $k = 1, \dots, \lfloor n/2 \rfloor$ there exists $F \in G_{n,n-2k}$ such that:*

$$P_F(K) \supseteq \frac{c}{\frac{n}{k}\log^2\left(e+\frac{n}{k}\right)} v_k^-(K)\, B_F$$

where $c > 0$ is an absolute constant.

For the proof of Theorem 7.1, we use a sort of converse to Carl's theorem (2.4) on the diameter of sections of a convex body satisfying 2-regular entropy estimates, which is due to V. Milman [23] (see also [9, Chapter 9]).

Lemma 7.2. *Let L be a symmetric convex body in $\mathbb{R}^n$. Then:*

$$\sqrt{k}\, c_k(L, B_2^n) \leqslant C \log(e + n/k) \sup_{k \leqslant m \leqslant n} \sqrt{m}\, e_m(L, B_2^n).$$

Remark 7.3. Clearly, by applying a linear transformation, the statement equally holds with B_2^n replaced by an arbitrary ellipsoid.

Proof of Theorem 7.1. Given $k = 1, \dots, \lfloor n/2 \rfloor$, let $\mathcal{E} = \mathcal{E}_{K,\alpha_k}$ denote Pisier's α_k-regular M-ellipsoid, for some $\alpha_k \in [1, 2)$ to be determined. Instead of directly using Pisier's estimate (2.6) on the Gelfand numbers as in the proof of Theorem 4.2 to deduce the existence of $E \in G_{n,n-k}$ so that:

$$P_E(K) \supseteq \frac{1}{P_{\alpha_k}} \left(\frac{k}{n}\right)^{1/\alpha_k} P_E(\mathcal{E}), \tag{7.1}$$

the starting point in [12] are the more traditional covering estimates (2.3):

$$\max\{e_k(K, \mathcal{E}), e_k(K^\circ, \mathcal{E}^\circ), e_k(\mathcal{E}, K), e_k(\mathcal{E}^\circ, K^\circ)\} \leqslant P_\alpha \left(\frac{n}{k}\right)^{1/\alpha_k}. \tag{7.2}$$

In [12], the following estimate was used (see [32, Theorem 5.14]):

$$c_k(K^\circ, \mathcal{E}^\circ) \leqslant C\sqrt{\frac{n}{k}} e_k(K^\circ, \mathcal{E}^\circ).$$

However, this estimate does not take into account the regularity of the covering. Consequently, a significantly improved estimate is obtained by employing Lemma 7.2 (and the subsequent remark) which exploits this regularity:

$$\begin{aligned}
\sqrt{k}\, c_k(K^\circ, \mathcal{E}^\circ) &\leqslant C \log(e + n/k) \sup_{k \leqslant m \leqslant n} \sqrt{m} e_m(K^\circ, \mathcal{E}^\circ) \\
&\leqslant C \log(e + n/k) \sup_{k \leqslant m \leqslant n} \sqrt{m}\, P_{\alpha_k} \left(\frac{n}{m}\right)^{1/\alpha_k}.
\end{aligned}$$

Even with this improvement, note that this is where the current approach incurs some unnecessary logarithmic price with respect to the approach in the previous sections: instead of using (7.1) directly, one uses (7.2) which Pisier obtains from (7.1) by applying Carl's theorem, and then uses the converse to Carl's theorem (Lemma 7.2) to pass back to Gelfand number estimates.

Using $\alpha_k = 2 - \frac{1}{\log(e+n/k)}$, we deduce that:

$$c_k(K^\circ, \mathcal{E}^\circ) \leqslant C\sqrt{\frac{n}{k}} \log^{3/2}(e+n/k),$$

or in other words, the existence of $E \in G_{n,n-k}$ such that:

$$P_E(K) \supseteq \frac{1}{C\sqrt{\frac{n}{k}}\log^{3/2}(e+n/k)} P_E(\mathcal{E}).$$

The rest of the proof is identical to that of Theorem 4.2. For the ellipsoid $\mathcal{E}' := P_E(\mathcal{E})$ we may always find a linear subspace $F \subseteq E$ of codimension k in E so that:

$$P_F(\mathcal{E}') \supseteq \inf_{H \in G_k(E)} \{\mathrm{vrad}(P_H(\mathcal{E}'))\} B_F.$$

Estimating $\mathrm{vrad}(P_H(\mathcal{E}')) = \mathrm{vrad}(P_H(\mathcal{E}))$ by comparing to $\mathrm{vrad}(P_H(K))$ via the dual covering estimate on $e_k(K,\mathcal{E})$ (note that there is no need to use the duality of entropy theorem here), we obtain:

$$\mathrm{vrad}(P_H(\mathcal{E}')) \geqslant \frac{1}{2e_k(K,\mathcal{E})}\mathrm{vrad}(P_H(K)) \geqslant \frac{1}{2C\sqrt{\frac{n}{k}}\log^{1/2}(e+n/k)}\mathrm{vrad}(P_H(K)).$$

Combining all of the above, we deduce the existence of $F \in G_{n,n-2k}$ so that:

$$P_F(K) \supseteq \frac{1}{C'\frac{n}{k}\log^2(e+n/k)}\mathrm{vrad}(P_H(K))B_F.$$

This concludes the proof. □

Having obtained a rather regular estimate on the Gelfand numbers, the next goal is to obtain an entropy estimate. To this end, one can use Carl's theorem (2.4) or (2.5), as we do in Section 4. The approach in [12] proceeds by employing an entropy extension theorem of Litvak, V. Milman, Pajor and Tomczak-Jaegermann [19]. We remark that this too may be avoided, by employing the following elementary covering estimate (see e.g. [9, Chapter 9]):

Lemma 7.4. *Let K be a symmetric convex body in $\mathbb{R}^n$ and assume that $B_2^n \subseteq \rho K$ for some $\rho \geqslant 1$. Let W be a subspace of $\mathbb{R}^n$ with $\dim W = m$ and $P_{W^\perp}(K) \supseteq B_{W^\perp}$. Then, we have*

$$N(B_2^n, 4K) \leqslant (3\rho)^m.$$

Finally, having a covering estimate at hand, the estimate on $M(K)$ is obtained by Dudley's entropy bound (1.6). Plugging in the lower bounds on $v_k^-(Z_q(\mu))$ given in Section 6, the results of [12] are recovered and improved.

As the reader may wish to check, the improved approach of this section over the arguments of [12] yields estimates which are almost as good as the ones obtained in Section 6, and only lose by logarithmic terms.

Acknowledgement. The first named author acknowledges support from the programme "ΑΡΙΣΤΕΙΑ ΙΙ" of the General Secretariat for Research and Technology of Greece. The second named author is supported by ISF (grant no. 900/10), BSF (grant no. 2010288), Marie-Curie Actions (grant no. PCIG10-GA-2011-304066) and the E. and J. Bishop Research Fund.

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