Operator-algebraic Aspects of Free Probability Theory

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- **von Neumann algebras**, which are $*$-subalgebras of $\mathcal{B}(H)$ closed in the topology of pointwise convergence.
  In the *abelian* case: $L_\infty(X, \mu)$ with μ a positive measure on X.

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- Is is true that $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$?

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for all $h \in G$.
Then, the induced map

$$\begin{aligned} \lambda : G &\to \mathcal{B}(\ell_2(G)) \\ g &\mapsto \lambda(g) \end{aligned}$$

is an injective unitary representation of G called the *left regular representation*. We similarly define the *right regular representation* $\rho : G \to \mathcal{B}(\ell_2(G))$.

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3. *tracial*, i.e. $\tau(TS) = \tau(ST)$, for all $T, S \in L(G)$.

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  This is known as the **Free Group Factor Isomorphism Problem**.
  Remains an **open** problem, but...

Theorem

Let A be a C^*-algebra generated by self-adjoint generators $a_1, \ldots, a_n$ and let B be another C^*-algebra generated by self-adjoint generators $b_1, \ldots, b_n$.

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$$\varphi : A \to \mathbb{C} \text{ and } \psi : B \to \mathbb{C} \, .$$

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If

$$\varphi\left(a_{i(1)}^{\varepsilon(1)} \cdot a_{i(2)}^{\varepsilon(2)} \cdot \ldots \cdot a_{i(k)}^{\varepsilon(k)}\right) = \psi\left(b_{i(1)}^{\varepsilon(1)} \cdot b_{i(2)}^{\varepsilon(2)} \cdot \ldots \cdot b_{i(k)}^{\varepsilon(k)}\right)$$

for all $k \in \mathbb{N}$,$1 \le i(1), \ldots, i(k) \le n$ and $\varepsilon(1), \ldots, \varepsilon(k) \in \{1, *\}$, then the mapping

$$a_i \mapsto b_i$$

extends to an (isometric) $*$-isomorphism between A and B.

As a result, if A is equipped with a faithful state φ, then the isomorphism class of A depends only on the **non-commutative distribution** of its generators $a_1, \ldots, a_n$, i.e. the family of complex numbers given by

$$\varphi\left(a_{i(1)}^{\epsilon(1)} \cdot a_{i(2)}^{\epsilon(2)} \cdots a_{i(k)}^{\epsilon(k)}\right)$$

where $1 \leq i(1), i(2), \ldots, i(k) \leq n$ and $\epsilon(1), \ldots, \epsilon(k) \in \{1, *\}$.

Definition

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We are interested in the *non-commutative distributions* of elements of A. For $a_1, \ldots, a_n \in A$, these are given by

$$\{\varphi(c_1 \cdots c_k) : k \geq 1, c_i \in \{a_1, \ldots, a_n, a_1^*, \ldots, a_n^*\} \text{ for all } i = 1, \ldots, k\}.$$

Examples

1. If (X, Σ, μ) is a (classical) probability space, then the pair (A, φ) where

$$A = L_\infty(X, \Sigma, \mu)$$

and

$$\varphi : A \to \mathbb{C}, \;\; \varphi(f) = \int_X f d\mu$$

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is a non-commutative probability space.

2. $(M_n(\mathbb{C}), tr)$, where tr is the canonical normalized trace.
3. If G is a group, then the algebras

$$\mathrm{C}^*_{\mathrm{red}}(G) \text{ and } L(G)$$

are non-commutative probability spaces when equipped with the canonical faithful tracial state τ.

Definition

A family $(A_i)_{i \in I}$ of unital subalgebras of (A, φ) is called **freely independent** if

$$\varphi(a_1 \cdots a_k) = 0$$

whenever

1. $a_j \in A_{i_j}$, for all $j = 1 \ldots, k$,
2. $\varphi(a_j) = 0$, for all $j = 1, \ldots, k$,
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This is a non-commutative concept; if a, b are classical random variables that are freely independent, then one of them must be almost surely a constant.

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If $\{G_i\}_{i \in I}$ is a family of subgroups of a group G, then $\{G_i\}_{i \in I}$ is free in G if and only if the family of algebras $\{L(G_i)\}_{i \in I}$ are freely independent in the non-commutative probability space $(L(G), \tau)$

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Theorem

If the family of algebras $(A_i)_{i\in I}$ is freely independent and generates A, then the state φ is uniquely determined by the restrictions $\varphi|_{A_i}$, $i \in I$.

Corollary

Let M be a von Neumann algebra and φ a faithful state on M. If M is generated by self-adjoint operators $x_1, \ldots, x_n$ that are freely independent and whose distributions are non-atomic, then $M \cong L(\mathbb{F}_n)$.

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The idea of Voiculescu is that the study of the free product of algebras should be done in analogy to classical independence and classical probability theory.

Free independence can be characterized by the **free cumulants**, which are given by a sequence of multilinear maps

$$\kappa_n : A^n \to \mathbb{C}$$

defined by

$$\kappa_n(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \dots, a_n) \cdot \mu(\pi, 1_n),$$

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by the following result:

Theorem (Speicher)

The following are equivalent:

1. *the algebras* $(A_i)_{i \in I}$ *are freely independent in* (A, φ)*,*
2. *mixed free cumulants vanish, i.e.*

$$\kappa_n(a_1, \dots, a_n) = 0,$$

whenever there exist at least two entries coming from different algebras.

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$$\lim_{n \to \infty} \mathbb{E}\left[tr\left(X_{i_1}^{(n)} \cdots X_{i_m}^{(n)} \right) \right] = \tau(x_{i_1} \cdots x_{i_m}),$$

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3. The **R-diagonal operators** are operators $a \in (A, \varphi)$ whose only non-vanishing free cumulants are of the form

$$\kappa_n(a, a^*, \dots, a, a^*) \text{ or } \kappa_n(a^*, a, \dots, a^*, a).$$

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2. *the distribution of a arises in the form $u \cdot p$, where u is a Haar unitary and u, p are free.*

If A is a C^*-algebra or von Neumann algebra, then associated to a state $\varphi : A \to \mathbb{C}$ is the Hilbert space $L_2(A, \varphi)$ obtained by completing A with respect to the inner product $< a, b >= \varphi(b^* a)$. A acts on $L_2(A, \varphi)$ via left multiplication.

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2. $\xi_1, \ldots, \xi_n \in \overline{\mathrm{alg}(x_1, \ldots, x_n)}^{||\cdot||_2} \subseteq L_2(M, \varphi)$.

The **free Fisher information** of $x_1, \ldots, x_n$ is defined as

$$\Phi^*(x_1, \ldots, x_n) = \begin{cases} \sum_{i=1}^n ||\xi_i||_2^2, & \text{if a conjugate system exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

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One then defines the non-microstates free entropy as an integral of the Fisher information of the tuple $x_1, \ldots, x_n$.

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If $x_1 = x_1^*, \ldots, x_n = x_n^* \in (M, \varphi)$ are such that $\Phi^*(x_1, \ldots, x_n) < +\infty$, then:

1. for $n \geq 2$, $W^*(x_1, \ldots, x_n)$ does not have property Γ and hence has trivial center (Dabrowski),
2. for every non-zero non-commutative polynomial P, there exists no non-zero self-adjoint element $w \in W^*(x_1, \ldots, x_n)$ such that $P(x_1, \ldots, x_n) \cdot w = 0$. In particular, the distribution of $P(x_1, \ldots, x_n)$ is not atomic (Mai, Speicher, Weber).

Theorem (Nica, Shlyakhtenko, Speicher)

The minimum

$$\min\{\Phi^*(a, a^*) : a^*a \textit{ has a prescribed distribution } \}$$

is attained when a is R-diagonal.

Bi-free probability theory was introduced by Voiculescu in 2013 as an extension of the free setting and involves the simultaneous study of left and right actions of algebras on reduced free product spaces.

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The underlying combinatorial objects are the lattices of *bi-non-crossing partitions*, which are obtained by applying permutations on non-crossing partitions that are naturally derived by keeping track of the positions where the left and right operators appear.
In the scalar setting, the notions of conjugate variables, Fisher information and entropy were extended to the bi-free context by Charlesworth and Skoufranis, using moment and bi-free cumulant formulae.

The theory of R-diagonal operators was recently developed in the bi-free setting, based on the $*$-alternating condition on bi-free cumulants.

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Definition

Let (A, φ) be a non-commutative probability space and $x, y \in A$. We say that the pair (x, y) is *bi-R-diagonal* if for every $n \in \mathbb{N}$, $\chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in A$ such that

$$a_i \in \begin{cases} \{x, x^*\}, & \text{if } \chi(i) = l \\ \{y, y^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

we have that

$$\kappa_\chi(a_1, \ldots, a_n) = 0$$

unless the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is of even length and alternating in $*$-terms and non-$*$-terms.

Theorem (K.)

1. *Let (x, y) and (z, w) be two bi-free pairs in (A, φ) such that (x, y) is bi-R-diagonal. Then, the pair (xz, wy) is also bi-R-diagonal.*

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2. *If (x, y) is bi-R-diagonal, then so is (x^n, y^n) for all n.*
3. *For $x, y \in (A, \varphi)$, the following are equivalent:*
   - A. *the pair (x, y) is bi-R-diagonal,*
   - B. *the joint distribution of the pair (x, y) arises in the form $(u_l z, wu_r)$, where (u_l, u_r) is a bi-Haar unitary that is bi-free from (z, w).*

Theorem (K., Skoufranis)

The minimum

$$\min\left\{\Phi^*(\{x,x^*\}\sqcup\{y,y^*\}) : \substack{\text{the joint distribution of}\\ (x,y)\text{is prescribed}}\right\}$$

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Thank you!

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The **non-microstates free entropy** is given by

$$\chi^*(X_1, \ldots, X_n) = \frac{1}{2}\int_0^\infty \left(\frac{n}{1+t} - \Phi^*(X_1 + \sqrt{t}S_1, \ldots, X_n + \sqrt{t}S_n) \right) dt + \frac{n}{2} log(2\pi e),$$

where $S_1, \ldots, S_n$ are freely independent semicircular operators that are free from $\{X_1, \ldots, X_n\}$.

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$$\{T \in M : TS = ST \ \forall S \in M\} = \mathbb{C}Id.$$

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The algebra $L(G)$ is a factor if and only if the group G is **i.c.c.** , which means that for all $g \neq e \in G$, the set

$$\{hgh^{-1} : h \in G\}$$

is infinite.

If G is an amenable, i.c.c. group, then $L(G) \cong \mathcal{R}$, where $\mathcal{R}$ is the hyperfinite II_1 factor (Connes).

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As a result, we are interested in non-amenable, i.c.c. groups.

Specifically, what can we say about $G = \mathbb{F}_2$?

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The Free Group Factor Isomorphism Problem

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The Free Group Factor Isomorphism Problem

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Remains open!

Theorem (Free Central Limit Theorem)

Let (A, φ) *be a non-commutative probability space and let* $(a_n)_{n\in\mathbb{N}} \subseteq A$ *be a sequence of self-adjoint, freely independent, identically distributed random variables. Assume that* $(a_n)_{n\in\mathbb{N}}$ *is standardized, in the sense that* $\varphi(a_n) = 0$ *and* $\varphi(a_n^2) = 1$ *for all* n*. Then*

$$\frac{a_1 + a_2 + \ldots + a_n}{\sqrt{n}} \xrightarrow{distr} s,$$

where s *is a semicircular random variable.*

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In a von Neumann algebra we can "deform" Haar unitaries into semicirculars.
In particular $L(\mathbb{F}_n)$ is generated by n freely independent semicircular random variables.