A large deviations approach to the geometry of random polytopes

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Dedicated to Professor Rolf Schneider on the occasion of his 65th birthday

1 Introduction and main results

The aim of this article is to present a general "large deviations approach" to the geometry of polytopes spanned by random points with independent coordinates. The origin of our work is in the study of the structure of ±1-polytopes, the convex hulls of subsets of the combinatorial cube $E_2^n = \{-1,1\}^n$. Understanding the complexity of this class of polytopes is important for the "polyhedral combinatorics" approach to combinatorial optimization, and was put forward by Ziegler in [20]. Many natural questions regarding the behavior of ±1–polytopes in high dimensions are open, since, for many important geometric parameters, low-dimensional intuition does not help to identify the extremal ±1-polytopes. The study of *random* ±1*-polytopes* sheds light to some of these questions, the main reason being that random behavior is often the extremal one.

A natural way to define random ±1-polytopes is to fix $N > n$ and to consider N independent random points $\vec{X}_1, \dots, \vec{X}_N$, uniformly distributed over E_2^n. Let $K_{n,N} = \mathrm{conv}\{\pm\vec{X}_1, \dots, \pm\vec{X}_N\}$ denote their absolute convex hull. This "Bernoulli model" of random polytopes was studied in [10]; the emphasis there was on the structure of the corresponding random normed spaces $X_{n,N}$ as N varies from "polynomial" to "exponential" in n. An observation in [10], demonstrating that random behavior is extremal, is that a random polytope $K_{n,N}$ has the largest possible volume among all ±1–polytopes with N vertices, at every scale of n and N. This is a consequence of the following fact: *if* $n \geqslant n_0$ *and if* $N \geqslant n(\ln n)^2$, *then*

$$K_{n,N} \supseteq c\left(\sqrt{\ln(N/n)}B_2^n \cap B_\infty^n\right)$$

with probability greater than $1 - e^{-n}$, *where* $c > 0$ *is an absolute constant*; here B_2^n is the Euclidean unit ball in $\mathbb{R}^n$ and B_∞^n is the unit cube $[-1,1]^n$.

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In [14], Litvak, Pajor, Rudelson and Tomczak–Jaegermann considered a more general class of symmetric random polytopes $K_{n,N}$, spanned by the rows of a matrix $\Gamma_{n,N} = (\xi_{ij})_{1\le i\le N,\, 1\le j\le n}$, whose entries ξ_{ij} are independent symmetric random variables satisfying the following conditions: $\|\xi_{ij}\|_{L^2} \geqslant 1$ and $\|\xi_{ij}\|_{L^{\psi_2}} \leqslant b$ for some $b \geqslant 1$, where $\|\cdot\|_{L^{\psi_2}}$ is the Orlicz norm corresponding to the function $\psi_2(t) = e^{t^2} - 1$. In this setting, which contains the Bernoulli model as well as the Gaussian model, the authors generalized and improved the estimates from [10] in two ways. First, they obtained estimates for N as small as $N = (1+\delta)n$, where $\delta > 0$ can be as small as $1/\ln n$. And second, they proved that, *for every* $0 < \beta < 1$, *the inclusion*

$$K_{n,N} \supseteq c(b)\left(\sqrt{\beta \ln(N/n)}B_2^n \cap B_\infty^n\right)$$

holds true with probability greater than $1 - \exp(-c_1 n^\beta N^{1-\beta}) - \exp(-c_2 N)$, *where* $c(b)$ *is a constant depending only on the parameter* b, *and* c_1 *and* c_2 *are constants depending on the underlying probabilistic model*. The approach in [14] is through random matrices; the authors obtain a lower bound of the order of $\sqrt{N}$ for the smallest singular value of the random matrix $\Gamma_{n,N}$ (under the assumptions above), with probability greater than $1 - \exp(-cN)$.

Finally, the study of ± 1-polytopes was continued by Mendelson, Pajor and Rudelson in [15], where it was shown that the behavior of random ± 1-polytopes is indeed extremal also for other geometric parameters, such as combinatorial dimension and entropy, at every scale of n and N.

Our approach in this paper has a different origin; namely the work of Dyer, Füredi and McDiarmid [4], which establishes a sharp threshold for the expected volume of random ± 1–polytopes (see Section 8 for the precise statement and a recent generalization of this result). The method introduced in [4] proved to be extremely useful and accurate, and plays a key role in the proof of the fact that there exist ± 1–polytopes with superexponential number of facets; it was first used for this purpose by Bárány and Pór in [2], and it was recently further exploited in [8] and [9]. We will call this the "large deviations approach". Our aim is to present a general version of this approach, and to compare the results to those obtained by the "random matrices approach", wherever possible.

We start with a description of the model and then state our main results.

1.1 The Model: Independent Coordinates

We first fix some standard notation. We work in $\mathbb{R}^n$ which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, by $\|\cdot\|_\infty$ the max-norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere; we also write B_∞^n for the unit ball of the norm $\|\cdot\|_\infty$. Volume, surface area and the cardinality of a finite set are denoted by $|\cdot|$ (this will cause no confusion). The boundary of a set $A \subset \mathbb{R}^n$ is denoted by ∂A. All logarithms are natural. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. The letters c, c', C, c_1, c_2 etc. denote absolute positive constants which may change from line to line.

We fix a Borel probability measure μ on the real line, and let X be a random variable, on some probability space $(\Omega, \mathcal{F}, P)$, with distribution μ, i.e., $\mu(B) := P(X \in B)$, $B \in \mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ the Borel σ-field of $\mathbb{R}$). We shall assume that μ is symmetric, i.e.,

$$\mu(B) = \mu(-B) \qquad \forall\, B \in \mathcal{B}(\mathbb{R}), \tag{1.1}$$

and that

$$\int_{\mathbb{R}} e^{tx}\, d\mu(x) = E\big(e^{tX}\big) < \infty \qquad \text{for all } t \text{ in an open interval.} \tag{1.2}$$

Given (1.1), condition (1.2) ensures that X has finite moments of all orders, and we shall assume throughout that μ is normalized to have

$$\mathrm{Var}(X) = 1; \tag{1.3a}$$

of course, by (1.1), we automatically also have that

$$E(X) = 0. \tag{1.3b}$$

Let also

$$x^* := \sup\{x \in \mathbb{R} \colon \mu([x, \infty)) > 0\} \tag{1.4}$$

and

$$p := \max_{x \in \mathbb{R}} P(X = x); \tag{1.5}$$

notice that, by our assumption that $\mathrm{Var}(X) \neq 0$, we have that

$$p < 1. \tag{1.6}$$

Thus in particular, the only condition that we impose on X is that it be ψ_1 (besides symmetry, which merely serves to simplify exposition). Recall that a variable X is said to be ψ_α, $\alpha > 0$, precisely when $\|X\|_{L^{\psi_\alpha}} := \inf\{t > 0 \colon E(\psi_\alpha(X/t)) \leqslant 1\} < \infty$ where ψ_α is the Orlicz function $\psi_\alpha(x) := \exp(|x|^\alpha) - 1$.

Let $X_1, \dots, X_n$ be independent and identically distributed random variables, defined on the product space $(\Omega^n, \mathcal{F}^{\otimes n}, P^n)$, each with distribution μ. Set $\vec{X} = (X_1, \dots, X_n)$ and, for a fixed N satisfying $N > n$, consider N independent copies $\vec{X}_1, \dots, \vec{X}_N$ of $\vec{X}$, defined on the product space $(\Omega^{nN}, \mathcal{F}^{\otimes nN}, \mathrm{Prob})$. This procedure defines the random polytope

$$K_N := \mathrm{conv}\big\{\vec{X}_1, \dots, \vec{X}_N\big\}. \tag{1.7}$$

The random polytope K_N is the principal object of study of this paper.

1.2 Main results

In order to state our results, we briefly introduce some terminology; see the next Section for details. Let $\varphi(t) := E\big(e^{tX}\big)$ $(t \in \mathbb{R})$ denote the moment generating function of X, $\psi(t) := \ln \varphi(t)$ its cumulant generating function, and let $J := \{t \in \mathbb{R} \colon \varphi(t) < \infty\}$. The function ψ is C^∞ on J° — the interior of J — and continuous on J. Furthermore, ψ is strictly convex and ψ' is strictly increasing on J°, and $\psi'(J^\circ)$ is an open interval contained in $I^\circ := (-x^*, x^*)$.

The Legendre transform of ψ is the function

$$\lambda(x) := \sup\{tx - \psi(t) \colon t \in \mathbb{R}\} \qquad (x \in \mathbb{R}).$$

For $\vec{x} = (x_1, \dots, x_n) \in (-x^*, x^*)^n$ set

$$\Lambda(\vec{x}) = \frac{1}{n} \sum_{i=1}^{n} \lambda(x_i).$$

Let $r^* = \sup\{\lambda(x) \colon x \in I^\circ\}$. For $0 \leqslant r < r^*$, define Λ_r by

$$\Lambda_r = \{\vec{x} \in (-x^*, x^*)^n \colon \Lambda(\vec{x}) \leqslant r\}.$$

Since λ is a convex function on I°, Λ_r is a convex body contained in $(-x^*, x^*)^n$.

The key quantity for our description of the typical K_N is the function q, defined for $\vec{x} \in (-x^*, x^*)^n$, by

$$q(\vec{x}) = \inf\Big\{P^n\big(\vec{X} \in H\big) \colon \vec{x} \in \partial H,\ H \text{ a closed half-space}\Big\}.$$

Using convexity arguments and the definition of q, one can prove the following two facts:

Fact 1.1. *Let $\gamma > 0$ and $0 < r < r^*$ be such that $\lambda(\gamma) \geqslant r$. Then*

$$1 - \operatorname{Prob}\big(K_N \supseteq \Lambda_r \cap \gamma B_\infty^n\big) \leqslant \binom{N}{n} p^{N-n} + 2\binom{N}{n}\big(1 - \inf q(\vec{x})\big)^{N-n},$$

where the inf *is over all $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$, and p is given by* (1.5).

Fact 1.2. *Let $\gamma > 0$ and $0 < r < r^*$ be such that $\lambda(\gamma) > r$. Then*

$$E\big(|\partial(\Lambda_r) \cap \gamma B_\infty^n \cap K_N|\big) \leqslant N \cdot \sup q(\vec{x}) \cdot |\partial(\Lambda_r) \cap \gamma B_\infty^n|,$$

where the sup *is over all $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$.*

Our main task is to give precise estimates for the quantities $\inf q(\vec{x})$ and $\sup q(\vec{x})$ over all $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$. We prove:

Theorem 1.3. *If $\gamma > 0$ is sufficiently small, there exists $n_0 = n_0(\gamma) \in \mathbb{N}$ with the following property: if $n \geqslant n_0$ and $\lambda(\gamma) \geqslant r$, then*

$$q(\vec{x}) \leqslant \exp\big(-rn - \tfrac{1}{2}\ln(rn) + c(\gamma)\big)$$

for every $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$, where $c(\gamma)$ is a constant depending only on γ and μ.

Theorem 1.4. *If $\gamma > 0$ is sufficiently small, there exists $n_0 = n_0(\gamma) \in \mathbb{N}$ with the following property: for $n \geqslant n_0$, and $\varepsilon \geqslant 3 \ln n/n$,*

$$q(\vec{x}) \geqslant \exp(-(r+\varepsilon)n)$$

for every $\vec{x} \in \Lambda_r \cap \gamma B_\infty^n$ and r in the range $0 < r \leqslant \lambda(\gamma)$.

Fix $N > n$ and define $\rho := (\ln N)/n$. Using the above we show that, with high probability, $K_N \cap \gamma B_\infty^n$ is "weakly sandwiched" between $\Lambda_{\rho-2\varepsilon} \cap \gamma B_\infty^n$ and $\Lambda_{\rho-\delta}$:

Theorem 1.5. *For γ sufficiently small, and N in the range $n^6 < N \leqslant e^{n\lambda(\gamma)}$,*

$$\mathrm{Prob}\big(K_N \supseteq \Lambda_{\rho-2\varepsilon} \cap \gamma B_\infty^n\big) \geqslant 1 - 2^{-n+1}$$

for all $\varepsilon \geqslant 3 \ln n/n$, and all sufficiently large n.

Theorem 1.6. *For γ sufficiently small, n large enough, and $n < N \leqslant e^{n\lambda(\gamma)}$,*

$$\mathrm{Prob}\big(\,|\partial(\Lambda_{\rho-\delta}) \cap \gamma B_\infty^n \cap K_N| \geqslant \alpha\, |\partial(\Lambda_{\rho-\delta}) \cap \gamma B_\infty^n|\,\big) \leqslant \frac{1}{\alpha}\frac{c(\gamma)}{(\ln N)^{1/6}},$$

for all $\delta \leqslant \frac{1}{3}(\ln\ln N)/n$ $(\delta \geqslant 0)$ and any $0 < \alpha < 1$, where $c(\gamma)$ is a constant depending only on γ and μ.

Using the method of proof of Theorem 1.5 one also obtains the following result:

Theorem 1.7. *For $\gamma > 0$ sufficiently small, and for any $\beta \in (0,1)$ and $\alpha \in (0, -\ln p)$ the following holds: for all n sufficiently large, and N in the range $n^{1+4/\beta} < N \leqslant n^{1+4/\beta} e^{n\beta\lambda(\gamma)}$,*

$$\mathrm{Prob}\big(K_N \supseteq \Lambda_\varrho \cap \gamma B_\infty^n\big) \geqslant 1 - e^{-\alpha N} - e^{-n^\beta N^{1-\beta}}$$

with $\varrho := \beta n^{-1} \ln(N/n) - 4n^{-1} \ln n$.

As will be seen in the next Section (Proposition 2.13), $n\Lambda(\vec{x}) \simeq \frac{1}{2}\,\|\vec{x}\|_2^2$ when $\|\vec{x}\|_\infty$ is small; more precisely, for any $\varepsilon > 0$, there exists $\gamma = \gamma_\mu(\varepsilon) > 0$ such that $\frac{1}{2}(1-\varepsilon)\,\|\vec{x}\|_2^2 \leqslant n\Lambda(\vec{x}) \leqslant \frac{1}{2}(1+\varepsilon)\,\|\vec{x}\|_2^2$ for all $\vec{x} \in \gamma B_\infty^n$. Thus for γ small, $\Lambda_\varrho \cap \gamma B_\infty^n \simeq \big(\sqrt{2n\varrho} B_2^n\big) \cap \gamma B_\infty^n = \big(\sqrt{2\beta \ln(N/n) - 8 \ln n}\, B_2^n\big) \cap \gamma B_\infty^n$. In this way, we give a different proof of [14, Theorem 4.2] and actually extend it to the case where the underlying distribution μ of the coordinates of the random vertex $\vec{X}$ is ψ_1.

We next turn to an application of a different flavor. For a polytope P in $\mathbb{R}^n$ with non-empty interior, we shall write $f_{n-1}(P)$ for the number of its facets, i.e., its $(n-1)$-dimensional faces. We then have the following:

Theorem 1.8. *There exist two positive constants a and b such that, for all sufficiently large n, and all N satisfying $n^6 < N \leqslant \exp(bn)$, one has that*

$$E[f_{n-1}(K_N)] \geqslant \left(\frac{\ln N}{a \ln n}\right)^{n/2}.$$

Using Theorem 1.8 for the special case where the distribution μ is the distribution $\mu(\{-1\}) = \mu(\{1\}) = \frac{1}{2}$, it was shown in [9] that there exist ± 1-polytopes with as many as $(cn/\ln n)^{n/2}$ facets, where $c > 0$ is a universal constant.

Our last application concerns a generalization of the result in [4]. Given a compactly supported (symmetric) probability measure μ, define

$$\kappa = \kappa(\mu) := \frac{1}{2x^*} \int_{-x^*}^{x^*} \lambda(x)\, dx. \tag{1.8}$$

In [7], we establish the following threshold for the expected volume of K_N, for a large class of distributions μ:

Theorem 1.9. *Let μ be an even, compactly supported, Borel probability measure on the real line and assume that $0 < \kappa(\mu) < \infty$. Then, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \sup\{(2x^*)^{-n} E(|K_N|)\colon N \leqslant \exp((\kappa - \varepsilon)n)\} = 0,$$

and whenever the distribution μ satisfies

$$\lim_{x \uparrow x^*} \frac{-\ln \mu\big([x,\infty)\big)}{\lambda(x)} = 1, \tag{1.9}$$

also

$$\lim_{n\to\infty} \inf\{(2x^*)^{-n} E(|K_N|)\colon N \geqslant \exp((\kappa + \varepsilon)n)\} = 1.$$

The main technical estimate needed to prove Theorem 1.9 is the following.

Proposition 1.10. *Assume that the probability distribution μ satisfies (1.9). Then, for every $\varepsilon > 0$, there exists $n_\mu(\varepsilon) \in \mathbb{N}$, depending only on ε and μ, such that for all $0 < r < \lambda(\alpha)$ and all $n \geqslant n_\mu(\varepsilon)$ we have that*

$$q_-(\Lambda_r) \geqslant \exp(-(1+\varepsilon)rn - \varepsilon n),$$

where $q_-(\Lambda_r) = \inf\{q(\vec{x}) : \vec{x} \in \partial(\Lambda_r)\}$.

Naturally, many of the arguments in this paper generalize the ones introduced in [4], [2], [8] and [9]. We chose to present complete proofs in the general setting in order to make the presentation self-contained, and to facilitate future references.

2 Large Deviations Preliminaries

In this Section we recall some basic facts concerning moment generating and cumulant generating functions. For more information on large deviations techniques the reader may wish to consult the books [3] and [18].

Let

$$\varphi(t) := E\big(e^{tX}\big) \qquad (t \in \mathbb{R}) \tag{2.1}$$

denote the moment generating function of X,

$$\psi(t) := \ln \varphi(t) \tag{2.2}$$

its cumulant generating function (or logarithmic moment generating function), and let

$$J := \{t \in \mathbb{R} \colon \varphi(t) < \infty\}\,. \tag{2.3}$$

Observe that, by Hölder's inequality, ψ is a convex function on J. Therefore, φ is also convex. By our assumption (1.2), i.e., that X is ψ_1, J is a non-degenerate interval (necessarily centered at zero, by assumption (1.1)). We shall write

$$t^* := \sup J, \tag{2.4}$$

whence $J^\circ = (-t^*, t^*)$.

Lemma 2.1. *$E(|X|^n) < \infty$ for all $n \in \mathbb{N}$, and*

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$$

for all t with $|t| < t^*$.

Lemma 2.2. *The function φ is C^∞ on J° and*

$$\frac{d^n}{dt^n}\varphi(t) = E\big(X^n e^{tX}\big) \qquad (n \in \mathbb{N}).$$

Furthermore, φ is continuous on J.

Note. In fact, by Fatou's lemma, when $t^* < \infty$, $\lim_{t \uparrow t^*} \varphi(t) = \varphi(t^*)$ also when $\varphi(t^*) = \infty$.

Definition 2.3. For $t \in J$ define the probability measure P_t on $(\Omega, \mathcal{F})$ by

$$P_t(A) := E\big(e^{tX - \psi(t)} \mathbf{1}_A\big) \qquad (A \in \mathcal{F}).$$

Define also $\mu_t(A) := P_t(X \in A)$ for $A \in \mathcal{B}(\mathbb{R})$. Notice that $P_0 = P$ and $\mu_0 = \mu$.

Lemma 2.4. *For $t \in J^\circ$, μ_t has finite moments of all orders, and*

$$E_t(X) = \psi'(t) \quad , \quad \mathrm{Var}_t(X) = \psi''(t) \quad , \quad E_t\big([X - \psi'(t)]^3\big) = \psi'''(t).$$

Corollary 2.5. *The function ψ is C^∞ on J° and continuous on J. Furthermore, ψ is strictly convex and ψ' is strictly increasing on J°, and $\psi'(J^\circ)$ is an open interval contained in $(-x^*, x^*)$.*

Proof. The fact that ψ is C^∞ on J° follows from Lemma 2.2 and the fact that φ is (strictly) positive. The continuity of ψ on J also follows from Lemma 2.2. Since

$$(\psi')'(t) = \psi''(t) = \mathrm{Var}_t(X) \geqslant 0,$$

by Lemma 2.4, ψ' is also non-decreasing and in fact strictly increasing if X is not constant almost surely. Finally, the continuity and strict monotonicity of ψ' imply that $\psi'(J^\circ)$ is an open interval when X is not constant; the inclusion $\psi'(J^\circ) \subseteq (-x^*, x^*)$ follows from the inequality $-x^* e^{tX} \leqslant X e^{tX} \leqslant x^* e^{tX}$, which holds with probability one for each fixed t, and the formula $\psi'(t) = E\big(Xe^{tX}\big)/E\big(e^{tX}\big)$ of Lemma 2.4. $\square$

We shall write

$$\psi'(-t^*) := \lim_{t \downarrow -t^*} \psi'(t) \quad \text{and} \quad \psi'(t^*) := \lim_{t \uparrow t^*} \psi'(t). \tag{2.5}$$

Thus $\psi'(J^\circ) = (\psi'(-t^*), \psi'(t^*))$. We shall also write $I := \psi'(J)$, whence

$$I^\circ = \psi'(J^\circ) = (\psi'(-t^*), \psi'(t^*)). \tag{2.6}$$

Remark 2.6. If $P(X = x) = 1$ for some $x \in \mathbb{R}$ (a case that we have explicitly excluded, see (1.6)), then obviously $\varphi(t) = e^{tx}$ and $\psi(t) = tx$ is linear, $-x^* = x^* = x$ and $I = \{x\}$, and $J = J^\circ = \mathbb{R}$.

Remark 2.7. The inclusion $\psi'(J^\circ) \subset (-x^*, x^*)$ may be proper. Consider, for example, the measure μ with density $f_p(x) := c_p\,|x|^{-p}\, e^{-|x|} \mathbf{1}_{[1,\infty)}(|x|)$ with respect to Lebesgue measure , where $p > 2$ and c_p is the normalizing constant making μ a probability measure.

Remark 2.8. If $t^* \in J$, then $\psi'(\pm t^*)$, as defined by (2.5), agrees with the one-sided derivative of ψ at $\pm t^*$; i.e., ψ has a continuous derivative on J.

Given that condition (1.6) is in force throughout the paper, we may give the following definition:

Definition 2.9. Define $h\colon I^\circ \to J^\circ$ by $h := (\psi')^{-1}$.

Lemma 2.10. *$h\colon I^\circ \to J^\circ$ is a strictly increasing C^∞ function, and*

$$h'(x) = \frac{1}{\psi''(h(x))}.$$

Definition 2.11. The Legendre transform of ψ is the function

$$\lambda(x) := \sup\{tx - \psi(t)\colon t \in \mathbb{R}\}, \qquad x \in \mathbb{R}.$$

Proposition 2.12. *(i)* $\lambda \geqslant 0$.

(ii) $x \in I^\circ,\ \lambda(x) = tx - \psi(t) \iff \psi'(t) = x,\ t \in J^\circ$; *hence*

$$\lambda(x) = xh(x) - \psi(h(x)) \qquad \textit{for } x \in I^\circ.$$

(*iii*) *λ is a strictly convex C^∞ function on I°, and*

$$\frac{d^n}{dx^n}\lambda(x) = \frac{d^{n-1}}{dx^{n-1}}h(x) \qquad n \in \mathbb{N}.$$

(*iv*) $\lambda(0) = 0$; *in particular, λ attains its unique minimum on I° at $x = 0$.*

(*v*) $\lambda(x) = \infty$ *for $x \in \mathbb{R} \setminus [-x^*, x^*]$.*

Proof. Assertion (i) follows from the fact that the function $t \mapsto tx - \psi(t)$ takes on the value 0 at $t = 0$.

For (ii) suppose first that $t \in J^\circ$ and $\psi'(t) = x$. Then $x \in I^\circ$, by definition of I°, and $s \mapsto sx - \psi(s)$ has a local extreme point at $s = t$, which by the strict concavity of this function on J° must be a global maximum; i.e.,

$$tx - \psi(t) = \max\{sx - \psi(s) \colon s \in J^\circ\} = \lambda(x),$$

the second equality being a consequence of the continuity of ψ on J and the fact that $\psi(s) = \infty$ for $s \in \mathbb{R} \setminus J$.

Conversely, suppose that $x \in I^\circ$ and that $\lambda(x) = tx - \psi(t)$ for some t. Since $sx - \psi(s) = -\infty$ for $s \in \mathbb{R} \setminus J$, and since $\lambda \geqslant 0$, we must have that $t \in J$. On the other hand, by definition of I°, there exists $s \in J^\circ$ with $\psi'(s) = x$, and by the argument of the preceding paragraph, we must then have that $\lambda(x) = sx - \psi(s)$. The strict concavity of the function $u \mapsto ux - \psi(u)$ and its continuity on J imply then that $s = t$, and hence $\psi'(t) = x$.

For (iii) observe that since ψ and h are C^∞, (ii) shows that λ also is. Furthermore,

$$\frac{d}{dx}\lambda(x) = h(x) + xh'(x) - \psi'(h(x))h'(x) = h(x),$$

by (ii) and the definition of h as $(\psi')^{-1}$, whence

$$\frac{d^n}{dx^n}\lambda(x) = \frac{d^{n-1}}{dx^{n-1}}h(x)$$

for all $n \in \mathbb{N}$. Finally,

$$\frac{d^2}{dx^2}\lambda(x) = h'(x) = \frac{1}{\psi''(h(x))} > 0$$

on I°; hence λ is strictly convex on I°.

For assertion (iv) observe first that $\psi'(0) = E(X)$, by Lemma 2.2. This implies that $h(E(X)) = 0$, and assertion (ii) then implies that $\lambda(E(X)) = 0$. By (1.3b) then, $\lambda(0) = 0$.

Finally, for (v) observe that, if $x^* < \infty$, then $\varphi(t) \leqslant e^{tx^*}$ for all $t \geqslant 0$. This implies that

$$tx - \psi(t) \geqslant t(x - x^*) \qquad \text{for } t \geqslant 0,\ x \in \mathbb{R},$$

and hence

$$\lambda(x) = \sup_{x\in\mathbb{R}}[tx - \psi(t)] \geqslant \lim_{t\to\infty}[t(x - x^*)] = \infty$$

for $x > x^*$. For $t \leqslant 0$ on the other hand, $\varphi(t) \leqslant e^{-tx^*}$, and hence $\lambda(x) \geqslant \lim_{t\to-\infty}[t(x + x^*)] = \infty$ when $x < -x^*$. □

Proposition 2.13. *For any* $a > 0$ *with* $[-a, a] \subset I^\circ$, *there exist constants* $c_1, \dots, c_6$ *in* $(0, \infty)$ *for which*:

(*i*) $c_1 \leqslant \psi''(h(x)) \leqslant c_2$ *for all* $x \in [-a, a]$.

(*ii*) $|h(x) - x| \leqslant c_3 x^2$ *for all* $x \in [-a, a]$.

(*iii*) $\left|\lambda(x) - \frac{1}{2}x^2\right| \leqslant c_4 |x|^3$ *for all* $x \in [-a, a]$.

(*iv*) $c_5 \leqslant \lambda(x)/[h(x)]^2 \leqslant c_6$ *for all* $x \in [-a, a]$.

Proof. The facts that ψ, and hence ψ'', is C^∞ and that $\psi'' > 0$ on J° imply that

$$c_1 = \min_{x\in[-a,a]} \psi''(h(x)) > 0 \quad \text{and} \quad c_2 = \max_{x\in[-a,a]} \psi''(h(x)) < \infty,$$

for any interval $[-a, a]$ contained in I°. Here we also use the fact that h is C^∞, hence continuous, on I°, whence $h([-a, a])$ is a compact subset of J° (in fact a symmetric interval, ψ', and hence h, being increasing and odd).

Next note that

$$\psi'(0) = E(X) \implies h(E(X)) = 0, \tag{2.7}$$

and

$$h'(E(X)) = \frac{1}{\psi''(h(E(X)))} = \frac{1}{\psi''(0)} = \frac{1}{\mathrm{Var}(X)}, \tag{2.8}$$

by Lemma 2.4, and by Proposition 2.12,

$$\lambda(E(X)) = E(X)h(E(X)) - \psi(h(E(X))) = 0, \tag{2.9}$$

$$\lambda'(E(X)) = h(E(X)) = 0, \tag{2.10}$$

$$\lambda''(E(X)) = h'(E(X)) = \frac{1}{\mathrm{Var}(X)}. \tag{2.11}$$

By our assumptions $E(X) = 0$, $\mathrm{Var}(X) = 1$, and Taylor's theorem,

$$h(x) = h(0) + h'(0)x + \tfrac{1}{2}x^2 h''(x_h) = 0 + x + \tfrac{1}{2}x^2 h''(x_h)$$

for some x_h between 0 and x, and since h'' is bounded on any interval $[-a, a]$ contained in I°, h being C^∞ on I°, assertion (ii) follows.

Similarly,

$$\begin{aligned}\lambda(x) &= \lambda(0) + x\lambda'(0) + \tfrac{1}{2}x^2\lambda''(0) + \tfrac{1}{6}x^3\lambda'''(x_\lambda)\\ &= \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3\lambda'''(x_\lambda)\end{aligned}$$

for some x_λ between 0 and x, and λ''' is bounded on any interval $[-a, a]$ contained in I°, λ being C^∞ on I°.

Finally, since h is strictly increasing and $h(0) = 0$, we have that $h(x) \neq 0$ for any $x \neq 0$. It follows that the function $x \mapsto \lambda(x)/[h(x)]^2$ is continuous on $I^\circ \setminus \{0\}$. Since also $\lambda(x)/[h(x)]^2 \to \frac{1}{2}$ as $x \to 0$, by (ii) and (iii), the function $x \mapsto \lambda(x)/[h(x)]^2$ extends to a continuous function on I°, f say. Notice also that $\lambda(x) > 0$ for $x \neq 0$, by Proposition 2.12 (i) and (iv), and therefore also $f(x) \neq 0$ for all $x \in I^\circ$. Therefore $[c_5, c_6] := f([-a, a])$ must be a compact interval contained in $(0, \infty)$. □

If the random variable X is bounded, i.e., if $x^* < \infty$ (recall the notation (1.4)), then clearly

$$\frac{E_t\big(|X - \psi'(t)|^3\big)}{E_t\big(|X - \psi'(t)|^2\big)} \leqslant 2x^*. \tag{2.12}$$

The ratio on the left hand side of (2.12) need not be bounded for unbounded X however; the Laplace distribution, with density $f(x) = e^{-\sqrt{2}|x|}/\sqrt{2}$ with respect to Lebesgue measure on $\mathbb{R}$, provides a counterexample. In the general case we have the following.

Lemma 2.14. *For $t \in J^\circ$ and $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \subseteq J^\circ$,*

$$\begin{aligned}\frac{E_t\big(|X - \psi'(t)|^3\big)}{E_t\big(|X - \psi'(t)|^2\big)} \leqslant{}& \frac{\psi''(t+\varepsilon)}{\psi''(t)}\,\frac{\varphi(t+\varepsilon)}{\varepsilon\,\varphi(t)} + \frac{\psi''(t-\varepsilon)}{\psi''(t)}\,\frac{\varphi(t-\varepsilon)}{\varepsilon\,\varphi(t)}\\ &+ \frac{|\psi'(t+\varepsilon) - \psi'(t)|^2}{\psi''(t)}\,\frac{\varphi(t+\varepsilon)}{\varepsilon\,\varphi(t)} + \frac{|\psi'(t-\varepsilon) - \psi'(t)|^2}{\psi''(t)}\,\frac{\varphi(t-\varepsilon)}{\varepsilon\,\varphi(t)} + |\psi'(t)|\,.\end{aligned}$$

Proof. Since

$$\begin{aligned}E_t\big(|X - \psi'(t)|^3\big) &\leqslant E_t\big(|X - \psi'(t)|^2\,|X|\big) + |\psi'(t)|\,E_t\big(|X - \psi'(t)|^2\big)\\ &= \frac{E\big(|X - \psi'(t)|^2\,|X|\,e^{tX}\big)}{\varphi(t)} + |\psi'(t)|\,\psi''(t),\end{aligned}$$

and $|X| \leqslant \varepsilon^{-1}e^{\varepsilon|X|} \leqslant \varepsilon^{-1}\big(e^{\varepsilon X} + e^{-\varepsilon X}\big)$ for any $\varepsilon > 0$, it follows that

$$\begin{aligned}E_t\big(|X - \psi'(t)|^3\big) \leqslant{}& \frac{1}{\varepsilon}\,\frac{E\big(|X - \psi'(t)|^2\,e^{(t+\varepsilon)X}\big)}{\varphi(t)}\\ &+ \frac{1}{\varepsilon}\,\frac{E\big(|X - \psi'(t)|^2\,e^{(t-\varepsilon)X}\big)}{\varphi(t)} + |\psi'(t)|\,\psi''(t)\end{aligned}$$

for any $\varepsilon > 0$ with $(t-\varepsilon, t+\varepsilon) \subseteq J^\circ$. Now

$$\begin{aligned} E\big(|X-\psi'(t)|^2\, e^{(t\pm\varepsilon)X}\big) &= E\big(|X-\psi'(t\pm\varepsilon)|^2\, e^{(t\pm\varepsilon)X}\big) \\ &\qquad + |\psi'(t\pm\varepsilon)-\psi'(t)|^2\, E\big(e^{(t\pm\varepsilon)X}\big) \\ &= \psi''(t\pm\varepsilon)\,\varphi(t\pm\varepsilon) + |\psi'(t\pm\varepsilon)-\psi'(t)|^2\, \varphi(t\pm\varepsilon), \end{aligned}$$

and hence

$$\begin{aligned} E_t\big(|X-\psi'(t)|^3\big) &\leqslant \psi''(t+\varepsilon)\frac{\varphi(t+\varepsilon)}{\varepsilon\,\varphi(t)} + \psi''(t-\varepsilon)\frac{\varphi(t-\varepsilon)}{\varepsilon\,\varphi(t)} \\ &+ |\psi'(t+\varepsilon)-\psi'(t)|^2\,\frac{\varphi(t+\varepsilon)}{\varepsilon\,\varphi(t)} + |\psi'(t-\varepsilon)-\psi'(t)|^2\,\frac{\varphi(t-\varepsilon)}{\varepsilon\,\varphi(t)} + |\psi'(t)|\,\psi''(t), \end{aligned}$$

from where the Lemma follows. □

Corollary 2.15. *For any interval $[-b,b]$ contained in J°, there exists $C \in (0,\infty)$ for which*

$$\frac{E_t\big(|X-\psi'(t)|^3\big)}{E_t\big(|X-\psi'(t)|^2\big)} \leqslant C \qquad \textit{for all } t \in [-b,b].$$

Proof. Apply Lemma 2.14 with ε such that $b+\varepsilon < t^*$, and use the facts that φ and ψ'' are bounded away from zero and infinity on the interval $[-(b+\varepsilon), b+\varepsilon] \subset J^\circ$, and that ψ' is bounded on $[-b,b]$. □

3 Weakly Sandwiching K_N

For $\vec{x} = (x_1, \dots, x_n) \in (-x^*, x^*)^n$ set

(3.1)
$$\Lambda(\vec{x}) = \frac{1}{n}\sum_{i=1}^{n} \lambda(x_i).$$

Let $r^* = \sup\{\lambda(x) \colon x \in I^\circ\}$. For $0 \leqslant r < r^*$, define Λ_r by

(3.2)
$$\Lambda_r = \{\vec{x} \in (-x^*, x^*)^n \colon \Lambda(\vec{x}) \leqslant r\}.$$

Since λ is an even convex function on I°, Λ_r is an origin symmetric convex body contained in $(-x^*, x^*)^n$.

We will show that there exists a constant $\gamma = \gamma(\mu) > 0$ with the following property: for any fixed $N > n$, one can find $r = r(N,n)$ and $\varepsilon = \varepsilon(n)$ such that $K_N \cap \gamma B_\infty^n$ is "weakly sandwiched" between $\Lambda_{r-\varepsilon} \cap \gamma B_\infty^n$ and Λ_r, in the following sense:

(i) $\Lambda_{r-\varepsilon} \cap \gamma B_\infty^n \subseteq K_N$ for the typical K_N;

(ii) most of the surface area of $\Lambda_r \cap \gamma B_\infty^n$ is missed by the typical K_N.

The key quantity for the proof of these two facts is introduced in the next definition.

Definition 3.1. For every $\vec{x} \in (-x^*, x^*)^n$, set

$$q(\vec{x}) := \inf \left\{ P^n\big(\vec{X} \in H\big) \colon \vec{x} \in H,\ H \text{ a closed half-space} \right\}.$$

Clearly,

$$q(\vec{x}) = \inf \left\{ P^n\big(\vec{X} \in H\big) \colon \vec{x} \in \partial H,\ H \text{ a closed half-space} \right\} \tag{3.3}$$

for any $\vec{x}$.

The function $q(\vec{x})$ was introduced in [4], and Fact 1.1, which we restate in the following Proposition, is a generalization of [4, Lemma 2.1 (b)] (see also [2, Lemma 4.2] and [8, Lemma 4.1]).

Proposition 3.2. *Let $\gamma > 0$ and $0 < r < r^*$ be such that $\lambda(\gamma) \geqslant r$. Then*

$$1 - \mathrm{Prob}\big(K_N \supseteq \Lambda_r \cap \gamma B_\infty^n\big) \leqslant \binom{N}{n} p^{N-n} + 2\binom{N}{n}\big(1 - \inf q(\vec{x})\big)^{N-n},$$

where the inf *is over all $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$, and p is given by* (1.5).

Proof. Fix r and γ as in the statement of the Proposition and write q_* for the infimum of $q(\vec{x})$ over $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$. Let also E denote the event that K_N has non-empty interior.

For every subset $J = \{j_1, \dots, j_n\}$ of $\{1, \dots, N\}$, of cardinality n, define the event E_J as follows: $\vec{X}_{j_1}, \dots, \vec{X}_{j_n}$ are affinely independent, and for one of the two closed half-spaces H_1, H_2 they determine, say H_i, we have simultaneously $K_N \subset H_i$ and $P^n\big(\vec{X} \notin H_i\big) \geqslant q_*$.

Consider now the event E where K_N has non-empty interior. If $(\Lambda_r \cap \gamma B_\infty^n) \nsubseteq K_N$, then there exists $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n \setminus K_N$. This is a consequence of the next claim, whose proof we postpone until the end of this proof:

Claim 3.3. The convex hull $\mathrm{conv}\big(\partial(\Lambda_r) \cap \gamma B_\infty^n\big)$ of $\partial(\Lambda_r) \cap \gamma B_\infty^n$ is $\Lambda_r \cap \gamma B_\infty^n$.

Since $\vec{x} \notin K_N$, there exists a facet F of K_N with the following property: one of the two closed half-spaces H_1 and H_2 determined by F contains K_N but does not contain $\vec{x}$. Thus, if H_i is this half-space, we have simultaneously $K_N \subset H_i$ and $P^n\big(\vec{X} \notin H_i\big) \geqslant q(\vec{x})$, and since $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$, we actually have that $P^n\big(\vec{X} \notin H_i\big) \geqslant q_*$. Since the hyperplane bounding H_i is determined by some affinely independent vertices $\vec{X}_{j_1}, \dots, \vec{X}_{j_n}$ of K_N (which lie in F), this shows that

$$E \cap \{(\Lambda_r \cap \gamma B_\infty^n) \nsubseteq K_N\} \subseteq \bigcup_J E_J. \tag{3.4}$$

By (3.4) we have that

$$\{\Lambda_r \cap \gamma B_\infty^n \nsubseteq K_N\} \subseteq E^c \cup \bigcup_J E_J. \tag{3.5}$$

It follows that

$$\operatorname{Prob}\big(\Lambda_r \cap \gamma B_\infty^n \nsubseteq K_N\big) \leqslant \operatorname{Prob}(E^c) + \sum_J \operatorname{Prob}(E_J)$$

$$= \operatorname{Prob}(E^c) + \binom{N}{n} \operatorname{Prob}(E'), \tag{3.6}$$

where $E' := E_{\{1,\dots,n\}}$.

It is not hard to see that

$$\operatorname{Prob}(E') \leqslant 2(1-q_*)^{N-n}. \tag{3.7}$$

Indeed, let E'' denote the event that $\vec{X}_1,\dots,\vec{X}_n$ are affinely independent. On E'', $\vec{X}_1,\dots,\vec{X}_n$ determine two closed half-spaces $H_i = H_i(\vec{X}_1,\dots,\vec{X}_n)$, $i=1,2$. Let E^i be the event that $\vec{X}_1,\dots,\vec{X}_n$ are affinely independent and $P^n\big(\vec{X} \notin H_i\big) \geqslant q_*$. Then, with Exp denoting expectation with respect to the measure Prob,

$$\begin{aligned}
\operatorname{Prob}(E') &\leqslant \sum_{i=1}^{2} \operatorname{Prob}\big(\{\vec{X}_{n+1},\dots,\vec{X}_N \in H_i\} \cap E^i\big)\\
&= \sum_{i=1}^{2} \operatorname{Exp}\big(\operatorname{Prob}\big(\{\vec{X}_{n+1},\dots,\vec{X}_N \in H_i\} \mid \vec{X}_1,\dots,\vec{X}_n\big)\mathbf{1}_{E^i}\big)\\
&\leqslant (1-q_*)^{N-n} \sum_{i=1}^{2} \operatorname{Prob}(E^i).
\end{aligned}$$

To obtain a bound on $\operatorname{Prob}(E^c)$ we argue as follows. If K_N has empty interior, there exists $J = \{j_1,\dots,j_n\} \subset \{1,\dots,N\}$ such that the set $\big\{\vec{X}_j : j \notin J\big\}$ is contained in the affine hull of $\big\{\vec{X}_j : j \in J\big\}$. We now claim the following:

Claim 3.4. If S has affine dimension smaller than n, then $P^n\big(\vec{X} \in S\big) \leqslant p$.

It follows that

$$\operatorname{Prob}(E^c) \leqslant \binom{N}{n} p^{N-n}. \tag{3.8}$$

This proves the Proposition, modulo the proofs of the two claims. □

Proof of Claim 3.3. Since obviously $\operatorname{conv}\big(\partial(\Lambda_r) \cap \gamma B_\infty^n\big) \subseteq \Lambda_r \cap \gamma B_\infty^n$, we only have to show the reverse inclusion. Let $\vec{x} \in \Lambda_r \cap \gamma B_\infty^n$, and assume that $\Lambda(\vec{x}) < r$ (otherwise there is nothing to prove). Since $\vec{x} \in \gamma B_\infty^n$, there exist $\lambda_1,\dots,\lambda_{2^n} \geqslant 0$ with $\sum_i \lambda_i = 1$ and

$$\vec{x} = \sum_i \lambda_i \vec{v}_i, \tag{3.9}$$

where $\vec{v}_1, \dots, \vec{v}_{2^n}$ denote the vertices of γB_∞^n. The condition $\lambda(\gamma) \geqslant r$ that we have assumed that r and γ satisfy implies that $\Lambda(\vec{v}_i) \geqslant r$ for all i; since we are assuming that $\Lambda(\vec{x}) < r$, there exist $t_i \in (0,1]$, such that $\Lambda(t_i\vec{v}_i + (1-t_i)\vec{x}) = r$. Write $\vec{y}_i := t_i\vec{v}_i + (1-t_i)\vec{x}$; then $\vec{y}_i \in \partial(\Lambda_r)$ and $\vec{v}_i = t_i^{-1}\vec{y}_i - t_i^{-1}(1-t_i)\vec{x}$ for all i. By (3.9) we then get that

$$\vec{x} = \left(1 + \sum_i \lambda_i \frac{1-t_i}{t_i}\right)^{-1} \sum_i \frac{\lambda_i}{t_i}\vec{y}_i,$$

and this is a convex combination of the points $\vec{y}_i \in \partial(\Lambda_r) \cap \gamma B_\infty^n$. □

Proof of Claim 3.4. Suppose that S is a Borel set contained in some hyperplane H. Then $H = \{\vec{y} \in \mathbb{R}^n \colon \langle \vec{u}, \vec{y} - \vec{x}\rangle = 0\}$ for some $\vec{u} = (u_1, \dots, u_n) \neq \vec{0}$ and $\vec{x} = (x_1, \dots, x_n)$. Suppose that $u_i \neq 0$. Then

$$P^n\big(\vec{X} \in S\big) \leqslant P^n\big(\vec{X} \in H\big) = P^n\left(X_i = x_i - u_i^{-1}\sum_{j\neq i} u_j(X_j - x_j)\right).$$

Since $P(X_i = x) \leqslant p$ for any $x \in \mathbb{R}$, the Claim follows. □

We next prove Fact 1.2 stated in the Introduction:

Proposition 3.5. *Let $\gamma > 0$ and $0 < r < r^*$ be such that $\lambda(\gamma) > r$. Then*

$$E\big(|\partial(\Lambda_r) \cap \gamma B_\infty^n \cap K_N|\big) \leqslant N \cdot \sup q(\vec{x}) \cdot |\partial(\Lambda_r) \cap \gamma B_\infty^n|\,,$$

where the sup *is over all $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$.*

Proof. Let $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$. If H is a closed half-space containing $\vec{x}$, and if $\vec{x} \in K_N = \operatorname{conv}\big\{\vec{X}_1, \dots, \vec{X}_N\big\}$, then there exists $i \leqslant N$ such that $\vec{X}_i \in H$ (otherwise we would have $\vec{x} \in K_N \subseteq H^c$, where H^c is the complementary half-space). It follows that

$$\operatorname{Prob}\big(\vec{x} \in K_N\big) \leqslant N \cdot P^n\big(\vec{X} \in H\big) \leqslant N \cdot q(\vec{x}).$$

Hence

$$\begin{aligned} E\big(|\partial(\Lambda_r) \cap \gamma B_\infty^n \cap K_N|\big) &\leqslant \int_{\partial(\Lambda_r)\cap\gamma B_\infty^n} \operatorname{Prob}\big(\vec{x} \in K_N\big)\, d\vec{x} \\ &\leqslant N \cdot \sup q(\vec{x}) \cdot |\partial(\Lambda_r) \cap \gamma B_\infty^n|\,, \end{aligned}$$

as asserted. □

Proposition 3.5 and an application of Markov's inequality show that, for every $\eta > 0$, we have

$$\operatorname{Prob}\big(|\partial(\Lambda_r) \cap \gamma B_\infty^n \cap K_N| \geqslant \eta\,|\partial(\Lambda_r) \cap \gamma B_\infty^n|\big) \leqslant \eta^{-1} \cdot N \cdot \sup q(\vec{x}). \tag{3.10}$$

The estimates in Propositions 3.2 and 3.5 will become useful if we can give sharp estimates for the quantities

$$q_*(r,\gamma) := \inf\{q(\vec{x})\colon x \in \partial(\Lambda_r) \cap \gamma B_\infty^n\} \tag{3.11}$$

and

$$q^*(r,\gamma) := \sup\{q(\vec{x})\colon x \in \partial(\Lambda_r) \cap \gamma B_\infty^n\}\,. \tag{3.12}$$

In fact, we will see that q is "almost constant" on $\partial(\Lambda_r) \cap \gamma B_\infty^n$, provided $\gamma = \gamma(\mu)$ is small enough. This main technical step is described in the next Section.

4 Large Deviations Estimates for $q(\vec{x})$

Given $x_1, \dots, x_n$ in I°, set

$$t_i := h(x_i) = \lambda'(x_i) \qquad (i \leqslant n). \tag{4.1}$$

In what follows, we will always assume that t_i and x_i are in this relation. Observe that $x_i = \psi'(t_i)$. We define the probability measure $P_{x_1,\dots,x_n}$ on $(\Omega^n, \mathcal{F}^{\otimes n})$, by

$$P_{x_1,\dots,x_n}(A) := E^n\left[\mathbf{1}_A \cdot \exp\left(\sum_{i=1}^n [t_i X_i - \psi(t_i)]\right)\right] \tag{4.2}$$

for $A \in \mathcal{F}^{\otimes n}$ (E^n denotes expectation with respect to the product measure P^n on $\mathcal{F}^{\otimes n}$). Direct computation shows the following.

Lemma 4.1. *Under $P_{x_1,\dots,x_n}$, the random variables $t_1X_1, \dots, t_nX_n$ are independent, with mean, variance and absolute central third moment given by*

$$\begin{aligned} E_{x_1,\dots,x_n}(t_iX_i) &= t_i\psi'(t_i) = t_ix_i, \\ E_{x_1,\dots,x_n}\left(|t_i(X_i - x_i)|^2\right) &= t_i^2\psi''(t_i), \\ E_{x_1,\dots,x_n}\left(|t_i(X_i - x_i)|^3\right) &= |t_i|^3\, E_{t_i}\left(|X - \psi'(t_i)|^3\right), \end{aligned}$$

respectively. □

Using Corollary 2.15, one immediately obtains the following bound on the ratio $E_{x_1,\dots,x_n}\left(|t_i(X_i - x_i)|^3\right)/E_{x_1,\dots,x_n}\left(|t_i(X_i - x_i)|^2\right)$ when the x_i stay in a closed interval contained in I°:

Lemma 4.2. *Assume that $[-a,a] \subset I^\circ$. Then there exists a constant $C = C(a,\mu)$ in $(0,\infty)$, depending only on a and μ, with the following property: if $|x_i| \leqslant a$, then*

$$\frac{E_{x_1,\dots,x_n}\left(|t_i(X_i - x_i)|^3\right)}{E_{x_1,\dots,x_n}\left(|t_i(X_i - x_i)|^2\right)} \leqslant C\,|t_i|\,,$$

for all $i = 1, \dots, n$ and all $n \in \mathbb{N}$. □

Notation 4.3. From now on, and until the end of this Section, we fix an a with $[-a,a] \subset I^\circ$, and denote by C the corresponding constant $C(a,\mu)$ of Lemma 4.2. We also fix

$$c := 6C. \tag{4.3}$$

Set

$$\sigma_n^2 := \sum_{i=1}^{n} E_{x_1,\dots,x_n}\big(|t_i(X_i - x_i)|^2\big) = \sum_{i=1}^{n} t_i^2 \psi''(t_i) \tag{4.4}$$

and

$$S_n := \frac{1}{\sigma_n} \sum_{i=1}^{n} t_i(X_i - x_i), \tag{4.5}$$

and let $F_n \colon \mathbb{R} \to \mathbb{R}$ be the cumulative distribution function of the random variable S_n under the probability law $P_{x_1,\dots,x_n}$:

$$F_n(x) := P_{x_1,\dots,x_n}(S_n \leqslant x) \qquad (x \in \mathbb{R}). \tag{4.6}$$

Write also μ_n for the probability measure on $\mathbb{R}$ defined by

$$\mu_n(-\infty, x] := F_n(x) \qquad (x \in \mathbb{R}). \tag{4.7}$$

Finally, set

$$\rho_n^{(3)} := \sum_{i=1}^{n} E_{x_1,\dots,x_n}\big(|t_i(X_i - x_i)|^3\big). \tag{4.8}$$

Lemma 4.2 shows that

$$\frac{\rho_n^{(3)}}{\sigma_n^2} \leqslant C \max_{1 \leqslant i \leqslant n} |t_i| \tag{4.9}$$

for all $n \in \mathbb{N}$, provided $x_1, \dots, x_n \in [-a,a]$. Notice also that

$$E_{x_1,\dots,x_n}(S_n) = 0 \quad \text{and} \quad \mathrm{Var}_{x_1,\dots,x_n}(S_n) = 1, \tag{4.10}$$

for any $x_1, \dots, x_n$.

Lemma 4.4. *For any $x_1, \dots, x_n$,*

$$P^n\left(\sum_{i=1}^{n} t_i(X_i - x_i) \geqslant 0\right) = \left[\int_{[0,\infty)} e^{-\sigma_n u}\, d\mu_n(u)\right] \exp\left(-\sum_{i=1}^{n} \lambda(x_i)\right). \tag{4.11}$$

Proof. We may write

$$P^n\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) = E_{x_1,\dots,x_n}\left[\mathbf{1}_{[0,\infty)}\left(\sum_{i=1}^n t_i(X_i - x_i)\right)\cdot \exp\left(-\sum_{i=1}^n t_i X_i\right)\right]\prod_{i=1}^n \varphi(t_i).$$

It follows that

$$P^n\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) = \int_{[0,\infty)} e^{-\sigma_n u}\, d\mu_n(u)\, \exp\left(\sum_{i=1}^n [\psi(t_i) - t_i x_i]\right),$$

and the result follows from Proposition 2.12. □

Write

$$\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad (x \in \mathbb{R}) \tag{4.12}$$

for the standard gaussian density,

$$\Phi(x) := \int_{-\infty}^x \phi(y)\, dy \qquad (x \in \mathbb{R}) \tag{4.13}$$

for the standard gaussian c.d.f., and ν for the standard gaussian probability measure: $\nu(-\infty, x] := \Phi(x)$, $x \in \mathbb{R}$. By the Berry–Esseen theorem [6, Theorem XVI.5.2] one has that

$$|F_n(x) - \Phi(x)| \leqslant 6\,\frac{\rho_n^{(3)}}{\sigma_n^3}$$

for all $x \in \mathbb{R}$, whence, by (4.9) and (4.3),

$$|F_n(x) - \Phi(x)| \leqslant \frac{c}{\sigma_n} \max_{1\leqslant i\leqslant n} |t_i| \tag{4.14}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, provided $x_1, \dots, x_n \in [-a, a]$.

Lemma 4.5. *The following inequality holds for any $x_1, \dots, x_n \in [-a, a]$, $n \in \mathbb{N}$:*

$$\left|\int_{(0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) - \int_{(0,\infty)} e^{-\sigma_n u}\, d\nu(u)\right| \leqslant \frac{2c}{\sigma_n} \max_{1\leqslant i\leqslant n} |t_i|\,.$$

Proof. Since

$$\begin{aligned}\int_{(0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) &= \int_0^\infty \mu_n\big(\{u\colon e^{-\sigma_n u}\mathbf{1}_{(0,\infty)}(u) \geqslant r\}\big)\, dr\\ &= \int_0^1 \mu_n\big(0, -\sigma_n^{-1}\log r\big]\, dr\\ &= \int_0^1 \big[F_n\big(-\sigma_n^{-1}\log r\big) - F_n(0)\big]\, dr,\end{aligned}$$

and similarly

$$\int_0^\infty e^{-\sigma_n u}\, d\nu(u) = \int_0^1 \big[\Phi\big(-\sigma_n^{-1}\log r\big) - \Phi(0)\big]\, dr,$$

the result follows from (4.14). $\square$

We will use the estimates

$$\frac{1}{x}\, m_1(x)\, e^{-x^2/2} \leqslant \int_x^\infty \phi(u)\, du \leqslant \frac{1}{x}\, m_2(x)\, e^{-x^2/2} \qquad (x>0), \tag{4.15}$$

where

$$m_1(x) = \frac{1}{\sqrt{2\pi}}\,\frac{2x}{x+\sqrt{x^2+4}} \qquad \text{and} \qquad m_2(x) = \frac{1}{\sqrt{2\pi}}\,\frac{4x}{3x+\sqrt{x^2+8}} \tag{4.16}$$

(see [11, p. 17] and, for the upper estimate, [19]). Since

$$\int_0^\infty e^{-\sigma_n u}\, d\nu(u) = \frac{e^{\sigma_n^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-(\sigma_n+u)^2/2}\, du = e^{\sigma_n^2/2}\int_{\sigma_n}^\infty \phi(u)\, du, \tag{4.17}$$

it follows that

$$\frac{m_1(\sigma_n)}{\sigma_n} \leqslant \int_0^\infty e^{-\sigma_n u}\, d\nu(u) \leqslant \frac{m_2(\sigma_n)}{\sigma_n}. \tag{4.18}$$

Combining (4.18) with Lemma 4.5, we obtain the estimates

$$\frac{m_1(\sigma_n)}{\sigma_n} - \frac{2c}{\sigma_n}\max_{1\leqslant i\leqslant n}|t_i| \leqslant \int_{(0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) \leqslant \frac{m_2(\sigma_n)}{\sigma_n} + \frac{2c}{\sigma_n}\max_{1\leqslant i\leqslant n}|t_i| \tag{4.19}$$

for $x_1,\dots,x_n \in [-a,a]$. Lemma 4.4 then yields the following estimates:

Theorem 4.6. *Let $x_1,\dots,x_n \in [-a,a]$ and $t_i = h(x_i)$, $i=1,\dots,n$. Then,*

$$P^n\left(\sum_{i=1}^n t_i(X_i-x_i) \geqslant 0\right) \geqslant \frac{1}{\sigma_n}\, e^{-n\Lambda(\vec{x})}\left(m_1(\sigma_n) - 2c\max_{1\leqslant i\leqslant n}|t_i|\right) \tag{4.20}$$

and

$$P^n\left(\sum_{i=1}^n t_i(X_i-x_i) \geqslant 0\right) \leqslant \frac{1}{\sigma_n}\, e^{-n\Lambda(\vec{x})}\left(m_2(\sigma_n) + 4c\max_{1\leqslant i\leqslant n}|t_i|\right). \tag{4.21}$$

Proof. Observe that the first factor on the right hand-side of (4.11) is

$$\int_{[0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) \geqslant \int_{(0,\infty)} e^{-\sigma_n u}\, d\mu_n(u). \tag{4.22}$$

Hence, we obtain the first inequality (4.20) by combining (4.11) with (4.19). For the second inequality, first observe that

$$\int_{[0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) = \int_{(0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) + P_{x_1,\dots,x_n}(S_n = 0),$$

and then use (4.14) to obtain,

$$P_{x_1,\dots,x_n}(S_n = 0) \leqslant F_n(\varepsilon) - F_n(-\varepsilon) \leqslant \Phi(\varepsilon) - \Phi(-\varepsilon) + \frac{2c}{\sigma_n} \max_{1\leqslant i\leqslant n} |t_i|$$

for all $\varepsilon > 0$. Thus

$$P_{x_1,\dots,x_n}(S_n = 0) \leqslant \frac{2c}{\sigma_n} \max_{1\leqslant i\leqslant n} |t_i|\,, \tag{4.23}$$

and the second inequality (4.21) follows now as well, using (4.11), (4.19) and (4.23) this time. □

Corollary 4.7. *There exists $\delta > 0$, and constants $c_1(\delta), \dots, c_4(\delta) \in (0,\infty)$, depending only on δ and μ, with the following property: if $x_1, \dots, x_n \in (-\delta,\delta)$, and $t_i = h(x_i)$, $i = 1, \dots, n$, then*

$$P^n\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) \geqslant \frac{c_1(\delta)}{\sqrt{n\Lambda(\vec{x})}}\, e^{-n\Lambda(\vec{x})} \left[m_1\Big(\sqrt{c_2(\delta) n\Lambda(\vec{x})}\Big) - 2c \max_{1\leqslant i\leqslant n} |t_i|\right]$$

and

$$P^n\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) \leqslant \frac{c_3(\delta)}{\sqrt{n\Lambda(\vec{x})}}\, e^{-n\Lambda(\vec{x})} \left[m_2\Big(\sqrt{c_4(\delta) n\Lambda(\vec{x})}\Big) + 4c \max_{1\leqslant i\leqslant n} |t_i|\right]$$

for all $n \in \mathbb{N}$, where $n\Lambda(\vec{x}) = \sum_{i=1}^n \lambda(x_i)$.

Proof. By Proposition 2.13, we can choose $\delta \in (0, a]$ such that

$$\tfrac{1}{4}x^2 \leqslant \lambda(x) \leqslant x^2 \quad \text{and} \quad \tfrac{1}{2}\,|x| \leqslant |h(x)| \leqslant 2\,|x| \tag{4.24}$$

for all $x \in (-\delta,\delta)$. Let $x_1, \dots, x_n \in (-\delta,\delta)$. Theorem 4.6 shows that

$$P^n\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) \geqslant \frac{1}{\sigma_n}\, e^{-n\Lambda(\vec{x})} \left(m_1(\sigma_n) - 2c \max_{1\leqslant i\leqslant n} |t_i|\right). \tag{4.25}$$

By (4.4), Proposition 2.13 and (4.24), we get

$$\sigma_n^2 = \sum_{i=1}^n t_i^2 \psi''(t_i) \leqslant 4c_2 \sum_{i=1}^n x_i^2 \leqslant 16 c_2 \sum_{i=1}^n \lambda(x_i) \tag{4.26}$$

and, similarly,

$$\sigma_n^2 = \sum_{i=1}^n t_i^2 \psi''(t_i) \geqslant \tfrac{1}{4} c_1 \sum_{i=1}^n x_i^2 \geqslant \tfrac{1}{4} c_1 \sum_{i=1}^n \lambda(x_i), \tag{4.27}$$

where c_1 and c_2 come from Proposition 2.13 (i), and depend only on δ and μ. Inserting these estimates into (4.25) yields the first inequality asserted in the Corollary. For the upper estimate we work in the same way, starting from (4.21) this time. □

Corollary 4.8. *There exist $\gamma > 0$, $k = k(\gamma) \in \mathbb{N}$ and $c(\gamma) > 0$, depending only on γ and μ, with the following property: for every $n \in \mathbb{N}$, if $x_1, \dots, x_n \in (-\gamma, \gamma)$ are such that $\sum_{i=1}^n \lambda(x_i) \geqslant k(\gamma)$, and if $t_i = h(x_i)$, $i = 1, \dots, n$, then*

$$P^n\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) \geqslant \frac{c(\gamma)}{\sqrt{n\Lambda(\vec{x})}}\, e^{-n\Lambda(\vec{x})}.$$

Proof. First fix a δ for which Corollary 4.7 holds true, and then choose $\gamma \in (0, \delta]$ so that $2ch(\gamma) \leqslant (2\sqrt{2\pi})^{-1}$; this is possible because $\lim_{s\to 0} h(s) = 0$.

We know that $m_1(x)$ increases to $(2\pi)^{-1/2}$ as $x \to \infty$; therefore, there exists $k = k(\gamma) \in \mathbb{N}$ such that

$$m_1\big(c_2(\delta)k(\gamma)\big) \geqslant \frac{5}{6\sqrt{2\pi}}.$$

From the first assertion of Corollary 4.7 we then easily conclude the result for $x_1 \dots, x_n \in (-\gamma, \gamma)$ with $n\Lambda(\vec{x}) = \sum_{i=1}^n \lambda(x_i) \geqslant k(\gamma)$. □

Definition 4.9. Given a Borel probability measure μ on the real line satisfying our assumptions (1.1)–(1.3), fix a constant δ for which the assertion of Corollary 4.7 and also (4.24) hold true, a constant $\gamma < \delta$ for which the assertion of Corollary 4.8 holds true, and set $\gamma_\mu := \min\big\{\gamma, \psi'(\tfrac{1}{4})\big\}$.

We can now estimate $q(\vec{x})$ for $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$. We start with Theorem 1.3:

Theorem 4.10. *For $\gamma \in (0, \gamma_\mu]$, there exists $n_0 = n_0(\gamma) \in \mathbb{N}$ with the following property: if $n \geqslant n_0$ and $r \leqslant \lambda(\gamma)$, then*

$$q(\vec{x}) \leqslant \exp\big(-rn - \tfrac{1}{2}\ln(rn) + c(\gamma)\big)$$

for every $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$, where $c(\gamma)$ is a constant depending only on γ and μ.

Proof. Let $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$. By the definition of $q(\vec{x})$ we have that

$$q(\vec{x}) \leqslant P^n\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right),$$

where $t_i = h(x_i)$, $i = 1, \dots, n$, and by Corollary 4.7 that

$$P^n\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) \leqslant \frac{c(\gamma)}{\sqrt{n\Lambda(\vec{x})}}\, e^{-n\Lambda(\vec{x})}.$$

Since $\Lambda(\vec{x}) = r$, the result follows. □

The upper estimate of Theorem 4.10 is complemented by the following lower estimate (this is the result stated as Theorem 1.4 in the introduction):

Theorem 4.11. *For $\gamma \in (0, \gamma_\mu]$, there exists $n_0 = n_0(\gamma) \in \mathbb{N}$ with the following property: for $n \geqslant n_0$, and $\varepsilon \geqslant 3 \ln n / n$,*

$$q(\vec{x}) \geqslant \exp(-(r+\varepsilon)n)$$

for every $\vec{x} \in \Lambda_r \cap \gamma B_\infty^n$ and $r \in (0, \lambda(\gamma)]$.

The proof of this will be based on Corollary 4.8 and on two additional facts. The first one generalizes a result of Montgomery-Smith from [16] (its proof appears in [14]):

Lemma 4.12. *For all $n \in \mathbb{N}$ and any $\vec{s} = (s_1, \dots, s_n) \in \mathbb{R}^n$, the inequality*

$$P^n\left(\sum_{i=1}^{n} s_i X_i \geqslant \tfrac{1}{2}\theta \, \|\vec{s}\|_2\right) \geqslant e^{-C(\mu)\theta^2}$$

holds for all $\theta > 0$ with $\theta \leqslant \|\vec{s}\|_2 \,/\, \|\vec{s}\|_\infty$, with $C(\mu) = 2\ln(2^7/\pi) + 4 \ln E\big(|X|^3\big)$.

Proof. This is Lemma 4.3 of [14] (with the constants replaced by the ones that actually appear in the proof), plus the additional fact that $h_L(\vec{s}) = \frac{1}{2}\theta \, \|\vec{s}\|_2$ for $0 < \theta \leqslant \|\vec{s}\|_2 \,/\, \|\vec{s}\|_\infty$, where $h_L(\vec{s}) = \sup\big\{\langle \vec{s}, \vec{v}\rangle : \vec{v} \in \frac{1}{2}(B_\infty^n \cap \theta B_2^n)\big\}$. □

The second one combines [2, Lemma 8.2] with a theorem of Bahadur and Ranga Rao [1]. Recall (2.6), defining I°.

Lemma 4.13. *Assume that $\gamma \in I^\circ$. There exists $m_0 = m_0(\gamma)$ such that, for all $m \geqslant m_0$, and any $s_1, \dots, s_m \in \mathbb{R}$ with $\sum_{i=1}^m s_i > 0$,*

$$P^m\left(\sum_{i=1}^{m} s_i(X_i - \gamma) \geqslant 0\right) \geqslant c(\gamma)\, m^{-3/2}\, e^{-m\lambda(\gamma)},$$

where the constant $c(\gamma) > 0$ depends only on γ and μ.

Proof. The first part of the argument in [2, Lemma 8.2] shows that

$$P^m\left(\sum_{i=1}^{m} s_i(X_i - \gamma) \geqslant 0\right) \geqslant \frac{1}{m}\, P^m\left(\sum_{i=1}^{m}(X_i - \gamma) \geqslant 0\right). \tag{4.28}$$

Indeed, for $i = 1, \dots, m$ define $\vec{s}_i := (s_i, \dots, s_m, s_1, \dots, s_{i-1})$, set $\vec{\mathbf{1}} := (1, \dots, 1)$, and notice that

$$P^m\left(\sum_{i=1}^{m} s_i(X_i - \gamma) \geqslant 0\right) = P^m\Big(\langle \vec{s}_i, \vec{X} - \gamma\vec{\mathbf{1}}\rangle \geqslant 0\Big) \qquad \text{for all } i = 1, \dots, m$$

(since $(X_1, \dots, X_n)$ and $(X_i, \dots, X_m, X_1, \dots, X_{i-1})$ have the same distribution, namely $\mu \times \dots \times \mu$). It follows that

$$
\begin{aligned}
P^m\left(\sum_{i=1}^m s_i(X_i - \gamma) \geqslant 0\right) &= \frac{1}{m} \sum_{i=1}^m P^m\Big(\big\langle \vec{s}_i, \vec{X} - \gamma\vec{\mathbf{1}}\big\rangle \geqslant 0\Big) \\
&\geqslant \frac{1}{m}\, P^m\left(\bigcup_{i=1}^m \Big\{\big\langle \vec{s}_i, \vec{X} - \gamma\vec{\mathbf{1}}\big\rangle \geqslant 0\Big\}\right) \\
&\geqslant \frac{1}{m} P^m\left((s_1 + \dots + s_n)\sum_{i=1}^m (X_i - \gamma) \geqslant 0\right) \\
&= \frac{1}{m} P^m\left(\sum_{i=1}^m (X_i - \gamma) \geqslant 0\right).
\end{aligned}
$$

By [1, Theorem 1] on the other hand, there exists a sequence b_m of positive numbers, such that

$$
\frac{\sqrt{2\pi m}}{b_m}\, e^{m\lambda(\gamma)}\, P^m\left(\sum_{i=1}^m (X_i - \gamma) \geqslant 0\right) \to 1 \qquad \text{as } m \to \infty,
$$

with $\ln b_m$ bounded, and hence b_m bounded away from 0. $\square$

Proof of Theorem 4.11. Fix $\gamma \in (0, \gamma_\mu]$, and let ε and r be as in the statement of the Theorem. We first assert the following:

Claim 4.14. It suffices to show that

$$
P^n\big(\vec{X} \in H\big) \geqslant e^{-(r+\varepsilon)n} \tag{4.29}
$$

for any closed half-space H whose bounding hyperplane supports $\Lambda_r \cap \gamma B^n_\infty$ and for which $(\Lambda_r \cap \gamma B^n_\infty) \cap H^\circ = \varnothing$ (i.e., H just "touches" $\Lambda_r \cap \gamma B^n_\infty$),

We postpone the proof of the claim until the end of this Section.

Next fix a closed half-space H of the form described in Claim 4.14. We will show (4.29) for this half-space.

Claim 4.15. There exists $\vec{x} \in H \cap (\partial(\Lambda_r) \cap \gamma B^n_\infty)$.

The proof of this claim is also postponed until the end of the Section.

Now fix $\vec{x} \in H \cap (\partial(\Lambda_r) \cap \gamma B^n_\infty)$. By symmetry we may assume that $\vec{x} = (x_1, \dots, x_n)$ with $0 \leqslant x_1 \leqslant \dots \leqslant x_n$. There exists $n_1 \in \{1, \dots, n\}$ such that $x_{n_1} < \gamma$ and $x_{n_1+1} = \gamma$. Set $\vec{t} = (t_1, \dots, t_n) = n\,\nabla\Lambda(\vec{x})$ and write $\vec{s} = (s_1, \dots, s_n)$ for the normal to the bounding hyperplane of H, so that

$$
H = \left\{\vec{y} \in \mathbb{R}^n \colon \sum_{i=1}^n s_i(y_i - x_i) \geqslant 0\right\}.
$$

By the form we have assumed of H, $\vec{s}$ lies in the normal cone to $\Lambda_r \cap \gamma B_\infty^n$ at $\vec{x}$; i.e., in the notation of [17],

$$\vec{s} \in N(\vec{x}, \Lambda_r \cap \gamma B_\infty^n) = \{\vec{v} \in \mathbb{R}^n \colon \langle \vec{v}, \vec{y} - \vec{x} \rangle \leqslant 0 \ \forall \vec{y} \in \Lambda_r \cap \gamma B_\infty^n\}\,.$$

According to [17, Theorem 2.2.1],

$$N(\vec{x}, \Lambda_r \cap \gamma B_\infty^n) = N(\vec{x}, \Lambda_r) + N(\vec{x}, \gamma B_\infty^n).$$

Now

$$N(\vec{x}, \Lambda_r) = \{t\, \nabla \Lambda(\vec{x}) \colon t \geqslant 0\}$$

and

$$\begin{aligned} N(\vec{x}, \gamma B_\infty^n) &= \{\vec{v} \in \mathbb{R}^n \colon \langle \vec{v}, \vec{y} - \vec{x} \rangle \leqslant 0 \ \forall \vec{y} \in \gamma B_\infty^n\} \\ &= \{\vec{v} \in \mathbb{R}^n \colon v_1 = \ldots = v_{n_1} = 0,\ v_{n_1+1}, \ldots, v_n \geqslant 0\}\,. \end{aligned}$$

It follows that we may assume that

$$s_i = t_i = \lambda'(x_i) = h(x_i) \quad \text{if } i \leqslant n_1 \quad \text{and} \quad s_i \geqslant t_i = \lambda'(\gamma) = h(\gamma) \quad \text{if } i > n_1.$$

Now write

$$\begin{aligned} P^n\big(\vec{X} \in H\big) &= P^n\left(\sum_{i=1}^{n} s_i(X_i - x_i) \geqslant 0\right) \\ &= P^n\left(\sum_{i=1}^{n_1} t_i(X_i - x_i) + \sum_{i=n_1+1}^{n} s_i(X_i - \gamma) \geqslant 0\right) \\ &\geqslant P^n\left(\sum_{i=1}^{n_1} t_i(X_i - x_i) \geqslant 0\right) \cdot P^n\left(\sum_{i=n_1+1}^{n} s_i(X_i - \gamma) \geqslant 0\right). \end{aligned} \tag{4.30}$$

We estimate the second probability in the last product using Lemma 4.13:

$$P^n\left(\sum_{i=n_1+1}^{n} s_i(X_i - \gamma) \geqslant 0\right) \geqslant \exp\big(-(n-n_1)\lambda(\gamma) - \tfrac{3}{2}\ln(n-n_1) - c_1(\gamma)\big). \tag{4.31}$$

To estimate the first probability we distinguish two cases:

Case 1: $\sum_{i=1}^{n_1} \lambda(x_i) \geqslant k(\gamma)$. We may then use Corollary 4.8 to estimate the first probability:

$$P^n\left(\sum_{i=1}^{n_1} t_i(X_i - x_i) \geqslant 0\right) \geqslant \exp\left(-\sum_{i=1}^{n_1} \lambda(x_i) - \tfrac{1}{2}\ln\sum_{i=1}^{n_1} \lambda(x_i) - c_2(\gamma)\right) \tag{4.32}$$

Combining (4.30) with (4.31) and (4.32), we obtain

$$\begin{aligned}
P^n\big(\vec{X}\in H\big) &\geqslant \exp\left(-\sum_{i=1}^{n_1}\lambda(x_i)-\tfrac{1}{2}\ln\sum_{i=1}^{n_1}\lambda(x_i)-c_2(\gamma)\right)\\
&\quad\times\ \exp\big(-(n-n_1)\lambda(\gamma)-\tfrac{3}{2}\ln(n-n_1)-c_1(\gamma)\big)\\
&\geqslant \exp\left(-\sum_{i=1}^{n}\lambda(x_i)-2\ln n-c(\gamma)\right)\\
&= \exp\big(-rn-2\ln n-c(\gamma)\big)\\
&\geqslant \exp(-(r+\varepsilon)n),
\end{aligned}$$

provided n is sufficiently large ($\ln n \geqslant c(\gamma)$).

CASE 2: $\sum_{i=1}^{n_1}\lambda(x_i)<k(\gamma)$. In this case we use Lemma 4.12 to estimate the first probability. We have that

$$P^n\left(\sum_{i=1}^{n_1}t_i(X_i-x_i)\geqslant 0\right)=P^n\left(\sum_{i=1}^{n_1}t_iX_i\geqslant \tfrac{1}{2}\theta\sqrt{\sum_{i=1}^{n_1}t_i^2}\right), \tag{4.33}$$

with

$$\theta = 2\,\frac{\sum_{i=1}^{n_1}t_ix_i}{\sqrt{\sum_{i=1}^{n_1}t_i^2}}. \tag{4.34}$$

To use Lemma 4.12 we need to check that

$$\theta\leqslant\frac{\sqrt{\sum_{i=1}^{n_1}t_i^2}}{\max_{1\leqslant i\leqslant n_1}t_i}. \tag{4.35}$$

Recall that, by the choice of γ_μ (cf. Definition 4.9), we have that $|x|\leqslant 2\,|h(x)|$ for all $x\in(-\gamma,\gamma)$ (cf. (4.24)). Hence $\sum_{i=1}^{n_1}t_ix_i\leqslant 2\sum_{i=1}^{n_1}t_i^2$, and therefore,

$$\theta\leqslant 4\sqrt{\sum_{i=1}^{n_1}t_i^2}\leqslant 4\,h(\gamma)\,\frac{\sqrt{\sum_{i=1}^{n_1}t_i^2}}{\max\limits_{1\leqslant i\leqslant n_1}t_i}, \tag{4.36}$$

since also $t_i=h(x_i)\leqslant h(\gamma)$ for all i, by the monotonicity of h. Since $\gamma\leqslant\psi'(\frac{1}{4})$ is small enough, we have that $h(\gamma)\leqslant\frac{1}{4}$ and θ satisfies (4.35).

By (4.33) and Lemma 4.12 we get the bound

$$P^n\left(\sum_{i=1}^{n_1}t_i(X_i-x_i)\geqslant 0\right)\geqslant e^{-C(\mu)\theta^2},$$

which upon using the Cauchy–Schwartz inequality and (4.24) yields the bound

$$P^n\left(\sum_{i=1}^{n_1}t_i(X_i-x_i)\geqslant 0\right)\geqslant\exp\left(-4C(\mu)\sum_{i=1}^{n_1}x_i^2\right)\geqslant\exp\left(-16C(\mu)\sum_{i=1}^{n_1}\lambda(x_i)\right).$$

Combining this with (4.30) and (4.31) again, we see that

$$\begin{aligned}
P^n\big(\vec{X} \in H\big) &\geqslant \exp\Bigg(-(n-n_1)\lambda(\gamma) - \tfrac{3}{2}\ln(n-n_1) - c_1(\gamma) - c_3(\gamma)\sum_{i=1}^{n_1}\lambda(x_i)\Bigg) \\
&\geqslant \exp\Bigg(-\sum_{i=1}^{n}\lambda(x_i) - \tfrac{3}{2}\ln(n-n_1) - c_1(\gamma) - |1-c_3(\gamma)|k(\gamma)\Bigg) \\
&= \exp\big(-rn - \tfrac{3}{2}\ln(n-n_1) - c'(\gamma)\big) \\
&\geqslant \exp(-(r+\varepsilon)n)
\end{aligned}$$

in this case as well, provided again that n is large enough to have $\ln n \geqslant \frac{2}{3}c'(\gamma)$.

The proof of the Theorem is now complete, modulo the proofs of the two claims. □

Proof of Claim 4.14. Let $\vec{x} \in \Lambda_r \cap \gamma B^n_\infty$ and let H be any closed half-space with $\vec{x} \in \partial H$. We can then write $H = \{\vec{y} \in \mathbb{R}^n \colon \sum_{i=1}^n s_i(y_i - x_i) \geqslant 0\}$ for some $\vec{s} = (s_1, \ldots, s_n) \neq \vec{0}$. For $z \geqslant 0$, define $H_z := \{\vec{y} \in \mathbb{R}^n \colon \sum_{i=1}^n s_i(y_i - x_i) \geqslant z\}$, and observe that the half-spaces H_z decrease as z increases. Notice also that $H_0 = H$. Let

$$z_* := \sup\{z \geqslant 0 \colon H_z \cap (\Lambda_r \cap \gamma B^n_\infty) \neq \varnothing\};$$

by compactness $z_* < \infty$, and we may set $H_* = H_{z_*}$. Since a decreasing intersection of nonempty compact sets is nonempty,

$$H_* \cap (\Lambda_r \cap \gamma B^n_\infty) = \bigcap_{0 \leqslant z < z_*} H_z \cap (\Lambda_r \cap \gamma B^n_\infty) \neq \varnothing,$$

while

$$H^\circ_* \cap (\Lambda_r \cap \gamma B^n_\infty) = \bigcup_{z > z_*} H^\circ_z \cap (\Lambda_r \cap \gamma B^n_\infty) = \varnothing.$$

Thus H_* is of the form described in the statement of the Claim. Furthermore, $H \supseteq H_*$, and hence $P^n\big(\vec{X} \in H\big) \geqslant P^n\big(\vec{X} \in H_*\big)$. By (3.3), this proves the Claim. □

Proof of Claim 4.15. By the form we have assumed of H (i.e., that it "touches" $\Lambda_r \cap \gamma B^n_\infty$), there exists $\vec{y} \in H \cap (\Lambda_r \cap \gamma B^n_\infty)$. If $\Lambda(\vec{y}) = r$, then $\vec{x} = \vec{y}$ is the sought for point. Assume next that $\Lambda(\vec{y}) < r$. Then $\vec{y} \in \partial(\gamma B^n_\infty)$ and $\vec{y}$ belongs to some face $F(\vec{y})$ of γB^n_∞ of minimal dimension. Notice that $F(\vec{y})$ must be contained in the hyperplane bounding H, and hence also be contained in H. By joining $\vec{y}$ with any vertex $\vec{v}$ of γB^n_∞ belonging to $F(\vec{y})$, we see that there exists $\vec{x} \in F(\vec{y})$ with $\Lambda(\vec{x}) = r$, by the intermediate value theorem, and since $\Lambda(\vec{y}) \leqslant r$ and $\Lambda(\vec{v}) = \lambda(\gamma) \geqslant r$. Since $F(\vec{y}) \subseteq \partial H$, it follows that $\vec{x} \in H \cap \partial(\Lambda_r) \cap \gamma B^n_\infty$. □

5 Geometry of K_N

In this Section we prove Theorems 1.5, 1.6 and 1.7. Throughout the Section γ will always be in the interval $(0, \gamma_\mu]$ (cf. Definition 4.9). Recall that γ_μ only depends on the underlying distribution μ of the coordinates of the random vertex $\vec{X}$. We also fix the following notation:

Notation 5.1.

$$\rho := \frac{\ln N}{n}.$$

We begin with Theorem 1.5:

Theorem 5.2. *For $\gamma \in (0, \gamma_\mu]$ and N in the range $n^6 < N \leqslant e^{n\lambda(\gamma)}$,*

$$\mathrm{Prob}\big(K_N \supseteq \Lambda_{\rho-2\varepsilon} \cap \gamma B_\infty^n\big) \geqslant 1 - 2^{-n+1}$$

for all $\varepsilon \geqslant 3 \ln n/n$, and all sufficiently large n.

Proof. By the choice of γ, we may apply Theorem 4.11; let $n_0 = n_0(\gamma)$ be the integer whose existence is asserted there, and let $n \geqslant n_0$. Fix also $\varepsilon = 3\ln n/n$. By Theorem 4.11,

$$\inf q(\vec{x}) \geqslant \exp(-(\rho - \varepsilon)n), \tag{5.1}$$

where the inf is over $\vec{x} \in \partial(\Lambda_{\rho-2\varepsilon}) \cap \gamma B_\infty^n$; notice that the condition $\lambda(\gamma) + 2\varepsilon \geqslant \rho > 2\varepsilon$, required to apply Theorem 4.11, is satisfied here, by our choice of N. Combining this with Proposition 3.2, we obtain that

(5.2)
$$1 - \mathrm{Prob}\big(K_N \supseteq \Lambda_{\rho-2\varepsilon} \cap \gamma B_\infty^n\big) \leqslant \binom{N}{n} p^{N-n} + 2\binom{N}{n}\big(1 - \exp(-(\rho-\varepsilon)n)\big)^{N-n}.$$

Claim 5.3. For n sufficiently large and $N > n^6$,

$$\binom{N}{n} p^{N-n} < 2^{-n}.$$

Proof of Claim 5.3. Since $\binom{N}{n} \leqslant (eN/n)^n$, it suffices to check that

$$1 + \ln\Big(\frac{N}{n}\Big) + \frac{N-n}{n}\ln p < -\ln 2. \tag{5.3}$$

Set $x := N/n$. Then, (5.3) is equivalent to

$$-(x-1)\ln p - \ln x > 1 + \ln p.$$

The claim follows from the facts that the function on the left-hand side increases to infinity as $x \to \infty$, and $x = N/n > n^5$. □

Claim 5.4. If n is large enough and $N \geqslant 2n$, then

$$2\binom{N}{n}\big(1-\exp(-(\rho-\varepsilon)n)\big)^{N-n} < 2^{-n}.$$

Proof of Claim 5.4. Since $1-x \leqslant e^{-x}$, it suffices to check that

$$\left(\frac{4eN}{n}\right)^n \exp\Big(-(N-n)e^{-(\rho-\varepsilon)n}\Big) < 1.$$

Observe that $e^{-(\rho-\varepsilon)n} = e^{\varepsilon n}/N$. Since $n\ln(4eN/n) \leqslant n^2\lambda(\gamma)$ (assume that $n \geqslant 4e$) and $(N-n)/N \geqslant \frac{1}{2}$, we want

$$2n^2\lambda(\gamma) < e^{\varepsilon n}.$$

This is satisfied when $\varepsilon = 3\ln n/n$ and n is sufficiently large. □

Combining (5.2) with the two Claims above, we obtain that

$$\mathrm{Prob}\big(K_N \supseteq \Lambda_{\rho-2\varepsilon} \cap \gamma B_\infty^n\big) \geqslant 1 - 2^{-n} - 2^{-n}$$

for n sufficiently large. □

We next complement Theorem 5.2 by showing that, for some sufficiently small $\delta \in (0,1)$, a fixed proportion of the surface area of $\Lambda_{\rho-\delta}$ lying in γB_∞^n is missed by the typical K_N, with high probability. This, in conjunction with Theorem 5.2, shows that, with high probability, $K_N \cap \gamma B_\infty^n$ is "weakly sandwiched" between $\Lambda_{\rho-2\varepsilon} \cap \gamma B_\infty^n$ and $\Lambda_{\rho-\delta}$. The following is Theorem 1.6 of the Introduction.

Theorem 5.5. *Fix $\gamma \in (0,\gamma_\mu]$. For n large enough and $n < N \leqslant e^{n\lambda(\gamma)}$,*

$$\mathrm{Prob}\big(\, |\partial(\Lambda_{\rho-\delta}) \cap \gamma B_\infty^n \cap K_N| \geqslant \alpha\, |\partial(\Lambda_{\rho-\delta}) \cap \gamma B_\infty^n|\,\big) \leqslant \frac{1}{\alpha}\frac{c(\gamma)}{(\ln N)^{1/6}},$$

for all $\delta \leqslant \frac{1}{3}(\ln\ln N)/n$ ($\delta \geqslant 0$) and any $0 < \alpha < 1$, where $c(\gamma)$ is a constant depending only on γ and μ.

Proof. Fix $\delta = \delta_n \leqslant \frac{1}{3}(\ln\ln N)/n$, $\delta \geqslant 0$. Proposition 3.5 shows that

$$E\big(|\partial(\Lambda_{\rho-\delta}) \cap \gamma B_\infty^n \cap K_N|\big) \leqslant N \cdot \sup q(\vec{x}) \cdot |\partial(\Lambda_{\rho-\delta}) \cap \gamma B_\infty^n| \tag{5.4}$$

where the sup is over all $\vec{x} \in \partial(\Lambda_{\rho-\delta}) \cap \gamma B_\infty^n$. Theorem 4.10 shows that

$$q(\vec{x}) \leqslant \exp\big(-(\rho-\delta)n - \tfrac{1}{2}\ln((\rho-\delta)n) + c(\gamma)\big) \tag{5.5}$$

for every $\vec{x} \in \partial(\Lambda_{\rho-\delta}) \cap \gamma B_\infty^n$, provided n is large enough. Since $\ln x \leqslant x-1$ for $x > 0$, the inequality

$$x - \tfrac{1}{3}\ln x \geqslant \tfrac{2}{3}x \tag{5.6}$$

holds for all $x > 0$. By (5.5) we then have that

$$\begin{aligned} N \cdot \sup q(\vec{x}) &\leqslant \exp\big(\delta n - \tfrac{1}{2}\ln(\ln N - \delta n) + c(\gamma)\big) \\ &\leqslant \exp\big(\tfrac{1}{3}\ln\ln N - \tfrac{1}{2}\ln(\ln N - \tfrac{1}{3}\ln\ln N) + c(\gamma)\big) \\ &\leqslant \exp\big(-\tfrac{1}{6}\ln\ln N + c'(\gamma)\big), \end{aligned}$$

the last inequality following from (5.6) for $x = \ln N$. Inserting this into (5.4) yields then that

$$E\big(|\partial(\mathit{\Lambda}_{\rho-\delta}) \cap \gamma B_\infty^n \cap K_N|\big) \leqslant c''(\gamma)(\ln N)^{-1/6}\, |\partial(\mathit{\Lambda}_{\rho-\delta}) \cap \gamma B_\infty^n|\,,$$

whence for any $0 < \alpha < 1$,

$$\mathrm{Prob}\big(\ |\partial(\mathit{\Lambda}_{\rho-\delta}) \cap \gamma B_\infty^n \cap K_N| \geqslant \alpha\, |\partial(\mathit{\Lambda}_{\rho-\delta}) \cap \gamma B_\infty^n|\ \big) \leqslant \frac{1}{\alpha}\frac{c''(\gamma)}{(\ln N)^{1/6}},$$

by Markov's inequality. □

Remark 5.6. An examination of the proof of Theorem 5.5 reveals that it remains valid whenever δ and N satisfy $(\ln N)/n - \lambda(\gamma) < \delta \leqslant \frac{1}{3}(\ln\ln N)/n$ (whence $0 < \rho - \delta < \lambda(\gamma)$). This may include negative values of δ.

Weaker inclusion, improved probability estimates

As already observed, Theorems 5.2 and 5.5 show that $K_N \cap \gamma B_\infty^n$ is "weakly sandwiched" between $\mathit{\Lambda}_{\rho-2\varepsilon} \cap \gamma B_\infty^n$ and $\mathit{\Lambda}_{\rho-\delta}$, with high probability. A result in the spirit of Theorem 5.2 also appears in [14], under different conditions on the underlying distribution μ of the coordinates of the random vertex $\vec{X}$:

Theorem 5.7 ([14], Theorem 4.2)**.** *Let ξ_{ij}, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant n$, be independent symmetric random variables satisfying*

$$1 \leqslant E(\xi_{ij}^2) \quad \text{and} \quad E\big(|\xi_{ij}|^3\big) \leqslant b \quad \text{for all } i, j,$$

and

$$P\big(\|\mathit{\Gamma}\| \geqslant a_1\sqrt{N}\big) \leqslant e^{-a_2 N}, \tag{5.7}$$

for some $b \geqslant 1$ and $a_1, a_2 > 0$, where $\mathit{\Gamma}$ is the random $N \times n$ matrix $\mathit{\Gamma} = (\xi_{ij})_{1\leqslant i\leqslant N,\ 1\leqslant j\leqslant n}$, and let $K'_N := \mathit{\Gamma}^* B_n^1$ be the absolute convex hull of the rows of $\mathit{\Gamma}$. Let $\beta \in (0,1)$. Then there exists a constant c_β such that for all N in the range $2^n \geqslant N \geqslant nc_\beta$,

$$\mathrm{Prob}\big(K'_N \supseteq 0.125(B_\infty^n \cap \varrho B_2^n)\big) \geqslant 1 - e^{-a_2 N} - e^{-0.2 n^\beta N^{1-\beta}},$$

with $\varrho = \sqrt{\beta \ln(N/n)/(12\ln(eb))}$.

In comparison, using the method of proof of Theorem 5.2 one obtains the following result (stated as Theorem 1.7 in the Introduction):

Theorem 5.8. *Let $\gamma \in (0, \gamma_\mu]$. For any $\beta \in (0,1)$ and $\alpha \in (0, -\ln p)$ the following holds: for all n sufficiently large, and $n^{1+4/\beta} < N \leqslant n^{1+4/\beta} e^{n\beta\lambda(\gamma)}$,*

$$\mathrm{Prob}\big(K_N \supseteq \Lambda_\varrho \cap \gamma B_\infty^n\big) \geqslant 1 - e^{-\alpha N} - e^{-n^\beta N^{1-\beta}}$$

with $\varrho := \beta n^{-1} \ln(N/n) - 4n^{-1} \ln n$.

Remark. As already observed in the Introduction,

$$\Lambda_\varrho \cap \gamma B_\infty^n \simeq \big(\sqrt{2n\varrho} B_2^n\big) \cap \gamma B_\infty^n = \big(\sqrt{2\beta \ln(N/n) - 8 \ln n}\, B_2^n\big) \cap \gamma B_\infty^n$$

for γ small enough, and of course

$$\sqrt{2\beta \ln(N/n) - 8 \ln n}\, B_2^n \supset 0.125 \sqrt{\beta \ln(N/n)/(12 \ln(eb))}\, B_2^n$$

for N in a range $n^a < N \leqslant n^a e^{\beta n \lambda(\gamma)}$ with a sufficiently large. On the other hand, Theorem 5.8 gives no information whatsoever for small values of N/n, e.g. when N is linear in n, as opposed to Theorem 5.7. Notice however, that in order to apply Theorem 5.7 one needs to verify condition (5.7), and it is at present unclear precisely when this condition is satisfied; in [14] it is verified for the case where the random variables ξ_{ij} are ψ_2 [14, Fact 2.4].

Proof of Theorem 5.8. We repeat the proof of Theorem 5.2. The argument in the proof of Claim 5.3 shows that, for any $0 < \alpha < -\ln p$, there exists a constant $c_\alpha \in (0, \infty)$ such that

$$\binom{N}{n} p^{N-n} < e^{-\alpha N} \tag{5.8}$$

whenever $N \geqslant n c_\alpha$. Since, for n sufficiently large,

$$1 - \mathrm{Prob}\big(K_N \supseteq \Lambda_\varrho \cap \gamma B_\infty^n\big) \leqslant \binom{N}{n} p^{N-n} + 2 \binom{N}{n} \big(1 - \exp(-(\varrho + \varepsilon)n)\big)^{N-n}$$

whenever $0 < \varrho \leqslant \lambda(\gamma)$, by Proposition 3.2 and Theorem 4.11, and where $\varepsilon = 3 \ln n/n$, it suffices to prove the following Claim.

Claim 5.9. Let $\beta \in (0,1)$. There exists $c_\beta \in (0, \infty)$ such that, for $N \geqslant n c_\beta$, one has that

$$2 \binom{N}{n} \big(1 - e^{-rn}\big)^{N-n} < e^{-n^\beta N^{1-\beta}} \tag{5.9}$$

for $r = \beta n^{-1} \ln(N/n) - n^{-1} \ln 2$.

Proof of Claim 5.9. Since $1-x \leqslant e^{-x}$ and $\binom{N}{n} \leqslant (eN/n)^n$, it suffices to verify that

$$2\left(\frac{eN}{n}\right)^n \exp\left(-e^{-rn}(N-n)\right) < e^{-n^\beta N^{1-\beta}}. \tag{5.10}$$

Setting $x := N/n$ we need to check that r satisfies

$$e^{rn} < \frac{x-1}{2+\log x + x^{1-\beta}}. \tag{5.11}$$

Observe that

$$\frac{x-1}{2+\log x + x^{1-\beta}} \sim x^\beta \tag{5.12}$$

as $x \to \infty$ ($\sim$ meaning that the ratio of the two sides tends to one as $x \to \infty$), so that

$$\frac{x-1}{2+\log x + x^{1-\beta}} > \tfrac{1}{2}x^\beta \tag{5.13}$$

when $x > c_\beta$ for an appropriate $c_\beta \in (0,\infty)$. This shows that (5.11) is satisfied for

$$r = \frac{\beta}{n}\ln\left(\frac{N}{n}\right) - \frac{\ln 2}{n} \tag{5.14}$$

when $N > nc_\beta$. □

The proof of Theorem 5.8 is now complete. □

6 Surface area of Λ_r

In this Section we prove the following Proposition:

Proposition 6.1. *There exists $R > 0$ with the following property: for any $\gamma \in I^\circ \cap J^\circ$, and all $r < c(\gamma)/R$ and $n \geqslant 4$, one has that*

$$|\partial(\Lambda_r) \cap \gamma B_\infty^n| \geqslant [c(\gamma)rn]^{(n-1)/2}\left|S^{n-1}\right|, \tag{6.1}$$

where $c(\gamma) \in (0,\infty)$ is a constant depending only on γ (and μ).

For the proof we first estimate the product curvature $\kappa(\vec{x})$ of the surface $\Lambda(\vec{x}) = r$ at a point $\vec{x} \in \gamma B_\infty^n$.

Lemma 6.2. *Let $0 < r < r^*$, and for $\vec{x} \in \partial(\Lambda_r)$ let $\kappa(\vec{x})$ denote the product curvature of the surface $\partial(\Lambda_r)$ at $\vec{x}$. Then, for every $\gamma \in I^\circ$, there exists a constant $c_1(\gamma) \in (0,\infty)$ (depending only on γ and μ) such that*

$$\kappa(\vec{x}) \leqslant [c_1(\gamma)\, rn]^{-(n-1)/2} \tag{6.2}$$

for all $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$.

Proof. Let $\nu(\vec{x}) = \nabla\Lambda(\vec{x}) / \|\nabla\Lambda(\vec{x})\|_2$ be the outward unit normal vector of Λ_r at $\vec{x}$. Following [17, Section 2.5], we write $T_{\vec{x}}\Lambda_r$ for the tangent space of Λ_r at $\vec{x}$, and consider the Weingarten map $W_{\vec{x}}\colon T_{\vec{x}}\Lambda_r \to T_{\vec{x}}\Lambda_r$. This is the restriction to $T_{\vec{x}}\Lambda_r$ of the differential $D_{\vec{x}}$ of the map $\vec{x} \mapsto \nu(\vec{x})$. Then $W_{\vec{x}}$ is symmetric and positive definite, therefore

$$\kappa(\vec{x}) = \det W_{\vec{x}} \leqslant \left(\frac{\operatorname{trace}(W_{\vec{x}})}{n-1}\right)^{n-1} \tag{6.3}$$

by the arithmetic-geometric means inequality. Let $(a_{ij})_{i,j=1}^n$ denote the matrix of $D_{\vec{x}}$ with respect to the standard basis of $\mathbb{R}^n$. It is easily checked that $\nu(\vec{x})$ is an eigenvector of the adjoint of $D_{\vec{x}}$, with corresponding eigenvalue 0; it follows from this and the fact that the eigenvalues of $W_{\vec{x}}$ are also eigenvalues of $D_{\vec{x}}$ and none of them is zero, that $\operatorname{trace}(W_{\vec{x}}) = \operatorname{trace}(D_{\vec{x}})$. A simple calculation also shows that

$$a_{ii} = \frac{\lambda''(x_i)\big(\|n\nabla\Lambda(\vec{x})\|_2^2 - [\lambda'(x_i)]^2\big)}{\|n\nabla\Lambda(\vec{x})\|_2^3} = \frac{h'(x_i)\big(\|\,\vec{t}\,\|_2^2 - [h(x_i)]^2\big)}{\|\,\vec{t}\,\|_2^3}. \tag{6.4}$$

Set

$$h^*(\gamma) := \sup_{x\in[-\gamma,\gamma]} h'(x). \tag{6.5}$$

It then follows from (6.4) that, if $\vec{x} \in \partial(\Lambda_r) \cap \gamma B_\infty^n$, then

$$\begin{aligned}
\frac{\operatorname{trace}(W_{\vec{x}})}{n-1} = \frac{\operatorname{trace}(D_{\vec{x}})}{n-1} &= \sum_{i=1}^n \frac{h'(x_i)\big(\|\,\vec{t}\,\|_2^2 - [h(x_i)]^2\big)}{(n-1)\,\|\,\vec{t}\,\|_2^3} \\
&\leqslant h^*(\gamma)\frac{n\,\|\,\vec{t}\,\|_2^2 - \sum_{i=1}^n t_i^2}{(n-1)\,\|\,\vec{t}\,\|_2^3} \\
&= \frac{h^*(\gamma)}{\|\,\vec{t}\,\|_2},
\end{aligned}$$

and (6.3) shows that

$$\kappa(\vec{x}) \leqslant \|\,\vec{t}\,\|_2^{-(n-1)}\,[h^*(\gamma)]^{n-1}. \tag{6.6}$$

Since also $\lambda(x_i) \leqslant c_6(\gamma)t_i^2$, by Proposition 2.13 (iv), we finally conclude (6.2) with $c_1(\gamma) = [c_6(\gamma)\,h^*(\gamma)^2]^{-1}$. □

Notation 6.3. For $r > 0$ set

$$M_r = \left\{\vec{u} \in S^{n-1}\colon \sqrt{n/r}\,\vec{u} \in B_\infty^n\right\}. \tag{6.7}$$

We shall need the following Lemma, which generalizes Lemmata 6.2 and 6.3 of [2].

Lemma 6.4. *(i) Let* $\vec{u} = (u_1, \ldots, u_n) \in S^{n-1}$*, set* $u_i^* := t^* u_i \|\vec{u}\|_\infty^{-1}$*,* $i = 1\ldots,n$*, with the convention* $0 \cdot \infty = 0$ *when* $t^* = \infty$ *and* $u_i = 0$*, and assume that*

$$0 < r < \frac{1}{n}\sum_{i=1}^{n} \lambda(\psi'(u_i^*)). \tag{6.8}$$

Then there exists a unique point $\vec{x}(\vec{u}, r) \in \partial(\Lambda_r)$ *such that*

$$\nabla\Lambda(\vec{x}(\vec{u}, r)) = \alpha(\vec{u}, r)\,\vec{u} \qquad (\alpha(\vec{u}, r) > 0) \tag{6.9}$$

is a positive multiple of $\vec{u}$*.*

(ii) Given $\gamma \in I^\circ \cap J^\circ$*, there exists a constant* $c_2(\gamma) \in (0, \infty)$ *(depending only on* γ *and* μ*) such that* $\vec{x}(\vec{u}, r)$ *is well defined and in the interior of* γB_∞^n *whenever* $0 < r < c_2(\gamma)/R$ *and* $\vec{u} \in M_R$ *(where* $R > 0$ *is arbitrary).*

Proof. Fix $\vec{u} = (u_1, \ldots, u_n) \in S^{n-1}$. For $s \in \left(0, t^* \|\vec{u}\|_\infty^{-1}\right)$, set

$$\vec{y}(\vec{u}, s) := (\psi'(su_1), \ldots, \psi'(su_n))\,. \tag{6.10}$$

Since $\lambda' = h$ and $h = (\psi')^{-1}$ (recall Proposition 2.12 and Definition 2.9), we have that

$$\nabla\Lambda(\vec{y}(\vec{u}, s)) = \frac{s}{n}\,\vec{u}. \tag{6.11}$$

Since the function

$$s \mapsto \Lambda(\vec{y}(\vec{u}, s)) = \frac{1}{n}\sum_{i=1}^{n} \lambda(\psi'(su_i)) \tag{6.12}$$

is continuous on the interval $\left(0, t^* \|\vec{u}\|_\infty^{-1}\right)$, takes on the value 0 at $s = 0$, and has limit $n^{-1}\sum_{i=1}^{n} \lambda(\psi'(u_i^*))$ as $s \to t^* \|\vec{u}\|_\infty^{-1}$, for each r satisfying (6.8), there exists a value $s(r)$ of s for which $\Lambda(\vec{y}(\vec{u}, s(r))) = r$; for such r we define $\vec{x}(\vec{u}, r) := \vec{y}(\vec{u}, s(r))$. Notice that

$$\frac{d}{ds}\Lambda(\vec{y}(\vec{u}, s)) = \frac{1}{n}\sum_{i=1}^{n} \lambda'(\psi'(su_i))\,\psi''(su_i)\,u_i > 0, \tag{6.13}$$

and hence the function defined by (6.12) is strictly increasing; thus $s(r)$ is unique. Notice further that the monotonicity of the function in (6.12) also implies that the constant $\alpha(\vec{u}, r)$ in (6.9), which by (6.11) is $s(r)/n$, is strictly increasing in r. This proves assertion (i)

For the proof of assertion (ii) fix $R > 0$, $\vec{u} \in M_R$, and let $\gamma > 0$ belong to $I^\circ \cap J^\circ$. Define $\vec{x} = (x_1, \ldots, x_n)$ by $x_i := \psi'(\gamma\sqrt{n/R}\,u_i)$, $i = 1, \ldots, n$; notice that, by our assumptions that $\gamma \in J^\circ$ and $\vec{u} \in M_R$, x_i is well defined for each i. Let also

$c_5 = c_5(\gamma)$ be as in Proposition 2.13 (iv) with $a = \psi'(\gamma)$. Then $\lambda(x_i) \geqslant c_5\gamma^2(n/R)u_i^2$ for each i and therefore

$$\Lambda(\vec{x}) \geqslant \frac{c_5\gamma^2}{R}. \tag{6.14}$$

Next let $\tilde{\Lambda}$ denote the function defined by (6.12). Then $\Lambda(\vec{x}) = \tilde{\Lambda}\big(\gamma\sqrt{n/R}\big)$ and therefore, if $r < c_5\gamma^2/R \leqslant \tilde{\Lambda}\big(\gamma\sqrt{n/R}\big)$, then $\vec{x}(\vec{u}, r)$ is well defined because $s(r)$ is well defined. Furthermore, $s(r) < \gamma\sqrt{n/R}$, and hence the i-th coordinate $x_i(\vec{u}, r)$ of $\vec{x}(\vec{u}, r)$ satisfies

$$|x_i(\vec{u}, r)| = s(r)\,|u_i| \leqslant \gamma\sqrt{\frac{n}{R}}\,|u_i| \leqslant \gamma. \tag{6.15}$$

In fact this inequality is strict unless $u_i = 0$, which shows that $\vec{x}(\vec{u}, r)$ belongs to the interior of γB_∞^n. □

Lemma 6.5. *There exists $R > 0$ such that*

$$|M_R| \geqslant e^{-(n-1)/2}\left|S^{n-1}\right| \tag{6.16}$$

for all $n \geqslant 4$.

Write γ_n for the standard Gaussian measure on $\mathbb{R}^n$ and σ_n for the rotation-invariant Borel probability measure on S^{n-1}. For the proof of Lemma 6.5 we shall use the following fact:

Fact 6.6. *If K is a symmetric convex body in $\mathbb{R}^n$, then*

$$\tfrac{1}{2}\,\sigma_n\big(S^{n-1} \cap \tfrac{1}{2}K\big) \leqslant \gamma_n(\sqrt{n}K) \leqslant \sigma_n(S^{n-1} \cap eK) + e^{-n/2}. \tag{6.17}$$

Proof of Fact 6.6. A proof appears in [12]. We sketch the proof of the right hand side inequality (which is the one we need). Observe that

$$\sqrt{n}\,K \subseteq \left(\sqrt{n/e^2}B_2^n\right) \cup C\left(\sqrt{n/e^2}S^{n-1} \cap \sqrt{n}K\right) \tag{6.18}$$

where, for $A \subseteq \sqrt{n/e^2}S^{n-1}$, we write $C(A)$ for the positive cone generated by A. It follows that

$$\gamma_n(\sqrt{n}K) \leqslant \gamma_n\left(\sqrt{n/e^2}B_2^n\right) + \sigma\left(\sqrt{n/e^2}S^{n-1} \cap \sqrt{n}K\right), \tag{6.19}$$

where σ denotes the rotation-invariant probability measure on $\sqrt{n/e^2}S^{n-1}$. Now

$$\sigma\left(\sqrt{n/e^2}S^{n-1} \cap \sqrt{n}K\right) = \sigma_n(S^{n-1} \cap eK), \tag{6.20}$$

and a direct computation shows that

$$\gamma_n\big(r\sqrt{n}B_2^n\big) \leqslant (r\sqrt{e})^n e^{-r^2 n/2} \tag{6.21}$$

for all $0 < r \leqslant 1$. It follows that

$$\gamma_n\left(\sqrt{n/e^2}B_2^n\right) \leqslant \exp(-n/2). \tag{6.22}$$

(6.19), (6.20) and (6.22) together show the right hand side inequality in (6.17). □

Proof of Lemma 6.5. Observe that

$$M_r = S^{n-1} \cap e\left(\sqrt{r/(e^2n)}\, C\right). \tag{6.23}$$

Hence, by the previous Fact,

$$\begin{aligned}\frac{|M_r|}{|S^{n-1}|} &= \sigma_n(M_r)\\ &\geqslant \gamma_n\big((\sqrt{r}/e)\, C\big) - e^{-n/2}\\ &= D\big(\sqrt{r}/e\big)^n - e^{-n/2},\end{aligned}$$

where

$$D(r) := \Phi(r) - \Phi(-r) = \frac{1}{\sqrt{2\pi}}\int_{-r}^{r} e^{-u^2/2}du. \tag{6.24}$$

Observe that $\big(\sqrt{e}+1\big)e^{-n/2} < e^{-n/4}$ for $n \geqslant 4$. Choose $R > 0$ so that

$$D\left(\sqrt{R}/e\right) > e^{-1/4}; \tag{6.25}$$

this is possible, since $\lim_{r\to+\infty} D(r) = 1$. Then,

$$D\left(\sqrt{R}/e\right)^n > \big(\sqrt{e}+1\big)e^{-n/2} \tag{6.26}$$

for $n \geqslant 4$, which completes the proof. □

We are now ready to finish the proof of Proposition 6.1.

Proof of Proposition 6.1. Let R be as in Lemma 6.5. For $\gamma \in I^\circ \cap J^\circ$, let $c_1(\gamma)$ and $c_2(\gamma)$ be as in Lemma 6.2 and Lemma 6.4, respectively. Let also $r > 0$ satisfy $r < c_2(\gamma)/R$. Writing $\vec{x}$ for $\vec{x}(\vec{u}, r)$ and expressing surface area in terms of product curvature (cf. [17, Theorem 4.2.4]), we can write

$$|\partial(\Lambda_r) \cap \gamma B_\infty^n| \geqslant \int_{M_R} \frac{1}{\kappa(\vec{x})}\, d\vec{u} \geqslant e^{-(n-1)/2}[c_1(\gamma)\, rn]^{(n-1)/2}\left|S^{n-1}\right|, \tag{6.27}$$

and the result follows. □

7 Facets of K_N

In this Section we prove Theorem 1.8. For a polytope P in $\mathbb{R}^n$ with non-empty interior, we shall write $f_{n-1}(P)$ for the number of its facets, i.e., its $(n-1)$-dimensional faces. We then have the following:

Theorem 7.1. *There exist two positive constants a and b such that, for all sufficiently large n, and all N satisfying $n^6 < N \leqslant \exp(bn)$, one has that*

$$E[f_{n-1}(K_N)] \geqslant \left(\frac{\ln N}{a \ln n}\right)^{n/2}. \tag{7.1}$$

For the proof of Theorem 7.1 we shall need the following auxiliary geometric lemma.

Lemma 7.2. *Let $\gamma > 0$ be in I° and assume that $r, \varepsilon > 0$ satisfy $r + \varepsilon < r^*$. If H is a half-space whose interior is disjoint from $\Lambda_r \cap \gamma B_\infty^n$, then*

$$|\partial(\Lambda_{r+\varepsilon}) \cap \gamma B_\infty^n \cap H| \leqslant [c(\gamma)\, \varepsilon n]^{(n-1)/2} \left|S^{n-1}\right|,$$

where $c(\gamma)$ is a constant depending only on γ (and μ).

Proof. Let H be a closed half-space whose interior is disjoint from $\Lambda_r \cap \gamma B_\infty^n$. We may without loss assume that ∂H is a supporting hyperplane for $\Lambda_r \cap \gamma B_\infty^n$ and then we may write

$$H = \{\vec{y} \in \mathbb{R}^n : \langle \vec{u}, \vec{y} - \vec{x} \rangle \geqslant 0\}$$

for some $\vec{u} \neq \vec{0}$ and $\vec{x} \in \partial(\Lambda_r \cap \gamma B_\infty^n)$. In fact we may assume that $\vec{x} \in \partial(\Lambda_r)$, whence $\Lambda(\vec{x}) = r$.

By symmetry we may assume that $0 \leqslant x_1 \leqslant \ldots \leqslant x_n$. If $x_i < \gamma$ for all i then we may take $\vec{u} = \nabla \Lambda(\vec{x})$. If $0 \leqslant x_1 \leqslant \ldots \leqslant x_k < \gamma = x_{k+1} = \ldots = x_n$, $\vec{u}$ must belong to the normal cone to $\Lambda_r \cap \gamma B_\infty^n$ at $\vec{x}$; we then may assume (see the proof of Theorem 4.11) that

$$u_i = \lambda'(x_i) \text{ for } 1 \leqslant i \leqslant k \quad \text{and} \quad u_j \geqslant \lambda'(x_j) \text{ for } k < j \leqslant n.$$

Let $\vec{y} \in H \cap \gamma B_\infty^n$; then, as $y_i - x_i \leqslant 0$ for $k < i \leqslant n$, we have that

$$\sum_{i=1}^{n} \lambda'(x_i)(y_i - x_i) \geqslant \sum_{i=1}^{n} u_i(y_i - x_i) \geqslant 0. \tag{7.2}$$

Suppose now that $\vec{y} \in \Lambda_{r+\varepsilon} \cap \gamma B_\infty^n \cap H$. By Taylor's theorem, there exist $\zeta_i \in [x_i \wedge y_i, x_i \vee y_i]$ ($a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$) such that

$$\begin{aligned}
\lambda(y_i) &= \lambda(x_i) + \lambda'(x_i)(y_i - x_i) + \tfrac{1}{2}\lambda''(\zeta_i)(y_i - x_i)^2 \\
&= \lambda(x_i) + h(x_i)(y_i - x_i) + \tfrac{1}{2}h'(\zeta_i)(y_i - x_i)^2 \\
&= \lambda(x_i) + h(x_i)(y_i - x_i) + \tfrac{1}{2}\frac{(y_i - x_i)^2}{\psi''(h(\zeta_i))}.
\end{aligned}$$

Since $\vec{x}, \vec{y} \in \gamma B_\infty^n$, we must have that $|\zeta_i| \leqslant \gamma$, and then $\psi''(h(\zeta_i)) \leqslant c_2$, where $c_2 = c_2(\gamma)$ is the constant in Proposition 2.13 (i) corresponding to $a = \gamma$. Thus

$$\lambda(y_i) \geqslant \lambda(x_i) + h(x_i)(y_i - x_i) + \frac{(y_i - x_i)^2}{2c_2}.$$

It follows that

$$\begin{aligned} \Lambda(\vec{y}) &\geqslant \Lambda(\vec{x}) + \langle \nabla \Lambda(\vec{x}), \vec{y} - \vec{x} \rangle + \frac{1}{2c_2 n} \sum_{i=1}^n (y_i - x_i)^2 \\ &= r + \langle \nabla \Lambda(\vec{x}), \vec{y} - \vec{x} \rangle + \frac{\|\vec{y} - \vec{x}\|_2^2}{2c_2 n} \\ &\geqslant r + \frac{\|\vec{y} - \vec{x}\|_2^2}{2c_2 n}, \end{aligned}$$

the last inequality by (7.2). On the other hand, since $\vec{y} \in \Lambda_{r+\varepsilon}$, we also have that

$$\Lambda(\vec{y}) \leqslant r + \varepsilon,$$

implying that

$$\|\vec{y} - \vec{x}\|_2 \leqslant \sqrt{2c_2 n \varepsilon}.$$

This shows that $\Lambda_{r+\varepsilon} \cap \gamma B_\infty^n \cap H$ is contained in a ball of radius $\sqrt{2c_2 n\varepsilon}$ around $\vec{x}$, and, by convexity, its surface area is at most $(2c_2 n\varepsilon)^{(n-1)/2} \left|S^{n-1}\right|$. □

Proof of Theorem 7.1. Fix $\gamma \in (0, \gamma_\mu]$. Let $c_1(\gamma)$ be the constant from Proposition 6.1 that corresponds to this γ, and set $b := \min\{c_1(\gamma), \lambda(\gamma)/R\}$.

Given N with $n^6 < N \leqslant e^{bn}$, recall the notation $\rho = (\ln N)/n$. From Theorem 5.2, and with $\varepsilon_n = 6 \ln n/n$, we have that

$$K_N \supseteq \Lambda_{\rho-\varepsilon} \cap \gamma B_\infty^n \tag{7.3}$$

holds with probability at least $1 - 2^{-n+1}$, for all sufficiently large n, and from Theorem 5.5 we also have that

$$|(\partial(\Lambda_\rho) \cap \gamma B_\infty^n) \setminus K_N| \geqslant \tfrac{1}{2} |\partial(\Lambda_\rho) \cap \gamma B_\infty^n| \tag{7.4}$$

with probability $1 - o_n(1)$. Thus the event, E_n say, where both (7.3) and (7.4) hold has probability at least $\frac{1}{2}$ when n is sufficiently large.

Now apply Lemma 7.2 with $r = \rho - \varepsilon_n$ and $\varepsilon = \varepsilon_n$: if F is a facet of K_N and H_F is the corresponding half-space which has interior disjoint from K_N, then

$$|\partial(\Lambda_\rho) \cap \gamma B_\infty^n \cap H_F| \leqslant [c_2(\gamma)\varepsilon_n n]^{(n-1)/2} \left|S^{n-1}\right|. \tag{7.5}$$

It follows that

$$\begin{aligned} f_{n-1}(K_N)\, [c_2(\gamma)\varepsilon_n n]^{(n-1)/2} \left|S^{n-1}\right| &\geqslant \sum_F |\partial(\Lambda_\rho) \cap \gamma B_\infty^n \cap H_F| \\ &\geqslant \left|\left(\partial(\Lambda_\rho) \cap \gamma B_\infty^n\right) \setminus K_N\right| \\ &\geqslant \tfrac{1}{2} |\partial(\Lambda_\rho) \cap \gamma B_\infty^n| \qquad (7.6) \end{aligned}$$

on E_n. Since $\rho \leqslant b = c_1(\gamma)/R$, Proposition 6.1 gives that

$$|\partial(\Lambda_\rho) \cap \gamma B_\infty^n| \geqslant [c_1(\gamma)\,\rho n]^{(n-1)/2} \left|S^{n-1}\right|, \tag{7.7}$$

and (7.6) and (7.7) yield the inequality

$$f_{n-1}(K_N)\,[c_2(\gamma)\varepsilon_n n]^{(n-1)/2} \geqslant \tfrac{1}{2}[c_1(\gamma)\,\rho n]^{(n-1)/2} \tag{7.8}$$

on E_n, for sufficiently large n. Since $\rho n = \ln N$ and $\varepsilon_n n = 6 \ln n$, this shows that

$$f_{n-1}(K_N) \geqslant \left(\frac{c(\gamma)\ln N}{\ln n}\right)^{n/2} \tag{7.9}$$

with probability greater than $\frac{1}{2}$, for all sufficiently large n. □

Call a polytope P in $\mathbb{R}^n$ a ± 1-polytope if its vertices are a subset of the vertices of the cube B_∞^n. Using Theorem 7.1 for the special case where the distribution μ is the distribution $\mu(\{-1\}) = \mu(\{1\}) = \frac{1}{2}$, we recover the result from [9] that there exist ± 1-polytopes with as many as $(cn/\ln n)^{n/2}$ facets, where $c > 0$ is a universal constant.

8 Threshold for the volume

In this final Section, which is only descriptive, we restrict ourselves to the case where μ is compactly supported; that is we assume that $x^* < \infty$ (recall (1.4)). Notice that (1.2) is then automatically satisfied. Furthermore, we cease to assume the normalization (1.3a) for μ (but assume that (1.6) holds).

In [7], and for a large class of distributions μ, we establish the following threshold for the expected volume of K_N using the "large deviations approach": for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup\{(2x^*)^{-n} E(|K_N|) \colon N \leqslant \exp((\kappa - \varepsilon)n)\} = 0 \tag{8.1}$$

and

$$\lim_{n\to\infty} \inf\{(2x^*)^{-n} E(|K_N|) \colon N \geqslant \exp((\kappa + \varepsilon)n)\} = 1. \tag{8.2}$$

In [4], Dyer, Füredi and McDiarmid studied the following two cases:
[DFM 1] If $\mu(\{1\}) = \mu(\{-1\}) = \frac{1}{2}$ then $\psi(t) = \ln(\cosh t)$. Then, $\lambda : (-1, 1) \to \mathbb{R}$ is given by

$$\lambda(x) = \tfrac{1}{2}(1+x)\ln(1+x) + \tfrac{1}{2}(1-x)\ln(1-x),$$

and (8.1)–(8.2) hold with $\kappa = \ln 2 - \frac{1}{2}$. This is the case of ± 1-polytopes.

[DFM 2] If μ is the uniform distribution on $[-1,1]$, then $\psi(t) = \ln(\sinh t/t)$, and (8.1)–(8.2) hold with

$$\kappa = \int_0^\infty \left(\frac{1}{u} - \frac{1}{e^u - 1}\right)^2 du.$$

Our result in [7] reads precisely as follows:

Theorem 8.1. *Let μ be an even, compactly supported, Borel probability measure on the real line and assume that $0 < \kappa(\mu) < \infty$. Then* (8.1) *holds for every $\varepsilon > 0$. Furthermore,* (8.2) *holds for every $\varepsilon > 0$, whenever the distribution μ satisfies*

$$\lim_{x \uparrow \alpha} \frac{-\ln P(X \geqslant x)}{\lambda(x)} = 1. \tag{8.3}$$

In [7] we provide an example which shows that (8.3) does not hold for every distribution μ. It can be verified for a large class of compactly supported distributions, however. We first recall the following definition (cf. [6, p. 276]). A measurable function $L\colon (0,\infty) \to (0,\infty)$ is *slowly varying* at zero if, for any $a > 0$, $L(ax)/L(x) \to 1$ as $x \downarrow 0$. Let us also agree that, for functions $f, g\colon J \to (0,\infty)$, where J is an interval in $\mathbb{R}$, and $u_0 \in \bar{J}$, $f(u) \sim g(u)$ as $u \to u_0$ shall mean that $\lim_{u\to u_0} f(u)/g(u) = 1$. Finally, recall that, by writing $f(u) \simeq g(u)$ as $u \to u_0$, we mean that there exist a neighborhood U of u_0, and constants $c_1 > 0$ and $c_2 < \infty$, such that $c_1 g(u) \leqslant f(u) \leqslant c_2 g(u)$ for $u \in U$. Theorem 8.1 is then complemented by the following result:

Theorem 8.2. *Condition* (8.3) *is satisfied in the following cases:*

(*i*) *When $P(X = x^*) > 0$.*

(*ii*) *When $P(X \geqslant x) \simeq (x^* - x)^a L(x^* - x)$ as $x \uparrow x^*$, with $a \geqslant 0$ and L slowly varying at zero.*

(*iii*) *When $-\ln P(X \geqslant x) \sim b(x^* - x)^{-a}$ as $x \uparrow x^*$, with $a, b > 0$.*

Notice that, in the presence of (8.3),

$$\kappa(\mu) < \infty \iff \int_{-x^*}^{x^*} -\ln P(X \geqslant x)\, dx < \infty.$$

This gives a criterion for the existence of a threshold for the volume, directly in terms of the distribution function of μ.

Notice that case (i) is subsumed by (ii) in Theorem 8.2 (take $a = 0$ and $L(x) = P(X \geqslant x^* - x)$ for all $x > 0$). Note also that [DFM 1] is covered by Theorem 8.2 (i), while [DFM 2] is covered by (ii) (take $a = 1$ and $L(x) = \frac{1}{2}$ for all $x > 0$). It is perhaps also worth mentioning that case (ii) also covers, for example, the case where $P(X \geqslant x^* - x)$ behaves like the Cantor function near the origin; in this case $a = \log_3 2$, $L \equiv 1$. Finally, we note that case (iii) covers the case where

$P(X \geqslant x^* - x)$ behaves like the distribution function of a positive stable random variable with index α in $(0, 1)$, near the origin.

Final Remark. After this work was completed, R. Latala [13] showed us an argument which establishes the following sharp version of Theorem 1.4: if $\gamma > 0$ is sufficiently small, and if $n \geqslant n_0(\gamma)$, then

$$q(\vec{x}) \geqslant \exp(-rn - \tfrac{1}{2}\ln(rn) - c(\gamma))$$

for every $\vec{x} \in \Lambda_r \cap \gamma B_\infty^n$ and r in the range $0 < r \leqslant \lambda(\gamma)$. In view of Theorem 1.3, this shows that $q(\vec{x})$ is "constant" on $\partial(\Lambda_r) \cap \gamma B_\infty^n$.

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