Methods for constructing Banach spaces with prescribed properties in their operator spaces

Pavlos Motakis

York University

7th MATH @ NTUA summer school in "Mathematical Analysis"

in honor of Spiros Argyros

June 28, 2024

Contents

Banach Spaces

Notation: Banach Spaces

- X, Y, Z, and W denote Banach spaces.

$$\ell_\infty = \Big\{x = (x(i))_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|x\|_\infty = \sup_{i\in\mathbb{N}} |x(i)| < \infty\Big\},$$

$$c_0 = \Big\{x = (x(i))_{i\in\mathbb{N}} \in \ell_\infty : \lim_{i\to\infty} x(i) = 0\Big\},$$

$$\ell_p = \Big\{x = (x(i))_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|x\|_p = \Big(\sum_{i=1}^{\infty} |x(i)|^p\Big)^{1/p} < \infty\Big\},\ 1 \leq p < \infty.$$

- X is separable if it contains a countable norm-dense subset.
  - ℓ_∞ is non-separable.
  - ℓ_p, $1 \leq p < \infty$, and c_0 are separable.

Convention: Unless stated otherwise, a Banach space is infinite-dimensional.

Notation: Banach Spaces

- X, Y, Z, and W denote Banach spaces.

$$\ell_\infty = \Big\{x = (x(i))_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|x\|_\infty = \sup_{i\in\mathbb{N}} |x(i)| < \infty\Big\},$$

$$c_0 = \Big\{x = (x(i))_{i\in\mathbb{N}} \in \ell_\infty : \lim_{i\to\infty} x(i) = 0\Big\},$$

$$\ell_p = \Big\{x = (x(i))_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|x\|_p = \Big(\sum_{i=1}^{\infty} |x(i)|^p\Big)^{1/p} < \infty\Big\},\ 1 \leq p < \infty.$$

- X is separable if it contains a countable norm-dense subset.
  - ℓ_∞ is non-separable.
  - ℓ_p, $1 \leq p < \infty$, and c_0 are separable.

Convention: Unless stated otherwise, a Banach space is infinite-dimensional.

Notation: Bounded Linear Operators

- T ,S, R and $K : X \to Y$ denote bounded linear operators.

$$\mathcal{L}(X,Y) = \Big\{T : X \to Y \text{ linear \& bounded}\Big\}$$

is a Banach space with

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}.$$

Notation: $\mathcal{L}(X) = \mathcal{L}(X,X)$ and $X^* = \mathcal{L}(X,\mathbb{R})$.

- X is reflexive if $X \equiv X^{**}$ canonically, i.e., $\widehat{X} = X^{**}$.
  - c_0, and ℓ_1 are non-reflexive.
  - ℓ_p, $1 < p < \infty$, is reflexive.

Notation: Bounded Linear Operators

- T ,S, R and $K : X \to Y$ denote bounded linear operators.

$$\mathcal{L}(X, Y) = \Big\{ T : X \to Y \text{ linear \& bounded} \Big\}$$

is a Banach space with

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}.$$

Notation: $\mathcal{L}(X) = \mathcal{L}(X, X)$ and $X^* = \mathcal{L}(X, \mathbb{R})$.

- X is reflexive if $X \equiv X^{**}$ canonically, i.e., $\widehat{X} = X^{**}$.
  - c_0, and ℓ_1 are non-reflexive.
  - ℓ_p, $1 < p < \infty$, is reflexive.

Notation: Compact Operators

- $K : X \to Y$ is called compact if for every bounded sequence $(x_i)_{i=1}^{\infty}$ in X, $(Kx_i)_{i=1}^{\infty}$ has a convergent subsequence.

Notation:

$$\mathcal{K}(X,Y) = \Big\{T \in \mathcal{L}(X,Y) \text{ compact}\Big\} \text{ and}$$
$$\mathcal{K}(X) = \mathcal{K}(X,X).$$

The Isomorphic Structure of the Subspaces of an X

- X and Y are C-isomorphic, or $X \simeq^C Y$, for $C \geq 1$, means:

  there exists a linear bijection $T : X \to Y$ with $\|T\|\|T^{-1}\| \leq C$.

  To disregard C, we say X and Y are isomorphic, or $X \simeq Y$.

  Question: What are the subspaces of an X up to isomorphism?

  Examples:

  - Let $X = \ell_2$. If $Y \subset \ell_2$ then $Y \simeq^1 \ell_2$.
  - Let $X = \ell_p$, $1 \leq p < \infty$, or $X = c_0$.
    If $Y \subset X$ then there exists $Z \subset Y$ such that $Z \simeq X$.
  - Let $X = L_p[0,1]$, $1 \leq p < \infty$.
    Then, ℓ_p and ℓ_2 are isomorphic to subspaces of X.

The Isomorphic Structure of the Subspaces of an X

- X and Y are C-isomorphic, or $X \simeq^C Y$, for $C \geq 1$, means:

  there exists a linear bijection $T : X \to Y$ with $\|T\|\|T^{-1}\| \leq C$.

  To disregard C, we say X and Y are isomorphic, or $X \simeq Y$.

  Question: What are the subspaces of an X up to isomorphism?

  Examples:

  - Let $X = \ell_2$. If $Y \subset \ell_2$ then $Y \simeq^1 \ell_2$.
  - Let $X = \ell_p$, $1 \leq p < \infty$, or $X = c_0$.
    If $Y \subset X$ then there exists $Z \subset Y$ such that $Z \simeq X$.
  - Let $X = L_p[0,1]$, $1 \leq p < \infty$.
    Then, ℓ_p and ℓ_2 are isomorphic to subspaces of X.

The Isomorphic Structure of the Subspaces of an X

- X and Y are C-isomorphic, or $X \simeq^C Y$, for $C \geq 1$, means:

  there exists a linear bijection $T : X \to Y$ with $\|T\|\|T^{-1}\| \leq C$.

  To disregard C, we say X and Y are isomorphic, or $X \simeq Y$.

  Question: What are the subspaces of an X up to isomorphism?

  Examples:

  - Let $X = \ell_2$. If $Y \subset \ell_2$ then $Y \simeq^1 \ell_2$.
  - Let $X = \ell_p$, $1 \leq p < \infty$, or $X = c_0$.
    If $Y \subset X$ then there exists $Z \subset Y$ such that $Z \simeq X$.
  - Let $X = L_p[0,1]$, $1 \leq p < \infty$.
    Then, ℓ_p and ℓ_2 are isomorphic to subspaces of X.

Isomorphic structure of subspaces

Question: For arbitrary X, is there $Y \subset X$ that is isomorphic to c_0 or some ℓ_p, $1 \leq p < \infty$?

- A counterexample was constructed by Tsirelson in 1974.

Isomorphic structure of subspaces

Question: For arbitrary X, is there $Y \subset X$ that is isomorphic to c_0 or some ℓ_p, $1 \leq p < \infty$?

- A counterexample was constructed by Tsirelson in 1974.

The Space $\mathcal{L}(X)$

- Let $X = \ell_p$, $1 \leq p < \infty$ or $X = c_0$.

  For $\varepsilon = (\varepsilon(i))_{i\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$ and a permutation $\pi : \mathbb{N} \to \mathbb{N}$.

  Define $T_{\varepsilon,\pi} \in \mathcal{L}(X)$ as follows: for $x = (x(i))_{i\in\mathbb{N}} \in X$,

$$T_{\varepsilon,\pi} x = \Big(\varepsilon(i) x\big(\pi^{-1}(i)\big)\Big)_{i\in\mathbb{N}}.$$

  Thus, $\mathcal{L}(X)$ is non-separable.

Constructing Operators: Arbitrary Banach Spaces

- Let X be an arbitrary infinite-dimensional Banach space.

(a) For $\lambda \in \mathbb{R}$, the scalar operator $\lambda I \in \mathcal{L}(X)$ is given by

$$\lambda I x = \lambda x.$$

(b) For $f \in X^*$, $y \in X$, the rank-one operator $f \otimes x \in \mathcal{L}(X)$ is given by

$$(f \otimes y)x = f(x)y.$$

Then $\|f \otimes x\| = \|f\|\|x\|$.

- For $f_1, \ldots, f_n \in X^*$ and $y_1, \ldots, y_n \in X$, the finite-rank operator

$$\sum_{i=1}^{n} f_i \otimes y_i \in \mathcal{L}(X).$$

- For $(f_i)_{i\in\mathbb{N}} \in X^*$ and $(y_i)_{i\in\mathbb{N}} \in X$, such that $\sum_{i=1}^{\infty} \|f_i\|\|y_i\| < \infty$, the nuclear operator

$$\sum_{i=1}^{\infty} f_i \otimes y_i \in \mathcal{L}(X).$$

Constructing Operators: Arbitrary Banach Spaces

- Let X be an arbitrary infinite-dimensional Banach space.

(a) For $\lambda \in \mathbb{R}$, the scalar operator $\lambda I \in \mathcal{L}(X)$ is given by

$$\lambda I x = \lambda x.$$

(b) For $f \in X^*$, $y \in X$, the rank-one operator $f \otimes x \in \mathcal{L}(X)$ is given by

$$(f \otimes y)x = f(x)y.$$

Then $\|f \otimes x\| = \|f\|\|x\|$.

- For $f_1, \ldots, f_n \in X^*$ and $y_1, \ldots, y_n \in X$, the finite-rank operator

$$\sum_{i=1}^{n} f_i \otimes y_i \in \mathcal{L}(X).$$

- For $(f_i)_{i\in\mathbb{N}} \in X^*$ and $(y_i)_{i\in\mathbb{N}} \in X$, such that $\sum_{i=1}^{\infty} \|f_i\|\|y_i\| < \infty$, the nuclear operator

$$\sum_{i=1}^{\infty} f_i \otimes y_i \in \mathcal{L}(X).$$

Constructing Operators: Arbitrary Banach Spaces

- Let X be an arbitrary infinite-dimensional Banach space.

(a) For $\lambda \in \mathbb{R}$, the scalar operator $\lambda I \in \mathcal{L}(X)$ is given by

$$\lambda I x = \lambda x.$$

(b) For $f \in X^*$, $y \in X$, the rank-one operator $f \otimes x \in \mathcal{L}(X)$ is given by

$$(f \otimes y)x = f(x)y.$$

Then $\|f \otimes x\| = \|f\|\|x\|$.

- For $f_1, \ldots, f_n \in X^*$ and $y_1, \ldots, y_n \in X$, the finite-rank operator

$$\sum_{i=1}^{n} f_i \otimes y_i \in \mathcal{L}(X).$$

- For $(f_i)_{i \in \mathbb{N}} \in X^*$ and $(y_i)_{i \in \mathbb{N}} \in X$, such that $\sum_{i=1}^{\infty} \|f_i\|\|y_i\| < \infty$, the nuclear operator

$$\sum_{i=1}^{\infty} f_i \otimes y_i \in \mathcal{L}(X).$$

Operators on an Arbitrary Banach Space

- If $\mathcal{N}(X) = \left\{T \in \mathcal{L}(X) \text{ nuclear}\right\}$,

$$\mathbb{R}I + \mathcal{N}(X) \subset \mathcal{L}(X).$$

By Hahn-Banach: $\dim(\mathcal{L}(X)) = \infty$.

Remark: $\mathcal{N}(X) \subset \mathcal{K}(X)$, and under "mild" assumptions on X,

$$\overline{\mathcal{N}(X)}^{\|\cdot\|} = \mathcal{K}(X).$$

Question: (Lindenstrauss, 1975)
Does there exist a Banach space X with the scalar-plus-compact property, i.e., such that

$$\mathbb{R}I + \mathcal{K}(X) = \mathcal{L}(X)?$$

- Such a space was constructed by Argyros and Haydon in 2011.

Operators on an Arbitrary Banach Space

- If $\mathcal{N}(X) = \left\{ T \in \mathcal{L}(X) \text{ nuclear} \right\}$,

$$\mathbb{R}I + \mathcal{N}(X) \subset \mathcal{L}(X).$$

By Hahn-Banach: $\dim(\mathcal{L}(X)) = \infty$.

Remark: $\mathcal{N}(X) \subset \mathcal{K}(X)$, and under "mild" assumptions on X,

$$\overline{\mathcal{N}(X)}^{\|\cdot\|} = \mathcal{K}(X).$$

Question: (Lindenstrauss, 1975)
Does there exist a Banach space X with the scalar-plus-compact property, i.e., such that

$$\mathbb{R}I + \mathcal{K}(X) = \mathcal{L}(X)?$$

- Such a space was constructed by Argyros and Haydon in 2011.

- If $\mathcal{N}(X) = \left\{ T \in \mathcal{L}(X) \text{ nuclear} \right\}$,

$$\mathbb{R}I + \mathcal{N}(X) \subset \mathcal{L}(X).$$

By Hahn-Banach: $\dim(\mathcal{L}(X)) = \infty$.

Remark: $\mathcal{N}(X) \subset \mathcal{K}(X)$, and under "mild" assumptions on X,

$$\overline{\mathcal{N}(X)}^{\|\cdot\|} = \mathcal{K}(X).$$

Question: (Lindenstrauss, 1975)
Does there exist a Banach space X with the scalar-plus-compact property, i.e., such that

$$\mathbb{R}I + \mathcal{K}(X) = \mathcal{L}(X)?$$

- Such a space was constructed by Argyros and Haydon in 2011.

Operators on an Arbitrary Banach Space

- If $\mathcal{N}(X) = \left\{T \in \mathcal{L}(X) \text{ nuclear}\right\}$,

$$\mathbb{R}I + \mathcal{N}(X) \subset \mathcal{L}(X).$$

By Hahn-Banach: $\dim(\mathcal{L}(X)) = \infty$.

Remark: $\mathcal{N}(X) \subset \mathcal{K}(X)$, and under "mild" assumptions on X,

$$\overline{\mathcal{N}(X)}^{\|\cdot\|} = \mathcal{K}(X).$$

Question: (Lindenstrauss, 1975)
Does there exist a Banach space X with the scalar-plus-compact property, i.e., such that

$$\mathbb{R}I + \mathcal{K}(X) = \mathcal{L}(X)?$$

- Such a space was constructed by Argyros and Haydon in 2011.

Fundamental Banach Spaces Concepts

Schauder Bases

- A Schauder basis of a Banach space is a sequence $(x_i)_{i=1}^{\infty}$ in X such that for every $x \in X$,

$$x = \sum_{i=1}^{\infty} a_i x_i,$$

for a unique $(a_i)_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$.

Example: For $X = \ell_p$, $1 \leq p < \infty$, or $X = c_0$ let, for $i \in \mathbb{N}$,

$$e_i = (0, 0, 0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{'th position}}}{1}, 0, \ldots).$$

Then $(e_i)_{i=1}^{\infty}$ is a Schauder basis of X because,

$$\text{if } x = (x(i))_{i=1}^{\infty} \in X \text{ then } x = \sum_{i=1}^{\infty} x(i) e_i.$$

Schauder Bases

- A Schauder basis of a Banach space is a sequence $(x_i)_{i=1}^{\infty}$ in X such that for every $x \in X$,

$$x = \sum_{i=1}^{\infty} a_i x_i,$$

for a unique $(a_i)_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$.

Example: For $X = \ell_p$, $1 \leq p < \infty$, or $X = c_0$ let, for $i \in \mathbb{N}$,

$$e_i = (0, 0, 0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{'th position}}}{1}, 0, \ldots).$$

Then $(e_i)_{i=1}^{\infty}$ is a Schauder basis of X because,

$$\text{if } x = (x(i))_{i=1}^{\infty} \in X \text{ then } x = \sum_{i=1}^{\infty} x(i) e_i.$$

- Two Schauder bases $(x_i)_{i=1}^\infty$ of X and $(y_i)_{i=1}^\infty$ of Y are called C-equivalent if $T : X \to Y$ given by

$$T\Big(\sum_{i=1}^{\infty} a_i x_i\Big) = \sum_{i=1}^{\infty} a_i y_i$$

is a C-isomorphism of X and Y.

Block Sequences

Let X have a Schauder basis $(x_i)_{i=1}^{\infty}$.

- The support of an $x = \sum_{i=1}^{\infty} a_i x_i$ in X is the set

$$\operatorname{supp}(x) = \{i \in \mathbb{N} : a_i \neq 0\}.$$

For $x \in X$ and $\varepsilon > 0$ there exists $y \in X$ with $\operatorname{supp}(y)$ finite

$$\|x - y\| < \varepsilon.$$

- A sequence $(y_i)_{i=1}^{\infty}$ in X is a block sequence if

$$\operatorname{supp}(y_1) < \operatorname{supp}(y_2) < \cdots < \operatorname{supp}(y_i) < \operatorname{supp}(y_{i+1}) < \cdots .$$

- For a block sequence $(y_i)_{i=1}^{\infty}$ in X, $Y = \overline{\langle \{y_i : i \in \mathbb{N}\} \rangle}$ is called a block subspace of X.

Block Sequences

Let X have a Schauder basis $(x_i)_{i=1}^\infty$.

- The support of an $x = \sum_{i=1}^\infty a_i x_i$ in X is the set

$$\mathrm{supp}(x) = \{i \in \mathbb{N} : a_i \neq 0\}.$$

For $x \in X$ and $\varepsilon > 0$ there exists $y \in X$ with $\mathrm{supp}(y)$ finite

$$\|x - y\| < \varepsilon.$$

- A sequence $(y_i)_{i=1}^\infty$ in X is a block sequence if

$$\mathrm{supp}(y_1) < \mathrm{supp}(y_2) < \cdots < \mathrm{supp}(y_i) < \mathrm{supp}(y_{i+1}) < \cdots.$$

- For a block sequence $(y_i)_{i=1}^\infty$ in X, $Y = \overline{\langle\{y_i : i \in \mathbb{N}\}\rangle}$ is called a block subspace of X.

Block Sequences

Let X have a Schauder basis $(x_i)_{i=1}^{\infty}$.

- The support of an $x = \sum_{i=1}^{\infty} a_i x_i$ in X is the set

$$\mathrm{supp}(x) = \{i \in \mathbb{N} : a_i \neq 0\}.$$

For $x \in X$ and $\varepsilon > 0$ there exists $y \in X$ with $\mathrm{supp}(y)$ finite

$$\|x - y\| < \varepsilon.$$

- A sequence $(y_i)_{i=1}^{\infty}$ in X is a block sequence if

$$\mathrm{supp}(y_1) < \mathrm{supp}(y_2) < \cdots < \mathrm{supp}(y_i) < \mathrm{supp}(y_{i+1}) < \cdots .$$

- For a block sequence $(y_i)_{i=1}^{\infty}$ in X, $Y = \overline{\langle\{y_i : i \in \mathbb{N}\}\rangle}$ is called a block subspace of X.

Block Sequences

Let X have a Schauder basis $(x_i)_{i=1}^{\infty}$.

- The support of an $x = \sum_{i=1}^{\infty} a_i x_i$ in X is the set

$$\operatorname{supp}(x) = \{i \in \mathbb{N} : a_i \neq 0\}.$$

For $x \in X$ and $\varepsilon > 0$ there exists $y \in X$ with $\operatorname{supp}(y)$ finite

$$\|x - y\| < \varepsilon.$$

- A sequence $(y_i)_{i=1}^{\infty}$ in X is a block sequence if

$$\operatorname{supp}(y_1) < \operatorname{supp}(y_2) < \cdots < \operatorname{supp}(y_i) < \operatorname{supp}(y_{i+1}) < \cdots .$$

- For a block sequence $(y_i)_{i=1}^{\infty}$ in X, $Y = \overline{\langle\{y_i : i \in \mathbb{N}\}\rangle}$ is called a block subspace of X.

Block Sequences

If X has a Schauder basis $(x_i)_{i=1}^{\infty}$, block subspaces saturate the infinite-dimensional subspaces Y of X.

Theorem: Let $Y \subset X$ and $\varepsilon > 0$.

Then Y contains an ε-perturbation of a block subspace W.

That is, there exists an isomorphism $T \in \mathcal{L}(X)$ with $\|I - T\| < \varepsilon$ and $T(W) \subset Y$.

Schreier's Space

Convex Combinations of Weakly Null Sequences

- A sequence $(x_i)_{i=1}^\infty$ in a Banach space is weakly null if, for every $f \in X^*$, $\lim_{i\to\infty} f(x_i) = 0$.

Theorem: (Mazur) $(x_i)_{i=1}^\infty$ is weakly null if and only if for every infinite $L \subset \mathbb{N}$ and $\varepsilon > 0$ there exist a finite $F \subset L$ and $(\lambda_i)_{i\in F}$ in $[0,1]$ such that $\sum_{i\in F}\lambda_i = 1$ and $\|\sum_{i\in F}\lambda_i x_i\| < \varepsilon$.

Question: (Banach - Saks, 1930s) Assume $(x_i)_{i=1}^\infty$ is weakly null and let $\varepsilon > 0$. Does there exist a finite $F \subset \mathbb{N}$ such that

$$\Big\|\sum_{i\in F}\frac{1}{\#F}x_i\Big\| < \varepsilon?$$

Theorem: (Schreier, 1932) There exists a weakly null sequence $(e_i)_{i=1}^\infty$ in some Banach space X_S such that for every finite $F \subset \mathbb{N}$,

$$\Big\|\sum_{i\in F}\frac{1}{\#F}e_i\Big\| \geq \frac{1}{2}.$$

Convex Combinations of Weakly Null Sequences

- A sequence $(x_i)_{i=1}^\infty$ in a Banach space is weakly null if, for every $f \in X^*$, $\lim_{i\to\infty} f(x_i) = 0$.

Theorem: (Mazur) $(x_i)_{i=1}^\infty$ is weakly null if and only if for every infinite $L \subset \mathbb{N}$ and $\varepsilon > 0$ there exist a finite $F \subset L$ and $(\lambda_i)_{i\in F}$ in $[0,1]$ such that $\sum_{i\in F} \lambda_i = 1$ and $\|\sum_{i\in F} \lambda_i x_i\| < \varepsilon$.

Question: (Banach - Saks, 1930s) Assume $(x_i)_{i=1}^\infty$ is weakly null and let $\varepsilon > 0$. Does there exist a finite $F \subset \mathbb{N}$ such that

$$\Big\|\sum_{i\in F} \frac{1}{\#F} x_i\Big\| < \varepsilon?$$

Theorem: (Schreier, 1932) There exists a weakly null sequence $(e_i)_{i=1}^\infty$ in some Banach space X_S such that for every finite $F \subset \mathbb{N}$,

$$\Big\|\sum_{i\in F} \frac{1}{\#F} e_i\Big\| \geq \frac{1}{2}.$$

Convex Combinations of Weakly Null Sequences

- A sequence $(x_i)_{i=1}^\infty$ in a Banach space is weakly null if, for every $f \in X^*$, $\lim_{i\to\infty} f(x_i) = 0$.

Theorem: (Mazur) $(x_i)_{i=1}^\infty$ is weakly null if and only if for every infinite $L \subset \mathbb{N}$ and $\varepsilon > 0$ there exist a finite $F \subset L$ and $(\lambda_i)_{i\in F}$ in $[0,1]$ such that $\sum_{i\in F} \lambda_i = 1$ and $\|\sum_{i\in F} \lambda_i x_i\| < \varepsilon$.

Question: (Banach - Saks, 1930s) Assume $(x_i)_{i=1}^\infty$ is weakly null and let $\varepsilon > 0$. Does there exist a finite $F \subset \mathbb{N}$ such that

$$\Big\|\sum_{i\in F} \frac{1}{\#F} x_i\Big\| < \varepsilon?$$

Theorem: (Schreier, 1932) There exists a weakly null sequence $(e_i)_{i=1}^\infty$ in some Banach space X_S such that for every finite $F \subset \mathbb{N}$,

$$\Big\|\sum_{i\in F} \frac{1}{\#F} e_i\Big\| \geq \frac{1}{2}.$$

Convex Combinations of Weakly Null Sequences

- A sequence $(x_i)_{i=1}^\infty$ in a Banach space is weakly null if, for every $f \in X^*$, $\lim_{i\to\infty} f(x_i) = 0$.

Theorem: (Mazur) $(x_i)_{i=1}^\infty$ is weakly null if and only if for every infinite $L \subset \mathbb{N}$ and $\varepsilon > 0$ there exist a finite $F \subset L$ and $(\lambda_i)_{i\in F}$ in $[0,1]$ such that $\sum_{i\in F}\lambda_i = 1$ and $\|\sum_{i\in F}\lambda_i x_i\| < \varepsilon$.

Question: (Banach - Saks, 1930s) Assume $(x_i)_{i=1}^\infty$ is weakly null and let $\varepsilon > 0$. Does there exist a finite $F \subset \mathbb{N}$ such that

$$\Big\|\sum_{i\in F}\frac{1}{\#F}x_i\Big\| < \varepsilon?$$

Theorem: (Schreier, 1932) There exists a weakly null sequence $(e_i)_{i=1}^\infty$ in some Banach space $X_\mathcal{S}$ such that for every finite $F \subset \mathbb{N}$,

$$\Big\|\sum_{i\in F}\frac{1}{\#F}e_i\Big\| \geq \frac{1}{2}.$$

Schreier's Construction

- The Schreier family:

$$\mathcal{S} = \Big\{F \subset \mathbb{N} : \ \#F \leq \min(F)\Big\}.$$

E.g., $\{5, 6, 12\} \in \mathcal{S}$ but $\{2, 5, 7\} \notin \mathcal{S}$.

- $c_{00} = \Big\{x = (x(i))_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \ x(i) \neq 0 \text{ for only finitely many } i\Big\}.$

- For $i \in \mathbb{N}$, and for $x = (x(i))_{i=1}^{\infty} \in c_{00}$, $F \subset \mathbb{N}$, let

$$Fx = \sum_{i \in F} x(i) e_i.$$

- Define a norm on c_{00} by letting, for $x = (x(i))_{i=1}^{\infty}$,

$$\|x\| = \sup\Big\{\sum_{i \in F} |x(i)| : F \in \mathcal{S}\Big\} = \sup_{F \in \mathcal{S}} \|Fx\|_{\ell_1},$$

and let $X_{\mathcal{S}}$ be the completion of c_{00} with $\|\cdot\|$.

Schreier's Construction

- The Schreier family:

$$\mathcal{S} = \Big\{ F \subset \mathbb{N} : \ \#F \leq \min(F) \Big\}.$$

E.g., $\{5, 6, 12\} \in \mathcal{S}$ but $\{2, 5, 7\} \notin \mathcal{S}$.

- $c_{00} = \Big\{ x = (x(i))_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \ x(i) \neq 0$ for only finitely many $i \Big\}$.

- For $i \in \mathbb{N}$, and for $x = (x(i))_{i=1}^{\infty} \in c_{00}$, $F \subset \mathbb{N}$, let

$$Fx = \sum_{i \in F} x(i) e_i.$$

- Define a norm on c_{00} by letting, for $x = (x(i))_{i=1}^{\infty}$,

$$\|x\| = \sup \Big\{ \sum_{i \in F} |x(i)| : F \in \mathcal{S} \Big\} = \sup_{F \in \mathcal{S}} \|Fx\|_{\ell_1},$$

and let $X_{\mathcal{S}}$ be the completion of c_{00} with $\|\cdot\|$.

Schreier's Construction

- The Schreier family:

$$\mathcal{S} = \Big\{ F \subset \mathbb{N} : \#F \le \min(F) \Big\}.$$

E.g., $\{5, 6, 12\} \in \mathcal{S}$ but $\{2, 5, 7\} \notin \mathcal{S}$.

- $c_{00} = \Big\{ x = (x(i))_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : x(i) \neq 0 \text{ for only finitely many } i \Big\}.$

- For $i \in \mathbb{N}$, and for $x = (x(i))_{i=1}^{\infty} \in c_{00}$, $F \subset \mathbb{N}$, let

$$Fx = \sum_{i \in F} x(i) e_i.$$

- Define a norm on c_{00} by letting, for $x = (x(i))_{i=1}^{\infty}$,

$$\|x\| = \sup \Big\{ \sum_{i \in F} |x(i)| : F \in \mathcal{S} \Big\} = \sup_{F \in \mathcal{S}} \|Fx\|_{\ell_1},$$

and let $X_{\mathcal{S}}$ be the completion of c_{00} with $\|\cdot\|$.

Schreier's Construction

- The Schreier family:

$$\mathcal{S} = \Big\{F \subset \mathbb{N} : \ \#F \leq \min(F)\Big\}.$$

E.g., $\{5, 6, 12\} \in \mathcal{S}$ but $\{2, 5, 7\} \notin \mathcal{S}$.

- $c_{00} = \Big\{x = (x(i))_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \ x(i) \neq 0 \text{ for only finitely many } i\Big\}.$

- For $i \in \mathbb{N}$, and for $x = (x(i))_{i=1}^{\infty} \in c_{00}$, $F \subset \mathbb{N}$, let

$$Fx = \sum_{i \in F} x(i) e_i.$$

- Define a norm on c_{00} by letting, for $x = (x(i))_{i=1}^{\infty}$,

$$\|x\| = \sup\Big\{\sum_{i \in F} |x(i)| : F \in \mathcal{S}\Big\} = \sup_{F \in \mathcal{S}} \|Fx\|_{\ell_1},$$

and let $X_{\mathcal{S}}$ be the completion of c_{00} with $\|\cdot\|$.

Schreier's Construction

Proposition: For every non-empty $F \subset \mathbb{N}$,

$$\Big\| \sum_{i \in F} \frac{1}{\#F} e_i \Big\| \geq \frac{1}{2}.$$

Lemma: For every finite $F \subset \mathbb{N}$ there exists $G \subset F$ with $G \in \mathcal{S}$ and $\#G \geq \#F/2$.

Proof of lemma: Write $F = \{k_1 < k_2 < \cdots < k_n\}$ and let $G = \{k_{\lfloor n/2 \rfloor + 1} < \cdots < k_n\}$.

Proof of proposition: Let $F \subset \mathbb{N}$ finite and take G as in the lemma. Then,

$$\Big\| \sum_{i \in F} \frac{1}{\#F} e_i \Big\| \geq \sum_{i \in G} \frac{1}{\#F} = \frac{\#G}{\#F} \geq \frac{1}{2}.$$

Schreier's Construction

Proposition: For every non-empty $F \subset \mathbb{N}$,

$$\Big\| \sum_{i \in F} \frac{1}{\#F} e_i \Big\| \geq \frac{1}{2}.$$

Lemma: For every finite $F \subset \mathbb{N}$ there exists $G \subset F$ with $G \in \mathcal{S}$ and $\#G \geq \#F/2$.

Proof of lemma: Write $F = \{k_1 < k_2 < \cdots < k_n\}$ and let $G = \{k_{\lfloor n/2 \rfloor + 1} < \cdots < k_n\}$.

Proof of proposition: Let $F \subset \mathbb{N}$ finite and take G as in the lemma. Then,

$$\Big\| \sum_{i \in F} \frac{1}{\#F} e_i \Big\| \geq \sum_{i \in G} \frac{1}{\#F} = \frac{\#G}{\#F} \geq \frac{1}{2}.$$

Schreier's Construction

Proposition: For every non-empty $F \subset \mathbb{N}$,

$$\Big\| \sum_{i \in F} \frac{1}{\#F} e_i \Big\| \geq \frac{1}{2}.$$

Lemma: For every finite $F \subset \mathbb{N}$ there exists $G \subset F$ with $G \in \mathcal{S}$ and $\#G \geq \#F/2$.

Proof of lemma: Write $F = \{k_1 < k_2 < \cdots < k_n\}$ and let $G = \{k_{\lfloor n/2 \rfloor + 1} < \cdots < k_n\}$.

Proof of proposition: Let $F \subset \mathbb{N}$ finite and take G as in the lemma. Then,

$$\Big\| \sum_{i \in F} \frac{1}{\#F} e_i \Big\| \geq \sum_{i \in G} \frac{1}{\#F} = \frac{\#G}{\#F} \geq \frac{1}{2}.$$

Schreier's Construction

Proposition: $(e_i)_{i=1}^{\infty}$ is weakly null in $X_{\mathcal{S}}$.

Lemma: Let $F_1 < \cdots < F_n$ be non-empty subsets of $\mathbb{N}$ such that, for $k = 1, \ldots, n-1$, $\#F_{k+1} \geq \max(F_k)$.

Then, for every $F \in \mathcal{S}$,

$$\sum_{k=1}^{n} \frac{\#(F \cap F_k)}{\#F_k} < 2.$$

Proof of proposition: For an infinite $L \subset \mathbb{N}$ and $\varepsilon > 0$ pick $n > 2/\varepsilon$ and $F_1, \ldots, F_n$ in L as in the lemma. Then,

$$\Big\|\frac{1}{n}\sum_{k=1}^{n}\Big(\sum_{i\in F_k}\frac{1}{\#F_k}e_i\Big)\Big\| = \frac{1}{n}\sup_{F\in\mathcal{S}}\Big(\sum_{k=1}^{n}\Big(\sum_{i\in F\cap F_k}\frac{1}{\#F_k}\Big)\Big)$$

$$= \frac{1}{n}\sup_{F\in\mathcal{S}}\sum_{k=1}^{n}\frac{\#(F\cap F_k)}{\#F_k} < \varepsilon.$$

Proposition: $(e_i)_{i=1}^{\infty}$ is weakly null in $X_{\mathcal{S}}$.

Lemma: Let $F_1 < \cdots < F_n$ be non-empty subsets of $\mathbb{N}$ such that, for $k = 1, \ldots, n-1$, $\#F_{k+1} \geq \max(F_k)$.
Then, for every $F \in \mathcal{S}$,

$$\sum_{k=1}^{n} \frac{\#(F \cap F_k)}{\#F_k} < 2.$$

Proof of proposition: For an infinite $L \subset \mathbb{N}$ and $\varepsilon > 0$ pick $n > 2/\varepsilon$ and $F_1, \ldots, F_n$ in L as in the lemma. Then,

$$\Big\| \frac{1}{n} \sum_{k=1}^{n} \Big(\sum_{i \in F_k} \frac{1}{\#F_k} e_i \Big) \Big\| = \frac{1}{n} \sup_{F \in \mathcal{S}} \Big(\sum_{k=1}^{n} \Big(\sum_{i \in F \cap F_k} \frac{1}{\#F_k} \Big) \Big)$$

$$= \frac{1}{n} \sup_{F \in \mathcal{S}} \sum_{k=1}^{n} \frac{\#(F \cap F_k)}{\#F_k} < \varepsilon.$$

Schreier's Construction

Proposition: $(e_i)_{i=1}^{\infty}$ is weakly null in $X_{\mathcal{S}}$.

Lemma: Let $F_1 < \cdots < F_n$ be non-empty subsets of $\mathbb{N}$ such that, for $k = 1, \ldots, n-1$, $\#F_{k+1} \geq \max(F_k)$.
Then, for every $F \in \mathcal{S}$,

$$\sum_{k=1}^{n} \frac{\#(F \cap F_k)}{\#F_k} < 2.$$

Proof of proposition: For an infinite $L \subset \mathbb{N}$ and $\varepsilon > 0$ pick $n > 2/\varepsilon$ and $F_1, \ldots, F_n$ in L as in the lemma. Then,

$$\Big\| \frac{1}{n} \sum_{k=1}^{n} \Big(\sum_{i \in F_k} \frac{1}{\#F_k} e_i \Big) \Big\| = \frac{1}{n} \sup_{F \in \mathcal{S}} \Big(\sum_{k=1}^{n} \Big(\sum_{i \in F \cap F_k} \frac{1}{\#F_k} \Big) \Big)$$

$$= \frac{1}{n} \sup_{F \in \mathcal{S}} \sum_{k=1}^{n} \frac{\#(F \cap F_k)}{\#F_k} < \varepsilon.$$

Remarks:

For $F = \{i_1, i_2, \ldots, i_n\} \in \mathcal{S}$, $(e_{i_k})_{k=1}^n$ in $X_{\mathcal{S}}$ is 1-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_1)$.

Every subspace of $X_{\mathcal{S}}$ contains a further subspace isomorphic to c_0.

Schreier's space $X_{\mathcal{S}}$ is isomorphic to a subspace of $C(\omega^\omega)$.

Tsirelson's Space

- Let X be an infinite-dimensional Banach space.

Question: Does X have a subspace isomorphic to c_0 or some ℓ_p, $1 \leq p < \infty$?

Theorem: (Dvoretsky, 1961) For every $n \in \mathbb{N}$, $C > 1$, X has an n-dimensional subspace that is C-isomorphic to $(\mathbb{R}^n, \|\cdot\|_2)$.

Theorem: (Krivine, 1976) If X has a Schauder basis there exists $1 \leq p \leq \infty$ such that:

for all $n \in \mathbb{N}$ and $C > 1$ there exists a finite block sequence $(x_i)_{i=1}^n$ in X that is C-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_p)$.

Theorem: (Tsirelson, 1974) There exists a Banach space T with no subspace isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$.

- Let X be an infinite-dimensional Banach space.

  Question: Does X have a subspace isomorphic to c_0 or some ℓ_p, $1 \leq p < \infty$?

  Theorem: (Dvoretsky, 1961) For every $n \in \mathbb{N}$, $C > 1$, X has an n-dimensional subspace that is C-isomorphic to $(\mathbb{R}^n, \|\cdot\|_2)$.

  Theorem: (Krivine, 1976) If X has a Schauder basis there exists $1 \leq p \leq \infty$ such that:

  for all $n \in \mathbb{N}$ and $C > 1$ there exists a finite block sequence $(x_i)_{i=1}^n$ in X that is C-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_p)$.

  Theorem: (Tsirelson, 1974) There exists a Banach space T with no subspace isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$.

Tsirelson's Space

- Let X be an infinite-dimensional Banach space.

  Question: Does X have a subspace isomorphic to c_0 or some ℓ_p, $1 \leq p < \infty$?

  Theorem: (Dvoretsky, 1961) For every $n \in \mathbb{N}$, $C > 1$, X has an n-dimensional subspace that is C-isomorphic to $(\mathbb{R}^n, \|\cdot\|_2)$.

  Theorem: (Krivine, 1976) If X has a Schauder basis there exists $1 \leq p \leq \infty$ such that:

  for all $n \in \mathbb{N}$ and $C > 1$ there exists a finite block sequence $(x_i)_{i=1}^n$ in X that is C-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_p)$.

  Theorem: (Tsirelson, 1974) There exists a Banach space T with no subspace isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$.

Tsirelson's Space

- Let X be an infinite-dimensional Banach space.

Question: Does X have a subspace isomorphic to c_0 or some ℓ_p, $1 \leq p < \infty$?

Theorem: (Dvoretsky, 1961) For every $n \in \mathbb{N}$, $C > 1$, X has an n-dimensional subspace that is C-isomorphic to $(\mathbb{R}^n, \|\cdot\|_2)$.

Theorem: (Krivine, 1976) If X has a Schauder basis there exists $1 \leq p \leq \infty$ such that:

for all $n \in \mathbb{N}$ and $C > 1$ there exists a finite block sequence $(x_i)_{i=1}^n$ in X that is C-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_p)$.

Theorem: (Tsirelson, 1974) There exists a Banach space T with no subspace isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$.

Tsirelson's Construction

- We define by induction a sequence of norms on c_{00}.
- For $x = (x(i))_{i=1}^{\infty}$ let $\|x\|_0 = \max_{i\in\mathbb{N}} |x(i)| = \|x\|_{c_0}$.
- If $\|\cdot\|_n$ is defined, for $x = (x(i))_{i=1}^{\infty}$ let

$$\|x\|_{n+1} = \max\left\{\|x\|_n, \sup\left(\frac{1}{2}\sum_{k=1}^{m}\|E_k x\|_n\right)\right\}.$$

  The supremum is over all $m \in \mathbb{N}$ and subsets $E_1 < \cdots < E_m$ of $\mathbb{N}$ with $\min(E_1) \geq m$, i.e., $\{\min(E_k) : 1 \leq k \leq m\} \in \mathcal{S}$.
- For $x \in c_{00}$ let $\|x\| = \lim_n \|x\|_n$ and T be the completion of c_{00} with $\|\cdot\|$.

Tsirelson's Construction

- We define by induction a sequence of norms on c_{00}.
- For $x = (x(i))_{i=1}^{\infty}$ let $\|x\|_0 = \max_{i\in\mathbb{N}} |x(i)| = \|x\|_{c_0}$.
- If $\|\cdot\|_n$ is defined, for $x = (x(i))_{i=1}^{\infty}$ let

$$\|x\|_{n+1} = \max\left\{\|x\|_n, \sup\left(\frac{1}{2}\sum_{k=1}^{m}\|E_k x\|_n\right)\right\}.$$

The supremum is over all $m \in \mathbb{N}$ and subsets $E_1 < \cdots < E_m$ of $\mathbb{N}$ with $\min(E_1) \geq m$, i.e., $\{\min(E_k) : 1 \leq k \leq m\} \in \mathcal{S}$.

- For $x \in c_{00}$ let $\|x\| = \lim_n \|x\|_n$ and T be the completion of c_{00} with $\|\cdot\|$.

- We define by induction a sequence of norms on c_{00}.
- For $x = (x(i))_{i=1}^{\infty}$ let $\|x\|_0 = \max_{i\in\mathbb{N}} |x(i)| = \|x\|_{c_0}$.
- If $\|\cdot\|_n$ is defined, for $x = (x(i))_{i=1}^{\infty}$ let

$$\|x\|_{n+1} = \max\left\{\|x\|_n, \sup\left(\frac{1}{2}\sum_{k=1}^{m}\|E_k x\|_n\right)\right\}.$$

  The supremum is over all $m \in \mathbb{N}$ and subsets $E_1 < \cdots < E_m$ of $\mathbb{N}$ with $\min(E_1) \geq m$, i.e., $\{\min(E_k) : 1 \leq k \leq m\} \in \mathcal{S}$.
- For $x \in c_{00}$ let $\|x\| = \lim_n \|x\|_n$ and T be the completion of c_{00} with $\|\cdot\|$.

Tsirelson's Construction

Remark: For $x \in c_{00}$

$$\|x\| = \max\left\{\|x\|_{c_0}, \sup\left(\frac{1}{2}\sum_{k=1}^{m}\|E_k x\|\right)\right\}.$$

The supremum is over all $m \in \mathbb{N}$ and subsets $E_1 < \cdots < E_m$ of $\mathbb{N}$ with $\min(E_1) \geq m$.

Corollary: For every finite block sequence $(x_i)_{i=1}^m$ with $m \leq \min \mathrm{supp}(x_1)$,

$$\frac{1}{2}\sum_{k=1}^{m}\|x_i\| \leq \left\|\sum_{i=1}^{m} x_i\right\| \leq \sum_{k=1}^{m}\|x_i\|.$$

That is, T is $(1/2)$-asymptotic ℓ_1.

Remark: For $x \in c_{00}$

$$\|x\| = \max\left\{\|x\|_{c_0}, \sup\left(\frac{1}{2}\sum_{k=1}^{m}\|E_k x\|\right)\right\}.$$

The supremum is over all $m \in \mathbb{N}$ and subsets $E_1 < \cdots < E_m$ of $\mathbb{N}$ with $\min(E_1) \geq m$.

Corollary: For every finite block sequence $(x_i)_{i=1}^m$ with $m \leq \min \operatorname{supp}(x_1)$,

$$\frac{1}{2}\sum_{k=1}^{m}\|x_i\| \leq \left\|\sum_{i=1}^{m} x_i\right\| \leq \sum_{k=1}^{m}\|x_i\|.$$

That is, T is $(1/2)$-asymptotic ℓ_1.

Tsirelson's Construction

Proposition: No block sequence of norm-one vectors in T is equivalent to the unit vector basis of c_0 or ℓ_p, $1 < p < \infty$.

Proof: Fix $1 < p < \infty$.

Assume that $(x_i)_{i=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_p.

Then there exists $C > 0$ such that for every $i_1 < \cdots < i_n$,

$$\Big\| \sum_{k=1}^{n} x_{k_i} \Big\| \le C \Big\| \sum_{k=1}^{n} e_{i_k} \Big\|_{\ell_p} = C n^{1/p}.$$

But also, if $\min \mathrm{supp}(x_{i_1}) \ge n$,

$$\Big\| \sum_{k=1}^{n} x_{i_k} \Big\| \ge \frac{1}{2} \sum_{k=1}^{n} \|x_{i_k}\| = \frac{n}{2}.$$

Therefore, $C \ge n^{1-1/p}/2 \to \infty$, as $n \to \infty$.

Tsirelson's Construction

Proposition: No block sequence of norm-one vectors in T is equivalent to the unit vector basis of c_0 or ℓ_p, $1 < p < \infty$.

Proof: Fix $1 < p < \infty$.

Assume that $(x_i)_{i=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_p.

Then there exists $C > 0$ such that for every $i_1 < \cdots < i_n$,

$$\Big\| \sum_{k=1}^{n} x_{k_i} \Big\| \leq C \Big\| \sum_{k=1}^{n} e_{i_k} \Big\|_{\ell_p} = Cn^{1/p}.$$

But also, if $\min \operatorname{supp}(x_{i_1}) \geq n$,

$$\Big\| \sum_{k=1}^{n} x_{i_k} \Big\| \geq \frac{1}{2} \sum_{k=1}^{n} \|x_{i_k}\| = \frac{n}{2}.$$

Therefore, $C \geq n^{1-1/p}/2 \to \infty$, as $n \to \infty$.

Tsirelson's Construction

Proposition: No block sequence of norm-one vectors in T is equivalent to the unit vector basis of c_0 or ℓ_p, $1 < p < \infty$.

Proof: Fix $1 < p < \infty$.

Assume that $(x_i)_{i=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_p.

Then there exists $C > 0$ such that for every $i_1 < \cdots < i_n$,

$$\Big\| \sum_{k=1}^{n} x_{k_i} \Big\| \leq C \Big\| \sum_{k=1}^{n} e_{i_k} \Big\|_{\ell_p} = C n^{1/p}.$$

But also, if $\min \operatorname{supp}(x_{i_1}) \geq n$,

$$\Big\| \sum_{k=1}^{n} x_{i_k} \Big\| \geq \frac{1}{2} \sum_{k=1}^{n} \|x_{i_k}\| = \frac{n}{2}.$$

Therefore, $C \geq n^{1-1/p}/2 \to \infty$, as $n \to \infty$.

Proposition: No block sequence of norm-one vectors in T is equivalent to the unit vector basis of ℓ_1.

Lemma: Let $\delta > 0$ and $(x_i)_{i=1}^{\infty}$ be a block sequence of norm-one vectors in T. For $N \geq \max \operatorname{supp}(x_1)/\delta$,

$$\left\| x_1 + \frac{1}{N} \sum_{i=2}^{N+1} x_i \right\| \leq 1 + \delta.$$

In particular,

$$\limsup_N \left\| x_1 + \frac{1}{N} \sum_{i=2}^{N+1} x_i \right\| \leq 1.$$

Tsirelson's Construction

Proposition: No block sequence of norm-one vectors in T is equivalent to the unit vector basis of ℓ_1.

Lemma: Let $\delta > 0$ and $(x_i)_{i=1}^{\infty}$ be a block sequence of norm-one vectors in T. For $N \geq \max \operatorname{supp}(x_1)/\delta$,

$$\left\| x_1 + \frac{1}{N} \sum_{i=2}^{N+1} x_i \right\| \leq 1 + \delta.$$

In particular,

$$\limsup_N \left\| x_1 + \frac{1}{N} \sum_{i=2}^{N+1} x_i \right\| \leq 1.$$

Tsirelson's Construction

Proposition: No block sequence of norm-one vectors in T is equivalent to the unit vector basis of ℓ_1.

Proof of Proposition: Assume $(x_i)_{i=1}^{\infty}$ is a block sequence of norm-one vectors C-equivalent to the unit vector basis of ℓ_1, i.e.,

$$\left\| x_1 + \frac{1}{N}\sum_{i=2}^{N+1} x_i \right\| \geq \frac{1}{C}\left\| e_1 + \frac{1}{N}\sum_{i=2}^{N+1} e_i \right\|_{\ell_1} = \frac{2}{C}.$$

By the Lemma,

$$\frac{2}{C} \leq \limsup_N \left\| x_1 + \frac{1}{N}\sum_{i=2}^{N+1} x_i \right\| \leq 1.$$

Thus, $C \geq 2$. This contradicts James' non-distrortion of ℓ_1.

Tsirelson's Construction

Proposition: No block sequence of norm-one vectors in T is equivalent to the unit vector basis of ℓ_1.

Proof of Proposition: Assume $(x_i)_{i=1}^{\infty}$ is a block sequence of norm-one vectors C-equivalent to the unit vector basis of ℓ_1, i.e.,

$$\left\| x_1 + \frac{1}{N}\sum_{i=2}^{N+1} x_i \right\| \geq \frac{1}{C}\left\| e_1 + \frac{1}{N}\sum_{i=2}^{N+1} e_i \right\|_{\ell_1} = \frac{2}{C}.$$

By the Lemma,

$$\frac{2}{C} \leq \limsup_N \left\| x_1 + \frac{1}{N}\sum_{i=2}^{N+1} x_i \right\| \leq 1.$$

Thus, $C \geq 2$. This contradicts James' non-distrortion of ℓ_1.

Tsirelson's Construction

Proposition: No block sequence of norm-one vectors in T is equivalent to the unit vector basis of ℓ_1.

Proof of Proposition: Assume $(x_i)_{i=1}^{\infty}$ is a block sequence of norm-one vectors C-equivalent to the unit vector basis of ℓ_1, i.e.,

$$\left\| x_1 + \frac{1}{N}\sum_{i=2}^{N+1} x_i \right\| \geq \frac{1}{C}\left\| e_1 + \frac{1}{N}\sum_{i=2}^{N+1} e_i \right\|_{\ell_1} = \frac{2}{C}.$$

By the Lemma,

$$\frac{2}{C} \leq \limsup_N \left\| x_1 + \frac{1}{N}\sum_{i=2}^{N+1} x_i \right\| \leq 1.$$

Thus, $C \geq 2$. This contradicts James' non-distrortion of ℓ_1.

Remarks:

Tsirelson's norm is the first saturated norm:

Every block sequence of norm-one vectors $(x_i)_{i=1}^{\infty}$ is weakly null, and for every $F \subset \mathbb{N}$,

$$\Big\| \sum_{i \in F} \frac{1}{\#F} x_i \Big\| \geq \frac{1}{4}.$$

Tsirelson's space is reflexive and asymptotic ℓ_1.

- These are "contradictory" properties.
- T displays hereditarily homegeneous block structure.

Remarks:

Tsirelson's norm is the first saturated norm:

Every block sequence of norm-one vectors $(x_i)_{i=1}^{\infty}$ is weakly null, and for every $F \subset \mathbb{N}$,

$$\Big\| \sum_{i \in F} \frac{1}{\#F} x_i \Big\| \geq \frac{1}{4}.$$

Tsirelson's space is reflexive and asymptotic ℓ_1.

- These are "contradictory" properties.
- T displays hereditarily homegeneous block structure.

Schlumprecht's Space

Schlumprecht's Construction

- In 1991 Schlumprecht constructed a Banach space $(S, \|\cdot\|)$ with hereditarily heterogeneous block structure.

- S is the completion of c_{00} with a norm such that for $x \in c_{00}$

$$\|x\| = \max\left\{\|x\|_{c_0}, \sup\left(\frac{1}{\log(m+1)}\sum_{k=1}^{m}\|E_k x\|\right)\right\},$$

where the supremum is over all $m \in \mathbb{N}$ and successive subsets $E_1, \ldots, E_m$ of $\mathbb{N}$.

Corollary: For every finite block sequence $(x_i)_{i=1}^n$,

$$\frac{1}{\log(n+1)}\sum_{k=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|.$$

Schlumprecht's Construction

- In 1991 Schlumprecht constructed a Banach space $(S, \|\cdot\|)$ with hereditarily heterogeneous block structure.

- S is the completion of c_{00} with a norm such that for $x \in c_{00}$

$$\|x\| = \max\left\{\|x\|_{c_0}, \sup\left(\frac{1}{\log(m+1)}\sum_{k=1}^{m}\|E_k x\|\right)\right\},$$

where the supremum is over all $m \in \mathbb{N}$ and successive subsets $E_1, \ldots, E_m$ of $\mathbb{N}$.

Corollary: For every finite block sequence $(x_i)_{i=1}^n$,

$$\frac{1}{\log(n+1)}\sum_{k=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|.$$

Schlumprecht's Construction

- In 1991 Schlumprecht constructed a Banach space $(S, \|\cdot\|)$ with hereditarily heterogeneous block structure.

- S is the completion of c_{00} with a norm such that for $x \in c_{00}$

$$\|x\| = \max\left\{\|x\|_{c_0}, \sup\left(\frac{1}{\log(m+1)}\sum_{k=1}^{m}\|E_k x\|\right)\right\},$$

where the supremum is over all $m \in \mathbb{N}$ and successive subsets $E_1, \ldots, E_m$ of $\mathbb{N}$.

Corollary: For every finite block sequence $(x_i)_{i=1}^n$,

$$\frac{1}{\log(n+1)}\sum_{k=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n}x_i\right\|.$$

Schlumprecht's Construction

Theorem: Let X be a block subspace S. For every $n \in \mathbb{N}$, there is a block sequence $(x_i)_{i=1}^n$ in X that is a 2-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_1)$.

Proof: Krivine's theorem and

$$\frac{\log(n+1)}{n^{1-1/p}} \to 0,$$

for all $p > 1$.

- For such a sequence, the vector

$$y = \frac{1}{n}\sum_{i=1}^{n} x_i$$

is a 2-normalized ℓ_1 average and it satisfies $1/2 \leq \|y\| \leq 1$.

Theorem: Let X be a block subspace S. For every $n \in \mathbb{N}$, there is a block sequence $(x_i)_{i=1}^n$ in X that is a 2-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_1)$.

Proof: Krivine's theorem and

$$\frac{\log(n+1)}{n^{1-1/p}} \to 0,$$

for all $p > 1$.

- For such a sequence, the vector

$$y = \frac{1}{n}\sum_{i=1}^{n} x_i$$

is a 2-normalized ℓ_1 average and it satisfies $1/2 \leq \|y\| \leq 1$.

Schlumprecht's Construction

- A block sequence $(y_i)_{i=1}^{\infty}$ such that, for each $i \in \mathbb{N}$, y_i is an 2-$\ell_1^{N_i}$-average and

$$\log(\log(N_{i+1})) > 2^{i+1} \max \operatorname{supp}(y_i)$$

is called a 2-rapidly increasing sequence (2-RIS).

Theorem: Let $(y_i)_{i=1}^{\infty}$ be 2-RIS in S. Then, for every $i_1 < \cdots < i_n \in \mathbb{N}$,

$$\frac{1}{2} \frac{n}{\log(n+1)} \leq \left\| \sum_{k=1}^{n} y_{i_k} \right\| \leq 4 \frac{n}{\log(n+1)}.$$

Schlumprecht's Construction

- A block sequence $(y_i)_{i=1}^{\infty}$ such that, for each $i \in \mathbb{N}$, y_i is an 2-$\ell_1^{N_i}$-average and

$$\log(\log(N_{i+1})) > 2^{i+1} \max \operatorname{supp}(y_i)$$

is called a 2-rapidly increasing sequence (2-RIS).

Theorem: Let $(y_i)_{i=1}^{\infty}$ be 2-RIS in S. Then, for every $i_1 < \cdots < i_n \in \mathbb{N}$,

$$\frac{1}{2} \frac{n}{\log(n+1)} \leq \left\| \sum_{k=1}^{n} y_{i_k} \right\| \leq 4 \frac{n}{\log(n+1)}.$$

Schlumprecht's Construction

Hereditary heterogeneity of Schlumprecht's space:

In every block subspace of S, there exist

- for every $n \in \mathbb{N}$, a block sequence $(x_i)_{i=1}^n$ that is 2-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_1)$, i.e., for $a_1, \ldots, a_n \in \mathbb{R}$,

$$\frac{1}{2}\sum_{i=1}^n |a_i| \le \Big\|\sum_{i=1}^n a_i x_i\Big\| \le \sum_{i=1}^n |a_i|,$$

and

- a 2-RIS $(y_i)_{i=1}^\infty$, i.e., for every $i_1 < \cdots < i_n \in \mathbb{N}$,

$$\frac{1}{2}\frac{n}{\log(n+1)} \le \Big\|\sum_{k=1}^n y_{i_k}\Big\| \le 4\frac{n}{\log(n+1)}.$$

Comment: Other types of sequences can be found.

Remark: S is a reflexive Banach space.

Schlumprecht's Construction

Hereditary heterogeneity of Schlumprecht's space:

In every block subspace of S, there exist

- for every $n \in \mathbb{N}$, a block sequence $(x_i)_{i=1}^n$ that is 2-equivalent to the unit vector basis of $(\mathbb{R}^n, \|\cdot\|_1)$, i.e., for $a_1, \ldots, a_n \in \mathbb{R}$,

$$\frac{1}{2}\sum_{i=1}^n |a_i| \leq \Big\|\sum_{i=1}^n a_i x_i\Big\| \leq \sum_{i=1}^n |a_i|,$$

  and

- a 2-RIS $(y_i)_{i=1}^\infty$, i.e., for every $i_1 < \cdots < i_n \in \mathbb{N}$,

$$\frac{1}{2}\frac{n}{\log(n+1)} \leq \Big\|\sum_{k=1}^n y_{i_k}\Big\| \leq 4\frac{n}{\log(n+1)}.$$

Comment: Other types of sequences can be found.

Remark: S is a reflexive Banach space.

The Gowers-Maurey Space

Unconditional Sequences

- A Schauder basis $(x_i)_{i=1}^{\infty}$ of a Banach space is called *C-unconditional*, for some $C \geq 1$, if for every $a_1, \ldots, a_n \in \mathbb{R}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$,

$$\Big\| \sum_{i=1}^{n} \varepsilon_i a_i x_i \Big\| \leq C \Big\| \sum_{i=1}^{n} a_i x_i \Big\|.$$

- Equivalently, for every $\varepsilon = (\varepsilon(i))_{i=1}^{\infty} \in \{-1, 1\}^{\mathbb{N}}$, $T_\varepsilon : X \to X$ with

$$T_\varepsilon \Big(\sum_{i=1}^{\infty} a_i x_i \Big) = \sum_{i=1}^{\infty} \varepsilon_i a_i x_i$$

is bounded and has norm at most C.

- A sequence $(y_i)_{i=1}^{\infty}$ in a Banach space X is an *unconditional sequence* if it is an unconditional Schauder basis of its closed linear span.

Unconditional Sequences

- A Schauder basis $(x_i)_{i=1}^{\infty}$ of a Banach space is called *C-unconditional*, for some $C \geq 1$, if for every $a_1, \ldots, a_n \in \mathbb{R}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$,

$$\Big\| \sum_{i=1}^{n} \varepsilon_i a_i x_i \Big\| \leq C \Big\| \sum_{i=1}^{n} a_i x_i \Big\|.$$

- Equivalently, for every $\varepsilon = (\varepsilon(i))_{i=1}^{\infty} \in \{-1, 1\}^{\mathbb{N}}$, $T_\varepsilon : X \to X$ with

$$T_\varepsilon\Big(\sum_{i=1}^{\infty} a_i x_i\Big) = \sum_{i=1}^{\infty} \varepsilon_i a_i x_i$$

is bounded and has norm at most C.

- A sequence $(y_i)_{i=1}^{\infty}$ in a Banach space X is an *unconditional sequence* if it is an unconditional Schauder basis of its closed linear span.

Unconditional Sequences

- A Schauder basis $(x_i)_{i=1}^\infty$ of a Banach space is called *C-unconditional*, for some $C \geq 1$, if for every $a_1, \ldots, a_n \in \mathbb{R}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$,

$$\Big\| \sum_{i=1}^n \varepsilon_i a_i x_i \Big\| \leq C \Big\| \sum_{i=1}^n a_i x_i \Big\|.$$

- Equivalently, for every $\varepsilon = (\varepsilon(i))_{i=1}^\infty \in \{-1, 1\}^{\mathbb{N}}$, $T_\varepsilon : X \to X$ with

$$T_\varepsilon \Big(\sum_{i=1}^\infty a_i x_i \Big) = \sum_{i=1}^\infty \varepsilon_i a_i x_i$$

is bounded and has norm at most C.

- A sequence $(y_i)_{i=1}^\infty$ in a Banach space X is an *unconditional sequence* if it is an unconditional Schauder basis of its closed linear span.

The Unconditional Basic Sequence Problem

- The unit vector bases of c_0, ℓ_p, $1 \leq p < \infty$, Schreier's, Tsirelson's, and Schlumprecht's spaces are unconditional.
- Every block sequence of a C-unconditional sequence is C-unconditional.
- The Schauder system of $C[0,1]$ and the Haar system of $L_1[0,1]$ are non-unconditional.

Question: Does every infinite-dimensional Banach space contain an unconditional sequence?

The Unconditional Basic Sequence Problem

- The unit vector bases of c_0, ℓ_p, $1 \leq p < \infty$, Schreier's, Tsirelson's, and Schlumprecht's spaces are unconditional.
- Every block sequence of a C-unconditional sequence is C-unconditional.
- The Schauder system of $C[0,1]$ and the Haar system of $L_1[0,1]$ are non-unconditional.

Question: Does every infinite-dimensional Banach space contain an unconditional sequence?

The Unconditional Basic Sequence Problem

- The unit vector bases of c_0, ℓ_p, $1 \leq p < \infty$, Schreier's, Tsirelson's, and Schlumprecht's spaces are unconditional.
- Every block sequence of a C-unconditional sequence is C-unconditional.
- The Schauder system of $C[0,1]$ and the Haar system of $L_1[0,1]$ are non-unconditional.

Question: Does every infinite-dimensional Banach space contain an unconditional sequence?

The Unconditional Basic Sequence Problem

- The unit vector bases of c_0, ℓ_p, $1 \leq p < \infty$, Schreier's, Tsirelson's, and Schlumprecht's spaces are unconditional.
- Every block sequence of a C-unconditional sequence is C-unconditional.
- The Schauder system of $C[0,1]$ and the Haar system of $L_1[0,1]$ are non-unconditional.

Question: Does every infinite-dimensional Banach space contain an unconditional sequence?

The Gowers-Maurey Space

Theorem: (Gowers - Maurey, 1993) There exists a reflexive Banach space X_{GM} without any unconditional sequences.

The Gowers-Maurey construction combines:

- The Schlumprecht construction and
- A Maurey-Rosenthal concept called a coding function.

The Gowers-Maurey Space

Theorem: (Gowers - Maurey, 1993) There exists a reflexive Banach space X_{GM} without any unconditional sequences.

The Gowers-Maurey construction combines:

- The Schlumprecht construction and
- A Maurey-Rosenthal concept called a coding function.

Hereditary heterogeneity in the Gowers-Maurey space yields:
In every block subspace of X_{GM}, there exist a 2-RIS $(y_i)_{i=1}^{\infty}$, i.e., for every $i_1 < \cdots < i_n \in \mathbb{N}$,

$$\frac{1}{2}\frac{n}{\log(n+1)} \leq \Big\| \sum_{k=1}^{n} y_{i_k} \Big\| \leq 4\frac{n}{\log(n+1)}.$$

For $(y_i)_{i=1}^{\infty}$ as above, the vector

$$z = \frac{\log(n+1)}{n}\sum_{i=1}^{n} y_i$$

is called a $(2, n)$-exact vector and it satisfies $1/2 \leq \|z\| \leq 4$.

The Gowers-Maurey Construction

Hereditary heterogeneity in the Gowers-Maurey space yields:

In every block subspace of X_{GM}, there exist a 2-RIS $(y_i)_{i=1}^{\infty}$, i.e., for every $i_1 < \cdots < i_n \in \mathbb{N}$,

$$\frac{1}{2}\frac{n}{\log(n+1)} \leq \left\| \sum_{k=1}^{n} y_{i_k} \right\| \leq 4\frac{n}{\log(n+1)}.$$

For $(y_i)_{i=1}^{\infty}$ as above, the vector

$$z = \frac{\log(n+1)}{n}\sum_{i=1}^{n} y_i$$

is called a $(2, n)$-exact vector and it satisfies $1/2 \leq \|z\| \leq 4$.

Theorem: Let X be a block subspace of X_{GM}. Then, for arbitrarily large $N \in \mathbb{N}$, there exists a block sequence $(z_i)_{i=1}^N$ in X such that each z_i is a $(2, n_i)$-exact vector and

$$\Big\| \sum_{i=1}^{n} z_i \Big\| \geq \frac{1}{2} \frac{N}{\sqrt{\log(N+1)}}$$

but

$$\Big\| \sum_{i=1}^{n} (-1)^i z_i \Big\| \leq 16 \frac{N}{\log(N+1)}.$$

In particular, no block sequence in X_{GM} is unconditional.

The Gowers-Maurey Construction

- An X is called indecomposable if for any bounded linear projection $P : X \to X$, either $\mathrm{image}(P)$ or $\mathrm{kernel}(P)$ is finite-dimensional.

Theorem: The space X_{GM} is hereditarily indecomposable (HI), i.e., every infinite-dimensional $Y \subset X_{\mathrm{GM}}$ is indecomposable.

Theorem: (Gowers, 1996) Every infinite-dimensional Banach space contains an unconditional sequence or an HI subspace.

Theorem: (Argyros - Felouzis, 2000 and Argyros - Raikoftsalis, 2012)

Every separable reflexive Banach space, e.g., ℓ_2, is isomorphic to a quotient of some reflexive HI space.

The Gowers-Maurey Construction

- An X is called indecomposable if for any bounded linear projection $P : X \to X$, either $\mathrm{image}(P)$ or $\mathrm{kernel}(P)$ is finite-dimensional.

Theorem: The space X_{GM} is hereditarily indecomposable (HI), i.e., every infinite-dimensional $Y \subset X_{\mathrm{GM}}$ is indecomposable.

Theorem: (Gowers, 1996) Every infinite-dimensional Banach space contains an unconditional sequence or an HI subspace.

Theorem: (Argyros - Felouzis, 2000 and Argyros - Raikoftsalis, 2012)

Every separable reflexive Banach space, e.g., ℓ_2, is isomorphic to a quotient of some reflexive HI space.

The Gowers-Maurey Construction

- An X is called indecomposable if for any bounded linear projection $P : X \to X$, either $\mathrm{image}(P)$ or $\mathrm{kernel}(P)$ is finite-dimensional.

Theorem: The space X_{GM} is hereditarily indecomposable (HI), i.e., every infinite-dimensional $Y \subset X_{\mathrm{GM}}$ is indecomposable.

Theorem: (Gowers, 1996) Every infinite-dimensional Banach space contains an unconditional sequence or an HI subspace.

Theorem: (Argyros - Felouzis, 2000 and Argyros - Raikoftsalis, 2012)

Every separable reflexive Banach space, e.g., ℓ_2, is isomorphic to a quotient of some reflexive HI space.

The Gowers-Maurey Construction

- An X is called indecomposable if for any bounded linear projection $P : X \to X$, either $\mathrm{image}(P)$ or $\mathrm{kernel}(P)$ is finite-dimensional.

Theorem: The space X_{GM} is hereditarily indecomposable (HI), i.e., every infinite-dimensional $Y \subset X_{\mathrm{GM}}$ is indecomposable.

Theorem: (Gowers, 1996) Every infinite-dimensional Banach space contains an unconditional sequence or an HI subspace.

Theorem: (Argyros - Felouzis, 2000 and Argyros - Raikoftsalis, 2012)

Every separable reflexive Banach space, e.g., ℓ_2, is isomorphic to a quotient of some reflexive HI space.

The Gowers-Maurey Construction

- A $T : X \to X$ is called strictly singular if:

  for every infinite-dimensional $Y \subset X$ there exists an infinite dimensional $Z \subset Y$ such that $T|_Z$ is compact.

  Denote
  $$\mathcal{SS}(X) = \Big\{ S \in \mathcal{L}(X) \text{ strictly singular} \Big\}.$$

  Always, $\mathcal{K}(X) \subset \mathcal{SS}(X)$.

  Theorem: $\mathcal{L}(X_{\mathrm{GM}}) = \mathbb{R}I + \mathcal{SS}(X_{\mathrm{GM}})$.

  Theorem: (Gowers-Maurey, 1993 and Ferenczi, 1996)
  For every HI space X, $\mathcal{L}(X)/\mathcal{SS}(X)$ is one, two, or four-dimensional.

The Gowers-Maurey Construction

- A $T : X \to X$ is called strictly singular if:

  for every infinite-dimensional $Y \subset X$ there exists an infinite dimensional $Z \subset Y$ such that $T|_Z$ is compact.

  Denote
  $$\mathcal{SS}(X) = \Big\{ S \in \mathcal{L}(X) \text{ strictly singular} \Big\}.$$

  Always, $\mathcal{K}(X) \subset \mathcal{SS}(X)$.

  Theorem: $\mathcal{L}(X_{\mathrm{GM}}) = \mathbb{R}I + \mathcal{SS}(X_{\mathrm{GM}})$.

  Theorem: (Gowers-Maurey, 1993 and Ferenczi, 1996)
  For every HI space X, $\mathcal{L}(X)/\mathcal{SS}(X)$ is one, two, or four-dimensional.

The Gowers-Maurey Construction

- A $T : X \to X$ is called strictly singular if:

  for every infinite-dimensional $Y \subset X$ there exists an infinite dimensional $Z \subset Y$ such that $T|_Z$ is compact.

  Denote
  $$\mathcal{SS}(X) = \Big\{ S \in \mathcal{L}(X) \text{ strictly singular} \Big\}.$$

  Always, $\mathcal{K}(X) \subset \mathcal{SS}(X)$.

  Theorem: $\mathcal{L}(X_{\mathrm{GM}}) = \mathbb{R}I + \mathcal{SS}(X_{\mathrm{GM}})$.

  Theorem: (Gowers-Maurey, 1993 and Ferenczi, 1996)
  For every HI space X, $\mathcal{L}(X)/\mathcal{SS}(X)$ is one, two, or four-dimensional.

The Argyros-Haydon Space

The Scalar-Plus-Compact Problem

Question: (Lindenstrauss, 1975)
Does there exist a Banach space X with the scalar-plus-compact property, i.e., such that $\mathcal{L}(X) = \mathbb{R}I + \mathcal{K}(X)$?

Comment: $\mathcal{L}(X_{\mathrm{GM}}) = \mathbb{R}I + \mathcal{SS}(X_{\mathrm{GM}})$.

Theorem: (Androulakis - Schlumprecht, 2001)
$\mathcal{K}(X_{\mathrm{GM}}) \subsetneq \mathcal{SS}(X_{\mathrm{GM}})$.

Theorem: (Argyros - Haydon, Acta Math. 2011) There exists an HI space $\mathfrak{X}_{\mathrm{AH}}$ with the scalar-plus-compact property.

The Scalar-Plus-Compact Problem

Question: (Lindenstrauss, 1975)
Does there exist a Banach space X with the scalar-plus-compact property, i.e., such that $\mathcal{L}(X) = \mathbb{R}I + \mathcal{K}(X)$?

Comment: $\mathcal{L}(X_{\mathrm{GM}}) = \mathbb{R}I + \mathcal{SS}(X_{\mathrm{GM}})$.

Theorem: (Androulakis - Schlumprecht, 2001)
$\mathcal{K}(X_{\mathrm{GM}}) \subsetneq \mathcal{SS}(X_{\mathrm{GM}})$.

Theorem: (Argyros - Haydon, Acta Math. 2011) There exists an HI space $\mathfrak{X}_{\mathrm{AH}}$ with the scalar-plus-compact property.

The Scalar-Plus-Compact Problem

Question: (Lindenstrauss, 1975)
Does there exist a Banach space X with the scalar-plus-compact property, i.e., such that $\mathcal{L}(X) = \mathbb{R}I + \mathcal{K}(X)$?

Comment: $\mathcal{L}(X_{\mathrm{GM}}) = \mathbb{R}I + \mathcal{SS}(X_{\mathrm{GM}})$.

Theorem: (Androulakis - Schlumprecht, 2001)
$\mathcal{K}(X_{\mathrm{GM}}) \subsetneq \mathcal{SS}(X_{\mathrm{GM}})$.

Theorem: (Argyros - Haydon, Acta Math. 2011) There exists an HI space $\mathfrak{X}_{\mathrm{AH}}$ with the scalar-plus-compact property.

The Argyros-Haydon Space

Theorem: (Argyros-Haydon, 2011) There exists an HI space $\mathfrak{X}_{\mathrm{AH}}$ that has the scalar-plus-compact property.

The Argyros-Haydon construction has two main components.

- The Gowers-Maurey construction, and thus it is an HI space.
- A Bourgain-Delbaen construction of a type of non-reflexive spaces called $\mathscr{L}_\infty$-spaces, and thus $\mathfrak{X}_{\mathrm{AH}}$ is a $\mathscr{L}_\infty$-space.

$\mathfrak{X}_{\mathrm{AH}}$ and the Invariant Subspace Problem

- For a Banach space X a $T \in \mathcal{L}(X)$ admits an invariant subspace if there exists a closed subspace $\{0\} \subsetneq Y \subsetneq X$ with $T(Y) \subset Y$.

- An X has the invariant subspace property (ISP) if every $T \in \mathcal{L}(X)$ admits an invariant subspace.

Question: (von Neumann) Does ℓ_2 have the invariant subspace property?

Theorem: (Aronszajn-Smith, 1954) For a Banach space X, every $K \in \mathcal{K}(X)$ admits an invariant subspace.

Conclusion: $\mathfrak{X}_{\mathrm{AH}}$ has the invariant subspace property, and it is the first known space with this property.

$\mathfrak{X}_{\rm AH}$ and the Invariant Subspace Problem

- For a Banach space X a $T \in \mathcal{L}(X)$ admits an invariant subspace if there exists a closed subspace $\{0\} \subsetneq Y \subsetneq X$ with $T(Y) \subset Y$.

- An X has the invariant subspace property (ISP) if every $T \in \mathcal{L}(X)$ admits an invariant subspace.

Question: (von Neumann) Does ℓ_2 have the invariant subspace property?

Theorem: (Aronszajn-Smith, 1954) For a Banach space X, every $K \in \mathcal{K}(X)$ admits an invariant subspace.

Conclusion: $\mathfrak{X}_{\rm AH}$ has the invariant subspace property, and it is the first known space with this property.

- For a Banach space X a $T \in \mathcal{L}(X)$ admits an invariant subspace if there exists a closed subspace $\{0\} \subsetneq Y \subsetneq X$ with $T(Y) \subset Y$.
- An X has the invariant subspace property (ISP) if every $T \in \mathcal{L}(X)$ admits an invariant subspace.

Question: (von Neumann) Does ℓ_2 have the invariant subspace property?

Theorem: (Aronszajn-Smith, 1954) For a Banach space X, every $K \in \mathcal{K}(X)$ admits an invariant subspace.

Conclusion: $\mathfrak{X}_{\mathrm{AH}}$ has the invariant subspace property, and it is the first known space with this property.

$\mathfrak{X}_{\mathrm{AH}}$ and the Invariant Subspace Problem

- For a Banach space X a $T \in \mathcal{L}(X)$ admits an invariant subspace if there exists a closed subspace $\{0\} \subsetneq Y \subsetneq X$ with $T(Y) \subset Y$.

- An X has the invariant subspace property (ISP) if every $T \in \mathcal{L}(X)$ admits an invariant subspace.

Question: (von Neumann) Does ℓ_2 have the invariant subspace property?

Theorem: (Aronszajn-Smith, 1954) For a Banach space X, every $K \in \mathcal{K}(X)$ admits an invariant subspace.

Conclusion: $\mathfrak{X}_{\mathrm{AH}}$ has the invariant subspace property, and it is the first known space with this property.

$\mathfrak{X}_{\mathrm{AH}}$ and the Invariant Subspace Problem

- For a Banach space X a $T \in \mathcal{L}(X)$ admits an invariant subspace if there exists a closed subspace $\{0\} \subsetneq Y \subsetneq X$ with $T(Y) \subset Y$.
- An X has the invariant subspace property (ISP) if every $T \in \mathcal{L}(X)$ admits an invariant subspace.

Question: (von Neumann) Does ℓ_2 have the invariant subspace property?

Theorem: (Aronszajn-Smith, 1954) For a Banach space X, every $K \in \mathcal{K}(X)$ admits an invariant subspace.

Conclusion: $\mathfrak{X}_{\mathrm{AH}}$ has the invariant subspace property, and it is the first known space with this property.

Spaces X with $\mathcal{L}(X) = \mathbb{R}I + \mathcal{K}(X)$

Theorem: (Argyros - Freeman - Haydon - Odell - Raikoftsalis - Schlumprecht - Zisimopoulou, 2012)

Every uniformly convex separable Banach space X is isomorphic to a subspace of a separable $\mathscr{L}_\infty$-space $\mathfrak{X}$ with the scalar-plus-compact property.

In particular, X is isomorphic to a subspace of a non-reflexive separable space with the invariant subspace property.

Question: Does there exist a reflexive infinite-dimensional Banach space with the scalar-plus-compact property?

Spaces X with $\mathcal{L}(X) = \mathbb{R}I + \mathcal{K}(X)$

Theorem: (Argyros - Freeman - Haydon - Odell - Raikoftsalis - Schlumprecht - Zisimopoulou, 2012)

Every uniformly convex separable Banach space X is isomorphic to a subspace of a separable $\mathscr{L}_\infty$-space $\mathfrak{X}$ with the scalar-plus-compact property.

In particular, X is isomorphic to a subspace of a non-reflexive separable space with the invariant subspace property.

Question: Does there exist a reflexive infinite-dimensional Banach space with the scalar-plus-compact property?

The Argyros-Haydon Method and Calkin Algebras

- For a Banach space X, $\mathcal{C}al(X) = \mathcal{L}(X)/\mathcal{K}(X)$ is a unital Banach algebra called the Calkin algebra of X.

  Question: For what unital Banach algebras $\mathcal{B}$ does there exist X such that $\mathcal{C}al(X) \simeq \mathcal{B}$

  In other words, what unital Banach algebras are Calkin algebras?

- For example, $\mathcal{C}al(\mathfrak{X}_{\mathrm{AH}}) \simeq \mathbb{R}$.

  Theorem: (Tarbard, 2012) $\ell_1(\mathbb{N}_0)$ is a Calkin algebra.

  Theorem: (M - Puglisi - Zisimopoulou, 2016 and M, 2024) Every separable $C(K)$ space is a Calkin algebra.

  Theorem: (M - Pelczar-Barwacz, 2024) The following spaces are Calking algebras:

  - ℓ_p, $1 \leq p < \infty$, e.g., ℓ_2,
  - L_p, $1 < p < \infty$,
  - Schlumprecht's space and Tsirelson's space.

The Argyros-Haydon Method and Calkin Algebras

- For a Banach space X, $\mathcal{Cal}(X) = \mathcal{L}(X)/\mathcal{K}(X)$ is a unital Banach algebra called the Calkin algebra of X.

  Question: For what unital Banach algebras $\mathcal{B}$ does there exist X such that $\mathcal{Cal}(X) \simeq \mathcal{B}$

  In other words, what unital Banach algebras are Calkin algebras?

- For example, $\mathcal{Cal}(\mathfrak{X}_{\mathrm{AH}}) \simeq \mathbb{R}$.

  Theorem: (Tarbard, 2012) $\ell_1(\mathbb{N}_0)$ is a Calkin algebra.

  Theorem: (M - Puglisi - Zisimopoulou, 2016 and M, 2024) Every separable $C(K)$ space is a Calkin algebra.

  Theorem: (M - Pelczar-Barwacz, 2024) The following spaces are Calking algebras:

  - ℓ_p, $1 \leq p < \infty$, e.g., ℓ_2,
  - L_p, $1 < p < \infty$,
  - Schlumprecht's space and Tsirelson's space.

The Argyros-Haydon Method and Calkin Algebras

- For a Banach space X, $\mathcal{C}al(X) = \mathcal{L}(X)/\mathcal{K}(X)$ is a unital Banach algebra called the Calkin algebra of X.

  Question: For what unital Banach algebras $\mathcal{B}$ does there exist X such that $\mathcal{C}al(X) \simeq \mathcal{B}$

  In other words, what unital Banach algebras are Calkin algebras?

- For example, $\mathcal{C}al(\mathfrak{X}_{\mathrm{AH}}) \simeq \mathbb{R}$.

  Theorem: (Tarbard, 2012) $\ell_1(\mathbb{N}_0)$ is a Calkin algebra.

  Theorem: (M - Puglisi - Zisimopoulou, 2016 and M, 2024) Every separable $C(K)$ space is a Calkin algebra.

  Theorem: (M - Pelczar-Barwacz, 2024) The following spaces are Calking algebras:

  - ℓ_p, $1 \leq p < \infty$, e.g., ℓ_2,
  - L_p, $1 < p < \infty$,
  - Schlumprecht's space and Tsirelson's space.

The Invariant Subspace Problem for Reflexive Spaces

The Invariant Subspace Problem for Reflexive Spaces

Question: (von Neumann) Does ℓ_2 have the invariant subspace property?

Theorem: (Enflo, 1987 and Read, 1984) There exist non-reflexive separable Banach spaces failing the ISP.

Theorem: (Argyros-Haydon, 2011) The non-reflexive separable space $\mathfrak{X}_{\mathrm{AH}}$ satisfies the ISP.

Question: (Read, 1989) Does there exist a separable reflexive space with the ISP?

The Invariant Subspace Problem for Reflexive Spaces

Question: (von Neumann) Does ℓ_2 have the invariant subspace property?

Theorem: (Enflo, 1987 and Read, 1984) There exist non-reflexive separable Banach spaces failing the ISP.

Theorem: (Argyros-Haydon, 2011) The non-reflexive separable space $\mathfrak{X}_{\mathrm{AH}}$ satisfies the ISP.

Question: (Read, 1989) Does there exist a separable reflexive space with the ISP?

The Invariant Subspace Problem for Reflexive Spaces

Question: (von Neumann) Does ℓ_2 have the invariant subspace property?

Theorem: (Enflo, 1987 and Read, 1984) There exist non-reflexive separable Banach spaces failing the ISP.

Theorem: (Argyros-Haydon, 2011) The non-reflexive separable space $\mathfrak{X}_{\mathrm{AH}}$ satisfies the ISP.

Question: (Read, 1989) Does there exist a separable reflexive space with the ISP?

Theorem: (Argyros - M, 2014) There exists a separable reflexive HI space $\mathfrak{X}_{\mathrm{ISP}}$ with the ISP.

This construction combines:

- The Tsirelson construction.
- An Odell-Schlumprecht concept called saturation under constraints.

Remark: Every infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ satisfies the ISP.

Theorem: (Argyros - M, 2014) There exists a separable reflexive HI space $\mathfrak{X}_{\mathrm{ISP}}$ with the ISP.

This construction combines:

- The Tsirelson construction.
- An Odell-Schlumprecht concept called saturation under constraints.

Remark: Every infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ satisfies the ISP.

The Invariant Subspace Problem for Reflexive Spaces

Theorem: (Argyros - M, 2014) There exists a separable reflexive HI space $\mathfrak{X}_{\mathrm{ISP}}$ with the ISP.

This construction combines:

- The Tsirelson construction.
- An Odell-Schlumprecht concept called saturation under constraints.

Remark: Every infinite-dimensional closed subspace of $\mathfrak{X}_{\mathrm{ISP}}$ satisfies the ISP.

The Invariant Subspace Problem for Reflexive Spaces

Theorem: $\mathfrak{X}_{\mathrm{ISP}}$ satisfies these main properties:

(1) $\mathcal{L}(\mathfrak{X}_{\mathrm{ISP}}) = \mathbb{R}I + \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$.

(2) For every $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$, ST is compact.

Theorem: (Lomonosov, 1973) Let $T, K \in \mathcal{L}(X)$, for some infinite-dimensional Banach space X. If

- K is compact and
- $TK = KT$

then T admits an invariant subspace.

Conclusion: Every $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ admits an invariant subspace.

Proof: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ and $\lambda \in \mathbb{R}$, $S \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ such that $T = \lambda I + S$.

S^2 is compact and $TS^2 = (\lambda I + S)S^2 = S^2(\lambda I + S) = S^2T$.

By Lomonosov's theorem T admits an invariant subspace.

The Invariant Subspace Problem for Reflexive Spaces

Theorem: $\mathfrak{X}_{\mathrm{ISP}}$ satisfies these main properties:

(1) $\mathcal{L}(\mathfrak{X}_{\mathrm{ISP}}) = \mathbb{R}I + \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$.

(2) For every $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$, ST is compact.

Theorem: (Lomonosov, 1973) Let $T, K \in \mathcal{L}(X)$, for some infinite-dimensional Banach space X. If

- K is compact and
- $TK = KT$

then T admits an invariant subspace.

Conclusion: Every $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ admits an invariant subspace.

Proof: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ and $\lambda \in \mathbb{R}$, $S \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ such that $T = \lambda I + S$.

S^2 is compact and $TS^2 = (\lambda I + S)S^2 = S^2(\lambda I + S) = S^2 T$.

By Lomonosov's theorem T admits an invariant subspace.

The Invariant Subspace Problem for Reflexive Spaces

Theorem: $\mathfrak{X}_{\mathrm{ISP}}$ satisfies these main properties:

(1) $\mathcal{L}(\mathfrak{X}_{\mathrm{ISP}}) = \mathbb{R}I + \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$.

(2) For every $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$, ST is compact.

Theorem: (Lomonosov, 1973) Let $T, K \in \mathcal{L}(X)$, for some infinite-dimensional Banach space X. If

- K is compact and
- $TK = KT$

then T admits an invariant subspace.

Conclusion: Every $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ admits an invariant subspace.

Proof: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ and $\lambda \in \mathbb{R}$, $S \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ such that $T = \lambda I + S$.

S^2 is compact and $TS^2 = (\lambda I + S)S^2 = S^2(\lambda I + S) = S^2 T$.

By Lomonosov's theorem T admits an invariant subspace.

The Invariant Subspace Problem for Reflexive Spaces

Theorem: $\mathfrak{X}_{\mathrm{ISP}}$ satisfies these main properties:

(1) $\mathcal{L}(\mathfrak{X}_{\mathrm{ISP}}) = \mathbb{R}I + \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$.

(2) For every $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$, ST is compact.

Theorem: (Lomonosov, 1973) Let $T, K \in \mathcal{L}(X)$, for some infinite-dimensional Banach space X. If

- K is compact and
- $TK = KT$

then T admits an invariant subspace.

Conclusion: Every $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ admits an invariant subspace.

Proof: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ and $\lambda \in \mathbb{R}$, $S \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ such that $T = \lambda I + S$.

S^2 is compact and $T\,S^2 = (\lambda I + S)S^2 = S^2(\lambda I + S) = S^2 T$.

By Lomonosov's theorem T admits an invariant subspace.

Theorem: $\mathfrak{X}_{\mathrm{ISP}}$ satisfies these main properties:

(1) $\mathcal{L}(\mathfrak{X}_{\mathrm{ISP}}) = \mathbb{R}I + \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$.

(2) For every $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$, ST is compact.

Theorem: (Lomonosov, 1973) Let $T, K \in \mathcal{L}(X)$, for some infinite-dimensional Banach space X. If

- K is compact and
- $TK = KT$

then T admits an invariant subspace.

Conclusion: Every $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ admits an invariant subspace.

Proof: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ and $\lambda \in \mathbb{R}$, $S \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ such that $T = \lambda I + S$.

S^2 is compact and $T\,S^2 = (\lambda I + S)S^2 = S^2(\lambda I + S) = S^2 T$.

By Lomonosov's theorem T admits an invariant subspace.

The Invariant Subspace Problem for Reflexive Spaces

Theorem: $\mathfrak{X}_{\mathrm{ISP}}$ satisfies these main properties:

(1) $\mathcal{L}(\mathfrak{X}_{\mathrm{ISP}}) = \mathbb{R}I + \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$.

(2) For every $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$, ST is compact.

Theorem: (Lomonosov, 1973) Let $T, K \in \mathcal{L}(X)$, for some infinite-dimensional Banach space X. If

- K is compact and
- $TK = KT$

then T admits an invariant subspace.

Conclusion: Every $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ admits an invariant subspace.

Proof: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ and $\lambda \in \mathbb{R}$, $S \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ such that $T = \lambda I + S$.

S^2 is compact and $T\,S^2 = (\lambda I + S)S^2 = S^2(\lambda I + S) = S^2 T$.

By Lomonosov's theorem T admits an invariant subspace.

Spreading Models

- A sequence $(x_n)_{n=1}^\infty$ in a Banach space generates a C-ℓ_1-spreading model if for all $n \leq i_1 < \cdots < i_n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$

$$C^{-1}\sum_{k=1}^n |a_k| \leq \Big\| \sum_{k=1}^n a_k x_{i_k} \Big\| \leq \sum_{k=1}^n |a_k|.$$

  We refer to such sequences as rank II sequences.

- A sequence $(x_n)_{n=1}^\infty$ in a Banach space generates a C-c_0-spreading model if for all $n \leq i_1 < \cdots < i_n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$

$$\max_{1\leq k\leq n} |a_k| \leq \Big\| \sum_{k=1}^n a_k x_{i_k} \Big\| \leq C \max_{1\leq k\leq n} |a_k|.$$

  We refer to such sequences as rank I sequences.

- We refer to sequences $(x_i)_{i=1}^\infty$ such that $\lim_n \|x_i\| = 0$ as rank 0 sequences.

Spreading Models

- A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space generates a C-ℓ_1-spreading model if for all $n \leq i_1 < \cdots < i_n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$

$$C^{-1} \sum_{k=1}^{n} |a_k| \leq \left\| \sum_{k=1}^{n} a_k x_{i_k} \right\| \leq \sum_{k=1}^{n} |a_k|.$$

  We refer to such sequences as rank II sequences.

- A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space generates a C-c_0-spreading model if for all $n \leq i_1 < \cdots < i_n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$

$$\max_{1 \leq k \leq n} |a_k| \leq \left\| \sum_{k=1}^{n} a_k x_{i_k} \right\| \leq C \max_{1 \leq k \leq n} |a_k|.$$

  We refer to such sequences as rank I sequences.

- We refer to sequences $(x_i)_{i=1}^{\infty}$ such that $\lim_n \|x_i\| = 0$ as rank 0 sequences.

Spreading Models

- A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space generates a C-ℓ_1-spreading model if for all $n \leq i_1 < \cdots < i_n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$

$$C^{-1} \sum_{k=1}^{n} |a_k| \leq \left\| \sum_{k=1}^{n} a_k x_{i_k} \right\| \leq \sum_{k=1}^{n} |a_k|.$$

  We refer to such sequences as rank II sequences.

- A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space generates a C-c_0-spreading model if for all $n \leq i_1 < \cdots < i_n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$

$$\max_{1 \leq k \leq n} |a_k| \leq \left\| \sum_{k=1}^{n} a_k x_{i_k} \right\| \leq C \max_{1 \leq k \leq n} |a_k|.$$

  We refer to such sequences as rank I sequences.

- We refer to sequences $(x_i)_{i=1}^{\infty}$ such that $\lim_n \|x_i\| = 0$ as rank 0 sequences.

Remark: Rank is stable under taking subsequences.

Theorem: Let $(x_i)_{i=1}^{\infty}$ be a weakly null sequence in $\mathfrak{X}_{\mathrm{ISP}}$. Then, it has a rank 0, a rank I or a rank II subsequence.

Theorem: In $\mathfrak{X}_{\mathrm{ISP}}$, there is a canonical way to construct a rank I sequence from a rank II sequence, and vice versa.

In particular, every subspace of $\mathfrak{X}_{\mathrm{ISP}}$ contains weakly null sequences of rank I and II.

Remark: Rank is stable under taking subsequences.

Theorem: Let $(x_i)_{i=1}^{\infty}$ be a weakly null sequence in $\mathfrak{X}_{\mathrm{ISP}}$. Then, it has a rank 0, a rank I or a rank II subsequence.

Theorem: In $\mathfrak{X}_{\mathrm{ISP}}$, there is a canonical way to construct a rank I sequence from a rank II sequence, and vice versa.

In particular, every subspace of $\mathfrak{X}_{\mathrm{ISP}}$ contains weakly null sequences of rank I and II.

Hereditary Heterogeneity of $\mathfrak{X}_{\mathrm{ISP}}$

Remark: Rank is stable under taking subsequences.

Theorem: Let $(x_i)_{i=1}^{\infty}$ be a weakly null sequence in $\mathfrak{X}_{\mathrm{ISP}}$. Then, it has a rank 0, a rank I or a rank II subsequence.

Theorem: In $\mathfrak{X}_{\mathrm{ISP}}$, there is a canonical way to construct a rank I sequence from a rank II sequence, and vice versa.

In particular, every subspace of $\mathfrak{X}_{\mathrm{ISP}}$ contains weakly null sequences of rank I and II.

Strictly Singular Operators on $\mathfrak{X}_{\text{ISP}}$

Theorem: Let $T \in \mathcal{L}(\mathfrak{X}_{\text{ISP}})$ be strictly singular. For a weakly null sequence $(x_i)_{i=1}^{\infty}$ of positive rank, $(Tx_i)_{i=1}^{\infty}$ has a subsequence of lesser rank.

Corollary: If $S, T \in \mathcal{SS}(\mathfrak{X}_{\text{ISP}})$ then ST is compact.

Proof: By reflexivity, it suffices to show that for a weakly null sequence $(x_i)_{i=1}^{\infty}$, $\lim_i \|STx_i\| = 0$.

Assume $(x_i)_{i=1}^{\infty}$, $(Tx_i)_{i=1}^{\infty}$ and $(STx_i)_{i=1}^{\infty}$ are of some rank.

- If $(x_i)_{i=1}^{\infty}$ is of rank 0, i.e. $\lim_i \|x_i\| = 0$, then $\lim_{i \in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^{\infty}$ is of rank I, then $(Tx_i)_{i=1}^{\infty}$ is of rank 0, i.e. $\lim_i \|Tx_i\| = 0$, and thus $\lim_{i \in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^{\infty}$ is of rank II, then $(Tx_i)_{i=1}^{\infty}$ is or rank I or 0.

  In either case, $(STx_i)_{i=1}^{\infty}$ is of rank 0, i.e., $\lim_i \|STx_i\| = 0$.

Strictly Singular Operators on $\mathfrak{X}_{\mathrm{ISP}}$

Theorem: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ be strictly singular. For a weakly null sequence $(x_i)_{i=1}^{\infty}$ of positive rank, $(Tx_i)_{i=1}^{\infty}$ has a subsequence of lesser rank.

Corollary: If $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ then ST is compact.

Proof: By reflexivity, it suffices to show that for a weakly null sequence $(x_i)_{i=1}^{\infty}$, $\lim_i \|STx_i\| = 0$.

Assume $(x_i)_{i=1}^{\infty}$, $(Tx_i)_{i=1}^{\infty}$ and $(STx_i)_{i=1}^{\infty}$ are of some rank.

- If $(x_i)_{i=1}^{\infty}$ is of rank 0, i.e, $\lim_i \|x_i\| = 0$, then $\lim_{i \in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^{\infty}$ is of rank I, then $(Tx_i)_{i=1}^{\infty}$ is of rank 0, i.e, $\lim_i \|Tx_i\| = 0$, and thus $\lim_{i \in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^{\infty}$ is of rank II, then $(Tx_i)_{i=1}^{\infty}$ is or rank I or 0.

  In either case, $(STx_i)_{i=1}^{\infty}$ is of rank 0, i.e., $\lim_i \|STx_i\| = 0$.

Strictly Singular Operators on $\mathfrak{X}_{\text{ISP}}$

Theorem: Let $T \in \mathcal{L}(\mathfrak{X}_{\text{ISP}})$ be strictly singular. For a weakly null sequence $(x_i)_{i=1}^{\infty}$ of positive rank, $(Tx_i)_{i=1}^{\infty}$ has a subsequence of lesser rank.

Corollary: If $S, T \in \mathcal{SS}(\mathfrak{X}_{\text{ISP}})$ then ST is compact.

Proof: By reflexivity, it suffices to show that for a weakly null sequence $(x_i)_{i=1}^{\infty}$, $\lim_i \|STx_i\| = 0$.

Assume $(x_i)_{i=1}^{\infty}$, $(Tx_i)_{i=1}^{\infty}$ and $(STx_i)_{i=1}^{\infty}$ are of some rank.

- If $(x_i)_{i=1}^{\infty}$ is of rank 0, i.e, $\lim_i \|x_i\| = 0$, then $\lim_{i\in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^{\infty}$ is of rank I, then $(Tx_i)_{i=1}^{\infty}$ is of rank 0, i.e, $\lim_i \|Tx_i\| = 0$, and thus $\lim_{i\in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^{\infty}$ is of rank II, then $(Tx_i)_{i=1}^{\infty}$ is or rank I or 0.

  In either case, $(STx_i)_{i=1}^{\infty}$ is of rank 0, i.e., $\lim_i \|STx_i\| = 0$.

Strictly Singular Operators on $\mathfrak{X}_{\mathrm{ISP}}$

Theorem: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ be strictly singular. For a weakly null sequence $(x_i)_{i=1}^\infty$ of positive rank, $(Tx_i)_{i=1}^\infty$ has a subsequence of lesser rank.

Corollary: If $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ then ST is compact.

Proof: By reflexivity, it suffices to show that for a weakly null sequence $(x_i)_{i=1}^\infty$, $\lim_i \|STx_i\| = 0$.

Assume $(x_i)_{i=1}^\infty$, $(Tx_i)_{i=1}^\infty$ and $(STx_i)_{i=1}^\infty$ are of some rank.

- If $(x_i)_{i=1}^\infty$ is of rank 0, i.e, $\lim_i \|x_i\| = 0$, then $\lim_{i\in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^\infty$ is of rank I, then $(Tx_i)_{i=1}^\infty$ is of rank 0, i.e, $\lim_i \|Tx_i\| = 0$, and thus $\lim_{i\in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^\infty$ is of rank II, then $(Tx_i)_{i=1}^\infty$ is or rank I or 0.

  In either case, $(STx_i)_{i=1}^\infty$ is of rank 0, i.e., $\lim_i \|STx_i\| = 0$.

Theorem: Let $T \in \mathcal{L}(\mathfrak{X}_{\text{ISP}})$ be strictly singular. For a weakly null sequence $(x_i)_{i=1}^\infty$ of positive rank, $(Tx_i)_{i=1}^\infty$ has a subsequence of lesser rank.

Corollary: If $S, T \in \mathcal{SS}(\mathfrak{X}_{\text{ISP}})$ then ST is compact.

Proof: By reflexivity, it suffices to show that for a weakly null sequence $(x_i)_{i=1}^\infty$, $\lim_i \|STx_i\| = 0$.

Assume $(x_i)_{i=1}^\infty$, $(Tx_i)_{i=1}^\infty$ and $(STx_i)_{i=1}^\infty$ are of some rank.

- If $(x_i)_{i=1}^\infty$ is of rank 0, i.e, $\lim_i \|x_i\| = 0$, then $\lim_{i\in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^\infty$ is of rank I, then $(Tx_i)_{i=1}^\infty$ is of rank 0, i.e, $\lim_i \|Tx_i\| = 0$, and thus $\lim_{i\in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^\infty$ is of rank II, then $(Tx_i)_{i=1}^\infty$ is or rank I or 0.
  In either case, $(STx_i)_{i=1}^\infty$ is of rank 0, i.e., $\lim_i \|STx_i\| = 0$.

Strictly Singular Operators on $\mathfrak{X}_{\mathrm{ISP}}$

Theorem: Let $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})$ be strictly singular. For a weakly null sequence $(x_i)_{i=1}^{\infty}$ of positive rank, $(Tx_i)_{i=1}^{\infty}$ has a subsequence of lesser rank.

Corollary: If $S, T \in \mathcal{SS}(\mathfrak{X}_{\mathrm{ISP}})$ then ST is compact.

Proof: By reflexivity, it suffices to show that for a weakly null sequence $(x_i)_{i=1}^{\infty}$, $\lim_i \|STx_i\| = 0$.

Assume $(x_i)_{i=1}^{\infty}$, $(Tx_i)_{i=1}^{\infty}$ and $(STx_i)_{i=1}^{\infty}$ are of some rank.

- If $(x_i)_{i=1}^{\infty}$ is of rank 0, i.e, $\lim_i \|x_i\| = 0$, then $\lim_{i\in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^{\infty}$ is of rank I, then $(Tx_i)_{i=1}^{\infty}$ is of rank 0, i.e, $\lim_i \|Tx_i\| = 0$, and thus $\lim_{i\in L} \|STx_i\| = 0$.
- If $(x_i)_{i=1}^{\infty}$ is of rank II, then $(Tx_i)_{i=1}^{\infty}$ is or rank I or 0.

  In either case, $(STx_i)_{i=1}^{\infty}$ is of rank 0, i.e., $\lim_i \|STx_i\| = 0$.

Thank you!