Methods for constructing Banach spaces with prescribed properties in their operator spaces

Definitions:

- $(x_i)_i$ is a **S-basis of X**: $\forall x \in X \; \exists! \, (a_i)_i \in \mathbb{R}^{\mathbb{N}}$ s.t. $x = \sum_{i=1}^{\infty} a_i x_i$.
- $(x_i)_i$ is **S-basic** if it is an S-basis of $\overline{\langle (x_i)_i \rangle}$.
- $(x_i)_i$ S-basic in X & $(y_i)_i$ S-basic in Y are **equivalent** if $x_i \mapsto y_i$ extends to a linear isomorphism between $\overline{\langle (x_i)_i \rangle}$ & $\overline{\langle (y_i)_i \rangle}$.

Theorem (Gowers-Maurey) $\exists$ X s.t. $\forall$ $T \in \mathcal{L}(X)$ $\exists \lambda \in \mathbb{R}$ s.t. $S = T - \lambda I$ is strictly singular.
S is strictly singular means that $\forall$ S-basic $(x_i)_i$ in X is not equivalent to $(Sx_i)_i$.

Theorem: $\exists$ $X = X_{MR}$ with an S-basis $(e_i)_i$ s.t. $\forall$ $T \in \mathcal{L}(X)$ $\exists \lambda \in \mathbb{R}$ s.t. $S = T - \lambda I$ has the following property: $\forall$ subsequence $(e_{i_k})_k$ of the S-basis is not equivalent to $(Se_{i_k})_k$.

Notation & Definitions

$c_{00} = \{ x = (x(i))_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ eventually null $\}$.

for $i \in \mathbb{N}$

$$e_i = e_i^* = (0,0,0,\dots,0,1,0,\dots)$$

ith position

$(e_i)_i$ & $(e_i^*)_i$ is the uvb of c_{00}.

Definition: A *norming set* is a $W \subset c_{00}$ s.t.

(i) $\forall i \in \mathbb{N}$, $e_i^* \in W$.

(ii) $\forall f = \sum_{i=1}^{\infty} f(i) e_i^* \in W$, $\|f\|_\infty = \max_{i \in \mathbb{N}} |f(i)| \leq 1$.

(iii) $\forall f = \sum_{i=1}^{\infty} f(i) e_i^* \in W$ and $n \in \mathbb{N}$

$$g = \sum_{i=1}^{n} f(i) e_i^* \in W.$$

For such W define $\|\cdot\|_W$ on c_{00} as follows:

for $f = \sum_{i=1}^{\infty} f(i) e_i^* \in W$ & $x = \sum_{i=1}^{\infty} x(i) e_i \in c_{00}$

let $f(x) = \sum_{i=1}^{\infty} f(i) x(i)$ and

$$\|x\|_W = \sup \{ |f(x)| : f \in W \}.$$

Let $X_W = \overline{(c_{00}, \|\cdot\|_W)}$. This is a B-space

& $(e_i)_i$ is an S-basis for X_W.

Examples:

(i) $W_0 = \{ \varepsilon e_i^* : \varepsilon \in \{-1,1\}, i \in \mathbb{N} \} \cup \{0\}.$

Then $X_{W_0} = c_0,$

(ii) $W_{\ell_1} = \{ \sum_{i=1}^{n} \varepsilon_i e_i^* : (\varepsilon_i)_{i=1}^{n} \in \{-1,1\}^n, n \in \mathbb{N} \}$
$\cup \{0\}.$

Then $X_{W_{\ell_1}} = \ell_1.$

Definition of $X = X_{MR}$

Fix $2 \leq m_1 < n_1 < m_2 < n_2 < m_3 < n_3 < \dots$

s.t. $\forall$ $j \in \mathbb{N}$:

- $m_{j+1} > \left(\sum_{i \leq j} m_i \right) \cdot 2^{j+1}$
- $\frac{m_j}{n_j} \cdot \frac{1}{2^{j+1}} \geq \sum_{i=j+1}^{\infty} \frac{m_i}{n_i}.$

Put $W_0 = \{ \pm e_i^* : i \in \mathbb{N} \}.$

For $j \in \mathbb{N}$ & $F \subset \mathbb{N}$ s.t. $1 \leq \# F \leq n_{2j}$
& $(\varepsilon_i)_{i \in F} \in \{-1,1\}^F$ we call

$$f = \frac{1}{m_{2j}} \sum_{i \in F} \varepsilon_i e_i^*$$

a type I functional of weight $w(f) = 1/m_{2j}.$

We will define type-II functionals.

Let $\mathcal{Q}$ = all finite sequences $(f_1, f_2, \dots, f_d)$ of type I functionals s.t.

$$\operatorname{supp}(f_1) < \operatorname{supp}(f_2) < \dots < \operatorname{supp}(f_d).$$

We fix a "coding function" $\sigma : \mathcal{Q} \to \mathbb{N}$, i.e., σ is an injection and furthermore

s.t. $\forall\ (f_1, \dots, f_d) \in \mathcal{Q}$,

$$m_{2\sigma(f_1, \dots, f_d)} > \|f_d\|_\infty^{-1} \cdot \operatorname{maxsupp}(f_d).$$

A sequence $(f_1, f_2, \dots, f_d)$ of type I functionals is called a $(2j-1)$-special sequence if:

- $\operatorname{supp}(f_1) < \operatorname{supp}(f_2) < \dots < \operatorname{supp}(f_d)$
- For $1 \leq i < d$,

$$w(f_{i+1}) = 1/m_{2\sigma(f_1, \dots, f_{i-1})}$$

- $d \leq n_{2j-1}$.

Remark: If $(f_1, \dots, f_d)$ is $(2j-1)$-special then if $w(f_d) = 1/m_{2i}$ then

$$\sigma^{-1}(i) = (f_1, \dots, f_{d-1}).$$

For a $(2j-1)$-special sequence $(f_1, \dots, f_d)$ of type I functionals we call

$$g = \frac{1}{m_{2j-1}} \sum_{i=1}^{d} f_i$$

a type II functional of weight $w(g) = \frac{1}{m_{2j-1}}$.

Let $W_{II} = \{ g$ type II functional of $w(g) = \frac{1}{m_{2j-1}}, j \in \mathbb{N} \}$.

Put $W = W_{MR} = W_0 \cup W_I \cup W_{II}$.
& $X = X_{MR} = X_W$.

Theorem (1) Let $T \in \mathcal{L}(X)$.
(i) $\lim_i \|Te_i - e_i^*(Te_i)e_i\|_\infty = 0$.
(ii) $\lim_i e_i^*(Te_i)$ exists in $\mathbb{R}$.

Corollary: If $T \in \mathcal{L}(X)$ & $\lambda = \lim_i e_i^*(Te_i)$ then $S = T - \lambda I$ has the property that $\lim_i \|Se_i\|_\infty = 0$.

Theorem (2) If $(x_i)_i$ is a bounded sequence in X such that $\|x_i\|_\infty \to 0$ then it is not equivalent to a subsequence of the basis.

Def: For $j \in \mathbb{N}$ & $F \subset \mathbb{N}$ with $\# F = n_{2j}$ & $(\varepsilon_i)_{i \in F} \in \{-1, 1\}^F$ we call

$$x = \frac{m_{2j}}{n_{2j}} \sum_{i \in F} \varepsilon_i e_i$$

we call this a $2j$-vector.

Remark 1: If $f = \frac{1}{m_{2j}} \sum_{i \in F} \varepsilon_i e_i^*$ then $f(x) = 1$.

Remark 2: If $f' = \frac{1}{m_{2j'}} \sum_{i \in E} \delta_i e_i^*$ is a type I functional then

$$f'(x) = \frac{1}{m_{2j'}} \frac{m_{2j}}{n_{2j}} \sum_{i \in E \cap F} \varepsilon_i \delta_i .$$

Key Lemma: $j_1 < j_2 < \dots < j_p \in \mathbb{N}$
$j_1' < j_2' < \dots < j_q' \in \mathbb{N}$.

Let $(x_k)_{k=1}^p$, $(f_\ell)_{\ell=1}^q$ be such that

x_k is an $2j_k$-vector $1 \leq k \leq p$
f_ℓ is a type I functional of $w(f_\ell) = \frac{1}{m_{2j_\ell'}}$.

Then $\sum_{k,m:} |f_\ell(x_k)| \leq 1$.

$j_k \neq j'_\ell$

Proof: Write $x_k = \frac{m_{2j_k}}{n_{2j_k}} \sum_{i \in E_k} \varepsilon_i e_i$ &

$$f_\ell = \frac{1}{m_{2j'_\ell}} \sum_{i \in F_\ell} \delta_i e_i^* .$$

Fix $1 \le \ell \le q$. Compute

$$\sum_{\substack{k: \\ j_k \neq j'_\ell}} |f_\ell(x_k)| = \qquad \left(\text{WS} \le \frac{1}{2^\ell}\right).$$

$$\le \sum_{\substack{k: \\ j_k \neq j'_\ell}} \frac{1}{m_{2j'_\ell}} \frac{m_{2j_k}}{n_{2j_k}} \sum_{i \in E_k \cap F_\ell} |\varepsilon_i \delta_i|$$

$$\le \sum_{\substack{k: \\ j_k \neq j'_\ell}} \frac{1}{m_{2j'_\ell}} \frac{m_{2j_k}}{n_{2j_k}} \#(E_k \cap F_\ell)$$

$$= \frac{1}{m_{2j'_\ell}} \left[\sum_{\substack{k: \\ j_k < j'_\ell}} \frac{m_{2j_k}}{n_{2j_k}} \#(E_k \cap F_\ell) + \sum_{\substack{k: \\ j_k > j'_\ell}} \frac{m_{2j_k}}{n_{2j_k}} \#(E_k \cap F_\ell) \right]$$

$$\le \underbrace{\frac{1}{m_{2j'_\ell}} \left[\sum_{\substack{k: \\ j_k < j'_\ell}} \frac{m_{2j_k}}{\cancel{n_{2j_k}}} \cancel{n_{2j_k}} \right.}_{1/2^\ell} + \underbrace{\left. \sum_{\substack{k: \\ j_k > j'_\ell}} \frac{m_{2j_k}}{n_{2j_k}} m_{2j'_\ell} \right]}_{1/2^\ell}$$

$\leq 1.$

Corollary:

- If x is a $2j$-vector then $\|x\| = 1$.
- $(e_i)_i$ is weakly null.

Definition:

(1) Let $j \in \mathbb{N}$ & $x = \frac{m_{2j}}{n_{2j}} \sum_{i \in E} \varepsilon_i e_i$ be a $2j$-vector and $f = \frac{1}{m_{2j}} \sum_{i \in F} \delta_i e_i^*$ be type I functional if $w(f) = 1/m_{2j}$.

- If $F = E$ & $\varepsilon_i = \delta_i$ for $i \in F$ then we call (x, f) a $(1, 2j)$-exact pair.
- If $F \cap E = \emptyset$ then we call (x, f) a $(0, 2j)$-exact pair.

(2) Let $\vartheta \in \{0, 1\}$ & let $j \in \mathbb{N}$. A sequence of exact pairs

$(x_1, f_1), (x_2, f_2), \ldots, (x_{n_{2j-1}}, f_{n_{2j-1}})$

is called a $(\vartheta, 2j-1)$-dependent sequence if
- $\forall\ 1 \leq k \leq n$ (x_k, f_k) is a $(\vartheta, 2j_k)$-exact pair
- $(f_1, f_2, \ldots, f_{n_{2j-1}})$ are a $(2j-1)$-special

sequence.

- $\mathrm{supp}(x_1) \cup \mathrm{supp}(f_1) < \mathrm{supp}(x_2) \cup \mathrm{supp}(f_2) < \dots$

Proposition: Let $\theta \in \{0, 1\}$ and a $(\theta, 2j-1)$-dependent sequence $(x_k, f_k)_{k=1}^{n_{2j-1}}$ & put

$$y = \frac{m_{2j-1}}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} x_k .$$

Then:

(i) If $\theta = 1$ then $\|y\| \geq 1$.

(ii) If $\theta = 0$ then $\|y\| \leq 2 \frac{m_{2j-1}}{n_{2j-1}}$.

Proof:

(i) Because $(f_k)_{k=1}^{n_{2j-1}}$ is $(2j-1)$-special then

$g = \frac{1}{m_{2j-1}} \sum_{k=1}^{n_{2j-1}} f_k$ is a type II functional

& thus $\|y\| \geq g(y) = 1$.

(ii) Fix $g \in W = W_0 \cup W_I \cup W_{II}$. We will show $|g(y)| \leq 2 \frac{m_{2j-1}}{n_{2j-1}}$.

If $g \in W_0 \cup W_I$ we omit this.

Assume $g = \frac{1}{m_{2j'-1}} \sum_{e=1}^{a} f'_e \in W_{II}$ &

$(f_1', f_2', \dots, f_q')$ is $(2j'-1)$-special.

Say each $x_k = \dfrac{m_{2j_k}}{n_{2j_k}} \sum_{i \in E_k} \varepsilon_i e_i$ is a $(2j_k)$-vector & f_k is of weight $w(f_k) = 1/m_{2j_k}$.

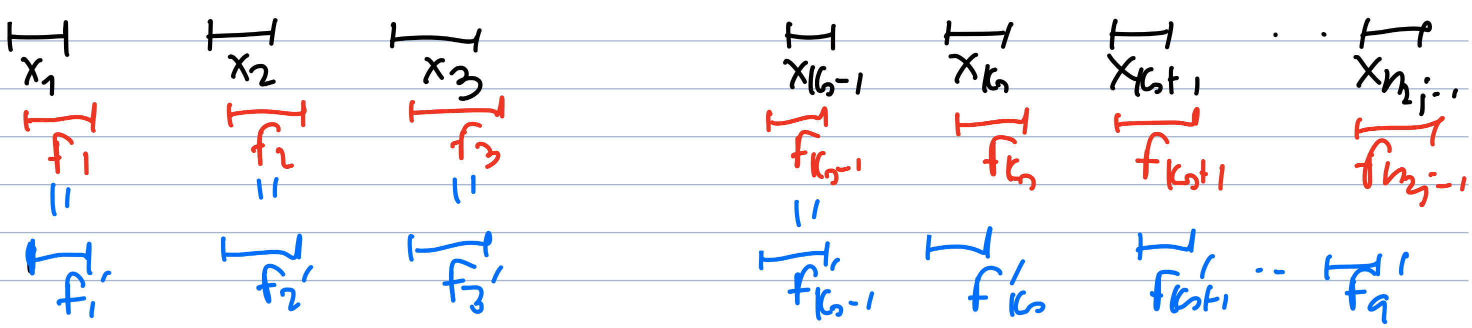
Put $l_0 = \max\{1 \le l \le q : w(f_l') \in \{w(f_1), \dots, w(f_{n_{2j-1}})\}\}$ (if it exists).

Choose $1 \le k_0 \le n_{2j-1}$ s.t. $w(f_{l_0}') = w(f_{k_0})$.

Then
$$\begin{aligned} w(f_{l_0}') &= 1/m_{2\sigma(f_1', \dots, f_{l_0-1}')} \\ &\parallel \\ w(f_{k_0}) &= 1/m_{2\sigma(f_1, \dots, f_{k_0-1})} \end{aligned}$$

$$\Rightarrow \sigma(f_1', \dots, f_{l_0-1}') = \sigma(f_1, \dots, f_{k_0-1}) \overset{\sigma\ 1\text{-}1}{\Rightarrow}$$

$$\Rightarrow (f_1', \dots, f_{l_0-1}') = (f_1, \dots, f_{k_0-1}).$$



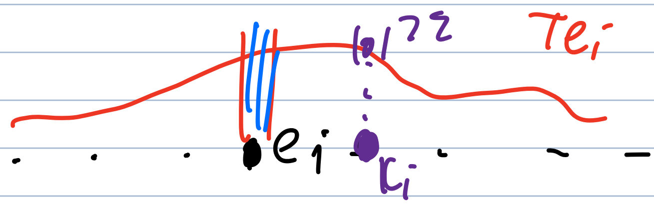
$$g(y) = \frac{1}{m_{2j'-1}} \, \frac{m_{2j-1}}{n_{2j-1}} \Big(\sum_{k=1}^{k_0-1} f_k'(x_k) + \underbrace{f_{k_0}'(x_{k_0})}_{\le 1} + \underbrace{1}_{\text{key lemma \& weight mismatch}} \Big)$$

$$\sum_{k=1}^{k_0-1} \underbrace{f_k''(x_k)}_{=0}$$

$$\le 2 \frac{1}{m_{2j'-1}} \, \frac{m_{2j-1}}{n_{2j-1}} \le 2 \, \frac{m_{2j-1}}{n_{2j-1}} .$$

Proposition: $T \in \mathcal{L}(X)$. Then

$$\lim_i \| Te_i - e_i^*(Te_i) e_i \|_\infty = 0.$$



Proof: Assume this is false, i.e., $\exists L \subset \mathbb{N}$ infinite & $\varepsilon > 0$ s.t. $\forall i \in L$

$$\| Te_i - e_i^*(Te_i) e_i \|_\infty > \varepsilon, \text{ i.e., } \exists k(i) \neq i$$

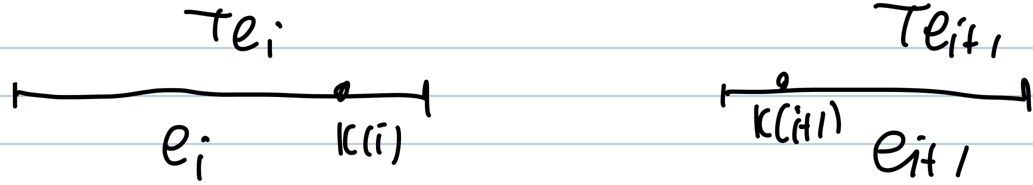
s.t. $|e_{k_i}^*(Te_i)| > \varepsilon$.

Wlog, because $(e_i) \xrightarrow{w} 0$ we may assume:

(i) $(Te_i)_{i \in L}$ is block.

(ii) $e_{k(i)}^*(Te_j) = \begin{cases} 0 & : i \neq j \\ \varepsilon & : i = j. \end{cases}$

(iii) $k(i) \in L$.



Fix $j \in \mathbb{N}$. We will construct inductively

$$f_1 = \frac{1}{m_{2j_1}} \sum_{i \in F_1} e^*_{k(i)}, \quad F_1 \subset L \text{ with } \# F_1 = n_{2j_1}.$$

$$f_2 = \frac{1}{m_{2j_2}} \sum_{i \in F_2} e^*_{k(i)}, \quad F_2 \subset L \text{ with } \# F_2 = n_{2j_2}.$$

$$\vdots$$

$$f_{n_{2j-1}} = \frac{1}{m_{2j_{2j-1}}} \sum_{i \in F_{n_{2j-1}}} e^*_{k(i)} \quad F_{n_{2j-1}} \subset L \text{ with}$$

$$\# F_{n_{2j-1}} = n_{2j_{2j-1}}$$

s.t. $(f_1, f_2, \ldots, f_{n_{2j-1}})$ is a $(2j-1)$-special sequence.

Observations: Define

$$x_1 = \frac{m_{2j_1}}{n_{2j_1}} \sum_{i \in F_1} e_i$$

$$x_2 = \frac{m_{2j_2}}{n_{2j_2}} \sum_{i \in F_2} e_i$$

$$\vdots$$

$$x_{n_{2j-1}} = \ldots$$

(i) (x_1, f_1), (x_2, f_2), ..., $(x_{n_{2j-1}}, f_{n_{2j-1}})$

is a $(0, 2j-1)$ - dependent sequence

$$\Rightarrow \quad y = \frac{m_{2j-1}}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} x_k , \qquad \|y\| \leq 2 \frac{m_{2j-1}}{n_{2j-1}} .$$

$$\text{(ii)} \quad \|Ty\| \geq \left(\frac{1}{m_{2j-1}} \sum_{k=1}^{n} f_k \right) (Ty) = \varepsilon .$$

$$\Rightarrow \quad \|T\| \geq \varepsilon \frac{n_{2j-1}}{2 m_{2j-1}} \quad , \quad \text{where } j \text{ arbitrary.}$$

This is absurd.