# Varopoulos extensions and Applications to Boundary Value Problems in rough domains

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7th MATH@NTUA summer school "Mathematical Analysis"
In honor of Spiros Argyros
School of Applied Mathematical and Physical Sciences
National Technical University of Athens
27 June – 3 July, 2024

*Based on joint works with a) Xavier Tolsa,*
*b) Bruno Poggi and Xavier Tolsa, and c) Thanasis Zacharopoulos*

MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD

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# The Dirichlet problem for continuous data

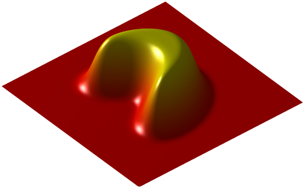
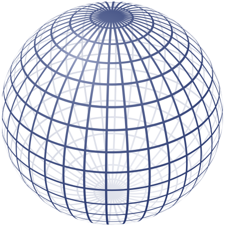
Fix $g \in C_c(\partial\Omega)$. The (continuous) Dirichlet problem for the Laplacian on $\Omega$ with boundary data $g$ is to find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

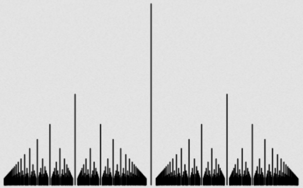
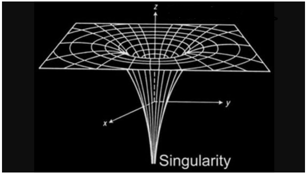
If $\Omega$ and its boundary are sufficiently nice, then this problem is always solvable for any $g \in C_c(\partial\Omega)$ (exact characterization by [Wiener 1924]).

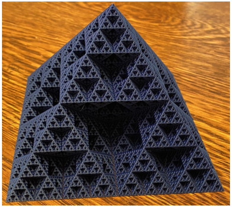
# Classical and contemporary considerations

Classical: Smooth data $g$ and boundary $\partial\Omega$

Contemporary: Singular, rough data $g$ and boundary $\partial\Omega$

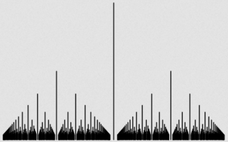



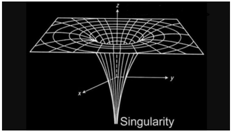


Kind offer of Bruno Poggi.

# Generalizing Dirichlet problem to singular data

Let $\Omega \subset \mathbb{R}^{n+1}$, $g \in L^p(\sigma)$,
$\sigma = \mathcal{H}^n|_{\partial\Omega}$ the surface measure.






How do we understand $u = g$ on $\partial\Omega$ when $g$ is singular?

Non-tangential convergence. We say

$$u \longrightarrow g \quad \textit{non-tangentially},$$

if

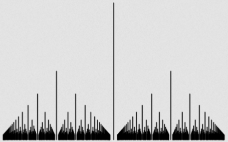
$$\lim_{\gamma(\xi) \ni x \to \xi} u(x) = g(\xi), \qquad \text{for } \sigma - \text{a.e. } \xi \in \partial\Omega,$$

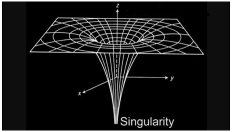
where

$$\gamma(\xi) := \{x \in \Omega \ : \ |x - \xi| < 2\,\mathrm{dist}(x, \partial\Omega)\}.$$

# A singular analogue of the maximum principle

Let $\Omega \subset \mathbb{R}^{n+1}$, $g \in L^p(\sigma)$, $\sigma$ the surface measure.






$$\gamma(\xi) := \{x \in \Omega \; : \; |x - \xi| < 2\,\mathrm{dist}(x, \partial\Omega)\}.$$

The non-tangential maximal function.

$$(\mathcal{N}u)(\xi) := \sup_{x \in \gamma(\xi)} |u(x)|, \quad \xi \in \partial\Omega,$$

$$\|\mathcal{N}u\|_{L^p(\partial\Omega,\sigma)} \leq C\|g\|_{L^p(\partial\Omega,\sigma)},$$

# The $L^p$-Dirichlet problem for ellitic PDEs

$$L = -\operatorname{div} A\nabla$$

$A$ satisfies ellipticity assumptions

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j, \qquad \|A\|_{L^\infty(\Omega)} \leq \frac{1}{\lambda}, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

We say that $(\mathrm{D}_{p'}^{L})$ is solvable if there exists $C > 0$ such that for each $g \in C_c(\partial\Omega)$, there exists a solution to the problem

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

with

$$\|\mathcal{N}(u)\|_{L^{p'}(\partial\Omega,\sigma)} \leq C\|g\|_{L^{p'}(\partial\Omega,\sigma)}.$$

# $L^p$-Regularity problem, i.e., the Dirichlet problem with data in the Sobolev space $\dot{W}^{1,p}$

For $x \in \Omega$, define

$$m_q(F)(x) := \left( \fint_{B_x} |F(y)|^q \, dy \right)^{1/q}, \quad \text{where } B_x := B(x, \delta(x)/4).$$

The Kenig-Pipher modified non-tangential maximal function is defined

$$\widetilde{\mathcal{N}}_q(u)(\xi) := \mathcal{N}(m_q(u))(\xi), \qquad \xi \in \partial\Omega.$$

Let $p > 1$, $L = -\operatorname{div} A\nabla$, $A$ a strongly elliptic, bounded matrix. We say that $(\mathrm{R}_p^L)$ is solvable if there exists $C > 0$ such that for each $f \in \mathrm{Lip}_c(\partial\Omega)$, there exists a solution to the problem

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega, \end{cases}$$

with

$$\|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{\dot{W}^{1,p}(\partial\Omega)}.$$

# Solvability of Regularity problem in $L^p$ for $\Delta$

1978 $C^1$ domains: Fabes, Jodeit, and Rivière showed that the regularity problem is solvable in $L^p$ for all $p \in (1, \infty)$.

1981 Lipschitz domains: Jerison and Kenig proved $L^2$ solvability of the Regularity problem using the so-called "Rellich inequality" $\|\partial_\nu u\|_{L^2(\partial\Omega)} \approx \|\nabla_t u\|_{L^2(\partial\Omega)}$.

1984 Lipschitz domains: Verchota showed $L^p$-solvability of the regularity problem for $1 < p \leq 2$ by showing invertibility of the single layer potentials.

1987 Lipschitz domains: Dahlberg and Kenig showed $L^p$-solvability of the regularity problem for $1 < p < 2 + \varepsilon$. Invertibility of layer potentials at the endpoint spaces as well (Hardy and BMO).

2010 $\varepsilon$-regular SKT domains: Hofmann, Mitrea, and Taylor showed for each fixed $p$, there exists $\varepsilon$ such that the regularity problem in $L^p$ is solvable.

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# Question of C. E. Kenig in 1991

## Question 1

*If* $\Omega \subset \mathbb{R}^{n+1}$ *is a chord-arc domain with connected boundary, is there* $p > 1$ *such that the regularity problem is solvable for the Laplacian?*

Chord-arc domains:

- $\partial\Omega$ is $n$-Ahlfors regular, i.e., $\sigma(B(\xi, r)) \approx r^n$, for every $\xi \in \partial\Omega$ and $r \in (0, 2\,\mathrm{diam}\,\partial\Omega)$.
- $\Omega$ and $\mathbb{R}^{n+1} \setminus \Omega$ have the corkscrew condition (quantitative, scale-invariant openness condition).
- $\Omega$ has the Harnack-chain condition (quantitative, scale-invariant connectivity condition).

# Question of C. E. Kenig was solved in greater generality.

## Theorem 2 (M.-Tolsa (2021))

*Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded corkscrew domain with $n$-Ahlfors regular boundary. Then*

*Solvability of $(\mathrm{D}_{p'}^{\Delta})$ on $\Omega$*

*implies*

*Solvability of $(\mathrm{R}_{p}^{\Delta})$ on $\Omega$,*

*where $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$. We use uniform rectifiability.*

## Theorem 3 (M.-Tolsa (2021))

*Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a corkscrew domain with $n$-Ahlfors regular boundary. Then, for every uniformly elliptic operator $L = -\operatorname{div} A\nabla$,*

*Solvability of $(\mathrm{R}_{p}^{L})$ on $\Omega$*

*implies*

*Solvability of $(\mathrm{D}_{p'}^{L})$.*

# The Hajłasz Sobolev Space $\dot{M}^{1,p}(\Sigma)$

Let $(\Sigma, \sigma)$ be a doubling metric space. For $f : \Sigma \to \mathbb{R}$, say $0 \leq g : \Sigma \to \mathbb{R}$ is a Hajłasz upper gradient of $f$ ($g \in D(f)$) if

$$|f(x) - f(y)| \leq |x - y|(g(x) + g(y)), \qquad \text{for } \sigma-\text{a.e. } x, y \in \Sigma.$$

$$\dot{M}^{1,p}(\Sigma) := \{f : f \text{ has a Hajłasz upper gradient in } L^p(\Sigma)\}.$$

$$\|f\|_{\dot{M}^{1,p}(\Sigma)} := \inf_{g \in D(f)} \|g\|_{L^p(\Sigma)}.$$

Denote $\nabla_{H,p} f$ the function $g$ that attains the infimum.

If $\Sigma := \partial\Omega$ satisfies the weak-$(1,p)$-Poincaré inequality, then

$$\|f\|_{\dot{W}^{1,p}(\partial\Omega)} \approx \|\nabla_t f\|_{L^p(\partial\Omega)}, \qquad \text{for each } f \in \mathrm{Lip}(\partial\Omega).$$

# Carleson measures

A signed Radon measure $\nu$ on $\overline{\Omega}$ is a Carleson measure if there exists $C > 0$ so that for any $\xi \in \partial\Omega$ and $r > 0$, we have that

$$\|\mathscr{C}(\nu)\|_{L^\infty(\partial\Omega)} \leq C.$$

We let $\|\nu\|_{\mathscr{C}}$ be the best possible constant $C$ in the inequality above.
Carleson's Theorem [Carleson, 1958]: If $\Omega = \mathbb{R}^{n+1}_+$, then for all $w \in L^\infty_{\text{loc}}(\Omega)$,

$$\int_\Omega |w| \, d\nu \lesssim \|\nu\|_{\mathscr{C}} \|\mathcal{N}(w)\|_{L^1(\partial\Omega)}.$$

So there is some $L^1 - L^\infty$ duality between $\mathcal{N}$ and $\mathscr{C}$.

## Question 4

*What would be the correct duality if we need to control* $\|\widetilde{\mathcal{N}}_q(w)\|_{L^p(\partial\Omega)}$, $q, p > 1$?

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# Generalizing Carleson's Theorem

If $d\mu = H\, dm$, $0 \leq H \in L^1_{\text{loc}}(\Omega)$, $m$ the Lebesgue measure on $\Omega$, then we define

$$\mathscr{C}(H)(\xi) := \sup_{r>0} \frac{1}{r^n} \int_{B(\xi,r)\cap\Omega} H\, dm.$$

The $\mathcal{N}$-$\mathscr{C}$ duality for $p > 1$ has been studied in $\mathbb{R}^{n+1}_+$ in [Coifman-Meyer-Stein, 1985], [Álvarez-Milman, 1987], and [Hytönen-Rosén, 2013].

Let $q \geq 1$ and $q'$ its Hölder conjugate. Define the $q'$−Carleson function of $H : \Omega \to \mathbb{R}^n, H \in L^{q'}_{\text{loc}}(\Omega)$ by

$$\mathscr{C}_{q'}(H)(\xi) := \sup_{r>0} \frac{1}{r^n} \int_{B(\xi,r)\cap\Omega} \left( ⨍_{B(x,\delta(x)/4)} |H(y)|^{q'}\, dy \right)^{\frac{1}{q'}} dx, \qquad \xi \in \partial\Omega.$$

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# Duality between $\mathcal{N}$ and $\mathscr{C}$

$$\widetilde{\mathcal{N}}_q(w)(\xi) := \sup_{x \in \gamma(\xi)} \left( \fint_{B(x,\delta(x)/4)} |w(y)|^q \, dy \right)^{1/q}, \qquad \xi \in \partial\Omega.$$

$$\mathscr{C}_{q'}(H)(\xi) := \sup_{r>0} \frac{1}{r^n} \int_{B(\xi,r)\cap\Omega} \left( \fint_{B(x,\delta(x)/4)} |H(y)|^{q'} \, dy \right)^{\frac{1}{q'}} dx, \qquad \xi \in \partial\Omega.$$

## Theorem 5 (Hytönen-Rosén ($\mathbb{R}^{n+1}_+$), M.-Poggi-Tolsa)

*Let $n \geq 1$, $\Omega \subset \mathbb{R}^{n+1}$ is a corkscrew domain with $n$-Ahlfors regular boundary. Suppose that either $\Omega$ is bounded, or that $\partial\Omega$ is unbounded. Let $p, q \in (1, \infty)$ and $p'$, $q'$ their Hölder conjugates. Then*

$$\int_\Omega |wH| \, dm \lesssim \|\mathscr{C}_{q'}(H)\|_{L^{p'}(\partial\Omega)} \|\widetilde{\mathcal{N}}_q(w)\|_{L^p(\partial\Omega)}, \quad w \in L^q_{\text{loc}}(\Omega), H \in L^{q'}_{\text{loc}}(\Omega),$$

$$\|\widetilde{\mathcal{N}}_q(w)\|_{L^p(\partial\Omega)} \lesssim \sup_{H : \|\mathscr{C}_{q'}(H)\|_{L^{p'}(\partial\Omega)} = 1} \left| \int_\Omega H w \, dm \right|, \quad w \in L^q_{\text{loc}}(\Omega),$$

Let $p > 1$, $L = -\operatorname{div} A\nabla$, $A$ a strongly elliptic, bounded matrix. We say that $(\mathrm{R}_p^L)$ is solvable if there exists $C > 0$ such that for each $f \in \mathrm{Lip}(\partial\Omega)$, there exists a solution to the problem

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega, \end{cases}$$

with

$$\|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{\dot{M}^{1,p}(\partial\Omega)}.$$

Recall $\delta = \mathrm{dist}(\cdot, \partial\Omega)$. $A$ is a DKP matrix if

$$\sup_{y \in B(x,\delta(x)/2)} \big(|\nabla A(y)|^2 \delta(y)\big)\, dx \text{ is a Carleson measure.}$$

$(\mathrm{D}_{p'})$ studied for DKP operators in [Kenig-Pipher, 2001].

# The connections between $(\mathrm{D}_{p'})$ and $(\mathrm{R}_p)$ for $L = -\operatorname{div} A\nabla$ on rough domains

[Dindoš-Pipher-Rule, 2017]:$(\mathrm{D}_{p'}^{L^*}) \implies (\mathrm{R}_p^L)$ if $\delta|\nabla A|^2$ is Carleson measure with small norm, and $\Omega$ is a bounded Lipschitz domain with small Lipschitz constant.

# $(\mathrm{D}_{p'}^{L^*}) \implies (\mathrm{R}_p^L)$ for DKP matrices on domains with unif. rect. boundaries

Recall $\delta = \mathrm{dist}(\cdot, \partial\Omega)$. $A$ is a DKP matrix if

$$\sup_{y \in B(x,\delta(x)/2)} \left(|\nabla A(y)|^2 \delta(y)\right) dx \text{ is a Carleson measure.}$$

## Theorem 6 (M.-Poggi-Tolsa, 2022)

*Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded Corkscrew domain with uniformly $n$-rectifiable boundary. Let $L = -\operatorname{div} A\nabla$, where $A$ is strongly elliptic, bounded, DKP matrix. Then*

*Solvability of $(\mathrm{D}_{p'}^{L^*})$ on $\Omega$*

*implies*

*Solvability of $(\mathrm{R}_p^L)$ on $\Omega$,*

*where $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$, and $L^* = -\operatorname{div} A^*\nabla$.*

# Related results

The problem was open even in the

Simultaneously and independently, M. Dindos, S. Hofmann, and J. Pipher showed the same result in Lipschitz graph domains using a different method (which cannot be generalized to more general domains).

J. Feneuil recently gave an alternative (and simpler) proof of [DHP] slightly improving the assumption on the matrix (weak-DKP condition).

# First reduction

To solve $(R_p^L)$ with data $f \in \mathrm{Lip}_c(\partial\Omega)$, we first let $u$ be the solution of the continuous Dirichlet problem for $L$ with data $f$. Recall that

$$\|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \sup_{F:\|\mathscr{C}_2(F)\|_{L^{p'}(\partial\Omega)}=1} \left| \int_\Omega \nabla u \cdot F \, dm \right|, \quad \nabla u \in L^2_{\mathrm{loc}}(\Omega),$$

If there exists a function $v \in W_0^{1,2}(\Omega)$ such that $L^* v = -\operatorname{div} F$ (weakly) then

$$\int_\Omega \nabla u \cdot F = \int_\Omega \nabla u \, A^* \nabla v - \langle \partial_{\nu_{A^*}} v, f \rangle = -\langle \partial_{\nu_{A^*}} v, f \rangle.$$

For the moment, consider $\partial_{\nu_A} u = \nu \cdot A \nabla u$ (eventually it will be the variational co-normal).

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$$\int_\Omega \nabla u \cdot F = \int_\Omega \nabla u \, A^* \nabla v - \langle \partial_{\nu_{A^*}} v, f \rangle = -\langle \partial_{\nu_{A^*}} v, f \rangle.$$

For the moment, consider $\partial_{\nu_A} u = \nu \cdot A \nabla u$ (eventually it will be the variational co-normal).

# First reduction

To solve $(R_p^L)$ with data $f \in \mathrm{Lip}_c(\partial\Omega)$, we first let $u$ be the solution of the continuous Dirichlet problem for $L$ with data $f$. Recall that

$$\|\widetilde{\mathcal{N}}_2(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \sup_{F:\|\mathscr{C}_2(F)\|_{L^{p'}(\partial\Omega)}=1} \Big| \int_\Omega \nabla u \cdot F \, dm \Big|, \quad \nabla u \in L^2_{\mathrm{loc}}(\Omega),$$

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# $L^{p'}$-solvability of the Poisson problem

Let $n \geq 2$, $\Omega \subset \mathbb{R}^{n+1}$ be a Corkscrew domain with $n$-Ahlfors regular boundary. Let $p > 1$, $p'$ its Hölder conjugate, and $L = -\operatorname{div} A\nabla$, $A$ elliptic, bounded. Write $2^* := \frac{2(n+1)}{n-1}$, $2_* = (2^*)' = \frac{2(n+1)}{n+3}$. Recall $\delta = \operatorname{dist}(\cdot, \partial\Omega)$.

## Theorem 7 (M.-Poggi-Tolsa)

*Assume that $(\mathrm{D}^L_{p'})$ holds in $\Omega$. Then for any $H, F \in L^\infty_c(\Omega)$, the weak solution $w \in Y^{1,2}_0(\Omega)$ to the problem*

$$\begin{cases} -\operatorname{div} A\nabla w = H - \operatorname{div} F, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

*satisfies the estimate*

$$\|\widetilde{\mathcal{N}}_{2^*}(w)\|_{L^{p'}(\partial\Omega)} \lesssim \|\mathscr{C}_{2_*}(\delta H)\|_{L^{p'}(\partial\Omega)} + \|\mathscr{C}_2(F)\|_{L^{p'}(\partial\Omega)}.$$

*Moreover, if $A$ is a DKP matrix, $\Omega$ is bounded and $H = 0$, then for any $\varphi \in \dot{M}^{1,p}(\partial\Omega)$,*

$$\langle \partial_{\nu_A} w, \varphi \rangle \lesssim \|\mathscr{C}_2(F)\|_{L^{p'}(\partial\Omega)} \|\varphi\|_{\dot{M}^{1,p}(\partial\Omega)}.$$

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## Theorem 7 (M.-Poggi-Tolsa)

*Assume that* $(\mathrm{D}_{p'}^{L})$ *holds in* $\Omega$. *Then for any* $H, F \in L_c^\infty(\Omega)$, *the weak solution* $w \in Y_0^{1,2}(\Omega)$ *to the problem*

$$\begin{cases} -\operatorname{div} A\nabla w = H - \operatorname{div} F, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

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*Moreover, if* $A$ *is a* DKP matrix, $\Omega$ *is bounded and* $H = 0$, *then for any* $\varphi \in \dot{M}^{1,p}(\partial\Omega)$,

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# The "Poisson-Dirichlet" and "Poisson-regularity" problems

We say that $(\mathrm{PD}_{p'}^{L})$ is solvable if there exists $C > 0$ so that for each $F, H \in L_c^\infty(\Omega)$, the unique weak solution $w \in Y_0^{1,2}(\Omega)$ to

$$\begin{cases} -\operatorname{div} A\nabla w = -\operatorname{div} F + H, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

satisfies the estimate

$$\|\widetilde{\mathcal{N}}_{2^*}(w)\|_{L^{p'}(\partial\Omega)} \leq C\|\mathscr{C}_2(F)\|_{L^{p'}(\partial\Omega)} + \|\mathscr{C}_{2_*}(\delta\, H)\|_{L^p(\partial\Omega)}.$$

We say that $(\mathrm{PR}_{p}^{L})$ is solvable if there exists $C > 0$ so that for each $F, H \in L_c^\infty(\Omega)$, the unique weak solution $v \in Y_0^{1,2}(\Omega)$ to

$$\begin{cases} -\operatorname{div} A\nabla v = -\operatorname{div} F + H, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

satisfies the estimate

$$\|\widetilde{\mathcal{N}}_2(\nabla v)\|_{L^p(\partial\Omega)} \leq C\|\mathscr{C}_2(F/\delta)\|_{L^{p'}(\partial\Omega)} + \|\mathscr{C}_{2_*}(H)\|_{L^p(\partial\Omega)}.$$

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# Characterizations of $(\mathrm{D}_{p'}^{L})$

## Theorem 8 (M., Poggi, Tolsa)

*Let $\Omega \subsetneq \mathbb{R}^{n+1}$, $n \geq 2$ be a domain satisfying the corkscrew condition and with $n$-Ahlfors regular boundary, such that either $\Omega$ is bounded, or $\partial\Omega$ is unbounded. Let $p \in (1, \infty)$, $p'$ its Hölder conjugate, and $L = -\operatorname{div} A\nabla$. The following are equivalent.*

- *(a)* $(\mathrm{D}_{p'}^{L})$ *is solvable in* $\Omega$.
- *(b)* $(\mathrm{PD}_{p'}^{L})$ *is solvable in* $\Omega$.
- *(c)* $(\mathrm{PD}_{p'}^{L})$ *is solvable in* $\Omega$ *for* $H \equiv 0$.
- *(d)* $(\mathrm{PR}_{p}^{L^*})$ *is solvable in* $\Omega$.
- *(e)* $(\mathrm{PR}_{p}^{L^*})$ *is solvable in* $\Omega$ *for* $F \equiv 0$.

# Comparison with the case that $L$ is the Laplacian

When $L = -\Delta$ we proved the 1-sided Rellich inequality for the solution of the continuous Dirichlet problem $u_f$ with boundary data $f \in \mathrm{Lip}_c(\partial\Omega)$:

$$\|\partial_\nu u\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\dot{M}^{1,p}(\partial\Omega)}$$

and then used the representation of $u$ via the difference of single and the double layer potentials

$$u_f(x) = \mathcal{D}(u|_{\partial\Omega})(x) - \mathcal{S}(\partial_\nu u|_{\partial\Omega})(x) \quad \text{for all } x \in \Omega.$$

and use the boundedness of the layer potentials.

When $L = -\operatorname{div} A\nabla$ and $A$ is DKP matrix, we do NOT have layer potential bounds. We still need a 1-sided Rellich-type inequality but this time it is for the Poisson Dirichlet problem.

$$\langle \partial_{\nu_A} w, \varphi \rangle \lesssim \|\mathscr{C}_2(F)\|_{L^{p'}(\partial\Omega)} \|\varphi\|_{\dot{M}^{1,p}(\partial\Omega)}.$$

In both case the important tools are the following:

a) A Corona decomposition of the domain in bounded Lipschitz domains (with small Lip constant), used in the construction of
b) A suitable version of Varopoulos extension of $M^{1,p}(\partial\Omega)$ functions.

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# Reduction to the construction of the extension

Define the bilinear form associated to the equation $L_A w = -\operatorname{div} F$ by

$$B_A(w,\Phi) = \int_\Omega A\nabla w \cdot \nabla\Phi + \int_\Omega F\cdot\nabla\Phi,$$

where $\varphi \in \mathrm{Lip}(\partial\Omega)$ and $\Phi \in \mathrm{Lip}(\overline{\Omega})$ with $\Phi|_{\partial\Omega} = \varphi$. The variational co-normal of $w$ is defined by

$$\langle \partial_{\nu_A} w, \varphi\rangle := \ell_w(\varphi) = B_A(w,\Phi).$$

Construct an extension $v_\phi$ such that

(i) $v_\phi \in W^{1,2}(\Omega)\cap C(\overline{\Omega})$ such that $v_\varphi|_{\partial\Omega} = \varphi$.

(ii) $\|\mathcal{N}_2(\nabla v_\varphi)\|_{L^p(\partial\Omega)} \lesssim \|\varphi\|_{\dot W^{1,p}(\partial\Omega)}$

and notice that $\varphi - \Phi \in W^{1,2}_0(\Omega)$ (test function) and so

$$B(w,\Phi) = B(w,\varphi-\Phi) + B(w,v_\phi) = B(w,v_\phi).$$

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$$\langle \partial_{\nu_A} w, \varphi \rangle = B(w, v_\phi) = \int_\Omega A\nabla w \cdot \nabla v_\phi + \int_\Omega F \cdot \nabla v_\phi =: I_1 + I_2.$$

By dualilty,

$$|I_2| \lesssim \|\mathscr{C}(F)\|_{L^{p'}(\partial\Omega)} \|\widetilde{\mathcal{N}}_2(\nabla v_\varphi\|_{L^p(\partial\Omega)} \lesssim \|\mathscr{C}(F)\|_{L^{p'}(\partial\Omega)} \|\varphi\|_{\dot{W}^{1,p}(\partial\Omega)}.$$

The desired bound for the $I_1$ term is

$$|I_1| \lesssim \|\widetilde{\mathcal{N}}_{2^*}(w)\|_{L^{p'}(\partial\Omega)} \|\varphi\|_{\dot{W}^{1,p}(\partial\Omega)} \lesssim \|\mathscr{C}(F)\|_{L^{p'}(\partial\Omega)} \|\varphi\|_{\dot{W}^{1,p}(\partial\Omega)}.$$

It follows by delicate estimates that are in the same spirit with the ones that M. and Tolsa used to show that the extension in the paper for the Laplacian satisfies

$$\|\mathscr{C}(\Delta v_\varphi)\|_{L^{p'}(\partial\Omega)} \lesssim \|\varphi\|_{\dot{W}^{1,p}(\partial\Omega)}.$$

In other words, it is an estimate

$$\|\mathscr{C}_2(L v_\varphi)\|_{L^{p'}(\partial\Omega)} \lesssim \|\varphi\|_{\dot{W}^{1,p}(\partial\Omega)}.$$

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# Construction of the auxiliary extension

Given a ball $B \subset \mathbb{R}^{n+1}$ centered in $\partial\Omega$ and an affine map $A : \mathbb{R}^{n+1} \to \mathbb{R}$, we consider the coefficient

$$\gamma_f(B) := \inf_A \left( |\nabla A| + \frac{1}{\sigma(B)} \int_B \frac{|f - A|}{r(B)} \, d\sigma \right),$$

where the infimum is taken over all affine maps $A : \mathbb{R}^{n+1} \to \mathbb{R}$.

Denote by $A_B$ the minimizer.

For every ball $B$ centered in $\partial\Omega$ with $r(B) \leq \operatorname{diam}(\Omega)$,

$$|\nabla A_B| \lesssim m_{B,\sigma}(\nabla_{H,p} f) := \sigma(B)^{-1} \int_B |\nabla_{H,p} f)| \, d\sigma. \tag{1}$$

If $B, B'$ are balls centered in $\partial\Omega$ such that $B \subset B'$ with $r(B) \approx r(B') \leq \operatorname{diam}(\Omega)$, then

$$|A_B(x) - A_{B'}(x)| \lesssim m_{B',\sigma}(\nabla_{H,p} f) \big( r(B) + \operatorname{dist}(x, B) \big). \tag{2}$$

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If $\varphi_P$ is a partition of unity subordinate to the Whitney decomposition of $\Omega$ in dyadic cubes $P \in \mathcal{W}(\Omega)$, then we define

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and $F = f$ on $\partial\Omega$.

We can show that:

(i) $F \in \mathrm{Lip}(\overline{\Omega})$ with $\mathrm{Lip}(F) \lesssim \mathrm{Lip}(f)$.

(ii) If $P_0$ is a Whitney cube in $\Omega$ and $b(P_0)$ the associated boundary cube s.t. $\ell(B(P_0)) \approx \ell(P_0)$ and $\mathrm{dist}(P_0, \partial\Omega) \approx \mathrm{dist}(P_0, b(P_0))$, then

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# Corona decomposition of $\Omega$

$$\Omega = \bigcup_{R \in \mathsf{Top}} \Omega_R \cup H$$

where

(i) $\Omega_R$ is a starlike Lipschitz domain, with sufficiently small constant in which $A$ is a DKP operator with sufficiently small constant.

(ii) $\overline{\Omega}_R \cap \overline{\Omega}_{R'} = \emptyset$,

(iii) for $\mathcal{H}^n$-a.e. $x \in \partial\Omega$, there exists a unique $\Omega_R$ such that $x \in \partial\Omega_R$,

(iv) Top has Carleson paacking condition, i.e., for every $S \in \mathcal{D}_\sigma$,

$$\sum_{R \in \mathsf{Top}: R \subset S} \sigma(R) \leq C\sigma(S)$$

(v) $H$ is a "buffer" region in the sense that

$$\sum_{P \in \mathcal{W}(\Omega) \cap H : b(P) \subset S} \sigma(b(P)) \leq C\sigma(S)$$

# Corona decomposition of $\Omega$

$$\Omega = \bigcup_{R \in \mathsf{Top}} \Omega_R \cup H$$

where

(i) $\Omega_R$ is a starlike Lipschitz domain, with sufficiently small constant in which $A$ is a DKP operator with sufficiently small constant.

(ii) $\overline{\Omega}_R \cap \overline{\Omega}_{R'} = \emptyset$,

(iii) for $\mathcal{H}^n$-a.e. $x \in \partial\Omega$, there exists a unique $\Omega_R$ such that $x \in \partial\Omega_R$,

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# Construction of the almost elliptic extension

Solve $(R_p)$ in each starlike Lipschitz domain $\Omega_j$ with boundary data $f_j := F|_{\partial\Omega_j} \in \mathrm{Lip}(\partial\Omega_j)$ producing $u_{f_j}$ extensions to $\Omega_j$ such that $Lu_{f_j} = 0$ in $\Omega_j$ and, for any $q \in (1, \infty)$,

$$\|\widetilde{\mathcal{N}}_{2,\Omega_j}(\nabla u_{f_j})\|_{L^q(\partial\Omega_j)} \lesssim \|\nabla_t f_j\|_{L^q(\partial\Omega_j)}.$$

The DESIRED EXTENSION is

$$v(x) = \begin{cases} u_{f_j}(x) & \text{, if } x \in \Omega_j \text{ for some } j \geq 1 \\ F(x) & \text{, if } x \in H. \end{cases}$$

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# Varopoulos extension of $L^p(\partial\Omega)$ functions

If $\Omega \subset \mathbb{R}^{n+1}$ is a corkscrew domain with $n$-Ahlfors regular boundary, the following holds:

## Theorem 9 (M.-Zacharopoulos)

*If $f \in \mathrm{Lip}_c(\partial\Omega)$, there exists a function $F : \overline{\Omega} \to \mathbb{R}$ such that*

*(i)* $F \in C^\infty(\Omega) \cap \mathrm{Lip}(\overline{\Omega})$,

*(ii)* $\|\mathscr{C}_s(\nabla F)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}$ *and* $\|\mathcal{N}(F)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}$, *for* $p \in (1, \infty]$,

*(iii)* $F|_{\partial\Omega} = f$ *continuously.*

Thank you for your attention!