Randomized isoperimetric inequalities

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Abstract

We discuss isoperimetric inequalities for convex sets. These include the classical isoperimetric inequality and that of Brunn-Minkowski, Blaschke-Santaló, Busemann-Petty and their various extensions. We show that many such inequalities admit stronger randomized forms in the following sense: for natural families of associated random convex sets one has stochastic dominance for various functionals such as volume, surface area, mean width and others. By laws of large numbers, these randomized versions recover the classical inequalities. We give an overview of when such stochastic dominance arises and its applications in convex geometry and probability.

1 Introduction

The focus of this paper is stochastic forms of isoperimetric inequalities for convex sets. To set the stage, we begin with two examples. Among the most fundamental isoperimetric inequalities is the Brunn-Minkowski inequality for the volume V_n of convex bodies $K, L \subseteq \mathbb{R}^n$,

$$V_n(K+L)^{1/n} \geqslant V_n(K)^{1/n} + V_n(L)^{1/n}, \tag{1.1}$$

where $K+L$ is the Minkowski sum $\{x+y : x \in K, y \in L\}$. The Brunn-Minkowski inequality is the cornerstone of the Brunn-Minkowski theory and its reach extends well beyond convex geometry; see Schneider's monograph [68] and Gardner's survey [27]. It is well-known that (1.1) provides a direct route to the classical isoperimetric inequality relating surface area S and volume,

$$\left(\frac{S(K)}{S(B)}\right)^{1/(n-1)} \geqslant \left(\frac{V_n(K)}{V_n(B)}\right)^{1/n}, \tag{1.2}$$

where B is the Euclidean unit ball. As equality holds in (1.1) if K and L are homothetic, it can be equivalently stated in isoperimetric form as follows:

$$V_n(K+L) \geqslant V_n(r_K B + r_L B), \tag{1.3}$$

where r_K, r_L denote the radii of Euclidean balls with the same volume as K, L, respectively, i.e., $r_K = (V_n(K)/V_n(B))^{1/n}$; for subsequent reference, with this notation, (1.2) reads

$$S(K) \geqslant S(r_K B). \tag{1.4}$$

Both (1.1) and (1.2) admit stronger empirical versions associated with random convex sets. Specifically, let $x_1, \ldots, x_N$ be independent random vectors (on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$) distributed according to the uniform density on a convex body $K \subseteq \mathbb{R}^n$, say, $f_K = \frac{1}{V_n(K)} \mathbb{1}_K$, i.e., $\mathbb{P}(x_i \in A) = \int_A f_K(x)dx$ for Borel sets $A \subseteq \mathbb{R}^n$. For each such K and $N > n$, we associate a random polytope

$$K_N = \operatorname{conv}\{x_1, \ldots, x_N\},$$

where conv denotes convex hull. Then the following stochastic dominance holds for the random polytopes K_{N_1}, L_{N_2} and $(r_K B)_{N_1}$, $(r_L B)_{N_2}$ associated with the bodies in (1.3): for all $\alpha \geqslant 0$,

$$\mathbb{P}\left(V_n(K_{N_1} + L_{N_2}) > \alpha\right) \geqslant \mathbb{P}\left(V_n((r_K B)_{N_1} + (r_L B)_{N_2}) > \alpha\right). \tag{1.5}$$

Integrating in α gives

$$\mathbb{E}V_n(K_{N_1} + L_{N_2}) \geqslant \mathbb{E}V_n((r_K B)_{N_1} + (r_L B)_{N_2}),$$

where $\mathbb{E}$ denotes expectation. By the law of large numbers, when $N_1, N_2 \to \infty$, the latter convex hulls converge to their ambient bodies and this leads to (1.3). Thus (1.1) is a global inequality which can be proved by a random approximation procedure in which stochastic dominance holds at each stage; for a different stochastic form of (1.1), see Vitale's work [72]. For the classical isoperimetric inequality, one has the following distributional inequality, for $\alpha \geqslant 0$,

$$\mathbb{P}\left(S(K_{N_1}) > \alpha\right) \geqslant \mathbb{P}\left(S((r_K B)_{N_1}) > \alpha\right). \tag{1.6}$$

The same integration and limiting procedure lead to (1.4). For fixed N_1 and N_2, the sets in the extremizing probabilities on the right-hand sides of (1.5) and (1.6) are not Euclidean balls, but rather sets that one generates using Euclidean balls. In particular, the stochastic forms are strictly stronger than the global inequalities (1.1) and (1.2).

The goal of this paper is to give an overview of related stochastic forms of isoperimetric inequalities. Both (1.1) and (1.2) hold for non-convex sets but we focus on stochastic dominance associated with convex sets. The underlying randomness, however, will not be limited to uniform distributions on convex bodies but will involve continuous distributions on $\mathbb{R}^n$. We will discuss a streamlined approach that yields stochastic dominance in a variety of inequalities in convex geometry and their applications. We pay particular attention to high-dimensional probability distributions and associated structures, e.g., random convex sets and matrices. Many of the results we discuss are from a series of papers [56], [57], along with D. Cordero-Erausquin, M. Fradelizi [24], S. Dann [25] and G. Livshyts [44]. We also present a few new results that fit in this framework and have not appeared previously.

Inequalities for the volume of random convex hulls in stochastic geometry have a rich history starting with Blaschke's resolution of Sylvester's famous four-point problem in the plane (see, e.g., [61], [18], [20], [28] for background and history). In particular, for planar convex bodies Blaschke proved that the random triangle K_3 (notation as above) satisfies

$$\mathbb{E}V_2(\Delta_3) \geqslant \mathbb{E}V_2(K_3) \geqslant \mathbb{E}V_2((r_K B_2)_3), \tag{1.7}$$

where Δ is a triangle in $\mathbb{R}^2$ with the same area as K and B_2 is the unit disk. Blaschke's proof of the lower bound draws on Steiner symmetrization, which is the basis for many related extremal inequalities, see, e.g,. [68], [28], [34]. More generally, shadow systems as put forth by Rogers and Shephard [69], [63] and developed by Campi and Gronchi, among others, play a fundamental role, e.g., [18], [21], [22], and will be defined and discussed further below. Finding maximizers in (1.7) for $n \geqslant 3$ has proved more difficult and is connected to the slicing problem, which we will not discuss here (see [13] for background).

A seminal result building on the lower bound in (1.7) is Busemann's random simplex inequality [16], [17]: for a convex body $K \subseteq \mathbb{R}^n$ and $p \geqslant 1$, the set $K_{o,n} = \mathrm{conv}\{o, x_1, \ldots, x_n\}$ (x_i's as above) satisfies

$$\mathbb{E}V_n(K_{o,n})^p \geqslant \mathbb{E}V_n((r_K B)_{o,n})^p. \tag{1.8}$$

This is a key ingredient in Busemann's intersection inequality,

$$\int_{S^{n-1}} V_{n-1}(K \cap \theta^\perp)^n d\sigma(\theta) \leq \int_{S^{n-1}} V_{n-1}((r_K B) \cap \theta^\perp)^n d\sigma(\theta), \tag{1.9}$$

where S^{n-1} is the unit sphere equipped with the Haar probability measure σ; (1.8) is also the basis for extending (1.9) to lower dimensional secitons as proved by Busemann and Straus [17] and Grinberg [32].

Inextricably linked to Busemann's random simplex inequality is the Busemann-Petty centroid inequality, proved by Petty [59]. The centroid body of a star body $K \subseteq \mathbb{R}^n$ is the convex body $Z(K)$ with support function given by

$$h(Z(K), y) = \frac{1}{V_n(K)} \int_K |\langle x, y \rangle| \, dx;$$

(star bodies and support functions are defined in §2) and it satisfies

$$V_n(Z(K)) \geqslant V_n((r_K B)).$$

The latter occupies a special role in the theory of affine isoperimetric inequalities; see Lutwak's survey [46].

One can view (1.8) as a result about convex hulls or about the random parallelotope $\sum_{i=1}^{n}[-x_i, x_i]$ (since $n! V_n(K_{o,n}) = |\det[x_1, \ldots, x_n]|$). Both viewpoints generalize: for convex hulls K_N with $N > n$, this was done by Groemer [33] and for Minkowski sums of $N \geqslant n$ random line segments by Bourgain, Meyer, Milman and Pajor [11]; these are stated in §5, where we discuss various extensions for different functionals and underlying randomness. These are the starting point for a systematic study of many related quantities.

In particular, convex hulls and zonotopes are natural endpoint families of sets in L_p-Brunn-Minkowski theory and its recent extensions. In the last twenty years, this area has seen significant developments. L_p analogues of centroid bodies are important for affine isoperimetric inequalities, e.g., [47], [48], [35] and are fundamental in concentration of volume in convex bodies, e.g,. [41], [42]. The L_p-version of the Busemann-Petty centroid inequality, due to Lutwak, Yang and Zhang [47], concerns the convex body $Z_p(K)$ defined by its support function

$$h^p(Z_p(K), y) = \frac{1}{V_n(K)} \int_K |\langle x, y\rangle|^p \, dx \tag{1.10}$$

and states that

$$V_n(Z_p(K)) \geqslant V_n(Z_p(r_K B)). \tag{1.11}$$

A precursor to (1.11) is due to Lutwak and Zhang [52] who proved that when K is origin-symmetric,

$$V_n(Z_p(K)^\circ) \leqslant V_n(Z_p(r_K B)^\circ). \tag{1.12}$$

When $p \to \infty$, $Z_p(K)$ converges to $Z_\infty(K) = K$ and (1.12) recovers the classical Blaschke-Santaló inequality [65],

$$V_n(K^\circ) \leqslant V_n((r_K B)^\circ). \tag{1.13}$$

The latter holds more generally for non-symmetric bodies with an appropriate choice of center. The analogue of (1.12) in the non-symmetric case was proved by Haberl and Schuster [35], to which we refer for further references and background on L_p-Brunn-Minkowski theory.

Inequalities (1.11) and (1.12) are fundamental inequalities in the L_p Brunn-Minkowski theory. Recently, such inequalities have been placed in a general framework involving Orlicz functions by Lutwak, Yang, and Zhang, e.g., [49], [50] and a closely related concept, due to Gardner, Hug and Weil [30], [29], termed M-addition, which we discuss in §5; for further extensions and background, see [10]. We treat stochastic forms of fundamental related inequalities. For example, we show that in (1.5) one can replace Minkowski addition by M-addition. With the help of laws of large numbers, this leads to a streamlined approach to many such inequalities.

The notion of M-addition fits perfectly with the random linear operator point of view which we have used in our work on this topic [56], [57]. For random vectors $x_1, \ldots, x_N$, we form the $n \times N$ random matrix $[x_1, \ldots, x_N]$ and view it as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$. If $C \subseteq \mathbb{R}^N$, then

$$[x_1, \ldots, x_N]C = \left\{ \sum_{i=1}^{N} c_i x_i : c = (c_i) \in C \right\}.$$

In particular, if $C = \text{conv}\{e_1, \ldots, e_N\}$, where $e_1, \ldots, e_N$ is the standard unit vector basis for $\mathbb{R}^N$, then

$$[x_1, \ldots, x_N]\text{conv}\{e_1, \ldots, e_N\} = \text{conv}\{x_1, \ldots, x_N\}.$$

Let B_p^N denote the closed unit ball in ℓ_p^N. If $C = B_1^N$, then

$$[x_1, \ldots, x_N]B_1^N = \text{conv}\{\pm x_1, \ldots, \pm x_N\}.$$

If $C = B_\infty^N$, then one obtains Minkowski sums,

$$[x_1, \ldots, x_N]B_\infty^N = \sum_{i=1}^{N} [-x_i, x_i].$$

We define the empirical analogue $Z_{p,N}(K)$ of the L_p-centroid body $Z_p(K)$ by its (random) support function

$$h^p(Z_{p,N}(K), y) = \frac{1}{N} \sum_{i=1}^{N} |\langle x_i, y \rangle|^p, \tag{1.14}$$

where $x_1, \ldots, x_N$ are independent random vectors with density $\frac{1}{V_n(K)} \mathbb{1}_K$; this can be compared with (1.10); in the matrix notation $Z_{p,N}(K) = N^{-1/p}[x_1, \ldots, x_N]B_q^N$, where $1/p + 1/q = 1$. In this framework, we will explain how uniform measures on Cartesian products of Euclidean balls arise as extremizers for

$$\mathbb{P}(\phi([X_1, \ldots, X_N]C) > \alpha) \tag{1.15}$$

and

$$\mathbb{P}(\phi(([X_1, \ldots, X_N]C)^\circ) > \alpha); \tag{1.16}$$

over the class of independent random vectors X_i with continuous distributions on $\mathbb{R}^n$ having bounded densities; here $C \subseteq \mathbb{R}^N$ is a compact convex set (sometimes with some additional symmetry assumptions) and ϕ an appropriate functional, e.g., volume, surface area, mean width, diameter, among others. Since the random sets in the extremizing probabilities are not typically balls but sets one generates using balls, there is no clear cut path to reduce distributional inequalities for (1.15) and (1.16) from one another via duality; for comparison, note that the Lutwak-Yang-Zhang inequality for L_p centroid bodies (1.11) implies the Lutwak-Zhang result for their polars (1.12) by the Blaschke-Santaló inequality since the extremizers in each case are balls (or ellipsoids).

The random operator approach allows one to interpolate between inequalities for families of convex sets, but such inequalities in turn yield information about random operators. For example, recall the classical Bieberbach inequality on the diameter of a convex body $K \subseteq \mathbb{R}^n$,

$$\mathrm{diam}(K) \geqslant \mathrm{diam}(r_K B). \tag{1.17}$$

A corresponding empirical form is given by

$$\mathbb{P}(\mathrm{diam}(K_N) > \alpha) \geqslant \mathbb{P}(\mathrm{diam}((r_K B)_N) > \alpha). \tag{1.18}$$

The latter identifies the extremizers of the distribution of certain operators norms. Indeed, if K is an origin-symmetric convex body and we set $K_{N,s} = \mathrm{conv}\{\pm x_1, \dots, \pm x_N\}$ $(x_i \in \mathbb{R}^n)$ then (1.18) still holds and we have the following for the $\ell_1^N \to \ell_2^n$ operator norm,

$$\mathrm{diam}(K_{N,s}) = 2 \left\| [x_1, \dots, x_N] : \ell_1^N \to \ell_2^n \right\|.$$

We show in §6 that if $\mathbf{X} = [X_1, \dots, X_N]$, where the X_i's are independent random vectors in $\mathbb{R}^n$ and have densities bounded by one, say, then for any N-dimensional normed space E, the quantity

$$\mathbb{P}\left(\|[X_1, \dots, X_N] : E \to \ell_2^n\| > \alpha\right)$$

is minimized when the columns X_i are distributed uniformly in the Euclidean ball $\widetilde{B}$ of volume one, centered at the origin. This can be viewed as an operator analogue of the Bieberbach inequality (1.17). When $n = 1$, $\mathbf{X}$ is simply a $1 \times N$ row vector and the latter extends to semi-norms. Thus if F is a subspace of $\mathbb{R}^n$, we get the following for random vectors $x \in \mathbb{R}^N$ with independent coordinates with densities bounded by one: the probability

$$\mathbb{P}(\|P_F x\|_2 > \alpha) \tag{1.19}$$

is minimized when x is sampled in the unit cube $[-1/2, 1/2]^N$ - products of "balls" in one dimension (here $\|\cdot\|_2$ is the Euclidean norm and

P_F is the orthogonal projection onto F). Combining (1.19) with a seminal result by Ball [3] on maximal volume sections of the cube, we obtain a new proof of a result of Rudelson and Vershynin [64] (which differs also from the proof in [44], our joint work G. Livshyts) on small ball probabilities of marginal densities of product measures; this is explained in §6.

As mentioned above, Busemann's original motivation for proving the random simplex inequality (1.8) was to bound suitable averages of volumes of central hyperplane sections of convex bodies (1.9). If $V_n(K) = 1$ and $\theta \in S^{n-1}$ then $V_{n-1}(K \cap \theta^{\perp})$ is the value of the marginal density of $\mathbb{1}_K$ on $[\theta] = \mathrm{span}\{\theta\}$ evaluated at 0, i.e. $\pi_{[\theta]}(\mathbb{1}_K)(0) = \int_{\theta^{\perp}} \mathbb{1}_K(x)dx$. Thus it is natural that marginal distributions of probability measures arise in this setting. One reason for placing Busemann-type inequalities in a probabilistic framework is that they lead to bounds for marginal distributions of random vectors not necessarily having independent coordinates, as in our joint work with S. Dann [25], which we discuss further in §5.

Lastly, we comment on some of the tools used to prove such inequalities. We make essential use of rearrangement inequalities such as that of Rogers [62], Brascamp, Lieb and Luttinger [12] and Christ [23]. These interface particularly well with Steiner symmetrization and shadow systems and other machinery from convex geometry. Another key ingredient is an inequality of Kanter [38] on stochastic dominance. In fact, we formulate the Rogers/Brascamp-Lieb-Luttinger inequality in terms of stochastic dominance using the notion of peaked measures as studied by Kanter [38] and Barthe [5], [6], among others. One can actually prove (1.19) directly using the Rogers/Brascamp-Lieb-Luttinger inequality and Kanter's theorem but we will show how these ingredients apply in a general framework for a variety of functionals. Similar techniques are used in proving analytic inequalities, e.g., for k-plane transform by Christ [23] and Baernstein and Loss [2]. Our focus is on phenomena in convex geometry and probability.

The paper is organized as follows. We start with definitions and background in §2. In §3, we discuss the rearrangement inequality of Rogers/Brascamp-Lieb-Luttinger and interpret it as a result about stochastic dominance for certain types of functions with a concavity property, called Steiner concavity, following Christ. In §4, we present examples of Steiner concave functions. In §5, we present general randomized inequalities. We conclude with applications to operators norms of random matrices and small deviations in §6.

2 Preliminaries

We work in Euclidean space $\mathbb{R}^n$ with the canonical inner-product $\langle \cdot, \cdot \rangle$ and Euclidean norm $\|\cdot\|_2$. As above, the unit Euclidean ball in $\mathbb{R}^n$ is $B = B_2^n$ and its volume is $\omega_n := |B_2^n|$; S^{n-1} is the unit sphere, equipped with the Haar probability measure σ. Let $G_{n,k}$ be the Grassmannian manifold of k-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,k}$.

A convex body $K \subseteq \mathbb{R}^n$ is a compact, convex set with non-empty interior. The set of all compact convex sets in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. For a convex body K we write $\widetilde{K}$ for the homothet of K of volume one; in particular, $\widetilde{B} = \omega_n^{-1/n} B$. Let $\mathcal{K}_\circ^n$ denote the class of all convex bodies that contain the origin in their interior. For $K, L \in \mathcal{K}^n$, the Minkowski sum $K + L$ is the set $\{x + y : x \in K, y \in L\}$; for $\alpha > 0$, $\alpha K = \{\alpha x : x \in K\}$. We say that K is origin-symmetric (or simply 'symmetric'), if $-x \in K$ whenever $x \in K$. For $K \in \mathcal{K}^n$, the support function of K is given by

$$h_K(x) = \sup\{\langle y, x \rangle \ : \ y \in K\} \quad (x \in \mathbb{R}^n).$$

The mean width of K is

$$w(K) = \int_{S^{n-1}} h_K(\theta) + h_K(-\theta) d\sigma(\theta) = 2 \int_{S^{n-1}} h_K(\theta) d\sigma(\theta).$$

Recall that the intrinsic volumes $V_1, \ldots, V_n$ are functionals on convex bodies which can be defined via the Steiner formula: for any convex body $K \subseteq \mathbb{R}^n$ and $\varepsilon > 0$,

$$V_n(K + \varepsilon B) = \sum_{j=0}^{n} \omega_{n-j} V_j(K) \varepsilon^{n-j};$$

here $V_0 \equiv 1$, V_1 is a multiple of the mean width, $2V_{n-1}$ is the surface area and V_n is the volume; see [68].

For compact sets C_1, C_2 in $\mathbb{R}^n$, we let $\delta^H(C_1, C_2)$ denote the Hausdorff distance:

$$\begin{aligned} \delta^H(C_1, C_2) &= \inf\{\varepsilon > 0 \ : \ C_1 \subseteq C_2 + \varepsilon B, C_2 \subseteq C_1 + \varepsilon B\} \\ &= \sup_{\theta \in S^{n-1}} |h_K(\theta) - h_L(\theta)|. \end{aligned}$$

A set $K \subseteq \mathbb{R}^n$ is star-shaped if it is compact, contains the origin in its interior and for every $x \in K$ and $\lambda \in [0, 1]$ we have $\lambda x \in K$. We call K a star-body if its radial function

$$\rho_K(\theta) = \sup\{t > 0 : t\theta \in K\} \quad (\theta \in S^{n-1})$$

is positive and continuous. Any positive continuous function $f : S^{n-1} \to \mathbb{R}$ determines a star body with radial function f.

Following Borell [8], [9], we say that a non-negative, non-identically zero, function ψ is γ-concave if: (i) for $\gamma > 0$, ϕ^γ is concave on $\{\psi > 0\}$, (ii) for $\gamma = 0$, $\log \psi$ is concave on $\{\psi > 0\}$; (iii) for $\gamma < 0$, ψ^γ is convex on $\{\psi > 0\}$. Let $s \in [-\infty, 1]$. A Borel measure μ on $\mathbb{R}^n$ is called s-concave if

$$\mu\left((1-\lambda)A + \lambda B\right) \geqslant \left((1-\lambda)\mu(A)^s + \lambda\mu(B)^s\right)^{\frac{1}{s}}$$

for all compact sets $A, B \subseteq \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$. For $s = 0$, one says that μ is log-concave and the inequality reads as

$$\mu\left((1-\lambda)A + \lambda B\right) \geqslant \mu(A)^{1-\lambda}\mu(B)^\lambda.$$

Also, for $s = -\infty$, the measure is called convex and the inequality is replaced by

$$\mu\left((1-\lambda)A + \lambda B\right) \geqslant \min\{\mu(A), \mu(B)\}.$$

An s-concave measure μ is always supported on some convex subset of an affine subspace E where it has a density. If μ is a measure on $\mathbb{R}^n$ absolutely continuous with respect to Lebesgue measure with density ψ, then it is s-concave if and only if its density ψ is γ-concave with $\gamma = \frac{s}{1-sn}$ (see [8], [9]).

Let A be a Borel subset of $\mathbb{R}^n$ with finite Lebesgue measure. The symmetric rearrangement A^* of A is the open ball with center at the origin, whose volume is equal to the measure of A. Since we choose A^* to be open, $\mathbf{1}_A^*$ is lower semicontinuous. The symmetric decreasing rearrangement of $\mathbf{1}_A$ is defined by $\mathbf{1}_A^* = \mathbf{1}_{A^*}$. We consider Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}_+$ which satisfy the following condition: for every $t > 0$, the set $\{x \in \mathbb{R}^n : f(x) > t\}$ has finite Lebesgue measure. In this case, we say that f vanishes at infinity. For such f, the symmetric decreasing rearrangement f^* is defined by

$$f^*(x) = \int_0^\infty \mathbb{1}_{\{f>t\}}(x)dt = \int_0^\infty \mathbb{1}_{\{f>t\}^*}(x)dt.$$

The latter should be compared with the "layer-cake representation" of f:

$$f(x) = \int_0^\infty 1_{\{f>t\}}(x)dt. \tag{2.1}$$

see [43, Theorem 1.13]. Note that the function f^* is radially-symmetric, decreasing and equimeasurable with f, i.e., $\{f > a\}$ and $\{f^* > a\}$ have the same volume for each $a > 0$. By equimeasurability one has that $\|f\|_p = \|f^*\|_p$ for each $1 \leqslant p \leqslant \infty$, where $\|\cdot\|_p$ denote the $L_p(\mathbb{R}^n)$-norm.

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function vanishing at infinity. For $\theta \in S^{n-1}$, we fix a coordinate system that $e_1 := \theta$. The Steiner symmetral $f(\cdot|\theta)$ of f with respect to $\theta^\perp := \{y \in \mathbb{R}^n : \langle y, \theta\rangle = 0\}$ is defined as follows: for $z := (x_2, \ldots, x_n) \in \theta^\perp$, we set $f_{z,\theta}(t) = f(t, x_2, \ldots, x_n)$ and define $f^*(t, x_2, \ldots, x_n|\theta) := (f_{z,\theta})^*(t)$. In other words, we obtain $f^*(\cdot|\theta)$ by rearranging f along every line parallel to θ. We will use the following fact, proved in [4]: If $g : \mathbb{R}^n \to \mathbb{R}_+$ is an integrable function with compact support, there exists a sequence of functions g_k, where $g_0 = g$ and $g_{k+1} = g_k^*(\cdot|\theta_k)$, for some $\theta_k \in S^{n-1}$, such that $\lim_{k\to\infty} \|g_k - g^*\|_1 = 0$. We refer the reader to the books [43], [71] or the introductory notes [14] for further background material on rearrangements of functions.

3 Inequalities for stochastic dominance

We start with a seminal inequality now known as the Rogers/Brascamp-Lieb-Luttinger inequality. It was observed by Madiman and Wang in [73] that Rogers proved the inequality in [62] but it is widely known as the Brascamp-Lieb-Luttinger inequality [12]. We will state it only for integrable functions since this is the focus in our paper.

Theorem 3.1. *Let $f_1, \ldots, f_M$ be non-negative integrable functions on $\mathbb{R}$ and $u_1, \ldots, u_M \in \mathbb{R}^N$. Then*

$$\int_{\mathbb{R}^N} \prod_{i=1}^{M} f_i(\langle x, u_i\rangle)dx \leqslant \int_{\mathbb{R}^N} \prod_{i=1}^{M} f_i^*(\langle x, u_i\rangle)dx. \tag{3.1}$$

We will write the above inequality in an equivalent form using the notion of peaked measures. The ideas behind this definition can be tracked back to Anderson [1] and Kanter [38], among others, but here we follow the terminology and notation of Barthe in [5], [6]. Let μ_1, μ_2 be finite Radon measures on $\mathbb{R}^n$ with $\mu_1(\mathbb{R}^n) = \mu_2(\mathbb{R}^n)$. We say that μ_1 is more peaked than μ_2 (and we write $\mu_1 \succ \mu_2$ or $\mu_2 \prec \mu_1$) if

$$\mu_1(K) \geqslant \mu_2(K) \tag{3.2}$$

for all symmetric convex bodies K in $\mathbb{R}^n$. If X_1, X_2 are random vectors in $\mathbb{R}^n$ with distributions μ_1 and μ_2, respectively, we write $X_1 \succ X_2$ if $\mu_1 \succ \mu_2$. Let f_1, f_2 two non-negative integrable functions on $\mathbb{R}^n$ with $\int f_1 = \int f_2$. We write $f_1 \succ f_2$ if the measures μ_i with densities f_i satisfy $\mu_1 \succ \mu_2$. It follows immediately from the definition that the relation $\succ$ is transitive. Moreover if $\mu_i \succ \nu_i$ and $t_i > 0$, $1 \leqslant i \leqslant N$ then $\sum_i t_i\mu_i \succ \sum_i t_i\nu_i$. Another consequence of the definition is that if $\mu \succ \nu$ and E is an k-dimensional subspace then the marginal of μ on E, i.e. $\mu \circ P_E^{-1}$, is more peaked than the marginal of ν on E.

To see this, take any symmetric convex body K in E and consider the infinite cylinder $C := K \times E^{\perp} \subseteq \mathbb{R}^n$. It is enough to check that $\mu(C) \geqslant \nu(C)$, and this is satisfied since C can be approximated from inside by symmetric convex bodies in $\mathbb{R}^n$. More generally, if $\mu \succ \nu$ then for every linear map T, we have

$$\mu \circ T \succ \nu \circ T, \tag{3.3}$$

where $\mu \circ T$ is the pushforward measure of μ through the map T.

Recall that $F : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave (quasi-convex) if for all s the set $\{x : F(x) > \alpha\}$ ($\{x : F(x) \leqslant s\}$) is convex.

Lemma 3.2. *Let μ_1, μ_2 be Radon measures on $\mathbb{R}^n$ and assume that $\mu_1 \succ \mu_2$. Then*

$$\int_{\mathbb{R}^n} F(x) d\mu_1(x) \geqslant \int_{\mathbb{R}^n} F(x) d\mu_2(x) \tag{3.4}$$

for all even non-negative quasi-concave functions F. In fact, the above statement is an equivalent formulation of the statement $\mu_1 \succ \mu_2$.

Proof. Assume first that F is even and quasi concave. Then by the layer-cake representation and Fubini's theorem we have that

$$\int_{\mathbb{R}^n} F(x) d\mu_1(x) = \int_0^\infty \int_{\{x:F(x)>s\}} d\mu_1(x) ds \geqslant$$
$$\int_0^\infty \int_{\{x:F(x)>s\}} d\mu_2(x) ds = \int_{\mathbb{R}^n} F(x) d\mu_1(x).$$

Note that if K is a symmetric convex body then $F := \mathbf{1}_K$ is even and quasi-concave and (3.4) becomes $\mu_1(K) \geqslant \mu_2(K)$. So (3.4) implies that $\mu_1 \succ \mu_2$. $\square$

We are now able to state the following equivalent formulation of the Rogers/Brascamp-Lieb-Luttinger inequality:

Proposition 3.3. *Let $f_1, \ldots, f_N$ be non-negative integrable functions on $\mathbb{R}$. Then*

$$\prod_{i=1}^{N} f_i \prec \prod_{i=1}^{N} f_i^*. \tag{3.5}$$

Let us explain why Theorem 3.1 implies Proposoition 3.3. Note first that without loss of generality we can replace the assumption "integrable" with "having integral 1." Let K be a symmetric convex body in $\mathbb{R}^N$. Then it can be approximated by intersections of symmetric slabs of the form

$$K_m := \bigcap_{i=1}^{m} \{x \in \mathbb{R}^N : |\langle x, u_i \rangle| \leqslant 1\}$$

for suitable $u_1, \ldots, u_m \in \mathbb{R}^N$. Note that $\mathbf{1}_{K_m} = \prod_{i=1}^m \mathbf{1}_{[-1,1]}(\langle \cdot, u_i \rangle)$. Apply (3.1) with $M = m + N$ and $u_{m+i} := e_i$, $i = 1, \ldots, N$. Then (since $\mathbf{1}_{K_m} \to \mathbf{1}_K$ in L_1), we get that

$$\int_K \prod_{i=1}^N f_i(x_i) dx \leqslant \int_K \prod_{i=1}^N f_i^*(x_i) dx. \tag{3.6}$$

Since K is an arbitrary symmetric convex body in $\mathbb{R}^N$, we get (3.5). The latter is an extension of a theorem of Anderson [1] and it is the basis of Christ's extension of the Rogers/Brascamp-Lieb-Luttinger inequality [23]; see also the thesis of Pfiefer [60] and work of Baernstein and Loss [2].

In the other direction, consider non-negative integrable functions $f_1, \ldots, f_m$ and let $u_1, \ldots, u_m$ be vectors in $\mathbb{R}^N$. Write $F(x) := \prod_{i=1}^m f_i(x_i)$ and $F_*(x) := \prod_{i=1}^m f_i^*(x_i)$. Let T be the $m \times N$ matrix with rows $u_1, \ldots, u_m$. Note that (3.5) implies that $F \prec F_*$. By (3.3) we also have that $F \circ T \prec F_* \circ T$ so that for any symmetric convex body $K \subseteq \mathbb{R}^N$, $\int_K F \circ T(x) dx \leq \int_K F_* \circ T(x) dx$, hence

$$\int_{\mathbb{R}^N} \prod_{i=1}^m f_i(\langle x, u_i \rangle) dx \leqslant \int_{\mathbb{R}^N} \prod_{i=1}^m f_i^*(\langle x, u_i \rangle) dx$$

which is (3.1).

Actually we will use the Rogers/Brascamp-Lieb-Luttinger inequality in the following form.

Corollary 3.4. *Let $f_1, \ldots, f_m$ be non-negative integrable functions on $\mathbb{R}$. Let $u_1, \ldots, u_m$ be non-zero vectors in $\mathbb{R}^N$ and let $F_1, \ldots, F_M$ be non-negative, even, quasi-concave functions on $\mathbb{R}^N$. Then*

$$\int_{\mathbb{R}^N} \prod_{j=1}^M F_j(x) \prod_{i=1}^m f_i(\langle x, u_i \rangle) dx \leqslant \int_{\mathbb{R}^N} \prod_{j=1}^M F_j(x) \prod_{i=1}^m f_i^*(\langle x, u_i \rangle) dx. \tag{3.7}$$

Also, if F is a non-negative, even, quasi-convex function on $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} F(x) \prod_{i=1}^N f_i(x_i) dx \geqslant \int_{\mathbb{R}^N} F(x) \prod_{i=1}^m f_i^*(x_i) dx. \tag{3.8}$$

Proof. (Sketch). Note that $\prod_{j=1}^M F_j(x)$ is again quasi-concave and even. So (3.7) follows from Proposition 3.3 and Lemma 3.2.

For the proof of (3.8) first notice that it is enough to prove in the case that $\int_{\mathbb{R}} f_i(t) dt = 1$, $1 \leqslant i \leqslant N$. Recall that for every $t > 0$, $\{F \leqslant t\}$ is convex and symmetric. Thus using Proposition 3.3 and

Lemma 3.2, we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} F(x)\prod_{i=1}^N f_i(x_i)dx \\
&\quad = \int_{\mathbb{R}^N}\left(\int_0^\infty \mathbf{1}_{\{F>t\}}(x)dt\right)\prod_{i=1}^N f_i(x_i)dx \\
&\quad = \int_0^\infty \int_{\mathbb{R}^N}(1-\mathbf{1}_{\{F\leqslant t\}})\prod_{i=1}^N f_i(x_i)dxdt \\
&\quad = \int_0^\infty \left(\int_{\mathbb{R}^N}\prod_{i=1}^N f_i^*(x_i)dx - \int_{\mathbb{R}^N}\mathbf{1}_{\{F\leqslant t\}})\prod_{i=1}^N f_i(x_i)dx\right)dt \\
&\quad \geqslant \int_0^\infty \left(\int_{\mathbb{R}^N}\prod_{i=1}^N f_i^*(x_i)dx - \int_{\mathbb{R}^N}\mathbf{1}_{\{F\leqslant t\}})\prod_{i=1}^N f_i^*(x_i)dx\right)dt \\
&\quad = \int_{\mathbb{R}^N} F(x)\prod_{i=1}^N f_i^*(x_i)dx.
\end{aligned}
$$

□

We say that a function f on $\mathbb{R}^n$ is unimodal if it is the increasing limit of a sequence of functions of the form,

$$\sum_{i=1}^m t_i \mathbf{1}_{K_i},$$

where $t_i \geqslant 0$ and K_i are symmetric convex bodies in $\mathbb{R}^n$. Even quasi-concave functions are unimodal and every even and non-increasing function on $\mathbb{R}_+$ is unimodal. In particular, for every integrable $f : \mathbb{R}^n \to \mathbb{R}_+$, f^* is unimodal. We will use the following lemma, which is essentially the bathtub principle (e.g., [43]).

Lemma 3.5. *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function.*

1. *If $g : \mathbb{R}_+ \to [0,1]$ is a measurable function and $\beta := \int_0^\infty f(t)t^{n-1}dt < \infty$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function, then*
$$\int_0^\infty \phi(t)f(t)t^{n-1}dt \geqslant \int_0^\infty \phi(t)h(t)t^{n-1}dt, \tag{3.9}$$
where $h := \mathbf{1}_{[0,(n\beta)^{\frac{1}{n}}]}$. If ϕ is non-increasing then the inequality in (3.9) is reversed.
2. *If $n = 1$, $\|f\|_1 = 1$, $\|f\|_\infty \leqslant 1$ and f is even, then $f^* \prec \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$.*
3. *If f is rotationally invariant, $\|f\|_1 = 1$, and $\|f\|_\infty \leqslant 1$, then for every star-shaped set $K \subseteq \mathbb{R}^n$, $\int_K f(x)dx \leqslant \int_K \mathbf{1}_{\widetilde{B}}(x)dx$.*

4. If $\|f\|_1 = 1$, $\|f\|_\infty \leqslant 1$, then $f^* \prec \mathbf{1}_{\widetilde{B}}$.

Proof. The proof of the first claim is standard, see e.g. [56, Lemma 3.5]. The second claim follows from the first, by choosing $n = 1$, $\beta = \frac{1}{2}$ and $\phi := \mathbf{1}_{[0,a]}$, $a > 0$. The third claim follows by applying (3.9) after writing the desired inequality in polar coordinates. The last claim follows immediately from the third. $\square$

A fundamental result on peaked measures is the following result of Kanter [38]

Theorem 3.6. *Let f_1, f_2 be functions on $\mathbb{R}^{n_1}$ such that $f_1 \succ f_2$ and f a unimodal function on $\mathbb{R}^{n_2}$. Then*

$$ff_1 \succ ff_2. \tag{3.10}$$

In particular, if f_i, g_i are unimodal functions on $\mathbb{R}^{n_i}$, $1 \leqslant i \leqslant M$ and $f_i \succ g_i$ for all i, then

$$\prod_{i=1}^{M} f_i \succ \prod_{i}^{M} g_i. \tag{3.11}$$

Proof. (Sketch) Without loss of generality, assume $\int f_1 = \int f_2 = \int f = 1$. Consider first the case where $f := \mathbf{1}_L$ for some symmetric convex body L in $\mathbb{R}^{n_2}$. Let K be a symmetric convex body in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The Prékopa-Leindler inequality implies that the symmetric function

$$F(x) := \int_{\mathbb{R}^{n_2}} \mathbf{1}_K(x,y)\mathbf{1}_L(y)dy$$

is log-concave. So, using Lemma 3.2, we have that

$$\int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \mathbf{1}_K(x,y) f_1(x) f(y) dx dy = \int_{\mathbb{R}^{n_1}} F(x) f_1(x) dx \geqslant$$

$$\int_{\mathbb{R}^{n_1}} F(x) f_2(x) dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \mathbf{1}_K(x,y) f_2(x) f(y) dx dy.$$

Again, by Lemma 3.2, we have that $ff_1 \succ ff_2$. The general case follows easily. $\square$

Theorem 3.6 and Lemma 3.5 immediately imply the following corollary.

Corollary 3.7. *Let $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}_+$ be probability densities of continuous distributions such that $\max_{i \leqslant M} \|f_i\|_\infty \leqslant 1$. If $n = 1$, then*

$$\prod_{i=1}^{m} f_i^* \prec \mathbf{1}_{Q_m} \tag{3.12}$$

where Q_m is the m-dimensional cube of volume 1 centered at 0. In the general case we have that

$$\prod_{i=1}^{m} f_i^* \prec \prod_{i=1}^{m} \mathbf{1}_{\widetilde{B}}. \tag{3.13}$$

3.1 Multidimensional case

Let f be a non-negative function on $\mathbb{R}^n$, $\theta \in S^{n-1}$ and $z \in \theta^{\perp}$. We write $f_{z,\theta}(t) := f_z(\theta) := f(z + t\theta)$. Let G be a non-negative function on the N-fold product $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$. Let $\theta \in S^{n-1}$ and let $Y := \{y_1, \ldots, y_N\} \subseteq \theta^{\perp} := \{y \in \mathbb{R}^n : \langle y, \theta \rangle = 0\}$. We define a function $G_Y : \mathbb{R}^N \to \mathbb{R}_+$ as

$$G_{Y,\theta}(t_1, \ldots, t_N) := G(y_1 + t_1\theta, \ldots, y_N + t_N\theta).$$

We say that $G : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}_+$ is Steiner concave if for every θ and $Y \subseteq \theta^{\perp}$ we have that $G_{Y,\theta}$ is even and quasi-concave. For example, if $N = n$, then negative powers of the absolute value of the determinant of an $n \times n$ matrix are Steiner concave since the determinant is a multilinear function of its columns (or rows). Our results depend on the following generalization of the Rogers and Brascamp-Lieb-Luttinger inequality due to Christ [23] (our terminology differs slightly from [23]).

Theorem 3.8. *Let $f_1, \ldots, f_N$ be non-negative integrable functions on $\mathbb{R}^n$, A an $\ell \times N$ matrix. Let $F^{(j)} : \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be Steiner concave functions $1 \leqslant j \leqslant M$ and let μ be a measure with a rotationally invariant quasi-concave density in $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{\ell=1}^{M} F^{(\ell)}(x_1, \ldots, x_\ell) \prod_{i=1}^{N} f_i \left(\sum_{j=1}^{\ell} a_{ij} x_j \right) d\mu(x_\ell) \ldots d\mu(x_1) \leqslant$$
$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{\ell=1}^{M} F^{(\ell)}(x_1, \ldots, x_\ell) \prod_{i=1}^{N} f_i^* \left(\sum_{j=1}^{\ell} a_{ij} x_j \right) d\mu(x_\ell) \ldots d\mu(x_1). \tag{3.14}$$

Proof. (Sketch) Note that the case $n = 1$, (3.14) is just (3.7). We consider the case $n > 1$. Let $u_i \in \mathbb{R}^{\ell}$ be the rows of the matrix A. Fix a direction $\theta \in S^{n-1}$ and let $y_1, \ldots, y_\ell \in \theta^{\perp}$ the (unique) vectors such that $x_j = y_j + t_j\theta$. Consider the function

$$h_i(\langle u_i, t \rangle) := f_i \left(\sum_{j=1}^{\ell} a_{ij}(y_j + t_j\theta) \right), \ 1 \leqslant i \leqslant N.$$

We defined the Steiner symmetral $f_i^*(\cdot|\theta) = h_i^*$ in the direction θ in §2. Then by Fubini's theorem we write each integral as an integral on $\theta^\perp$ and $[\theta] = \mathrm{span}\{\theta\}$, for each fixed $y_1, \ldots, y_\ell$ we apply (3.7) for the functions h_i and the quasi-concave functions $F^{(\ell)}_{Y,\theta}$. (Recall the definition of Steiner concavity). Using Fubini's theorem again, we have proved that

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{\ell=1}^{M} F^{(\ell)}(x_1, \ldots, x_\ell) \prod_{i=1}^{N} f_i \left(\sum_{j=1}^{\ell} a_{ij} x_j \right) d\mu(x_\ell) \ldots d\mu(x_1) \leqslant$$

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{\ell=1}^{M} F^{(\ell)}(x_1, \ldots, x_\ell) \prod_{i=1}^{N} f_i^* \left(\sum_{j=1}^{\ell} a_{ij} x_j | \theta \right) d\mu(x_\ell) \ldots d\mu(x_1). \tag{3.15}$$

In [12] it has been proved that the function f^* can be approximated (in the L_1 metric) by a suitable sequence of Steiner symmetrizations. This leads to (3.14). □

Let F be a Steiner concave function. Notice that the function $\tilde{F} := \mathbf{1}_{\{F>\alpha\}}$ is also Steiner concave. Indeed, if $\theta \in S^{n-1}$ and $Y \subseteq \theta^\perp$ and notice that $\tilde{F}_{Y,\theta}(t) = 1$ if and only if $F_{Y,\theta}(t) > \alpha$. Since F is Steiner concave, $\tilde{F}_{Y,\theta}$ is the indicator function of a symmetric convex set. So $\tilde{F}$ is also Steiner concave. Thus we have the following corollary.

Corollary 3.9. *Let $F : \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be a Steiner concave function and let $f_i : \mathbb{R}^n \to \mathbb{R}_+$ be non-negative functions with $\|f_i\|_1 = 1$ for $1 \leqslant i \leqslant N$. Let ν be the (product) probability measure defined on $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$ with density $\prod_i f_i$ and let ν^* have density $\prod_i f_i^*$. Then for each $\alpha > 0$,*

$$\nu \left(\{ F(x_1, \ldots, x_N) > \alpha \} \right) \leqslant \nu^* \left(\{ F(x_1, \ldots, x_N) > \alpha \} \right). \tag{3.16}$$

Moreover if $G : \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ is a Steiner convex function, then

$$\nu \left(\{ F(x_1, \ldots, x_N) > \alpha \} \right) \geqslant \nu^* \left(\{ F(x_1, \ldots, x_N) > \alpha \} \right). \tag{3.17}$$

Proof. We apply (3.14) for μ the Lebesgue measure $\ell = N$, A the identity matrix, $M = 1$ and for the function $\tilde{F}$ (as defined above). This proves (3.16). Working with the function $1 - \tilde{F}$ as in the proof of (3.8) we get (3.17). □

3.2 Cartesian products of balls as extremizers

In the last section, we discussed how in the presence of Steiner concavity, one can replace densities by their symmetric decreasing rearrangements. Among products of bounded, radial, decreasing densities, the uniform measure on Cartesian products of balls arises in extremal inequalities under several conditions and we discuss two of them in this section.

We will say that a function $F : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}_+$ is coordinate-wise decreasing if for any $x_1, \ldots, x_N \in \mathbb{R}^n$, and $0 \leqslant s_i \leqslant t_i, 1 \leqslant i \leqslant N$,

$$F(s_1x_1, \ldots, s_Nx_N) \geqslant F(t_1x_1, \ldots, t_Nx_N). \tag{3.18}$$

The next proposition can be proved by using Fubini's theorem iteratively and Lemma 3.5 (as in [24]).

Proposition 3.10. *Let $F : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}_+$ be a function that is coordinate-wise decreasing. If $g_1, \ldots, g_N : \mathbb{R}^n \to \mathbb{R}_+$ are rotationally invariant densities with $\max_{i \leqslant N} \|g_i\|_\infty \leqslant 1$, then*

$$\int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} F(x_1, \ldots, x_N) \prod_{i=1}^N g_i(x_i) dx_N \ldots dx_1 \tag{3.19}$$

$$\leqslant \int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} F(x_1, \ldots, x_N) \prod_{i=1}^N \mathbf{1}_{\widetilde{B}}(x_i) dx_N \ldots dx_1. \tag{3.20}$$

Using Corollary 3.7, we get the following.

Proposition 3.11. *Let $F : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}_+$ be a function that is quasi-concave and even. Assume that $g_1, \ldots, g_N$ are each less peaked than $\mathbb{1}_{\widetilde{B}}$. Then*

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} F(x_1, \ldots, x_N) \prod_{i=1}^N g_i(x_i) dx_N \ldots dx_1 \tag{3.21}$$

$$\leqslant \int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} F(x_1, \ldots, x_N) \prod_{i=1}^N \mathbf{1}_{\widetilde{B}}(x_i) dx_N \ldots dx_1. \tag{3.22}$$

4 Examples of Steiner concave and convex functions

As discussed in the previous section, the presence of Steiner concavity (or convexity) allows one to prove extremal inequalities when the extremizers are rotationally invariant. The requisite Steiner concavity is present for many functionals associated with random structures. As we will see, in many important cases, verifying the Steiner concavity condition is not a routine matter but rather depends on fundamental inequalities in convex geometry. In this section we give several non-trivial examples of Steiner concave (or Steiner convex) functions and we describe the variety of tools that are involved.

4.1 Shadow systems and mixed volumes

Shadow systems were defined by Shephard [70] and developed by Rogers and Shephard [63], and Campi and Gronchi, among others; see, e.g., [18], [20], [19], [21], [66]. Let C be a closed convex set in $\mathbb{R}^{n+1}$. Let $(e_1, \ldots, e_{n+1})$ be an orthonormal basis of $\mathbb{R}^{n+1}$ and write $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}e_{n+1}$ so that $\mathbb{R}^n = e_{n+1}^{\perp}$. Let $\theta \in S^{n-1}$. For $t \in \mathbb{R}$ let P_t be the projection onto $\mathbb{R}^n$ parallel to $e_{n+1} - t\theta$: for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$P_t(x + se_{n+1}) = x + ts\theta.$$

Set $K_t = P_t C \subseteq \mathbb{R}^n$. Then the family (K_t) is a shadow system of convex sets, where t varies in an interval on the real line. Shephard [69] proved that for each $1 \leqslant j \leqslant n$,

$$[0, 1] \ni t \mapsto V_j(P_t C)$$

is a convex function; see work of Campi and Gronchi, e.g., [22], [19] for further background and references. Here we consider the following N-parameter variation, which can be reduced to the one-parameter case.

Proposition 4.1. *Let n, N be postive integers and C be a compact convex set in $\mathbb{R}^n \times \mathbb{R}^N$. Let $\theta \in S^{n-1} \subseteq \mathbb{R}^n$. For $t \in \mathbb{R}^N$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^N$, we define $P_t(x, y) = x + \langle y, t\rangle\theta$. Then*

$$\mathbb{R}^N \ni t \mapsto V_j(P_t C)$$

is a convex function.

Proof. (Sketch) Fix s and t in $\mathbb{R}^N$. It is sufficient to show that

$$[0, 1] \ni \lambda \mapsto V_j(P_{s+\lambda(t-s)}C)$$

is convex. Note that $\lambda \mapsto P_{s+\lambda(s-t)}C$ is a one-parameter shadow system and we can apply Shephard's result above; for an alternate argument, following Groemer [33], see [56]. $\square$

Corollary 4.2. *Let C be a compact convex set in $\mathbb{R}^N$. Then*

$$(\mathbb{R}^n)^N \ni (x_1, \ldots, x_N) \mapsto V_j([x_1, \ldots, x_N]C)$$

is Steiner convex on $\mathbb{R}^N$. Moreover, if C is 1-unconditional then the latter function is coordinate-wise increasing analogous to definition (3.18)*. In particular,*

$$(\mathbb{R}^n)^N \ni (x_1, \ldots, x_N) \mapsto V_j([x_1, \ldots, x_N]C) \tag{4.1}$$

is Steiner convex and coordinate-wise increasing.

Proof. Let $\theta \in S^{n-1}$ and $y_i \in \theta^\perp$ for $i = 1, \ldots, N$. Write $x_i = y_i + t_i\theta$. Let $\mathcal{C} = [y_1 + e_{n+1}, \ldots, y_N + e_{n+N}]C$. Then $\mathcal{C}$ is a compact convex set in $\mathbb{R}^n \times \mathbb{R}^N$ which is symmetric with respect to $\theta^\perp$ in $\mathbb{R}^{n+N}$ since $[y_1 + e_{n+1}, \ldots, y_N + e_{n+N}]C \subseteq \theta^\perp$. Let $P_t : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ be defined as in Proposition 4.1. Then

$$P_t([y_1 + e_{n+1}, \ldots, y_N + e_{n+N}]C = [y_1 + t_1\theta, \ldots, y_N + t_N\theta]C.$$

We apply the previous proposition to obtain the convexity claim. Now for each $\theta \in S^{n-1}$ and $y_1, \ldots, y_N \in \theta^\perp$, the sets $[y_1 + t_1\theta, \ldots, y_N + t_N\theta]C$ and $[y_1 - t_1\theta, \ldots, y_N - t_N\theta]C$ reflections of one another and so the evenness condition holds as well. The coordinate-wise monotonicity holds since one has the following inclusion when C is 1-unconditional: for $0 \leqslant s_i \leqslant t_i$,

$$[s_1x_1, \ldots, s_Nx_N]C \subseteq [t_1x_1, \ldots, t_Nx_N]C.$$

$\square$

4.2 Dual setting

Here we discuss the following dual setting involving the polar dual of a shadow system. Rather than looking at projections of a fixed higher-dimensional convex set as in the previous section, this involves intersections with subspaces. We will invoke a fundamental inequality concerning sections of symmetric convex sets, known as Busemann's inequality [15]. This leads to a randomized version of an extension of the Blaschke-Santaló inequality to the class of convex measures (defined in §2). For this reason we will need the following extension of Busemann's inequality to convex measures from our joint work with D. Coredero-Erausquin and M. Fradelizi [24]; this builds on work by Ball [4], Bobkov [7], Kim, Yaskin and Zvavitch [39]).

Theorem 4.3. *(Busemann Theorem for convex measures). Let ν be a convex measure with even density $\psi \in \mathbb{R}^n$. Then the function Φ defined on $\mathbb{R}^n$ by $\Phi(0) = 0$ and for $z \neq 0$,*

$$\Phi(z) = \frac{\|z\|_2}{\int_{z^\perp} \psi(x)dx}$$

is a norm.

The latter inequality is the key to the following theorem from [24] which extends the result of Campi-Gronchi [21] to the setting of convex measures; the approach taken in [21] was the starting point for our work in this direction.

Proposition 4.4. *Let ν be a measure on $\mathbb{R}^n$ with a density ψ which is even and γ-concave on $\mathbb{R}^n$ for some $\gamma \geqslant -\frac{1}{n+1}$. Let $(K_t) := P_t C$ be an N-parameter shadow system of origin symmetric convex sets with respect to an origin symmetric body $C \subseteq \mathbb{R}^n \times \mathbb{R}^N$. Then the function $\mathbb{R}^N \ni\mapsto t \to \nu(K_t^\circ)^{-1}$ is convex.*

This result and the assumption on the symmetries of C and ν, leads to the following corollary. The proof is similar to that given in [24].

Corollary 4.5. *Let $r \geqslant 0$, C be an origin-symmetric convex set in $\mathbb{R}^N$. Let ν be a radial measure on $\mathbb{R}^n$ with a density ψ which is $-1/(n+1)$-concave on $\mathbb{R}^n$. Then the function*

$$G(x_1, \ldots, x_N) = \nu(([x_1 \ldots x_N]C + rB_2^N)^\circ)$$

is Steiner concave. Moreover if C is 1-unconditional then the function G is coordinate-wise decreasing.

Remark. The present setting is limited to origin-symmetric convex bodies. The argument of Campi and Gronchi [21] leading to the Blaschke-Santaló inequality has been extended to the non-symmetric case by Meyer and Reisner in [54]. It would be interesting to see an asymmetric version for random sets as it would give an empirical form of the Blaschke-Santaló inequality and related inequalities, e.g,. [35] in the asymmetric case.

4.3 Minkowski addition and extensions

In this section, we recall several variations of Minkowski addition that are the basis of L_p-Brunn-Minkowski theory and its extensions. L_p-addition as originally defined by Firey [26] of convex sets K and L with the origin in their interior is given by

$$h^p_{K+_pL}(x) = h^p_K(x) + h^p_L(x).$$

The L_p-Brunn-Minkowski inequality of Firey states that

$$V_n(K +_p L)^{1/n} \geqslant V_n(K)^{1/n} + V_n(L)^{1/n}. \tag{4.2}$$

A more recent pointwise definition that applies to compact sets K and L is due to Lutwak, Yang and Zhang [51]

$$K +_p L = \{(1-t)^{1/q} + t^{1/q}y : x \in K, y \in L, 0 \leqslant t \leqslant 1\}; \tag{4.3}$$

they proved that with the latter definition (4.2) extends to compact sets.

A general framework incorporating the latter as well as more general notions in the Orlicz setting initiated by Lutwak, Yang and Zhang

[49], [50], was studied by Gardner, Hug and Weil [30], [29]. Let M be an arbitrary subset of $\mathbb{R}^m$ and define the *M-combination* $\oplus_M(K^1,\dots,K^m)$ of arbitrary sets $K^1,\dots,K^m$ in $\mathbb{R}^n$ by

$$\begin{aligned}\oplus_M(K^1,\dots,K^m) &= \left\{\sum_{i=1}^m a_i x^{(i)} : x^{(i)} \in K^i, (a_1,\dots,a_m)\in M\right\}\\ &= \bigcup_{(a_i)\in M}(a_1K^1+\ldots+a_mK^m).\end{aligned}$$

Gardner, Hug, and Weil [30] develop a general framework for addition operations on convex sets which model important features of the Orlicz-Brunn-Minkowski theory. The notion of M-addition is closely related to linear images of convex sets in this paper. In particular, if $C = M$ and $K^1 = \{x_1\},\dots,K^m = \{x_m\}$, where $x_1,\dots,x_m \in \mathbb{R}^n$, then $[x_1,\dots,x_m]C = \oplus_M(\{x_1\},\dots,\{x_m\})$.

As a sample result we mention just the following from [30] (see Theorem 6.1 and Corollary 6.4.

Theorem 4.6. *Let M be a convex set in $\mathbb{R}^m$, $m \geqslant 2$.*

i. If M contained in the positive orthant and $K^1,\dots,K^m$ are convex sets in $\mathbb{R}^n$, then $\oplus_M(K^1,\dots,K^m)$ is a convex set.

ii. If M is 1-unconditional and $K^1,\dots,K^m$ are origin-symmetric convex sets, then $\oplus_M(K^1,\dots,K^m)$ is an origin symmetric convex set.

For several examples we mention the following:

(i) If $M = \{(1,1)\}$ and K^1 and K^2 are convex sets, then $K^1 \oplus_M K^2 = K_1 + K_2$, i.e., $\oplus_M$ is the usual Minkowski addition.

(ii) If $M = B_q^N$ with $1/p + 1/q = 1$, and K^1 and K^2 are origin symmetric convex bodies, then $K^1 \oplus_M K^2 = K^1 +_p K^2$, i.e., $\oplus_M$ corresonds to L_p-addition as in (4.3).

(iii) There is a close connection between Orlicz addition as defined in [49], [50] and M-addition, as shown in [29]. In fact, we define Orlicz addition in terms of the latter as it interfaces well with our operator approach. As an example, let $\psi : [0,\infty)^2 \to [0,\infty)$ be convex, increasing in each argument, and $\psi(0,0) = 0$, $\psi(1,0) = \psi(0,1) = 1$. Let K and L be origin-symmetric convex bodies and let $M = B_\psi^\circ$, where $B_\psi = \{(t_1,t_2) \in [-1,1]^2 : \psi(|t_1|,|t_2|) \leqslant 1\}$. Then we define $K +_\psi L$ to be $K \oplus_M L$.

Let $N_1,\dots,N_m$ be positive integers. For each $i = 1,\dots,m$, consider collections of vectors $\{x_{i1},\dots,x_{iN_i}\} \subseteq \mathbb{R}^n$ and let $C_1,\dots,C_m$ be convex sets with $C_i \subseteq \mathbb{R}^{N_i}$. Then for any $M \subseteq \mathbb{R}^{N_1+\ldots+N_m}$,

$$
\begin{aligned}
\oplus_M&([x_{11},\ldots,x_{1N_1}]C_1,\ldots,[x_{m1},\ldots,x_{mN_m}]C_m)\\
&=\left\{\sum_{i=1}^m a_i\left(\sum_{j=1}^{N_i} c_{ij}x_{ij}\right) : (a_i)_i \in M, (c_{ij})_j \in C_i\right\}\\
&=\left\{\sum_{i=1}^m\sum_{j=1}^{N_i} a_i c_{ij}x_{ij} : (a_i)_i \in M, (c_{ij})_j \in C_i\right\}\\
&=[x_{11},\ldots,x_{1N_1},\ldots,x_{m1},\ldots,x_{mN_m}](\oplus_M(C_1',\ldots,C_m')),
\end{aligned}
$$

where C_i' is the natural embedding of C_i into $\mathbb{R}^{N_1+\ldots+N_m}$. Thus the M-combination of families of sets of the form $[x_{i1},\ldots,x_{iN_i}]C_i$ fits exactly in the framework considered in this paper. In particular, the j-th intrinsic volume of latter set is a Steiner convex function by Corollary 4.2.

For subsequent reference we note one special case of the preceding identities. Let $C_1 = \operatorname{conv}\{e_1,\ldots,e_{N_1}\}$ and $C_2 = \operatorname{conv}\{e_1,\ldots,e_{N_2}\}$. Then we identify C_1 with $C_1' = \operatorname{conv}\{e_1,\ldots,e_{N_1}\}$ in $\mathbb{R}^{N_1+N_2}$ and C_2 with $C_2' = \operatorname{conv}\{e_{N_1+1},\ldots,e_{N_1+N_2}\} \subseteq \mathbb{R}^{N_1+N_2}$. If $x_1,\ldots,x_{N_1}$, $x_{N_1+1},\ldots,x_{N_1+N_2} \in \mathbb{R}^n$, then

$$
\begin{aligned}
\operatorname{conv}&\{x_1,\ldots,x_{N_1}\} \oplus_M \operatorname{conv}\{x_{N_1+1},\ldots,x_{N_1+N_2}\}\\
&=[x_1,\ldots,x_{N_1}]C_1 \oplus_M [x_{N_1+1},\ldots,x_{N_1+N_2}]C_2\\
&=[x_1,\ldots,x_{N_1},x_{N_1+1},\ldots,x_{N_1+N_2}](C_1' \oplus_M C_2').
\end{aligned}
$$

This will be used in §5.

4.4 Unions and intersections of Euclidean balls

Here we consider Euclidean balls $B(x_i,R)=\{x\in\mathbb{R}^n : |x-x_i|\leqslant r\}$ of a given radius $r>0$ with centers with centers $x_1,\ldots,x_N\in\mathbb{R}^n$.

Theorem 4.7. *For each $1\leqslant j\leqslant n$, the function*

$$
(\mathbb{R}^n)^N \ni (x_1,\ldots,x_N)\mapsto V_j\left(\bigcap_{i=1}^N B(x_i,r)\right) \tag{4.4}
$$

is Steiner concave. Moreover, it is quasi-concave and even on $(\mathbb{R}^n)^N$.

Proof. Let F be the function in (4.4). Let $\mathbf{u}=(u_1,\ldots,u_N)\in(\mathbb{R}^n)^N$ and $\mathbf{v}=(v_1,\ldots,v_N)\in(\mathbb{R}^n)^N$ belong to the support of F. One checks the following inclusion,

$$
\bigcap_{i=1}^N B\left(\frac{u_i+v_i}{2},r_i\right) \supseteq \frac{1}{2}\bigcap_{i=1}^N B(u_i,r_i)+\frac{1}{2}\bigcap_{i=1}^N B(v_i,r_i),
$$

and then applies the concavity of $K \mapsto V_j(K)^{1/j}$, which is a consequence of the Alexandrov-Fenchel inequalities. $\square$

Remark. The latter theorem is also true when V_j is replaced by a function which is monotone with respect to inclusion, rotation-invariant and quasi-concave with respect to Minkowski addition; see [58].

The latter can be compared with the following result for the convex hull of unions of Euclidean balls.

Theorem 4.8. *The function*

$$(\mathbb{R}^n)^N \ni (x_1, \ldots, x_N) \mapsto V_j\left(\operatorname{conv}\left(\bigcup_{i=1}^N B(x_i, r)\right)\right)$$

is Steiner convex.

Proof. Since

$$\operatorname{conv}\left(\bigcup_{i=1}^N B(x_i, r)\right) = \operatorname{conv}\{x_1, \ldots, x_N\} + B(0, r),$$

we can apply the same projection argument as in the proof of Corollary 4.2; see also work of Pfiefer [60] for a direct argument. $\square$

4.5 Operator norms

Steiner convexity is also present for operator norms from an arbitrary normed space into ℓ_2^n.

Proposition 4.9. *Let E be an N-dimensional normed space. For $x_1, \ldots, x_N \in \mathbb{R}^n$, let $\mathbf{X} = [x_1, \ldots, x_N]$. Then the operator norm*

$$(\mathbb{R}^n)^N \ni \mathbf{X} \mapsto \|\mathbf{X} : E \to \ell_2^n\| \tag{4.5}$$

is Steiner convex.

Proof. Denote the map in (4.5) by G. Then G is convex and hence the restriction to any line is convex. In particular, if $z \in S^{n-1}$ and $y_1, \ldots, y_N \in z^\perp$, then the function $G_Y : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$G_Y(t_1, \ldots, t_N) = G(y_1 + t_1 z_1, \ldots, y_N + t_N z_N)$$

is convex. To show that G_Y is even, we use the fact that $y_1, \ldots, y_N \in z^\perp$ to get for any $\lambda \in \mathbb{R}^N$,

$$\left\|\sum \lambda_i (y_i + t_i z)\right\|_2^2 \quad = \quad \left\|\sum \lambda_i (y_i - t_i z)\right\|_2^2,$$

hence $G_Y(t) = G_Y(-t)$. $\square$

5 Stochastic forms of isoperimetric inequalities

We now have all the tools to prove the randomized inequalities mentioned in the introduction and others. We will first prove two general theorems on stochastic dominance and then show how these imply a variety of randomized inequalities. At the end of the section, we discuss some examples of a different flavor.

For the next two theorems, we assume we have the following sequences of independent random vectors defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$; recal that $\widetilde{B} = \omega_n^{-1/n} B$.

1. $X_1, X_2, \ldots$, sampled according to densities $f_1, f_2, \ldots$ on $\mathbb{R}^n$, respectively (which will be chosen accordingly to the functional under consideration).
2. $X_1^*, X_2^*, \ldots$, sampled according to $f_1^*, f_2^*, \ldots$, respectively.
3. $Z_1, Z_2 \ldots$ sampled uniformly in $\widetilde{B}$.

We use $\mathbf{X}$ to denote the $n \times N$ random matrix $\mathbf{X} = [X_1 \ldots X_N]$. Similarly, $\mathbf{X}^* = [X_1^* \ldots X_N^*]$ $\mathbf{Z} = [Z_1 \ldots Z_N]$.

Theorem 5.1. *Let C be a compact convex set in $\mathbb{R}^N$ and $1 \leqslant j \leqslant n$. Then for each $\alpha \geqslant 0$,*

$$\mathbb{P}(V_j(\mathbf{X}C) > \alpha) \geqslant \mathbb{P}(V_j(\mathbf{X}^*C) > \alpha). \tag{5.1}$$

Moreover, if C is 1-unconditional and $\|f_i\|_\infty \leqslant 1$ for $i = 1, \ldots, N$, then for each $\alpha \geqslant 0$,

$$\mathbb{P}(V_j(\mathbf{X}C) > \alpha) \geqslant \mathbb{P}(V_j(\mathbf{Z}C) > \alpha). \tag{5.2}$$

Proof. By Corollary 4.2, we have Steiner convexity. Thus we may apply Corollary 3.9 to obtain (5.1). If C is unconditional, then Proposition 3.10 applies so we can conclude (5.2). □

Theorem 5.2. *Let C be an origin symmetric convex body in $\mathbb{R}^N$. Let ν be a radial measure on $\mathbb{R}^n$ with a density ψ which is $-1/(n+1)$-concave on $\mathbb{R}^n$. Then for each $\alpha \geqslant 0$,*

$$\mathbb{P}(\nu((\mathbf{X}C)^\circ) > \alpha) \leqslant \mathbb{P}(\nu(\mathbf{X}^*C)^\circ > \alpha). \tag{5.3}$$

Moreover, if C is 1-unconditional and $\|f_i\|_\infty \leqslant 1$ for $i = 1, \ldots, N$, then for each $\alpha \geqslant 0$,

$$\mathbb{P}(\nu((\mathbf{X}C)^\circ) > \alpha) \leqslant \mathbb{P}(\nu((\mathbf{Z}C)^\circ) > \alpha). \tag{5.4}$$

Proof. By Corollary 4.5, the function is Steiner concave. Thus we may apply Corollary 3.9 to obtain (5.3). If C is unconditional, then Proposition 3.10 applies so we can conclude (5.4). □

We start by explicitly stating some of the results mentioned in the introduction. We will first derive consequences for points sampled in convex bodies or compact sets $K \subseteq \mathbb{R}^n$. In this case, we have immediate distributional inequalities as $(\frac{1}{V_n(K)}\mathbb{1}_K)^* = \frac{1}{V_n(r_K B)}\mathbb{1}_{r_K B}$, even without the unconditionality assumption on C. The case of compact sets deserves special mention for comparison to classical inequalities.

1. Busemann random simplex inequality. As mentioned the Busemann random simplex inequality says that if $K \subseteq \mathbb{R}^n$ is a compact set with $V_n(K) > 0$ and $K_{o,n} = \mathrm{conv}\{o, X_1, \ldots, X_n\}$, where $X_1, \ldots, X_n$ are i.i.d. random vectors with density $f_i = \frac{1}{V_n(K)}\mathbb{1}_K$, then for $p \geqslant 1$,

$$\mathbb{E}V_n(K_{o,n})^p \geqslant \mathbb{E}V_n((r_K B)_{o,n})^p. \tag{5.5}$$

In our notation, $X_1^*, \ldots, X_n^*$ have density $\frac{1}{V_n(r_K B)}\mathbb{1}_{r_K B}$. For $C = \mathrm{conv}\{o, e_1, \ldots, e_n\}$, we have $K_{n,o} = \mathrm{conv}\{o, X_1, \ldots, X_n\}$. Thus the stochastic dominance of Theorem 5.1 implies (5.5) for all $p > 0$.

2. Groemer's inequality for random polytopes. Let $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$, where the X_i's are as in the previous example. An inequality of Groemer [33] states that for $p \geqslant 1$,

$$\mathbb{E}V_n(K_N)^p \geqslant \mathbb{E}V_n((r_K B)_N)^p; \tag{5.6}$$

this was extended by Tsolomitis and Giannopoulos for $p \in (0,1)$ in [31]. Let $C = \mathrm{conv}\{e_1, \ldots, e_N\}$ so that $K_N = [X_1, \ldots, X_N]C$ and $(r_K B)_N = [X_1^*, \ldots, X_N^*]C$. Then (5.6) follows from Theorem 5.1.

3. Bourgain-Meyer-Milman-Pajor inequality for random zonotopes. Let $Z_{1,N}(K) = \sum_{i=1}^N [-X_i, X_i]$, with X_i as above. Bourgain, Meyer, Milman, and Pajor [11] proved that for $p > 0$,

$$\mathbb{E}V_n(Z_{1,N}(K))^p \geqslant \mathbb{E}V_n(Z_{1,N}(r_K B))^p \tag{5.7}$$

With the notation of the previous examples, $Z_{1,N}(K) = [X_1, \ldots, X_N]B_\infty^N$. Thus Theorem 5.1 implies (5.7).

4. Inequalities for intrinsic volumes. For completeness, we record here how one obtains the stochastic form of the isoperimetric inequality (1.6). In fact, we state a stochastic form of the following extended isoperimetric inequality for convex bodies $K \subseteq \mathbb{R}^n$: for $1 \leqslant j \leqslant n$,

$$V_j(K) \geqslant V_j(r_K B). \tag{5.8}$$

The latter is a consequence of the Alexandrov-Fenchel inequalities, e.g., [68]. With K_N as above, a stochastic form (5.8) is the following: for $\alpha \geqslant 0$,

$$\mathbb{P}(V_j(K_N) > \alpha) \geqslant \mathbb{P}(V_j((r_K B)_N) > \alpha), \tag{5.9}$$

which is immediate from Theorem 5.1. For expectations, results of this type for intrinsic volumes were proved by Pfiefer [60] and Hartzoulaki and the first named author [36].

For further information on the previous inequalities and others we refer the reader to the paper of Campi and Gronchi [20] and the references therein. We have singled out these four as particular examples of M-additions (defined in the previous section). For example, if $C = \mathrm{conv}\{e_1, \ldots, e_N\}$, we have

$$K_N = \oplus_C(\{X_1\}, \ldots, \{X_N\}).$$

Similarly, for $C = B_\infty^N$,

$$\oplus_C([-X_1, X_1], \ldots, [-X_N, X_N]).$$

One can also intertwine the above operations and others. For example, if $C = \mathrm{conv}\{e_1, e_1 + e_2, e_1 + e_2 - e_3\}$. Then

$$[X_1, X_2, X_3]C = \mathrm{conv}\{X_1, X_1 + X_2, X_1 + X_2 - X_3\}$$

and Theorem 5.1 applies to such sets as well. The randomized Brunn-Minkowski inequality (1.5) is just one example of mixing two operations - convex hull and Minkowski summation. In the next example, we state a sample stochastic form of the Brunn-Minkowski inequality for M-addition in which (1.5) is just a special case; all of the previous examples also fit in this framework for additional summands. For other Brunn-Minkowski type inequalities for M-addition, see [30], [29].

5. Brunn-Minkowski type inequalities. Let K and L be convex bodies in $\mathbb{R}^n$ and let $M \subseteq \mathbb{R}^2$ be convex and contained in the positive orthant. Then the following Brunn-Minkowski type inequality holds for each $1 \leqslant j \leqslant n$,

$$V_j(K \oplus_M L) \geqslant V_j(r_K B \oplus_M r_L B). \tag{5.10}$$

We first formulate a suitable stochastic form of the latter. Let $K_{N_1} = \mathrm{conv}\{X_1, \ldots, X_{N_1}\}$, where $X_1, \ldots, X_{N_1}$ have density $f_i = \frac{1}{V_n(K)}\mathbb{1}_K$ and $L_{N_2} = \mathrm{conv}\{X_{N_1+1}, \ldots, X_{N_1+N_2}\}$, and $X_{N_1+1}, \ldots, X_{N_1+N_2}$ have density $f_i = \frac{1}{V_n(L)}\mathbb{1}_L$. Then for $\alpha > 0$,

$$\mathbb{P}\left(V_j(K_{N_1} \oplus_M L_{N_2}) > \alpha\right) \geqslant \mathbb{P}\left(V_j((r_K B)_{N_1} \oplus_M (r_L B)_{N_2}) > \alpha\right). \tag{5.11}$$

To see that (5.11) holds, set

$$C_1 = \mathrm{conv}\{e_1, \dots, e_{N_1}\}, \quad C_2 = \mathrm{conv}\{e_1, \dots, e_{N_2}\}.$$

Identifying C_1 with $C_1' = \mathrm{conv}\{e_1, \dots, e_{N_1}\}$ in $\mathbb{R}^{N_1+N_2}$ and C_2 with $C_2' = \mathrm{conv}\{e_{N_1+1}, \dots, e_{N_1+N_2}\} \subseteq \mathbb{R}^{N_1+N_2}$ as in §4.3, we have

$$\begin{aligned} K_{N_1} \oplus_M L_{N_2} &= [X_1, \dots, X_{N_1}]C_1 \oplus_M [X_{N_1+1}, \dots, X_{N_1+N_2}]C_2 \\ &= [X_1, \dots, X_{N_1}, X_{N_1+1}, \dots, X_{N_1+N_2}](C_1' \oplus_M C_2'). \end{aligned}$$

Write $\mathbf{X_1} = [X_1, \dots, X_{N_1}]$ and $\mathbf{X_2} = [X_{N_1+1}, \dots, X_{N_1+N_2}]$, and $\mathbf{X_1^*} = [X_1^*, \dots, X_{N_1}^*]$ and $\mathbf{X_2^*} = [X_{N_1+1}^*, \dots, X_{N_1+N_2}^*]$. In block matrix form, we have

$$K_{N_1} \oplus_M L_{N_2} = [\mathbf{X_1},\ \mathbf{X_2}](C_1' \oplus_M C_2').$$

Similarly,

$$(r_K B)_{N_1} \oplus_M (r_L B)_{N_2} = [\mathbf{X_1^*},\ \mathbf{X_2^*}](C_1' \oplus_M C_2'),$$

and so Theorem 5.1 implies (5.11). To prove (1.5), we simply take $M = \{(1,1)\}$ and $j = n$ in (5.11). Inequality (5.10) follows from (5.11) when $N_1, N_2 \to \infty$. For simplicity of notation, we have stated this for only two sets and C_1, C_2 as above.

For another example involving a law of large numbers, we turn to the following, stated in the symmetric case for simplicity.

6. Orlicz-Busemann-Petty centroid inequality. Let $\psi : [0,\infty) \to [0,\infty)$ be a Young function, i.e., convex, strictly increasing with $\psi(0) = 0$. Let f be a bounded probability density of a continuous distribution on $\mathbb{R}^n$. Define the Orlicz-centroid body $Z_\psi(f)$ associated to ψ by its support function

$$h(Z_\psi(f), y) = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \psi\left(\frac{|\langle x, y\rangle|}{\lambda}\right) f(x)dx \leqslant 1 \right\}.$$

Let $r_f > 0$ be such that $\|f\|_\infty \mathbb{1}_{r_f B}$ is a probability density. Then

$$V_n(Z_\psi(f)) \geqslant V_n(Z_\psi(\|f\|_\infty \mathbb{1}_{r_f B}). \tag{5.12}$$

Here we assume that $h(Z_\psi(f), y)$ is finite for each $y \in S^{n-1}$ and so $h(Z_\psi(f), \cdot)$ defines a norm and hence is the support function of the symmetric convex body $Z_\psi(f)$. When f is the indicator of a convex body, (5.12) was proved by Lutwak, Yang and Zhang [49] (where it was also studied for more general functions ψ); it was extended to star bodies by Zhu [74]; the version for probability densities and the randomized version below is from [56]; an extension of (5.12) to the asymmetric case was carried out by Huang and He [37].

The empirical analogue of (5.12) arises by considering the following finite-dimensional origin-symmetric Orlicz balls

$$B_{\psi,N} := \left\{ t = (t_1, \dots, t_N) \in \mathbb{R}^N : \frac{1}{N} \sum_{i=1}^{N} \psi(|t_i|) \leqslant 1 \right\}.$$

with associated Orlicz norm $\|t\|_{B_{\psi/N}} := \inf\{\lambda > 0 : t \in \lambda B_{\psi,N}\}$, which is the support function for $B^\circ_{\psi,N}$. For independent random vectors $X_1, \dots, X_N$ distributed according to f, we let

$$Z_{\psi,N}(f) = [X_1, \dots, X_N] B^\circ_{\psi,N}.$$

Then for $y \in S^{n-1}$,

$$h(Z_{\psi,N}(f), y) = \|(\langle X_1, y\rangle, \dots, \langle X_N, y\rangle)\|_{B_{\psi/N}}.$$

Applying Theorem 5.1 for $C = B^\circ_{\psi,N}$, we get that for $1 \leqslant j \leqslant n$ and $\alpha \geqslant 0$,

$$\mathbb{P}(V_j(Z_{\psi,N}(f)) > \alpha) \geqslant \mathbb{P}(V_j(Z_{\psi,N}(\|f\|_\infty \mathbb{1}_{r_f \widetilde{B}})) > \alpha). \tag{5.13}$$

Using the law of large numbers, one may check that

$$Z_{\psi,N}(f) \to Z_\psi(f) \tag{5.14}$$

almost surely in the Hausdorff metric (see [56]); when $\psi(x) = x^p$ and $f = \frac{1}{V_n(K)} \mathbb{1}_K$, $Z_{\psi,N}(f) = Z_{p,N}(K)$ as defined in the introduction; in this case, the convergence in (5.14) immediate by the classical law of large numbers (compare (1.10) and (1.14)). By integrating (5.13) and sending $N \to \infty$, we thus obtain (5.12).

We now turn to the dual setting.

7. Blaschke-Santaló type inequalities. The Blaschke-Santaló inequality states that if K is a symmetric convex body in $\mathbb{R}^n$, then

$$V_n(K^\circ) \leqslant V_n((r_K B)^\circ). \tag{5.15}$$

This was proved by Blaschke for $n = 2, 3$ and in general by Santaló [65]; see also Meyer and Pajor's proof by Steiner symmetrization [53] and [68], [28] for further background; origin symmetry in (5.15) is not needed but we discuss the randomized version only in the symmetric case. One can obtain companion results for all of the inequalities mentioned so far with suitable choices of symmetric convex bodies C. Let ν be a radially decreasing measure as in Theorem 5.2. Let $C = B_1^N$

and set $K_{N,s} = [X_1, \ldots, X_N]B_1^N$, where X_i has density $f_i = \frac{1}{V_n(K)}\mathbb{1}_K$. Then for $\alpha > 0$,

$$\mathbb{P}(\nu((K_{N,s})^\circ) > \alpha) \leqslant \mathbb{P}(\nu(((r_K B)_{N,s})^\circ) > \alpha).$$

Similarly, if K and L are origin-symmetric convex bodies and $M \subseteq \mathbb{R}^2$ is unconditional, then for $\alpha > 0$,

$$\mathbb{P}(\nu((K_{N_1,s} \oplus_M L_{N_1,s})^\circ) > \alpha) \leqslant \mathbb{P}(\nu(((r_K B)_{N_1,s} \oplus_M (r_L B)_{N_1,s})^\circ) > \alpha). \tag{5.16}$$

We also single out the polar dual of the last example on Orlicz-Busemann-Petty centroid bodies. Let ψ and $B_{\psi,N}$ be as above. Then

$$\mathbb{P}(\nu(Z_{\psi,N}^\circ(f)) > \alpha) \geqslant \mathbb{P}(\nu(Z_{\psi,N}^\circ(\|f\|_\infty \mathbb{1}_{r_f B})) > \alpha).$$

For a particular choice of ψ we arrive at the following example, which has not appeared in the literature before and deserves an explicit mention.

8. Level sets of the logarithmic Laplace transform. For a continuous probability distribution with a symmetric bounded density f, recall that the logarithmic Laplace transform is defined by

$$\Lambda(f, y) = \log \int_{\mathbb{R}^n} \exp\left(\langle x, y\rangle\right) f(x)dx.$$

For such f and $p > 0$, we define an origin-symmetric convex body $\Lambda_p(f)$ by

$$\Lambda_p(f) = \{y \in \mathbb{R}^n : \Lambda_f(y) \leqslant p\}.$$

The empirical analogue is defined as follows: for independent random vectors $X_1, \ldots, X_N$ with density f, set

$$\Lambda_{p,N}(f) = \left\{ y \in \mathbb{R}^n : \frac{1}{N}\sum_{i=1}^N \psi(|\langle X_i, y\rangle|) \leqslant e^p \right\}.$$

If we set $\psi_p(x) = e^{-p}(e^x - 1)$ then $([X_1, \ldots, X_N]B_{\psi_p,N}^\circ)^\circ = \Lambda_{p,N}(f)$. Then we have the following stochastic dominance

$$\mathbb{P}(\nu(\Lambda_{p,N}(f)) > \alpha) \leqslant \mathbb{P}(\nu(\Lambda_{p,N}(\|f\|_\infty \mathbb{1}_{r_f B})) > \alpha),$$

where r_f satisfies $\|f\|_\infty \mathbb{1}_{r_f B} = 1$. When $N \to \infty$, we get

$$\nu(\Lambda_p(f)) \leqslant \nu(\Lambda_p(\|f\|_\infty \mathbb{1}_{r_f B}).$$

The latter follows from the law of large numbers as in [56, Lemma 5.4] and the argument given in [24, §5].

For log-concave densities, the level sets of the logarithmic Laplace transform are known to be isomorphic to the duals to the L_p-centroid bodies; see work of Latała and Wojtaszczyk [42], or Klartag and E. Milman [41]; these bodies are essential in establishing concentration properties of log-concave measures, e.g., [55], [40], [13].

9. Ball-polyhedra. All of the above inequalities are volumetric in nature. For convex bodies, they all reduce to comparisons of bodies of a given volume. For an example of a different flavor, we have the following inequality involving random ball polyhedra, for $R > 0$,

$$\mathbb{P}\left(V_j\left(\bigcap\nolimits_{i=1}^{N} B(X_i, R)\right) \geqslant \alpha\right) \leqslant \mathbb{P}\left(V_j\left(\bigcap\nolimits_{i=1}^{N} B(Z_i, R)\right) > \alpha\right).$$

When the X_i's are sampled according to a particular density f associated with a convex body K, the latter leads to the following generalized Urysohn inequality,

$$V_j(K) \leqslant V_j((w(K))/2)B),$$

where $w(K)$ is the mean width of K, see [58]; the latter is not a volumetric inequality when $j < n$. The particular density f is the uniform measure a star-shaped set $A(K, R)$ defined by specifying its radial function $\rho_{A(K,R)}(\theta) = R - h_K(-\theta)$; Steiner symmetrization of $A(K, R)$ preserves the mean-width of K (for large R) so the volumetric techniques here lead to a stochastic dominance inequality for mean width.

We have focused this discussion on stochastic dominance. It is sometimes useful to relax the probabilistic formulation and instead consider the quantities above in terms of bounded integrable functions. We give one such example.

10. Functional forms. The following functional version Buseman's random simplex inequality (1.8) is useful for marginal distributions of high-dimensional probability distributions; this is from joint work with S. Dann [25]. Let $f_1, \dots, f_k$ be non-negative, bounded, integrable functions such that $\|f_i\|_1 > 0$ for each $i = 1, \dots, k$. For $p \in \mathbb{R}$, set

$$g_p(f_1, \dots, f_k) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} V_k(\mathrm{conv}\{0, x_1, \dots, x_k\})^p \prod_{i=1}^{k} f_i(x_i) dx_1 \dots dx_k.$$

Then for $p > 0$,

$$g_p(f_1, \dots, f_k) \geqslant \left(\prod_{i=1}^{k} \frac{\|f_i\|_1^{1+p/n}}{\omega_n^{1+p/n} \|f_i\|_\infty^{p/n}}\right) g_p(\mathbb{1}_{B_2^n}, \dots, \mathbb{1}_{B_2^n}).$$

The latter is just a special case of a general functional inequality [25]. Following Busemann's argument, we obtain the following. Let $1 \leqslant k \leqslant n-1$ and let f be a non-negative, bounded integrable function on $\mathbb{R}^n$. Then

$$\int_{G_{n,k}} \frac{\left(\int_E f(x)dx\right)^n}{\|f|_E\|_\infty^{n-k}} d\nu_{n,k}(E) \leqslant \frac{\omega_k^n}{\omega_n^k} \left(\int_{\mathbb{R}^n} f(x)dx\right)^k;$$

when $f = \mathbb{1}_K$ this recovers the inequality of Busemann and Straus [17] and Grinberg [32] extending (1.9). Schneider proved an analogue of the latter on the affine Grassmannian [67], which can also be extended to a sharp isoperimetric inequality for integrable functions [25]. The functional versions lead to small ball probabilities for projections of random vectors that need not have independent coordinates.

6 An application to operator norms of random matrices

In the last section we gave examples of functionals on random convex sets which are minorized or majorized by the uniform measure on the Cartesian product of Euclidean balls. In some cases the associated distribution function can be accurately estimated. For example, passing to the complements in (5.2), we get for $\alpha \geqslant 0$,

$$\mathbb{P}(V_n(\mathbf{X}C) \leqslant \alpha) \leqslant \mathbb{P}(V_n(\mathbf{Z}C) \leqslant \alpha), \tag{6.1}$$

where $\mathbf{X}$ and $\mathbf{Z}$ are as in Theorem 5.1. When $C = B_1^N$, i.e., for random symmetric convex hulls, we have estimated the quantity on the righthand side of (6.1) in [57] for all α less than an absolute contant (sufficiently small), at least when $N \leqslant e^n$. (The reason for the restriction is that we compute this for Gaussian matrices and the comparison to the uniform measure on the Cartesian products of balls is only valid in this range). This leads to sharp bounds for small deviation probabilities for the volume of random polytopes that were known before only for certain sub-gaussian distributions. The method of [57] applies more broadly. In this section we will focus on the case of the operator norm of a random matrix with independent columns.

By combining Corollary 3.9, and Propositions 3.11 and 4.9, we get the following result, which is joint work with G. Livshyts [45].

Theorem 6.1. *Let $N, n \in \mathbb{N}$. Let E be an N-dimensional normed space. Then the random matrices $\mathbf{X}, \mathbf{X}^*$ and $\mathbf{Z}$ (as in §5) satisfy the following for each $\alpha \geqslant 0$,*

$$\mathbb{P}\left(\|\mathbf{X} : E \to \ell_2^n\| \leqslant \alpha\right) \leqslant \mathbb{P}\left(\|\mathbf{X}^* : E \to \ell_2^2\| \leqslant \alpha\right). \tag{6.2}$$

Moreover, if $\|f_i\|_\infty \leqslant 1$ *for each* $i = 1, \ldots, N$*, then*

$$\mathbb{P}\left(\|\mathbf{X} : E \to \ell_2^n\| \leqslant \alpha\right) \leqslant \mathbb{P}\left(\|\mathbf{Z} : E \to \ell_2^2\| \leqslant \alpha\right).$$

As before, the latter result reduces the problem to computations for matrices $\mathbf{Z}$ with columns sampled in the Euclidean ball of volume one. For the important case of the operator norm, i.e., $E := \ell_2^N$, then one can provide the following bound for $\varepsilon > 0$,

$$\mathbb{P}(\|\mathbf{Z}\|_{2\to 2} \leqslant \varepsilon\sqrt{N}) \leqslant (c_1\varepsilon)^{nN-1},$$

where c_1, c_2 are absolute constants. Consequently, we get small ball probability bounds for the operator norm under minimal assumptions on the columns of $\mathbf{X}$.

For $1 \times N$ matrices, the latter theorem reduces to small-ball probabilities for norms a random vector x in $\mathbb{R}^N$ distributed according to a density of the form $\prod_{i=1}^N f_i$ where each f_i is a density on the real line. In particular, if $\|f_i\|_\infty \leq 1$ for each $i = 1, \ldots, N$, then for any norm $\|\cdot\|$ on $\mathbb{R}^N$ (the norm for the dual of E), we have for $\varepsilon > 0$,

$$\mathbb{P}\left(\|x\| \leqslant \varepsilon\right) \leqslant \mathbb{P}\left(\|z\| \leqslant \varepsilon\right), \tag{6.3}$$

where z is a random vector in the cube $[-1/2, 1/2]^N$ - the uniform measure on Cartesian products of "balls" in 1-dimension. In fact, by approximation from within, the same result holds if $\|\cdot\|$ is a semi-norm. Thus if x and z are as above, for each $\varepsilon > 0$ we have

$$\mathbb{P}(\|P_E x\|_2 \leqslant \varepsilon\sqrt{k}) \leqslant \mathbb{P}(\|P_E z\|_2 \leqslant \varepsilon\sqrt{k}) \leqslant (2\sqrt{\pi e}\varepsilon)^k, \tag{6.4}$$

where the last inequality uses a result of Ball [3]. In this way we recover the result of Rudelson and Vershynin from [64], who proved (6.4) with a bound of the form $(C\varepsilon)^k$ for some absolute constant C. Using the Rogers/Brascamp-Lieb-Luttinger inequality and Kanter's theorem, one can also obtain the sharp constant of $\sqrt{2}$ for marginal densities, which was first computed in [44] by adapting Ball's arguments from [3].

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