Random ball-polyhedra and inequalities for intrinsic volumes

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Abstract

We prove a randomized version of the generalized Urysohn inequality relating mean-width to the other intrinsic volumes. To do this, we introduce a stochastic approximation procedure that sees each convex body K as the limit of intersections of Euclidean balls of large radii and centered at randomly chosen points. The proof depends on a new isoperimetric inequality for the intrinsic volumes of such intersections. If the centers are i.i.d. and sampled according to a bounded continuous distribution, then the extremizing measure is uniform on a Euclidean ball. If one additionally assumes that the centers have i.i.d. coordinates, then the uniform measure on a cube is the extremizer. We also discuss connections to a randomized version of the extended isoperimetric inequality and symmetrization techniques.

1 Introduction

We prove a new randomized version of a classical inequality for intrinsic volumes. For context, we start by recalling two such inequalities and a known randomized version of one of them. The intrinsic volumes $V_1, \ldots, V_n$ are functionals on convex bodies which can be defined via the Steiner formula: for any convex

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body $K \subseteq \mathbb{R}^n$ and $\varepsilon > 0$,

$$|K + \varepsilon B| = \sum_{j=0}^{n} \omega_{n-j} V_j(K) \varepsilon^{n-j},$$

where $|\cdot|$ denotes n-dimensional Lebesgue measure, $B = B_2^n$ is the unit Euclidean ball in $\mathbb{R}^n$, ω_{n-j} is the volume of B_2^{n-j}, and $V_0 \equiv 1$; V_1 is a multiple of the mean-width, $2V_{n-1}$ is the surface area and $V_n = |\cdot|$ is the volume. The V_j's satisfy the extended isoperimetric inequality: for $1 \leqslant j < n$,

$$\left(\frac{V_n(K)}{V_n(B)}\right)^{1/n} \leqslant \left(\frac{V_j(K)}{V_j(B)}\right)^{1/j}; \tag{1.1}$$

as well as the generalized Urysohn inequality: for $1 < j \leqslant n$,

$$\left(\frac{V_j(K)}{V_j(B)}\right)^{1/j} \leqslant \frac{V_1(K)}{V_1(B)}. \tag{1.2}$$

The classical isoperimetric inequality corresponds to $j = n-1$ in (1.1); Urysohn's inequality to $j = n$ in (1.2) (or $j = 1$ in (1.1)). The Alexandrov-Fenchel inequality for mixed volumes (e.g. [21]) implies both (1.1) and (1.2). Alternatively, symmetrization methods can be used. For example, Steiner symmetrization, which preserves $V_n(K)$ and decreases $V_j(K)$ ($j < n$), can be used to prove (1.1); a general framework for such inequalities, building on work of Rogers and Shephard [20], is discussed by Campi and Gronchi in [6]. On the other hand, Minkowski symmetrization, which fixes $V_1(K)$ but increases $V_j(K)$ ($1 < j \leqslant n$), can be used to prove (1.2) (§2 contains definitions of these symmetrizations; here "decrease" and "increase" are meant in the non-strict sense).

A known randomized version of (1.1) is due to the first-named author and Hartzoulaki [12]. In a slightly more general form, using [17], the latter can be stated as follows. Assume that $|K| = |B|$ and sample independent random vectors $X_1, \ldots, X_N$ according to the uniform density $\frac{1}{|K|}\mathbb{1}_K$, i.e., $\mathbb{P}(X_i \in A) = \frac{1}{|K|}\int_A \mathbb{1}_K(x)dx$ for Borel sets $A \subset \mathbb{R}^n$. Additionally, sample independent random vectors $Z_1, \ldots, Z_N$ according to $\frac{1}{|B|}\mathbb{1}_B$. Then for all $1 \leqslant j \leqslant n$ and $t > 0$,

$$\mathbb{P}\Big(V_j(\mathrm{conv}\{X_1, \ldots, X_N\}) \geqslant t\Big) \geqslant \mathbb{P}\Big(V_j(\mathrm{conv}\{Z_1, \ldots, Z_N\}) \geqslant t\Big), \tag{1.3}$$

where conv denotes the convex hull. Integrating in t yields

$$\mathbb{E}V_j(\mathrm{conv}\{X_1,\dots,X_N\}) \geqslant \mathbb{E}V_j(\mathrm{conv}\{Z_1,\dots,Z_N\}). \qquad (1.4)$$

By the law of large numbers, the latter convex hulls converge to their respective ambient bodies and thus when $N \to \infty$,

$$V_j(K) \geqslant V_j(B) \quad \text{whenever} \quad V_n(K) = V_n(B),$$

which is equivalent to (1.1). Thus (1.1) can be seen as a global inequality which arises through a random approximation procedure in which stochastic domination holds at each stage. In fact, (1.3) holds not just for the convex hull but for a variety of other (linear, convex) operations and one can sample points according to continuous distributions on $\mathbb{R}^n$ (see [17]). Our recent focus has been on V_n. For example, such distributional inequalities are useful for small deviation inequalities for the volume of random sets [18]; inequalities in the dual setting, obtained in joint work with Fradelizi and Cordero-Erausquin [8], lead to a stochastic version of the Blaschke-Santalo inequality and the L_p-versions of Lutwak and Zhang [15].

Our aim here is to present a stochastic version of (1.2) of a different type - using intersections of Euclidean balls. In [3], Bezdek, Lángi, Naszódi, and Papez study the intersection of finitely many (unit) Euclidean balls, called *ball-polyhedra*, and lay out a broad framework for their study; they treat analogues of classical theorems in convexity such as those of Caratheodory and Steinitz, and they study their facial structure. Motivation arises, in part, from the Kneser-Poulsen Conjecture on the monotonicity of the volume of intersections (or unions) of Euclidean balls under contractions of their centers; see e.g. Bezdek's expository monograph [2]. Ball-polyhedra are also of their own inherent geometric interest since for large radii they resemble intersections of half-spaces, i.e., convex polyhedra, and hence all convex bodies - this is our motivation. We consider intersections of balls whose centers X_i are sampled independently according to a continuous distribution, i.e., a density $f : \mathbb{R}^n \to [0,\infty)$ with $\int_{\mathbb{R}^n} f(x)dx = 1$ so that $\mathbb{P}(X_i \in A) = \int_A f(x)dx$ for Borel sets $A \subset \mathbb{R}^n$. In what follows, by a *probability density* we always mean that of a continuous distribution. Different random models associated with ball-polyhedra have been studied by Csikós [9], Ambrus, Kevei and Vígh [1] and Fodor, Kevei and Vígh [10].

Our first result is the following isoperimetric inequality for intrinsic volumes; here $B(x,r)$ is the closed Euclidean ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$ (so $B = B(0,1)$).

Theorem 1.1. *Let $N, n \geqslant 1$ and $R > 0$. Let f be a probablity density on $\mathbb{R}^n$ that is bounded by one. Consider independent random vectors $X_1, \ldots, X_N$ sampled according to f and $Z_1, \ldots, Z_N$ according to $\mathbb{1}_{B(0,r_n)}$ where $r_n > 0$ is chosen so that $|B(0,r_n)| = 1$. Then for all $1 \leqslant j \leqslant n$ and $t > 0$,*

$$\mathbb{P}\left(V_j\left(\bigcap_{i=1}^N B(X_i,R)\right) > t\right) \leqslant \mathbb{P}\left(V_j\left(\bigcap_{i=1}^N B(Z_i,R)\right) > t\right). \tag{1.5}$$

In particular,

$$\mathbb{E}V_j\left(\bigcap_{i=1}^N B(X_i,R)\right) \leqslant \mathbb{E}V_j\left(\bigcap_{i=1}^N B(Z_i,R)\right). \tag{1.6}$$

In §5, we show that the latter theorem implies (1.2) which we formulate as follows for comparison purposes: if $K \subseteq \mathbb{R}^n$ is a convex body, then for each $1 < j \leqslant n$,

$$V_j(K) \leq V_j(B) \quad \text{whenever} \quad V_1(K) = V_1(B). \tag{1.7}$$

In general, Steiner symmetrization of K is not useful for comparing $V_1(K)$ and $V_j(K)$, $j < n$, (since it decreases both). Nevertheless, as in our previous work [17], Theorem 1.1 is based on Steiner symmetrization (and rearrangement inequalities). The essential difference here is that we apply such techniques to auxillary sets associated to K, which we then use to generate random ball-polyhedra that approximate K. This method is useful for comparing convex bodies of a given mean-width. In fact, the technique also applies to Wulff shapes, a topic which has received increased attention recently in Brunn-Minkowski theory; see work of Böröczky, Lutwak, Yang and Zhang [4] and Schuster and Weberndorfer [22].

As mentioned above, (1.1) and (1.2) share a common result - Urysohn's inequality. Since we have two different randomzied inequalities that lead to Urysohn's inequality, namely for random ball-polyhedra by taking $j = n$ in (1.6), and for random convex hulls by taking $j = 1$ in (1.4), it is natural to investigate the relationship between the two randomized forms. It turns out that the random ball-polyhedra version implies the random convex hull version. This is a consequence of a result of Gorbovickis [11], which has been used to establish the Kneser-Poulsen conjecture for large radii (see §5.3).

Lastly, we also consider random ball polyhedra with independently chosen centers $X_i = (X_{i1}, \ldots, X_{in}) \in \mathbb{R}^n$ having independent coordinates and bounded densities, say by one. In this case, the uniform density on the unit cube $Q_n = [-1/2, 1/2]^n$ is the extremizer.

Theorem 1.2. *Let $N, n \geqslant 1$ and $R > 0$. Let $h(x) = \prod_{i=1}^{n} h_i(x_i)$, where each h_i is a probability density on $\mathbb{R}$ that is bounded by one. Consider independent random vectors $X_1, \ldots, X_N$ sampled according to h and $Y_1, \ldots, Y_N$ according to $\mathbb{1}_{Q_n}$. Then for all $1 \leqslant j \leqslant n$ and $t > 0$,*

$$\mathbb{P}\left(V_j\left(\bigcap_{i=1}^{N} B(X_i, R)\right) > t\right) \leqslant \mathbb{P}\left(V_j\left(\bigcap_{i=1}^{N} B(Y_i, R)\right) > t\right). \tag{1.8}$$

In particular,

$$\mathbb{E} V_j\left(\bigcap_{i=1}^{N} B(X_i, R)\right) \leqslant \mathbb{E} V_j\left(\bigcap_{i=1}^{N} B(Y_i, R)\right). \tag{1.9}$$

The paper is organized as follows: we recall definitions in §2. Theorems 1.1 and 1.2 are proved in §3. Wulff shapes and (non-random) ball polyhedra are discussed in §4. In §5, we derive the generalized Urysohn inequality (1.2), discuss a connection to Minkowski symmetrizations, and compare the two random versions of Urysohn's inequality.

2 Preliminaries

We work in Euclidean space $\mathbb{R}^n$ with the canonical inner-product $\langle \cdot, \cdot \rangle$, Euclidean norm $|\cdot|$; we also use $|\cdot|$ (or V_n) for volume. As above, the unit Euclidean ball in $\mathbb{R}^n$ is $B = B_2^n$ and its volume is $\omega_n := |B_2^n|$; S^{n-1} is the unit sphere, equipped with the Haar probability measure σ.

A convex body $K \subseteq \mathbb{R}^n$ is a compact, convex set with non-empty interior. The set of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. For

$K, L \in \mathcal{K}^n$, the Minkowski sum $K + L$ is the set $\{x + y : x \in K, y \in L\}$; for $\alpha > 0$, $\alpha K = \{\alpha x : x \in K\}$. We say that K is symmetric if it is origin-symmetric, i.e., $-x \in K$ whenever $x \in K$. For $K \in \mathcal{K}^n$, the support function of K is given by

$$h_K(x) = \sup\{\langle y, x\rangle \ : \ y \in K\} \quad (x \in \mathbb{R}^n).$$

The mean-width of K is

$$\begin{aligned} w(K) &= \int_{S^{n-1}} h_K(\theta) + h_K(-\theta) d\sigma(\theta) \\ &= 2\int_{S^{n-1}} h_K(\theta) d\sigma(\theta). \end{aligned}$$

If $K \in \mathcal{K}^n$ and $u \in S^{n-1}$, the Minkowski symmetral of K about $u^\perp$ is the convex body

$$M_u(K) = \frac{K + R_u(K)}{2},$$

where R_u is the reflection about $u^\perp$. The Steiner symmetral of a convex body will be defined later, and more generally for functions.

For compact sets C_1, C_2 in $\mathbb{R}^n$, we let $\delta^H(C_1, C_2)$ denote the Hausdorff distance:

$$\delta^H(C_1, C_2) \ = \ \inf\{\varepsilon > 0 \ : \ C_1 \subseteq C_2 + \varepsilon B_2^n, C_2 \subseteq C_1 + \varepsilon B_2^n\}$$

Let $\mathcal{K}_\circ^n$ denote the class of all convex bodies that contain the origin in their interior. We will make use of the following fact (see, e.g., [21, §1.8]): If $K, L, K_1, K_2, \ldots \in \mathcal{K}_\circ^n$ satisfy $K_N \xrightarrow{\delta_H} K$ as $N \to \infty$, then

$$K_N \cap L \xrightarrow{\delta_H} K \cap L \quad \text{ as } N \to \infty. \tag{2.1}$$

A set $K \subseteq \mathbb{R}^n$ is star-shaped if it is compact, contains the origin in its interior and for every $x \in K$ and $\lambda \in [0, 1]$ we have $\lambda x \in K$. We call K a star-body if its radial function

$$\rho_K(\theta) = \sup\{t > 0 : t\theta \in K\} \quad (\theta \in S^{n-1})$$

is positive and continuous. Any positive continuous function $f : S^{n-1} \to \mathbb{R}$ determines a star body with radial function f.

For non-negative functions f and g on $[0,\infty)$, we write $f(r) = O(g(r))$ as $r \to \infty$ if there exists $M > 0$ and $r_0 >$ such that $f(r) \leqslant Mg(r)$ for all $r \geqslant r_0$; we write $f(r) = o(g(r))$ if $f(r)/g(r) \to 0$ as $r \to \infty$.

We say that a non-negative function f on $\mathbb{R}^n$ is quasi-concave if $\{x \in \mathbb{R}^n : f(x) > t\}$ is convex for each $t \geqslant 0$.

For Borel sets $A \subseteq \mathbb{R}^n$ with $|A| < \infty$, the volume-radius $\mathrm{vr}(A)$ is the radius of a Euclidean ball with the same volume as A; the symmetric rearrangement A^* of A is the (open) Euclidean ball of radius $\mathrm{vr}(A)$. The symmetric decreasing rearrangement of 1_A is defined by $(1_A)^* := 1_{A^*}$. If $f : \mathbb{R}^n \to \mathbb{R}^+$ is an integrable function, we define its symmetric decreasing rearrangement f^* by

$$f^*(x) = \int_0^\infty 1^*_{\{f>t\}}(x)dt = \int_0^\infty 1_{\{f>t\}^*}(x)dt.$$

The latter should be compared with the "layer-cake representation" of f:

$$f(x) = \int_0^\infty 1_{\{f>t\}}(x)dt; \tag{2.2}$$

see [14, Theorem 1.13]. The function f^* is radially-symmetric, decreasing and equimeasurable with f, i.e., $\{f > \alpha\}$ and $\{f^* > \alpha\}$ have the same volume for each $\alpha > 0$. By equimeasurability one has $\|f\|_p = \|f^*\|_p$ for each $1 \leqslant p \leqslant \infty$, where $\|\cdot\|_p$ denotes the $L_p(\mathbb{R}^n)$-norm. For a nonnegative, integrable function f on $\mathbb{R}^n$, the rearrangement f^* can be reached by a sequence of *Steiner symmetrals* $f^*(\cdot|\theta)$, which correspond to symmetrization in dimension one in the direction $\theta \in S^{n-1}$; namely $f^*(\cdot|\theta)$ is obtained by rearranging f along every line parallel to θ. The function $f^*(\cdot|\theta)$ is symmetric with respect to $\theta^\perp$. We refer the reader to the book [14] for further background material on rearrangements of functions.

3 Extremal inequalities for random ball-polyhedra

In this section we prove a more general version of Theorem 1.1. It concerns a family of functions $\phi : \mathcal{K}^n \to [0,\infty)$ satisfying the following three conditions:

(a) *Minkowski-concave*: for all $K, L \in \mathcal{K}^n$ and $\lambda \in (0,1)$,

$$\phi(1-\lambda)K + \lambda L) \geqslant (1-\lambda)\phi(K) + \lambda\phi(L);$$

(b) *monotone*: $\phi(K) \leqslant \phi(L)$ whenever $K, L \in \mathcal{K}^n$ satisfy $K \subseteq L$;

(c) *rotation-invariant*: $\phi(UK) = \phi(K)$ for all orthogonal transformations U of $\mathbb{R}^n$ and $K \in \mathcal{K}^n$.

It is known that $V_j(\cdot)^{1/j}$ satisfies each of the latter conditions (e.g., [21]).

Theorem 3.1. *Let $N, n \geqslant 1$ and $r_1, \dots, r_N \in (0, \infty)$. Assume that $\phi : \mathcal{K}^n \to [0, \infty)$ satisfies (a), (b) and (c). Let $f_1, \dots, f_N$ be probability densities on $\mathbb{R}^n$. Consider independent random vectors $X_1, \dots, X_N$ and $X_1^*, \dots, X_N^*$ such that X_i is distributed according to f_i and X_i^* according to f_i^*, for $i = 1, \dots, N$. Then for any $t \geqslant 0$,*

$$\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) \geqslant t\right) \leqslant \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i^*, r_i)\right) \geqslant t\right). \tag{3.1}$$

Furthermore, assume each f_i is bounded. Let $Z_1, \dots, Z_N$ be independent random vectors with Z_i distributed according to $a_i \mathbb{1}_{b_i B}$, where $a_i = \|f_i\|_\infty$ and b_i satisfies $\int_{\mathbb{R}^n} a_i \mathbb{1}_{b_i B} dx = 1$, for $i = 1, \dots, N$. Then

$$\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) \geqslant t\right) \leqslant \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(Z_i, r_i)\right) \geqslant t\right). \tag{3.2}$$

As in [17], [18], we use the rearrangement inequality of Rogers [19] and Brascamp-Lieb-Luttinger [5]; in particular, the following variant due to Christ [7].

Theorem 3.2. *Let $F : (\mathbb{R}^n)^N = \otimes_{i=1}^N \mathbb{R}^n \to [0, \infty)$. Then*

$$\begin{aligned} \int_{(\mathbb{R}^n)^N} & F(x_1, \dots, x_N) f_1(x_1) \cdots f_N(x_N)\, dx_1 \dots dx_N \\ & \leqslant \int_{(\mathbb{R}^n)^N} F(x_1, \dots, x_N) f_1^*(x_1) \cdots f_N^*(x_N)\, dx_1 \dots dx_N \end{aligned} \tag{3.3}$$

holds for any integrable $f_1, \dots, f_N : \mathbb{R}^n \to [0, \infty)$ whenever F satisfies the following condition: for every $z \in S^{n-1} \subseteq \mathbb{R}^n$ and for every $Y = (y_1, \dots, y_N) \subseteq (z^\perp)^N \subseteq (\mathbb{R}^n)^N$, the function $F_{z,Y} : \mathbb{R}^N \to [0, \infty)$ defined by

$$F_{z,Y}(t) := F(y_1 + t_1 z, \dots, y_N + t_N z). \tag{3.4}$$

is even and quasi-concave.

Remark 3.3. (i) When $n = 1$, the condition on F in the latter theorem reduces to $F : \mathbb{R}^N \to [0, \infty)$ being even and quasi-concave.

(ii) The proof of the latter theorem relies on the fact that such integrals are increased when the f_i's are replaced by their Steiner symmetrals $f_i^*(\cdot|\theta)$. When repeated in suitable directions θ, they yield the symmetric decreasing rearrangements f_i^*. We refer the reader to [7] or [17], [8] for the details.

We also combine the latter with a theorem of Kanter [13]. If f and g are probability densities on $\mathbb{R}^n$ such that $\int_K f(x)dx \leqslant \int_K g(x)dx$ for every symmetric convex set $K \subset \mathbb{R}^n$, we say that f is less peaked than g. Furthemore, we say that f is unimodal if it is quasi-concave and even.

Theorem 3.4. *Let $n, N \geqslant 1$. Let $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ be unimodal probability densities on $\mathbb{R}^n$. Assume that f_i is less peaked than g_i for each $i = 1, \ldots, N$. Then $\prod_{i=1}^n f_i$ is less peaked than $\prod_{i=1}^n g_i$.*

We will also use the following basic lemma (it can be proved using, e.g., [8, Lemma 4.3]).

Lemma 3.5. *Any probability density on $\mathbb{R}$ that is bounded by one is less peaked than $\mathbb{1}_{[-1/2,1/2]}$. Any radial probability density on $\mathbb{R}^n$ that is bounded by one is less peaked than $\mathbb{1}_{B(0,r_n)}$ where r_n satisfies $|B(0, r_n)| = 1$.*

The requisite concavity needed to apply Theorem 3.2 is a consequence of the following lemma.

Lemma 3.6. *Let $N, n \geqslant 1$ and $r_1, \ldots, r_N \in (0, \infty)$. Assume that $\phi : \mathcal{K}^n \to [0, \infty)$ satisfies (a) and (b) and $\phi(K) = \phi(-K)$ for each $K \in \mathcal{K}^n$. Set*

$$F(x_1, \ldots, x_N) = \phi\left(\bigcap_{i=1}^N B(x_i, r_i)\right).$$

Then F is even and concave on its support. Additionally, assume that ϕ satisfies condition (c). If $z \in S^{n-1}$ and $y_1, \ldots, y_N \in z^\perp$ and $F_{z,Y} : \mathbb{R}^N \to [0, \infty)$ is defined by

$$F_{z,Y}(t) := \phi\left(\bigcap_{i=1}^N B(y_i + t_i z, r_i)\right),$$

then $F_{z,Y}$ is even and concave on its support.

Proof. The function F is clearly even on $(\mathbb{R}^n)^N$. For the concavity claim, let $\mathbf{u} = (u_1, \ldots, u_N) \in (\mathbb{R}^n)^N$ and $\mathbf{v} = (v_1, \ldots, v_N) \in (\mathbb{R}^n)^N$ belong to the support of F. We will first show that

$$\begin{aligned} &\bigcap_{i=1}^N B\left(\frac{u_i + v_i}{2}, r_i\right) \\ &\supseteq \frac{1}{2} \bigcap_{i=1}^N B(u_i, r_i) + \frac{1}{2} \bigcap_{i=1}^N B(v_i, r_i). \end{aligned}$$

Let $w_1, w_2 \in \mathbb{R}^n$ and assume $|w_1 - u_i| \leqslant r_i$ and $|w_2 - v_i| \leqslant r_i$ for $i = 1, \ldots, N$. Then for $i = 1, \ldots, N$,

$$\begin{aligned} &\left|\frac{w_1 + w_2}{2} - \left(\frac{u_i + v_i}{2}\right)\right| \\ &\leqslant \frac{1}{2}|w_1 - u_i| + \frac{1}{2}|w_2 - v_i| \\ &\leqslant r_i, \end{aligned}$$

which shows the inclusion. By monotonicity and concavity of ϕ, we have

$$\begin{aligned} F((\mathbf{u}+\mathbf{v})/2) &= \phi\left(\bigcap_{i=1}^N B\left(\frac{u_i + v_i}{2}, r_i\right)\right) \\ &\geqslant \phi\left(\frac{1}{2} \bigcap_{i=1}^N B(u_i, r_i) + \frac{1}{2} \bigcap_{i=1}^N B(v_i, r_i)\right) \\ &\geqslant \frac{1}{2}\phi\left(\bigcap_{i=1}^N B(u_i, r_i)\right) + \frac{1}{2}\phi\left(\bigcap_{i=1}^N B(v_i, r_i)\right) \\ &= \frac{1}{2}F(\mathbf{u}) + \frac{1}{2}F(\mathbf{v}). \end{aligned}$$

Therefore, F is concave on its support.

The second concavity claim follows from the fact that the restriction of a concave function to a line is itself concave. Finally, let $z \in S^{n-1}$ and $y_1, \ldots, y_N \in z^\perp$. Let R_z denote the reflection about $z^\perp$. Then

$$\begin{aligned} R_z\left(\bigcap_{i=1}^N B(y_i + t_i z, r_i)\right) &= \bigcap_{i=1}^N R_z(r_i B(0,1) + (y_i + t_i z)) \\ &= \bigcap_{i=1}^N (r_i B(0,1) + (y_i - t_i z)) \\ &= \bigcap_{i=1}^N B(y_i - t_i z, r_i). \end{aligned}$$

Since ϕ satisfies (c), we have

$$F_{z,Y}(t) = \phi\left(\bigcap_{i=1}^N B(y_i + t_i z, r_i)\right)$$

$$
\begin{aligned}
&= \phi\left(\bigcap_{i=1}^N B(y_i - t_i z, r_i)\right)\\
&= F_{z,Y}(-t).
\end{aligned}
$$

□

Proof of Theorem 3.1. Let F be as in Lemma 3.6. For $t > 0$, set $H = \mathbb{1}_{\{F>t\}}$. Let $z \in S^{n-1}$ and $Y = (y_1, \ldots, y_N) \in (z^\perp)^N$. Let $F_{z,Y}$ and $H_{z,Y}$ be as defined in Lemma 3.6. Note that $\mathbb{1}_{\{F_{z,Y}>t\}} = H_{z,Y}$. By Lemma 3.6, $F_{z,Y}$ is an even, concave function. It follows that $H_{z,Y}$ is even and quasi-concave. Therefore we can apply Theorem 3.2 to obtain

$$
\begin{aligned}
&\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > t\right)\\
&\quad = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} H(x_1, \ldots, x_N) \prod_{i=1}^N f_i(x_i) dx_1 \ldots dx_N\\
&\quad \leqslant \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} H(x_1, \ldots, x_N) \prod_{i=1}^N f_i^*(x_i) dx_1 \ldots dx_N\\
&\quad = \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i^*, r_i)\right) > t\right),
\end{aligned}
$$

which proves (3.1).

We will first prove (3.2) under the additional assumption that $\|f_i\|_\infty = 1$ for $i = 1, \ldots, N$. Furthermore, by the first part of the proof we may assume that each f_i is radial and decreasing, hence unimodal. By Lemma 3.5, f_i is less peaked than $\mathbb{1}_{B(0,r_n)}$. Since $H = \mathbb{1}_{\{F>t\}}$ is the indicator function of a symmetric convex set in $(\mathbb{R}^n)^N$, Theorem 3.4 yields

$$
\begin{aligned}
&\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > t\right)\\
&\quad = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} H(x_1, \ldots, x_N) \prod_{i=1}^N f_i(x_i) dx_1 \ldots dx_N\\
&\quad \leqslant \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} H(x_1, \ldots, x_N) \prod_{i=1}^N \mathbb{1}_{B(0,r_n)}(x_i) dx_1 \ldots dx_N\\
&\quad = \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(Z_i, r_i)\right) > t\right).
\end{aligned}
$$

The general case follows by a change of variables; note that we make no assumption of homogeneity of ϕ in the following argument. For $i = 1,\dots,N$, let $c_i = \|f_i\|_\infty^{-1/n}$ and set

$$\bar{f}_i(x) = \frac{f_i(c_i x)}{\int_{\mathbb{R}^n} f_i(c_i y)\,dy} = \frac{f_i(c_i x)}{\|f_i\|_\infty}.$$

Then $\|\bar{f}_i\|_1 = \|\bar{f}_i\|_\infty = 1$ for $i = 1,\dots,N$. We apply what we just proved with $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_N$ and $H(c_1\cdot,\dots,c_N\cdot)$ (which remains the indicator of a symmetric convex set)

$$\begin{aligned}
&\mathbb{P}\Big(\phi\Big(\textstyle\bigcap_{i=1}^N B(X_i,r_i)\Big) > t\Big)\\
&\quad= \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} H(x_1,\dots,x_N)\textstyle\prod_{i=1}^n f_i(x_i)\,dx_1\dots dx_N\\
&\quad= \textstyle\prod_{i=1}^N \|f_i\|_\infty \displaystyle\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} H(c_1y_1,\dots,c_ny_N)\prod_{i=1}^N \frac{c_i^n f_i(c_iy_i)}{\|f_i\|_\infty}\,dy_1\dots dy_N\\
&\quad= \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} H(c_1y_1,\dots,c_ny_N)\textstyle\prod_{i=1}^N \bar{f}_i(y_i)\,dy_1\dots dy_N\\
&\quad\leqslant \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} H(c_1y_1,\dots,c_ny_N)\textstyle\prod_{i=1}^N \mathbb{1}_{r_nB}(y_i)\,dy_1\dots dy_N\\
&\quad= \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} H(x_1,\dots,x_N)\textstyle\prod_{i=1}^N \|f_i\|_\infty \mathbb{1}_{c_ir_nB}(x_i)\,dx_1\dots dx_N\\
&\quad= \mathbb{P}\Big(\phi\Big(\textstyle\bigcap_{i=1}^N B(Z_i,r_i)\Big) > t\Big),
\end{aligned}$$

where, as above, $r_n = \omega_n^{-1/n}$. This proves (3.2) as claimed with $b_i = c_ir_n$, for $i = 1,\dots,N$. □

Now we turn to a generalization of Theorem 1.2.

Theorem 3.7. *Let $N, n \geqslant 1$ and $r_1,\dots,r_N \in (0,\infty)$. Assume that $\phi : \mathcal{K}^n \to [0,\infty)$ satisfies (a) and (b). Let $h_1,\dots,h_N$ be probability densities on $\mathbb{R}^n$ with $h_i(x) = \prod_{j=1}^n h_{ij}(x_j)$ and each h_{ij} is a probability density on $\mathbb{R}$ that is bounded by one. Consider independent random vectors $X_1,\dots,X_N$ and $Y_1,\dots,Y_N$ such that X_i is distributed according to h_i and Y_i according to $\mathbb{1}_{Q_n}$, for $i = 1,\dots,N$. Then for any $t \geqslant 0$,*

$$\mathbb{P}\Big(\phi\Big(\textstyle\bigcap_{i=1}^N B(X_i,r_i)\Big) \geqslant t\Big) \leqslant \mathbb{P}\Big(\phi\Big(\textstyle\bigcap_{i=1}^N B(Y_i,r_i)\Big) \geqslant t\Big). \qquad (3.5)$$

Proof. Note that each h_{ij}^* is less peaked than $\mathbb{1}_{[-1/2,1/2]}$, hence by Theorem 3.4, $\prod_{i=1}^N \prod_{j=1}^n h_{ij}^*$ is less peaked than $\prod_{i=1}^N \mathbb{1}_{Q_n}$. Let F

be as in Lemma 3.6, $t > 0$ and $H = \mathbb{1}_{\{F>t\}}$. For $x_i \in \mathbb{R}^n$ we write $x_i = (x_{i1}, \ldots, x_{in})$. Since F is even and concave on its support, we can apply Theorem 3.2 (considering F as a concave function on $\mathbb{R}^{nN}$ as in Remark 3.3(i)) and Theorem 3.4 to obtain

$$\begin{aligned}
&\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > t\right) \\
&\quad = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} H(x_1, \ldots, x_N) \prod_{i=1}^N \prod_{j=1}^n h_{ij}(x_{ij}) dx_1 \ldots dx_N \\
&\quad \leqslant \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} H(x_1, \ldots, x_N) \prod_{i=1}^N \prod_{j=1}^n h_{ij}^*(x_i) dx_1 \ldots dx_N \\
&\quad \leqslant \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} H(x_1, \ldots, x_N) \prod_{i=1}^N \mathbb{1}_{Q_n}(x_i) dx_1 \ldots dx_N \\
&\quad = \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(Y_i, r_i)\right) > t\right).
\end{aligned}$$

□

Remark 3.8. One can adapt the latter argument to treat densities h_{ij} that are not necessarily bounded by the same value. In this case, h_{ij} is less peaked than $\|h_{ij}\|_\infty \mathbb{1}_{[-\frac{1}{2\|h_{ij}\|_\infty}, \frac{1}{2\|h_{ij}\|_\infty}]}$. Then the corresponding extremizers would be uniform measures on suitable coordinate boxes.

4 Wulff shapes and ball-polyhedra

In this section we recall the definition of the Wulff shape and show that it can be approximated by (non-random) ball-polyhedra of large radii; for background on Wulff shapes in Brunn-Minkowski theory and further references, see [21].

If $f : S^{n-1} \to \mathbb{R}$ is a positive continuous function, the Wulff shape $W(f)$ is defined by

$$W(f) = \bigcap_{\theta \in S^{n-1}} H^-(\theta, f(\theta)), \tag{4.1}$$

where

$$H^-(\theta, f(\theta)) = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leqslant f(\theta)\}. \tag{4.2}$$

Then $W(f)$ is a convex body with the origin in its interior. If K is a convex body with support function h_K, then $W(h_K) = K$.

With f as above and $R > \sup_{\theta \in S^{n-1}} f(\theta)$, we introduce a star body $A(f,R)$ by specifying its radial function:

$$\rho_{A(f,R)}(-\theta) = R - f(\theta) \qquad (\theta \in S^{n-1}). \tag{4.3}$$

The role of $A(f,R)$ is described in the following proposition; as mentioned, $\mathrm{vr}(A(f,R)) = (|A(f,R)|/\omega_n)^{1/n}$ is the radius of a Euclidean ball with the same volume as $A(f,R)$.

Proposition 4.1. *Let $f : S^{n-1} \to \mathbb{R}$ be continuous, $R > \sup_{\theta \in S^{n-1}} f(\theta)$ and $A(f,R)$ as in* (4.3). *Then, in the Hausdorff metric,*

$$W(f) = \lim_{R\to\infty} \bigcap\nolimits_{x \in A(f,R)} B(x,R), \tag{4.4}$$

and

$$R - \mathrm{vr}(A(f,R)) \geqslant \int_{S^{n-1}} f(\theta) d\sigma(\theta); \tag{4.5}$$

moreover, equality holds as $R \to \infty$.

The proof of the proposition relies on the following lemmas.

Lemma 4.2. *Let $N, n \geq 1$, $x_1, \ldots, x_N \in \mathbb{R}^n$ and set $P = \mathrm{conv}\{x_1, \ldots, x_N\}$. Then for each $r > 0$,*

$$\bigcap\nolimits_{x \in P} B(x,r) = \bigcap\nolimits_{i=1}^{N} B(x_i, r). \tag{4.6}$$

Proof of Lemma 4.2. Let $y \in \bigcap_{i=1}^{N} B(x_i, r)$ so that $|y - x_i| \leqslant r$ for each $i = 1, \ldots, N$. Let $x \in P$ and write $x = \sum_{i=1}^{N} \alpha_i x_i$, where $\alpha_1, \ldots, \alpha_N \geqslant 0$ and $\sum_{i=1}^{N} \alpha_i = 1$. Then

$$|y - x| = |\textstyle\sum_{i=1}^{N} \alpha_i y - \sum_{i=1}^{N} \alpha_i x_i| \leqslant \sum_{i=1}^{N} \alpha_i |y - x_i| \leqslant r,$$

hence $y \in \bigcap_{x \in P} B(x,r)$. The reverse inclusion is trivial. □

Lemma 4.3. *Let $f : S^{n-1} \to$ be positive and continuous. Then*

$$W(f) = \lim_{R\to\infty} \bigcap\nolimits_{\theta \in S^{n-1}} B(-(R - f(\theta)\theta, R), \tag{4.7}$$

where the convergence is in the Hausdorff metric.

Proof of Lemma 4.3. Fix $\theta \in S^{n-1}$ and $R > \sup_{\theta \in S^{n-1}} f(\theta)$. Note that

$$B(-(R - f(\theta))\theta, R) \subseteq H^{-}(\theta, f(\theta)). \tag{4.8}$$

Indeed, if $x \notin H^-(\theta, f(\theta))$, then $\langle x, \theta\rangle > f(\theta)$, hence $|x| > f(\theta)$ so

$$\begin{aligned} &|x + (R - f(\theta))\theta|^2 \\ &\quad > |x|^2 + 2(R - f(\theta))f(\theta) + R^2 - 2Rf(\theta) + f^2(\theta) \\ &\quad > R^2. \end{aligned}$$

Therefore $x \notin B(-(R - f(\theta))\theta, R)$ which establishes (4.8). It follows that

$$\bigcap_{\theta\in S^{n-1}} B(-(R - f(\theta))\theta, R) \subseteq W(f). \tag{4.9}$$

Next note that

$$\begin{aligned} &B(-(R - f(\theta))\theta, R) \\ &\quad \supseteq \{x \in W(f) : |x + (R - f(\theta))\theta|^2 \leqslant R^2\} \\ &\quad = \left\{x \in W(f) : |x|^2 + 2(R - f(\theta))\langle x, \theta\rangle \leqslant 2Rf(\theta) - f^2(\theta)\right\} \\ &\quad = \left\{x \in W(f) : \langle x, \theta\rangle \leqslant \frac{2Rf(\theta) - f^2(\theta) - |x|^2}{2(R - f(\theta))}\right\} \\ &\quad = \left\{x \in W(f) : \left\langle x, \frac{\theta}{f(\theta)}\right\rangle \leqslant \frac{1 - \frac{f(\theta)}{2R} - \frac{|x|^2}{2Rf(\theta))}}{1 - \frac{f(\theta)}{R}}\right\} \\ &\quad \supseteq \left\{x \in W(f) : \left\langle x, \frac{\theta}{f(\theta)}\right\rangle \leqslant 1 - O(1/R)\right\}, \end{aligned}$$

where the implied constants in $O(1/R)$ depend only on the in-radius and out-radius of $W(f)$, hence on the minimum and maximum values of f. As $R \to \infty$, the latter set converges to $W(f) \cap H^-(\theta, f(\theta))$. Moreover, the convergence is uniform in θ. For R sufficiently large, each of the latter sets belongs to $\mathcal{K}^n_\circ$ so we may apply (2.1) to get

$$\begin{aligned} \lim_{R\to\infty} \bigcap_{\theta\in S^{n-1}} B(-(R - f(\theta))\theta, R) &\supseteq W(f) \cap \bigcap_{\theta\in S^{n-1}} H^-(\theta, f(\theta)) \\ &= W(f), \end{aligned}$$

which, combined with (4.9) completes the proof. □

Proof of Proposition 4.1. The map $\theta \mapsto -(R - f(\theta))\theta$ is a bijection between S^{n-1} and the boundary $\partial A(f, R)$ of $A(f, R)$. Therefore

$$\bigcap_{\theta\in S^{n-1}} B(-(R - f(\theta))\theta, R) = \bigcap_{x\in\partial A(f,R)} B(x, R) \tag{4.10}$$

$$= \bigcap_{x\in A(f,R)} B(x,R), \qquad (4.11)$$

where the last equality is simply Lemma 4.2 applied on each line segment

$$P(\theta) = \mathrm{conv}\{\rho_{A(K,R)}(\theta), \rho_{A(K,R)}(-\theta)\} \quad (\theta \in S^{n-1}).$$

Thus equality (4.4) follows from Lemma 4.3. Since $A(f,R)$ is a star body, we can use polar coordinates and Jensen's inequalty to get

$$\begin{aligned} \mathrm{vr}(A(f,R)) &= \left(\int_{S^{n-1}} \rho_{A(f,R)}(-\theta)^n d\sigma(\theta)\right)^{1/n} \\ &= \left(\int_{S^{n-1}} (R - f(\theta))^n d\sigma(\theta)\right)^{1/n} \\ &\geqslant R - \int_{S^{n-1}} f(\theta) d\sigma(\theta). \end{aligned}$$

Writing $\|f\|_1 = \int_{S^{n-1}} f(\theta)d\sigma(\theta)$, we can prove the equality in the latter by Taylor expansion:

$$\begin{aligned} &\mathrm{vr}(A(f,R)) \\ &\quad = R\left(\int_{S^{n-1}} \left(1 - \frac{nf(\theta)}{R} + O(1/R^2)\right) d\sigma(\theta)\right)^{1/n} \\ &\quad = R\left(1 - \frac{n\|f\|_1}{R} + O(1/R^2)\right)^{1/n} \\ &\quad = R\exp\left(\frac{1}{n}\log\left(1 - \frac{n\|f\|_1}{R} + O(1/R^2)\right)\right) \\ &\quad = R\exp\left(\frac{1}{n}\left(-\frac{n\|f\|_1}{R} + O(1/R^2)\right)\right) \\ &\quad = R\exp\left(-\frac{\|f\|_1}{R} + O(1/R^2)\right) \\ &\quad = R\left(1 - \frac{\|f\|_1}{R} + O(1/R^2)\right) \\ &\quad = R - \|f\|_1 + O(1/R). \end{aligned}$$

□

5 Forms of Urysohn's inequality

5.1 Derivation of the generalized Urysohn inequality

Proposition 5.1. *Theorem 3.1 implies the generalized Urysohn inequality* (1.2).

Proof. Let K be a convex body in $\mathbb{R}^n$. For $R > \sup_{\theta\in S^{n-1}} h_K(\theta)$, let $A(h_K,R)$ be the star-shaped set defined in (4.3). The volume radius of $A(h_K,R)$ is $r := \mathrm{vr}(A(h_K,R)) = \omega_n^{-1/n}|A(K,R)|^{1/n}$. Consider independent random vectors $X_1, X_2, \ldots$ in $A(h_K,R)$ sampled according to $\frac{1}{|A(h_K,R)|}\mathbb{1}_{A(h_K,R)}$. Sample also $Z_1, Z_2, \ldots$ according to $\frac{1}{|A(h_k,R)|}\mathbb{1}_{rB}$. For each j, $V_j(\cdot)^{1/j}$ satisfies the assumptions of Theorem 3.1. Thus for each N,

$$\mathbb{E}V_j\left(\bigcap_{i=1}^N B(X_i,R)\right)^{1/j} \leqslant \mathbb{E}V_j\left(\bigcap_{i=1}^N B(Z_i,R)\right)^{1/j}. \tag{5.1}$$

As $N\to\infty$, we have

$$\bigcap_{i=1}^N B(X_i,R) \to \bigcap_{i=1}^\infty B(X_i,R)$$

in δ^H with probability one (see, e.g., [21, Lemma 1.8.2]). Setting $\delta = \delta^H(\{X_i\}_{i=1}^\infty, A(h_K,R))$, we have

$$\bigcap_{x\in A(h_K,R)} B(x,R) \subset \bigcap_{i=1}^\infty B(X_i,R) \subset \bigcap_{x\in A(h_K,R)} B(x,R+\delta).$$

On the other hand, $\delta = 0$ almost surely (see, e.g., [16, Proposition 6.17], which is stated more generally for convergence of random closed sets in the Fell topology but for compact subsets of the compact set $A(h_K,R)$ this coincides with convergence in δ^H). The same argument applies to the $Z_i's$ and rB. For each j, V_j is continuous with respect to convergence of convex bodies in δ^H. Thus as $N\to\infty$ in (5.1), we get

$$V_j\left(\bigcap_{x\in A(h_K,R)} B(x,R)\right) \leqslant V_j\left(\bigcap_{z\in rB} B(z,R)\right). \tag{5.2}$$

Note that

$$\bigcap_{z\in rB} B(z,R) = B(0, R - \mathrm{vr}(A(h_K,R))).$$

By Proposition 4.1, we have

$$K = \lim_{R\to\infty} \bigcap_{x\in A(h_K,R)} B(x,R)$$

in δ^H, and

$$\lim_{R\to\infty} R - \mathrm{vr}(A(h_K,R)) \to \int_{S^{n-1}} h_K(\theta)d\sigma(\theta) = w(K)/2,$$

where $w(K)$ is the mean-width of K. Thus when $R \to \infty$ in (5.2), we get $V_j(K) \leqslant V_j(B(0,w(K)/2)$, which is equivalent to the generalized Urysohn inequality (1.2) (since $w(K)$ is a multiple of $V_1(K)$ and $B(0,w(K)/2)$ is a ball of the same mean-width as K). ☐

The latter proof ultimately rests on Steiner symmetrization of the set $A(h_K,R)$ (see Remark 3.3). Since Minkowski symmetrization of K is also useful for proving the generalized Urysohn inequality, it is natural to investigate its effect on $A(h_k,R)$. It turns out that one can obtain (5.2) via Minkowski symmetrization of K as well. Since this further illuminates the use of $A(h_K,R)$, we discuss it in the next subsection.

5.2 Relation to Minkowski symmetrization

It will be convenient to identify convex bodies with their support functions and write $A(K,R)$ rather than $A(h_K,R)$ (defined in (4.3)). If K and L are convex bodies, the equality $h_{(K+L)/2} = (h_K + h_L)/2$ implies

$$\rho_{A(\frac{K+L}{2},R)} = \frac{1}{2}(\rho_{A(K,R)} + \rho_{A(L,R)}).$$

In particular, if $u \in S^{n-1}$ and $M_u(K)$ is the Minkowski symmetral of K about $u^\perp$, then $A(M_u(K),R)$ is the star-body with radial function $\frac{1}{2}(\rho_{A(K,R)} + \rho_{A(R_u(K),R)})$. Using (4.10) and (4.11)

$$\begin{aligned}
&\bigcap\nolimits_{x\in A(M_u(K),R)} B(x,R)\\
&\quad = \bigcap\nolimits_{\theta\in S^{n-1}} B\Big(\rho_{M_u(K)}(\theta),R\Big)\\
&\quad \supseteq \frac{1}{2}\bigcap\nolimits_{\theta\in S^{n-1}} B(\rho_{A(K,R)}(\theta),R) + \frac{1}{2}\bigcap\nolimits_{\theta\in S^{n-1}} B(\rho_{A(L,R)}(\theta),R)\\
&\quad = \frac{1}{2}\bigcap\nolimits_{x\in A(K,R)} B(x,R) + \frac{1}{2}\bigcap\nolimits_{x\in A(R_u(K),R)} B(x,R).
\end{aligned}$$

Since the latter two sets are reflections of each other, we can apply the Brunn-Minkowski inequality to get

$$V_j\Big(\bigcap\nolimits_{x\in A(M_u(K),R)} B(x,R)\Big) \geq V_j\Big(\bigcap\nolimits_{x\in A(K,R)} B(x,R)\Big). \tag{5.3}$$

It is known that given a convex body K, there is a sequence of directions so that successive Minkowski symmerizations about those directions converge to a Euclidean ball with the same mean-width as K (e.g., [21]). Combining this with inequality (5.3), we get another proof of (5.2), which can be interpreted as a (non-random) version of the generalized Urysohn inequality for ball-polyhedra.

5.3 Connection between random ball-polyhedra and random convex hulls

As mentioned already, the inequality for random ball-polyhedra obtained by taking $j = n$ in (1.6) implies Urysohn's inequality, and so does the inequality for random convex hulls when $j = 1$ in (1.4). Here we show that the former implies the latter. The proof uses the following theorem of Gorbovickis [11, Theorem 4].

Theorem 5.2. *Let $x_1 \dots, x_N \in \mathbb{R}^n$ where $n \geq 2$. Then the following asymptotic equality holds as $R \to \infty$:*

$$\left|\left(\bigcap_{i=1}^N B(x_i, R)\right)\right| = \omega_n R^n - n\omega_n w(\mathrm{conv}\{x_1, \dots, x_N\}) R^{n-1} + o(R^{n-1}). \tag{5.4}$$

Assume that K is a convex body in $\mathbb{R}^n$ with $|K| = |B|$. Sample independent random vectors $X_1, \dots, X_N$ in K and $Z_1, \dots, Z_N$ in B according to their respective uniform probability measures. For each fixed value of $X_1, \dots, X_N$, Theorem 5.2 implies

$$n\omega_n w(\mathrm{conv}\{X_1, \dots, X_N\}) = R - R^{-(n-1)} \left|\left(\bigcap_{i=1}^N B(X_i, R)\right)\right| + o(1), \tag{5.5}$$

as $R \to \infty$. By compactness of K, we can use dominated convergence to conclude

$$n\omega_n \mathbb{E} w(\mathrm{conv}\{X_1, \dots, X_N\}) = R - R^{-(n-1)} \mathbb{E} \left|\left(\bigcap_{i=1}^N B(X_i, R)\right)\right| + \mathbb{E} o(1),$$

as $R \to \infty$. By continuity of the volume of the intersection and the mean-width, the quantity $\mathbb{E} o(1)$ is also of the form $o(1)$. The same argument applies to $Z_1, \dots, Z_N$. By Theorem 3.1, we get

$$\mathbb{E} w(\mathrm{conv}\{X_1, \dots, X_N\}) \geqslant \mathbb{E} w(\mathrm{conv}\{Z_1, \dots, Z_N\}),$$

which is equivalent to the $j = 1$ case in (1.4).

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