A central limit theorem for projections of the cube

Grigoris Paouris * Peter Pivovarov † J. Zinn ‡

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Abstract

We prove a central limit theorem for the volume of projections of the cube $[-1,1]^N$ onto a random subspace of dimension n, when n is fixed and $N \to \infty$. Randomness in this case is with respect to the Haar measure on the Grassmannian manifold.

1 Main result

The focus of this paper is the volume of random projections of the cube $B_\infty^N = [-1,1]^N$ in $\mathbb{R}^N$. To fix the notation, let $n \geqslant 1$ be an integer and for $N \geqslant n$, let $G_{N,n}$ denote the Grassmannian manifold of all n-dimensional linear subspaces of $\mathbb{R}^N$. Equip $G_{N,n}$ with the Haar probability measure $\nu_{N,n}$, which is invariant under the action of the orthogonal group. Suppose that $(E(N))_{N\geqslant n}$ is a sequence of random subspaces with $E(N)$ distributed according to $\nu_{N,n}$. We consider the random variables

$$Z_N = |P_{E(N)} B_\infty^N|, \tag{1.1}$$

where $P_{E(N)}$ denotes the orthogonal projection onto $E(N)$ and $|\cdot|$ is n-dimensional volume, when n is fixed and $N \to \infty$. We show that Z_N satisfies the following central limit theorem.

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Theorem 1.1.

$$\frac{Z_N - \mathbb{E}Z_N}{\sqrt{\mathrm{var}(Z_N)}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty. \tag{1.2}$$

Here $\xrightarrow{d}$ denotes convergence in distribution and $\mathcal{N}(0,1)$ a standard Gaussian random variable with mean 0 and variance 1. Our choice of scaling for the cube is immaterial as the quantity in (1.2) is invariant under scaling and translation of $[-1,1]^N$.

Gaussian random matrices play a central role in the proof of Theorem 1.1, as is often the case with results about random projections onto subspaces $E \in G_{N,n}$. Specifically, we let G be an $n \times N$ random matrix with independent columns $g_1, \ldots, g_N$ distributed according to standard Gaussian measure γ_n on $\mathbb{R}^n$, i.e.,

$$d\gamma_n(x) = (2\pi)^{-n/2} e^{-\|x\|_2^2/2} dx.$$

We view G as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$. If $C \subset \mathbb{R}^N$ is any convex body, then

$$|GC| = \det(GG^*)^{\frac{1}{2}} |P_E C|, \tag{1.3}$$

where $E = \mathrm{Range}(G^*)$ is distributed uniformly on $G_{N,n}$. Moreover, $\det(GG^*)^{1/2}$ and $|P_E C|$ are independent. The latter fact underlies the Gaussian representation of intrinsic volumes, as proved by B. Tsirelson in [23] (see also [27]); it is also used in R. Vitale's probabilistic derivation of the Steiner formula [26]. Passing between Gaussian vectors and random orthogonal projections is useful in a variety of contexts, e.g., [12], [15], [1], [5], [6], [13], [8], [17]. As we will show, however, it is a delicate matter to use (1.3) to prove limit theorems, especially with the normalization required in Theorem 1.1. Our path will involve analyzing asymptotic normality of $|GB_\infty^N|$ before dealing with the quotient $|GB_\infty^N| / \det(GG^*)^{1/2}$.

The set

$$GB_\infty^N = \left\{ \sum_{i=1}^{N} \lambda_i g_i : |\lambda_i| \leqslant 1, i = 1, \ldots, N \right\}$$

is a random zonotope, i.e., a Minkowski sum of the random segments $[-g_i, g_i] = \{\lambda g_i : |\lambda| \leqslant 1\}$. By the well-known zonotope volume formula (e.g. [14]), $X_N = |GB_\infty^N|$ satisfies

$$X_N = 2^n \sum_{1 \leqslant i_1 < \ldots < i_n \leqslant N} |\det[g_{i_1} \cdots g_{i_n}]|, \tag{1.4}$$

where $\det[g_{i_1} \cdots g_{i_n}]$ is the determinant of the matrix with columns $g_{i_1}, \ldots, g_{i_n}$. The quantity

$$U_N = \frac{1}{\binom{N}{n}} \sum_{1 \leqslant i_1 < \ldots < i_n \leqslant N} |\det[g_{i_1} \cdots g_{i_n}]|$$

is a U-statistic and central limit theorems for U-statistics go back to W. Hoeffding [11]. In fact, formula (1.4) for X_N is simply a special case of Minkowski's theorem on mixed volumes of convex sets (see §2). In [25], R. Vitale proved a central limit theorem for Minkowski sums of more general random convex sets, using mixed volumes and U-statistics (discussed in detail below). In particular, it follows from Vitale's results that X_N satisfies a central limit theorem, namely,

$$\frac{X_N - \mathbb{E}X_N}{s_{N,n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{1.5}$$

where $s_{N,n}$ is a certain conditional standard deviation (see Theorem 3.3). Using Vitale's result and a more recent randomization inequality for U-statistics [7, Chapter 3], we show in §4 that X_N satisfies a central limit theorem with the canonical normalization:

$$\frac{X_N - \mathbb{E}X_N}{\sqrt{\operatorname{var}(X_N)}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty. \tag{1.6}$$

It is tempting to think that the latter central limit theorem for X_N easily yields Theorem 1.1. However, for a family of convex bodies $C = C_N \subset \mathbb{R}^N$, $N = n, n+1, \ldots$, asymptotic normality of $|GC|$ is not sufficient to conclude that $|P_{E(N)}C|$ is asymptotically normal. For example, if $C = B_2^N$, then $|GB_2^N| = \det(GG^*)^{1/2}|B_2^n|$ is asymptotically normal (e.g., [2, Theorems 4.2.3, 7.5.3]), however $|P_{E(N)}B_2^N|$ is constant.

In fact, as we show in Proposition 4.4, both X_N and $\det(GG^*)^{1/2}$ contribute to asymptotic normality of $Z_N = |P_{E(N)}B_\infty^N|$, a technical difficulty that requires careful analysis. In particular, the aforementioned randomization inequality from [7, Chapter 3] is invoked again to deal with the canonical normalization for Z_N in Theorem 1.1. As a by-product, we also obtain the limiting behavior of the variance of Z_N as $N \to \infty$.

We mention that when $n = 1$, Theorem 1.1 implies that if (θ_N) is a sequence of random vectors with θ_N distributed uniformly on the sphere S^{N-1}, then the ℓ_1-norm $\|\cdot\|_1$ (the support function of the cube) satisfies

$$\frac{\|\theta_N\|_1 - \mathbb{E}\|\theta_N\|_1}{\sqrt{\mathrm{var}(\|\theta_N\|_1)}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty.$$

The central limit theorem for X_N in (1.6) can be seen as a counter-part to a recent result of I. Bárány and V. Vu [4] for convex hulls of Gaussian vectors. In particular, when $n \geqslant 2$ the quantity $V_N = |\mathrm{conv}\{g_1, \ldots, g_N\}|$ satisfies

$$\frac{V_N - \mathbb{E}V_N}{\sqrt{\mathrm{var}(V_N)}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty;$$

see the latter article for the corresponding Berry-Esseen type estimate. The latter result is one of several recent deep central limit theorems in stochastic geometry concerning random convex hulls, e.g., [19], [28], [3]. The techniques used in this paper are different and the main focus here is to understand the Grassmannian setting.

Lastly, for a thorough exposition of the properties of the cube, see [29].

2 Preliminaries

The setting is $\mathbb{R}^n$ with the usual inner-product $\langle \cdot, \cdot \rangle$ and Euclidean norm $\|\cdot\|_2$; n-dimensional Lebesgue measure is denoted by $|\cdot|$. For sets $A, B \subset \mathbb{R}^n$ and scalars $\alpha, \beta \in \mathbb{R}$, we define $\alpha A + \beta B$ by usual scalar multiplication and Minkowski addition: $\alpha A + \beta B = \{\alpha a + \beta b : a \in A, b \in B\}$.

2.1 Mixed volumes

The mixed volume $V(K_1,\ldots,K_n)$ of compact convex sets $K_1,\ldots,K_n$ in $\mathbb{R}^n$ is defined by

$$V(K_1,\ldots,K_n) = \frac{1}{n!}\sum_{j=1}^{n}(-1)^{n+j}\sum_{i_1<\ldots<i_j}\left|K_{i_1}+\ldots+K_{i_j}\right|.$$

By a theorem of Minkowski, if $t_1,\ldots,t_N$ are non-negative real numbers then the volume of $K = t_1K_1+\ldots+t_NK_N$ can be expressed as

$$|K| = \sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N} V(K_{i_1},\ldots,K_{i_n})t_{i_1}\cdots t_{i_n}. \tag{2.1}$$

The coefficients $V(K_{i_1},\ldots,K_{i_n})$ are non-negative and invariant under permutations of their arguments. When the K_i's are origin-symmetric line segments, say $K_i = [-x_i,x_i] = \{\lambda x_i : |\lambda| \leqslant 1\}$, for some $x_1,\ldots,x_n \in \mathbb{R}^n$, we simplify the notation and write

$$V(x_1,\ldots,x_n) = V([-x_1,x_1],\ldots,[-x_n,x_n]). \tag{2.2}$$

We will make use of the following properties:

(i) $V(K_1,\ldots,K_n) > 0$ if and only if there are line segments $L_i \subset K_i$ with linearly independent directions.

(ii) If $x_1,\ldots,x_n \in \mathbb{R}^n$, then

$$n!V(x_1,\ldots,x_n) = 2^n|\det[x_1\cdots x_n]|, \tag{2.3}$$

where $\det[x_1\cdots x_n]$ denotes the determinant of the matrix with columns $x_1,\ldots,x_n$.

(iii) $V(K_1,\ldots,K_n)$ is increasing in each argument (with respect to inclusion).

For further background we refer the reader to [21, Chapter 5] or [10, Appendix A].

A *zonotope* is a Minkowski sum of line segments. If $x_1,\ldots,x_N$ are vectors in $\mathbb{R}^n$, then

$$\sum_{i=1}^{N}[-x_i,x_i] = \left\{\sum_{i=1}^{N}\lambda_i x_i : |\lambda_i| \leqslant 1,\ i = 1,\ldots,N\right\}.$$

Alternatively, a zonotope can be seen as a linear image of the cube $B_\infty^N = [-1,1]^N$. If $x_1,\ldots,x_N \in \mathbb{R}^n$, one can view the $n\times N$ matrix $X = [x_1 \cdots x_N]$ as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$; in this case, $XB_\infty^N = \sum_{i=1}^N [-x_i, x_i]$.

By (2.1) and properties (i) and (ii) of mixed volumes, the volume of $\sum_{i=1}^N [-x_i, x_i]$ satisfies

$$\left|\sum_{i=1}^N [-x_i, x_i]\right| = 2^n \sum_{1\leqslant i_1<\ldots<i_n\leqslant N} |\det[x_{i_1}\cdots x_{i_n}]|. \tag{2.4}$$

Note that for $x_1,\ldots,x_n \in \mathbb{R}^n$,

$$|\det[x_1\cdots x_n]| = \|x_1\|_2 \|P_{F_1^\perp} x_2\|_2 \cdots \|P_{F_{n-1}^\perp} x_n\|_2, \tag{2.5}$$

where $F_k = \operatorname{span}\{x_1,\ldots,x_k\}$ for $k = 1,\ldots,n-1$ (which can be proved using Gram-Schmidt orthogonalization, e.g., [2, Theorem 7.5.1]).

We will also use the Cauchy-Binet formula. Let $x_1,\ldots,x_N \in \mathbb{R}^n$ and let X be the $n\times N$ matrix with columns $x_1,\ldots,x_N$, i.e., $X = [x_1\cdots x_N]$. Then

$$\det(XX^*)^{\frac{1}{2}} = \sum_{1\leqslant i_1<\ldots<i_n\leqslant N} \det[x_{i_1}\cdots x_{i_n}]^2; \tag{2.6}$$

for a proof, see, e.g., [9, §3.2].

2.2 Slutsky's theorem

We will make frequent use of Slutsky's theorem on convergence of random variables (see, e.g., [22, §1.5.4]).

Theorem 2.1. *Let (X_N) and (α_N) be sequences of random variables. Suppose that $X_N \xrightarrow{d} X_0$ and $\alpha_N \xrightarrow{\mathbb{P}} \alpha_0$, where α_0 is a finite constant. Then*

$$X_N + \alpha_N \xrightarrow{d} X_0 + \alpha_0$$

and

$$\alpha_N X_N \xrightarrow{d} \alpha_0 X_0.$$

Slutsky's theorem also applies when the X_N's take values in $\mathbb{R}^k$ and satisfy $X_N \xrightarrow{d} X_0$ and (A_N) is a sequence of $m\times k$ random matrices such that $A_N \xrightarrow{\mathbb{P}} A_0$ and the entries of A_0 are constants. In this case, $A_N X_N \xrightarrow{d} A_0 X_0$.

3 U-statistics

In this section, we give the requisite results from the theory of U-statistics needed to prove asymptotic normality of X_N and Z_N stated in the introduction. For further background on U-statistics, see e.g. [22], [20], [7].

Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with values in a measurable space $(S, \mathcal{S})$. Let $h : S^m \to \mathbb{R}$ be a measurable function. For $N \geqslant m$, the U-statistic of order m with kernel h is defined by

$$U_N = U_N(h) = \frac{(N-m)!}{N!} \sum_{(i_1,\dots,i_m)\in I_N^m} h(X_{i_1}, \dots, X_{i_m}), \tag{3.1}$$

where

$$I_N^m = \left\{(i_1, \dots, i_m) : i_j \in \mathbb{N}, 1 \leqslant i_j \leqslant N, i_j \neq i_k \text{ if } j \neq k\right\}.$$

When h is symmetric, i.e., $h(x_1, \dots, x_m) = h(x_{\sigma(1)}, \dots, x_{\sigma(m)})$ for every permutation σ of m elements, we can write

$$U_N = U(X_1, \dots, X_N) = \frac{1}{\binom{N}{m}} \sum_{1 \leqslant i_1 < \dots < i_m \leqslant N} h(X_{i_1}, \dots, X_{i_m}); \tag{3.2}$$

here the sum is taken over all $\binom{N}{m}$ subsets $\{i_1, \dots, i_m\}$ of $\{1, \dots, N\}$.

Using the latter notation, we state several well-known results, due to Hoeffding (see, e.g., [22, Chapter 5]).

Theorem 3.1. *For $N \geqslant m$, let U_N be a statistic with kernel $h : S^m \to \mathbb{R}$. Set $\zeta = \mathrm{var}(\mathbb{E}[h(X_1, \dots, X_m)|X_1])$.*

(1) The variance of U_N satisfies

$$\mathrm{var}(U_N) = \frac{m^2\zeta}{N} + O(N^{-2}) \text{ as } N \to \infty.$$

(2) If $\mathbb{E}|h(X_1, \dots, X_m)| < \infty$, then $U_N \xrightarrow{a.s.} \mathbb{E}U_N$ as $N \to \infty$.

(3) If $\mathbb{E}h^2(X_1, \dots, X_m) < \infty$ and $\zeta > 0$, then

$$\sqrt{N}\left(\frac{U_N - \mathbb{E}U_N}{m\sqrt{\zeta}}\right) \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty.$$

The corresponding Berry-Esseen type bounds are also available (see, e.g,. [22, page 193]), stated here in terms of the function

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2} ds.$$

Theorem 3.2. *With the preceding notation, suppose that* $\xi = \mathbb{E}|h(X_1,\ldots,X_m)|^3 < \infty$ *and*

$$\zeta = \operatorname{var}(\mathbb{E}[h(X_1,\ldots,X_m)|X_1]) > 0.$$

Then

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\sqrt{N} \left(\frac{U_N - \mathbb{E}U_N}{m\sqrt{\zeta}} \right) \leqslant t \right) - \Phi(t) \right| \leqslant \frac{c\xi}{(m^2\zeta)^{\frac{3}{2}}\sqrt{N}},$$

where $c > 0$ *is an universal constant.*

3.1 U-statistics and mixed volumes

Let $\mathcal{C}_n$ denote the class of all compact, convex sets in $\mathbb{R}^n$. A topology on $\mathcal{C}_N$ is induced by the Hausdorff metric

$$\delta^H(K,L) = \inf\{\delta > 0 : K \subset L + \delta B_2^n, L \subset K + \delta B_2^n\},$$

where B_2^n is the Euclidean ball of radius one. A random convex set is a Borel measurable map from a probability space into $\mathcal{C}_n$. A key ingredient in our proof is the following theorem for Minkowski sums of random convex sets due to R. Vitale [25]; we include the proof for completeness.

Theorem 3.3. *Let* $n \geqslant 1$ *be an integer. Suppose that* $K_1, K_2, \ldots$ *are i.i.d. random convex sets in* $\mathbb{R}^n$ *such that* $\mathbb{E}\sup_{x\in K_1} \|x\|_2 < \infty$. *Set* $V_N = |\sum_{i=1}^N K_i|$ *and suppose that* $\mathbb{E}V(K_1,\ldots,K_n)^2 < \infty$ *and furthermore that* $\zeta = \operatorname{var}(\mathbb{E}[V(K_1,\ldots,K_n)|K_1]) > 0$. *Then*

$$\sqrt{N}\left(\frac{V_N - \mathbb{E}V_N}{(N)_n n\sqrt{\zeta}} \right) \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty,$$

where $(N)_n = \frac{N!}{(N-n)!}$.

Proof. Taking $h : (\mathcal{C}_n)^n \to \mathbb{R}$ to be $h(K_1,\ldots,K_n) = V(K_1,\ldots,K_n)$ and using (2.1), we have

$$\frac{1}{(N)_n} V_N = U_N + \frac{1}{(N)_n} \sum_{(i_1,\ldots,i_n)\in J} V(K_{i_1},\ldots,K_{i_n}) \tag{3.3}$$

where

$$U_N = \frac{1}{(N)_n} \sum_{(i_1,\ldots,i_n)\in I_N^n} V(K_{i_1},\ldots,K_{i_n}),$$

and $J = \{1,\ldots,N\}^n \setminus I_N^n$. Note that $|J|/(N)_n = O(\frac{1}{N})$ and thus the second term on the right-hand side of (3.3) tends to zero in probability. Applying Theorem 3.1(3) and Slutsky's theorem leads to the desired conclusion. □

In the special case when the K_i's are line segments, say $K_i = [-X_i, X_i]$ where $X_1, X_2, \ldots$ are i.i.d. random vectors in $\mathbb{R}^n$, the assumptions in the latter theorem can be readily verified by using (2.3). Furthermore, if the X_i's are rotationally-invariant, the assumptions simplify further as follows (essentially from [25], stated here in a form that best serves our purpose).

Corollary 3.4. *Let $X = R\theta$ be a random vector such that θ is uniformly distributed on the sphere S^{n-1} and $R \geqslant 0$ is independent of θ and satisfies $\mathbb{E}R^2 < \infty$ and $\mathrm{var}(R) > 0$. For each $i = 1, 2, \ldots$, let $X_i = R_i\theta_i$ be independent copies of X. Let $D_n = |\det[\theta_1 \cdots \theta_n]|$ and set*

$$\zeta_1 = 4^n \mathrm{var}(R) \mathbb{E}^{2(n-1)} R \mathbb{E}^2 D_n.$$

Then $V_N = |\sum_{i=1}^N [-X_i, X_i]|$ satisfies

$$\sqrt{N}\left(\frac{V_N - \mathbb{E}V_N}{\binom{N}{n} n \sqrt{\zeta_1}}\right) \to \mathcal{N}(0,1) \text{ as } N \to \infty.$$

Proof. Plugging $X_i = R_i\theta_i$, $i = 1, \ldots, n$, into (2.3) gives

$$n! V(X_1, \ldots, X_n) = 2^n R_1 \cdots R_n D_n. \tag{3.4}$$

By (2.5),

$$D_n = \|\theta_1\|_2 \|P_{F_1^\perp}\theta_2\|_2 \cdots \|P_{F_{n-1}^\perp}\theta_n\|_2, \tag{3.5}$$

with $F_k = \mathrm{span}\{\theta_1, \ldots, \theta_k\}$ for $k = 1, \ldots, n-1$. In particular, $D_n \leqslant 1$ and thus (3.4) implies

$$\mathbb{E}V(X_1, \ldots, X_n)^2 \leqslant \frac{4^n}{(n!)^2} \mathbb{E}^n R^2 < \infty.$$

Using (3.4) once more, together with (3.5), we have

$$n! \mathbb{E}[V(X_1, \ldots, X_n) | X_1] = 2^n R_1 \mathbb{E}R_2 \cdots \mathbb{E}R_n \mathbb{E}D_n; \tag{3.6}$$

here we have used the fact that $\mathbb{E}\|P_{F_k^\perp}\theta_{k+1}\|_2$ depends only on the dimension of F_k (which is equal to k a.s.) and that $\|\theta_1\|_2 = 1$ a.s. By (3.6) and our assumption $\text{var}(R) > 0$, we can apply Theorem 3.3 with

$$\zeta = \text{var}(\mathbb{E}[V(X_1,\ldots,X_n)|X_1]) = \frac{\zeta_1}{(n!)^2} > 0,$$

where ζ_1 is defined in the statement of the corollary. □

For further information on Theorem 3.3, including a CLT for the random sets themselves, or the case when $\zeta = 0$, see [25] or [16, Pg 232]; see also [24].

Corollary 3.4 implies the first central limit theorem for X_N stated in the introduction (1.5). However, to recover the central limit theorem for X_N in (1.6), involving the variance $\text{var}(X_N)$ and not a conditional variance, some additional tools are needed.

3.2 Randomization

In this subsection, we discuss a randomization inequality for U-statistics. It will be used for variance estimates, the proof of the central limit theorem for X_N in (1.6) and it will also play a crucial role in the proof of Theorem 1.1.

Using the notation at the beginning of §3, suppose that $h : (\mathbb{R}^n)^m \to \mathbb{R}$ satisfies $\mathbb{E}|h(X_1,\ldots,X_m)| < \infty$ and let $1 < r \leqslant m$. Following [7, Definition 3.5.1], we say that h is degenerate of order $r-1$ if

$$\mathbb{E}_{X_r,\ldots,X_m} h(x_1,\ldots,x_{r-1},X_r,\ldots,X_m) = \mathbb{E}h(X_1,\ldots,X_m)$$

for all $x_1,\ldots,x_{r-1} \in \mathbb{R}^n$, and the function

$$S^r \ni (x_1,\ldots,x_r) \mapsto \mathbb{E}_{X_{r+1},\ldots,X_m} h(x_1,\ldots,x_r,X_{r+1},\ldots,X_m)$$

is non-constant. If h is not degenerate of any positive order r, we say it is non-degenerate or degenerate of order 0. We will make use of the following randomization theorem, which is a special case of [7, Theorem 3.5.3].

Theorem 3.5. *Let $1 \leqslant r \leqslant m$ and $p \geqslant 1$. Suppose that $h : S^m \to \mathbb{R}$ is degenerate of order $r-1$ and $\mathbb{E}|h(X_1,\ldots,X_m)|^p < \infty$. Set*

$$f(x_1,\ldots,x_m) = h(x_1,\ldots,x_m) - \mathbb{E}h(X_1,\ldots,X_m).$$

Let $\varepsilon_1, \dots, \varepsilon_N$ denote i.i.d. Rademacher random variables, independent of $X_1, \dots, X_N$. Then

$$\mathbb{E}\Big| \sum_{(i_1,\dots,i_m)\in I_N^m} f(X_{i_1}, \dots, X_{i_m})\Big|^p \simeq_{m,p} \mathbb{E}\Big| \sum_{(i_1,\dots,i_m)\in I_N^m} \varepsilon_{i_1} \cdots \varepsilon_{i_r} f(X_{i_1}, \dots, X_{i_m})\Big|^p.$$

Here $A \simeq_{m,p} B$ means $C'_{m,p}A \leqslant B \leqslant C''_{m,p}A$, where $C'_{m,p}$ and $C''_{m,p}$ are constants that depend only on m and p.

Corollary 3.6. *Let μ be probability measure on $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure. Suppose that $X_1, \dots, X_N$ are i.i.d. random vectors distributed according to μ. Let $p \geqslant 2$ and suppose $\mathbb{E}|\det[X_1 \cdots X_n]|^p < \infty$. Define $f : (\mathbb{R}^n)^n \to \mathbb{R}$ by*

$$f(x_1, \dots, x_n) = |\det[x_1 \cdots x_n]| - \mathbb{E}|\det[X_1 \cdots X_n]|.$$

Then

$$\mathbb{E}\Big| \sum_{1\leqslant i_1<\dots<i_n\leqslant N} f(X_{i_1}, \dots, X_{i_n})\Big|^p \leqslant C_{n,p} N^{p(n-\frac{1}{2})} \mathbb{E}|f(X_1, \dots, X_n)|^p,$$

where $C_{n,p}$ is a constant that depends on n and p.

Proof. Since μ is absolutely continuous, $\dim(\mathrm{span}\{X_1, \dots, X_k\}) = k$ a.s. for $k = 1, \dots, n$. Moreover, $f(ax_1, \dots, x_n) = |a| f(x_1, \dots, x_n)$ for any $a \in \mathbb{R}$, hence f is non-degenerate (cf. (2.5)). Thus we may apply Theorem 3.5 with $r = 1$:

$$\begin{aligned}\mathbb{E}\Big| \sum_{1\leqslant i_1<\dots<i_n\leqslant N} n! f(X_{i_1}, \dots, X_{i_n})\Big|^p &= \mathbb{E}\Big| \sum_{(i_1,\dots,i_n)\in I_N^n} f(X_{i_1}, \dots, X_{i_n})\Big|^p \\ &\leqslant C_{n,p} \mathbb{E}\Big| \sum_{(i_1,\dots,i_n)\in I_N^n} \varepsilon_{i_1} f(X_{i_1}, \dots, X_{i_n})\Big|^p.\end{aligned}$$

Suppose now that $X_1, \dots, X_N$ are fixed. Taking expectation in $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ and appling Khintchine's inequality and then Hölder's inequality twice, we have

$$
\begin{aligned}
&\mathbb{E}_\varepsilon\Big|\sum_{(i_1,\dots,i_n)\in I_N^n}\varepsilon_{i_1} f(X_{i_1},\dots,X_{i_n})\Big|^p\\
&= \mathbb{E}_\varepsilon\Big|\sum_{i_1=1}^{N}\varepsilon_{i_1}\sum_{\substack{(i_2,\dots,i_n)\\(i_1,\dots,i_n)\in I_N^n}} f(X_{i_1},\dots,X_{i_n})\Big|^p\\
&\leqslant C\Big|\sum_{i_1=1}^{N}\Big(\sum_{\substack{(i_2,\dots,i_n)\\(i_1,\dots,i_n)\in I_N^n}} f(X_{i_1},\dots,X_{i_n})\Big)^2\Big|^{\frac{p}{2}}\\
&\leqslant C\left(\binom{N-1}{n-1}(n-1)!\right)^{\frac{p}{2}}\Big|\sum_{(i_1,\dots,i_n)\in I_N^n} f(X_{i_1},\dots,X_{i_n})^2\Big|^{\frac{p}{2}}\\
&\leqslant C\left(\binom{N-1}{n-1}(n-1)!\right)^{\frac{p}{2}}\left(\binom{N}{n}n!\right)^{\frac{p-2}{2}}\sum_{(i_1,\dots,i_n)\in I_N^n}|f(X_{i_1},\dots,X_{i_n})|^p,
\end{aligned}
$$

where C is an absolute constant. Taking expectation in the X_i's gives

$$
\begin{aligned}
&\mathbb{E}\Big|\sum_{(i_1,\dots,i_n)\in I_N^n}\varepsilon_{i_1} f(X_{i_1},\dots,X_{i_n})\Big|^p\\
&\leqslant \left(\binom{N-1}{n-1}(n-1)!\right)^{\frac{p}{2}}\left(\binom{N}{n}n!\right)^{\frac{p-2}{2}}\binom{N}{n}n!\,\mathbb{E}|f(X_1,\dots,X_n)|^p.
\end{aligned}
$$

The proposition follows as stated by using the estimate $\binom{N}{n}\leqslant (eN/n)^n$. □

4 Proof of Theorem 1.1

As explained in the introduction, our first step is identity (1.3), the proof of which is included for completeness.

Proposition 4.1. *Let $N\geqslant n$ and let G be an $n\times N$ random matrix with i.i.d. standard Gaussian entries. Let $C\subset\mathbb{R}^N$ be a convex body. Then*

$$|GC| = \det(GG^*)^{\frac{1}{2}}\,|P_E C|, \tag{4.1}$$

where $E = \mathrm{Range}(G^*)$. *Moreover,* E *is distributed uniformly on* $G_{N,n}$ *and* $\det(GG^*)^{\frac{1}{2}}$ *and* $|P_E C|$ *are independent.*

Proof. Identity (4.1) follows from polar decomposition; see, e.g., [17, Theorem 2.1(iii)]. To prove that the two factors are independent, we note that if U is an orthogonal transformation, we have $\det(GG^*)^{1/2} = \det((GU)(GU)^*)^{1/2}$; moreover, G and GU have the same distribution. Thus if U is a random orthogonal transformation distributed according to the Haar measure, we have for $s, t \geqslant 0$,

$$\begin{aligned}
&\mathbb{P}_{\otimes\gamma_n}\left(\det(GG^*)^{1/2} \leqslant s, |P_{\mathrm{Range}(G^*)}C| \leqslant t\right) \\
&\quad = \mathbb{P}_{\otimes\gamma_n} \otimes \mathbb{P}_U \left(\det(GG^*)^{1/2} \leqslant s, |P_{\mathrm{Range}(U^*G^*)}C| \leqslant t\right) \\
&\quad = \mathbb{E}_{\otimes\gamma_n}\left(\mathbb{1}_{\{\det(GG^*)^{1/2} \leqslant s\}} \mathbb{E}_U \mathbb{1}_{\{|P_{U^*\mathrm{Range}(G^*)}C| \leqslant t\}}\right) \\
&\quad = \mathbb{P}_{\otimes\gamma_n}\left(\det(GG^*)^{1/2} \leqslant s\right) \nu_{N,n}\left(E \in G_{N,n} : |P_E C| \leqslant t\right).
\end{aligned}$$

□

Taking $C = B_\infty^N$ in (4.1), we set

$$X_N = \left|GB_\infty^N\right| = 2^n \sum_{1 \leqslant i_1 < \ldots < i_n \leqslant N} |\det[g_{i_1} \cdots g_{i_n}]| \tag{4.2}$$

(cf. (2.4)),

$$Y_N = \det(GG^*)^{\frac{1}{2}} = \left(\sum_{1 \leqslant i_1 < \ldots < i_n \leqslant N} \det[g_{i_1} \cdots g_{i_m}]^2\right)^{\frac{1}{2}} \tag{4.3}$$

(cf. (2.6)), and

$$Z_N = \left|P_E B_\infty^N\right|, \tag{4.4}$$

where E is distributed according to $\nu_{N,n}$ on $G_{N,n}$. Then $X_N = Y_N Z_N$, where Y_N and Z_N are independent. In order to prove Theorem 1.1, we start with several properties of X_N and Y_N.

Proposition 4.2. *Let* X_N *be as defined in (4.2).*

(1) For each $p \geqslant 2$,

$$\mathbb{E}|X_N - \mathbb{E}X_N|^p \leqslant C_{n,p} N^{p(n-\frac{1}{2})}.$$

(2) *The variance of* X_N *satisfies*

$$\frac{\mathrm{var}(X_N)}{N^{2n-1}} \to c_n \text{ as } N \to \infty,$$

where c_n *is a positive constant that depends only on* n*.*

(3) X_N *is asymptotically normal; i.e.,*

$$\frac{X_N - \mathbb{E}X_N}{\sqrt{\mathrm{var}(X_N)}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty.$$

Proof. Statement (1) follows from Corollary 3.6. To prove (2), let g be a random vector distributed according to γ_n. Then Corollary 3.4 with $\zeta_1 = 4^n \mathrm{var}(\|g\|_2)\mathbb{E}^{2(n-1)}\|g\|_2\mathbb{E}^2 D_n$ yields

$$\sqrt{N}\left(\frac{X_N - \mathbb{E}X_N}{\binom{N}{n} n\sqrt{\zeta_1}}\right) \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty. \tag{4.5}$$

On the other hand, by part (1) we have

$$\frac{\mathbb{E}|X_N - \mathbb{E}X_N|^4}{N^{4n-2}} \leqslant C_{n,p}.$$

This implies that the sequence $(X_N - \mathbb{E}X_N)/N^{n-\frac{1}{2}}$ is uniformly integrable, hence

$$\frac{\sqrt{\mathrm{var}(X_N)}}{N^{-\frac{1}{2}}\binom{N}{n} n\sqrt{\zeta_1}} \to 1 \text{ as } N \to \infty.$$

Part (3) now follows from (4.5) and Slutsky's theorem. □

We now turn to $Y_N = \det(GG^*)^{\frac{1}{2}}$. It is well-known that

$$Y_N = \chi_N \chi_{N-1} \cdot \ldots \cdot \chi_{N-n+1}, \tag{4.6}$$

where $\chi_k = \sqrt{\chi_k^2}$ and the χ_k^2's are independent chi-squared random variables with k degrees of freedom, $k = N, \ldots, N-n+1$ (see, e.g., [2, Chapter 7]). Consequently,

$$\mathbb{E}Y_N^2 = \frac{N!}{(N-n)!} = N^n\left(1 - \frac{1}{N}\right)\cdots\left(1 - \frac{n-1}{N}\right).$$

Additionally, we will use the following basic properties of Y_N.

Proposition 4.3. *Let Y_N be as defined in (4.3).*

(1) For each $p \geqslant 2$,

$$\mathbb{E}|Y_N^2 - \mathbb{E}Y_N^2|^p \leqslant C_{n,p} N^{p(n-\frac{1}{2})}.$$

(2) The variance of Y_N satisfies

$$\frac{\mathrm{var}(Y_N)}{N^{n-1}} \to \frac{n}{2} \text{ as } N \to \infty.$$

(3) Y_N^2 is asymptotically normal; i.e.,

$$\sqrt{N}\left(\frac{Y_N^2}{N^n} - 1\right) \xrightarrow{d} \mathcal{N}(0, 2n) \text{ as } N \to \infty.$$

Proof. To prove part (1), we apply Corollary 3.6 to Y_N^2.

To prove part (2), we use (4.6) and define $Y_{N,n}$ by $Y_{N,n} = Y_N = \chi_N \chi_{N-1} \cdot \ldots \cdot \chi_{N-n+1}$ and procede by induction on n. Suppose first that $n = 1$ so that $Y_{N,1} = \chi_N$. By the concentration of Gaussian measure (e.g., [18, Remark 4.8]), there is an absolute constant c_1 such that $\mathbb{E}|\chi_N - \mathbb{E}\chi_N|^4 < c_1$ for all N, which implies that the sequence $(\chi_N - \mathbb{E}\chi_N)_N$ is uniformly integrable. By the law of large numbers $\chi_N/\sqrt{N} \to 1$ a.s. and hence $\mathbb{E}\chi_N/\sqrt{N} \to 1$, by uniform integrability. Note that

$$\begin{aligned}
\chi_N - \mathbb{E}\chi_N &= \frac{\chi_N^2 - \mathbb{E}^2\chi_N}{\chi_N + \mathbb{E}\chi_N} \\
&= \frac{\sqrt{N}}{\chi_N + \mathbb{E}\chi_N}\frac{\chi_N^2 - N}{\sqrt{N}} + \frac{\sqrt{N}}{\chi_N + \mathbb{E}\chi_N}\frac{N - \mathbb{E}^2\chi_N}{\sqrt{N}}.
\end{aligned}$$

By Slutsky's theorem and the classical central limit theorem,

$$\frac{\sqrt{N}}{\chi_N + \mathbb{E}\chi_N}\frac{\chi_N^2 - N}{\sqrt{N}} \xrightarrow{d} \frac{1}{2}\mathcal{N}(0,2) \text{ as } N \to \infty,$$

while

$$\frac{\sqrt{N}}{\chi_N + \mathbb{E}\chi_N}\frac{N - \mathbb{E}^2\chi_N}{\sqrt{N}} \to 0 \text{ (a.s.) as } N \to \infty,$$

since $\mathrm{var}(\chi_N) = N - \mathbb{E}^2\chi_N < c_1^{1/2}$. Thus

$$\chi_N - \mathbb{E}\chi_N \xrightarrow{d} \frac{1}{2}\mathcal{N}(0,2) = \mathcal{N}(0,\frac{1}{2}) \text{ as } N \to \infty.$$

Appealing again to uniform integrability of $(\chi_N - \mathbb{E}\chi_N)_N$, we have

$$\mathrm{var}(Y_{N,1}) = \mathbb{E}|\chi_N - \mathbb{E}\chi_N|^2 \to \frac{1}{2} \text{ as } N \to \infty.$$

Assume now that

$$\frac{\mathrm{var}(Y_{N-1,n-1})}{N^{n-2}} \to \frac{n-1}{2} \text{ as } N \to \infty.$$

Note that

$$\begin{aligned}
\mathrm{var}(Y_{N,n}) &= \mathbb{E}\chi_N^2\mathbb{E}Y_{N-1,n-1}^2 - \mathbb{E}^2\chi_N\mathbb{E}^2Y_{N-1,n-1} \\
&= \mathbb{E}(\chi_N^2 - \mathbb{E}^2\chi_N)\mathbb{E}Y_{N-1,n-1}^2 + \mathbb{E}^2\chi_N(\mathbb{E}Y_{N-1,n-1}^2 - \mathbb{E}^2Y_{N-1,n-1}) \\
&= \mathrm{var}(\chi_N)\mathbb{E}Y_{N-1,n-1}^2 + \mathbb{E}^2\chi_N\,\mathrm{var}(Y_{N-1,n-1}).
\end{aligned}$$

We conclude the proof of part (2) with

$$\frac{\mathrm{var}(\chi_N)\mathbb{E}Y_{N-1,n-1}^2}{N^{n-1}} \to \frac{1}{2},$$

and, using the inductive hypothesis,

$$\frac{\mathbb{E}^2\chi_N\,\mathrm{var}(Y_{N-1,n-1})}{N^{n-1}} \to \frac{n-1}{2}.$$

Lastly, statement (3) is well-known (see, e.g., [2, §7.5.3]). □

The next proposition is the key identity for Z_N. To state it we will use the following notation:

$$\Delta_{n,p}^p = \mathbb{E}|\det[g_1 \cdots g_n]|^p. \tag{4.7}$$

Explicit formulas for $\Delta_{n,p}^p$ are well-known and follow from identity (2.5); see, e.g., [2, pg 269].

Proposition 4.4. *Let X_N, Y_N and Z_N be as above (cf. (4.2) - (4.4)). Then*

$$\frac{Z_N - \mathbb{E}Z_N}{N^{\frac{n-1}{2}}} = \alpha_{N,n}\frac{X_N - \mathbb{E}X_N}{N^{n-\frac{1}{2}}} - \beta_{N,n}\frac{Y_N^2 - \mathbb{E}Y_N^2}{N^{n-\frac{1}{2}}} - \delta_{N,n}, \tag{4.8}$$

where

(i) $\alpha_{N,n} \xrightarrow{a.s.} 1$ *as* $N \to \infty$*;*

(ii) $\beta_{N,n} \xrightarrow{a.s.} \beta_n = \frac{2^{n-1}\Delta_{n,1}}{\Delta_{n,2}^2}$ *as* $N \to \infty$*;*

(iii) $\delta_{N,n} \xrightarrow{a.s.} 0$ *as* $N \to \infty$*.*

Moreover, for all $p \geqslant 1$*,*

$$\sup_{N \geqslant n+4p-1} \max(\mathbb{E}|\alpha_{N,n}|^p, \mathbb{E}|\beta_{N,n}|^p, \mathbb{E}|\delta_{N,n}|^p) \leqslant C_{n,p}.$$

The latter proposition is the first step in passing from the quotient $Z_N = X_N/Y_N$ to the normalization required in Theorem 1.1. The fact that $N^{n-\frac{1}{2}}$ appears in both of the denominators on the right-hand side of (4.8) indicates that both X_N and Y_N^2 must be accounted for in order to capture the asymptotic normality of Z_N.

Proof. Write

$$\begin{aligned}
Z_N - \mathbb{E}Z_N &= \frac{X_N}{Y_N} - \frac{\mathbb{E}X_N}{\mathbb{E}Y_N} \\
&= \frac{X_N - \mathbb{E}X_N}{Y_N} - \left(\frac{\mathbb{E}X_N}{\mathbb{E}Y_N} - \frac{\mathbb{E}X_N}{Y_N}\right) \\
&= \frac{X_N - \mathbb{E}X_N}{Y_N} - \frac{(Y_N^2 - \mathbb{E}Y_N^2 + \mathrm{var}(Y_N))\mathbb{E}X_N}{Y_N(Y_N + \mathbb{E}Y_N)\mathbb{E}Y_N} \\
&= \frac{X_N - \mathbb{E}X_N}{Y_N} - \frac{(Y_N^2 - \mathbb{E}Y_N^2)\mathbb{E}X_N}{Y_N(Y_N + \mathbb{E}Y_N)\mathbb{E}Y_N} - \frac{\mathrm{var}(Y_N)\mathbb{E}X_N}{Y_N(Y_N + \mathbb{E}Y_N)\mathbb{E}Y_N}.
\end{aligned}$$

Thus

$$\frac{Z_N - \mathbb{E}Z_N}{N^{\frac{n-1}{2}}} = \alpha_{N,n}\left(\frac{X_N - \mathbb{E}X_N}{N^{n-\frac{1}{2}}}\right) - \beta_{N,n}\left(\frac{Y_N^2 - \mathbb{E}Y_N^2}{N^{n-\frac{1}{2}}}\right) - \delta_{N,n},$$

which shows that (4.8) holds with

$$\alpha_{N,n} = \frac{N^{\frac{n}{2}}}{Y_N}, \quad \beta_{N,n} = \frac{N^{\frac{n}{2}}\mathbb{E}X_N}{Y_N(Y_N + \mathbb{E}Y_N)\mathbb{E}Y_N}, \quad \delta_{N,n} = \beta_{N,n}\frac{\mathrm{var}(Y_N)}{N^{n-\frac{1}{2}}}.$$

Using the factorization of Y_N in (4.6) and applying the SLLN for each χ_k ($k = N, \ldots, N-n+1$), we have

$$\frac{Y_N}{\sqrt{\frac{N!}{(N-n)!}}} \xrightarrow{a.s.} 1 \text{ as } N \to \infty,$$

and hence

$$\alpha_{N,n} = \frac{N^{n/2}}{Y_N} \stackrel{a.s.}{\to} 1 \text{ as } N \to \infty.$$

By the Cauchy-Binet forumula (2.6) and the SLLN for U-statistics (Theorem 3.1(2)), we have

$$\frac{1}{\binom{N}{n}} Y_N^2 \stackrel{a.s.}{\to} \Delta_{n,2}^2 \text{ as } N \to \infty.$$

Thus

$$\beta_{N,n} = \frac{2^n \binom{N}{n} \Delta_{n,1}}{Y_N^2 (1 + \frac{\mathbb{E}Y_N}{Y_N})} \frac{N^{n/2}}{\mathbb{E}Y_N} \stackrel{a.s.}{\to} \frac{2^n \Delta_{n,1}}{2\Delta_{n,2}^2} \text{ as } N \to \infty.$$

By Proposition 4.3(2) and Slutsky's theorem, we also have $\delta_{N,n} \stackrel{a.s.}{\to} 0$ as $N \to \infty$. To prove the last assertion, we note that for $1 \leqslant p \leqslant (N-n+1)/2$,

$$\mathbb{E}\left(\frac{N^{\frac{n}{2}}}{Y_N}\right)^p \leqslant C_{n,p},$$

where $C_{n,p}$ is a constant that depends on n and p only (see, e.g., [17, Lemma 4.2]). □

Proof of Theorem 1.1. To simplify the notation, for $I = \{i_1, \ldots, i_n\} \subset \{1, \ldots, N\}$, write $d_I = |\det[g_{i_1} \cdots g_{i_n}]|$. Applying Proposition 4.4, we can write

$$\frac{Z_N - \mathbb{E}Z_N}{N^{\frac{n-1}{2}}} = \frac{\binom{N}{n}}{N^{n-\frac{1}{2}}}(U_N - \mathbb{E}U_N) + A_{N,n} - B_{N,n} - \delta_{N,n},$$

where

$$U_N = \frac{1}{\binom{N}{n}} \sum_{|I|=n} (2^n d_I - \beta_n d_I^2),$$

$$A_{N,n} = (\alpha_{N,n} - 1)\left(\frac{X_N - \mathbb{E}X_N}{N^{n-\frac{1}{2}}}\right),$$

and

$$B_{N,n} = (\beta_{N,n} - \beta_n)\left(\frac{Y_N^2 - \mathbb{E}Y_N^2}{N^{n-\frac{1}{2}}}\right).$$

Set $I_0 = \{1, \ldots, n\}$. Applying Theorem 3.1(3) with

$$\zeta = \operatorname{var}(\mathbb{E}[(2^n d_{I_0} - \beta_n d_{I_0}^2) | g_1]), \tag{4.9}$$

yields

$$\sqrt{N}\left(\frac{U_N - \mathbb{E}U_N}{n\sqrt{\zeta}}\right) \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty.$$

By Proposition 4.4, $\alpha_{N,n} \overset{a.s.}{\to} 1$, $\beta_{N,n} \overset{a.s.}{\to} \beta_n$ and $\delta_{N,n} \overset{a.s.}{\to} 0$; moreover, each of the latter sequences is uniformly integrable. Thus by Hölder's inequality and Proposition 4.2(1)

$$\mathbb{E}|A_{N,n}| \le (\mathbb{E}|\alpha_{N,n} - 1|^2)^{1/2} C_n \to 0 \text{ as } N \to \infty.$$

Similarly, using Proposition 4.3(1),

$$\mathbb{E}|B_{N,n}| \le (\mathbb{E}|\beta_{N,n} - \beta_n|^2)^{1/2} C_n \to 0 \text{ as } N \to \infty.$$

By Slutsky's theorem and the fact that $\binom{N}{n}/N^n \to 1/n!$ as $N \to \infty$, we have

$$\frac{n!(Z_N - \mathbb{E}Z_N)}{N^{\frac{n-1}{2}} n\sqrt{\zeta}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty. \tag{4.10}$$

To conclude the proof of the theorem, it is sufficient to show that

$$\frac{n!\sqrt{\operatorname{var}(Z_N)}}{N^{\frac{n-1}{2}} n\sqrt{\zeta}} \to 1 \text{ as } N \to \infty. \tag{4.11}$$

Once again we appeal to uniform integrability: by Proposition 4.4,

$$\frac{|Z_N - \mathbb{E}Z_N|}{N^{\frac{n-1}{2}}} \leqslant 2^n |\alpha_{N,n}| \frac{|X_N - \mathbb{E}X_N|}{N^{n-\frac{1}{2}}} + |\beta_{N,n}| \frac{|Y_N^2 - \mathbb{E}Y_N^2|}{N^{n-\frac{1}{2}}} + |\delta_{N,n}|.$$

By Hölder's inequality and Propositions 4.2(1), 4.3(1) and 4.4,

$$\sup_{N \geqslant n+8p-1} \left| \frac{Z_N - \mathbb{E}Z_N}{N^{\frac{n-1}{2}}} \right|^p \leqslant C_{n,p},$$

which, combined with (4.10), implies (4.11). □

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Grigoris Paouris: grigoris@math.tamu.edu
Department of Mathematics, Texas A&M University
College Station, TX, 77843-3368

Peter Pivovarov: pivovarovp@missouri.edu
Mathematics Department, University of Missouri
Columbia, MO, 65211

Joel Zinn: jzinn@math.tamu.edu
Department of Mathematics, Texas A&M University
College Station, TX, 77843-3368