On Dvoretzky's theorem for subspaces of L_p

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Abstract

We prove that for any $p > 2$ and every n-dimensional subspace X of L_p, the Euclidean space ℓ_2^k can be $(1+\varepsilon)$-embedded into X with $k \geq c_p \min\{\varepsilon^2 n, (\varepsilon n)^{2/p}\}$, where $c_p > 0$ is a constant depending only on p.

1 Introduction

In the present note we discuss the classical result of A. Dvoretzky in almost spherical sections of normed spaces in the case for subspace of L_p, $2 < p < \infty$. Dvoretzky gave an affirmative answer in a question of Grothendieck which was motivated by the well known Dvoretzky-Rogers lemma from [6]. In particular, Grothendieck asked if any finite-dimensional normed spaces has lower dimensional subspace which is almost Euclidean and the dimension grows with respect to the dimension of the ambient space. Dvoretzky proved in [5] that: Given k positive integer and $\varepsilon \in (0,1)$ there exists $N = N(k,\varepsilon)$ with the following property: For every $n \geq N$ and any n-dimensional normed space X there exists k-dimensional subspace F which $(1+\varepsilon)$-isomorphic to ℓ_2^k, the Euclidean space of dimension k. In modern functional analytic language this means that every infinite-dimensional Banach space contains ℓ_2^n's uniformly. Dvoretzky's proof was providing $N(k,\varepsilon) \geq \exp(c\varepsilon^{-2}k^2 \log k)$, for some absolute constant $c > 0$ (here and elsewhere in this paper c and C denote positive absolute constants). However, the aforementioned estimate is not optimal. The optimal dependence on the dimension, was proved later by V. Milman in his groundbreaking work [15]. The estimate obtained is $N(k,\varepsilon) \geq \exp(ck\varepsilon^{-2}\log\frac{1}{\varepsilon})$. Equivalently, this states that for any $\varepsilon \in (0,1)$ there exists a function $c(\varepsilon) > 0$ with the following property: for every n-dimensional normed space X there exists $k \geq c(\varepsilon)\log n$ and linear map $T : \ell_2^k \to X$ with $\|x\|_2 \leq \|Tx\|_X \leq (1+\varepsilon)\|x\|_2$ for all $x \in \ell_2^k$ – we say that ℓ_2^k can be $(1+\varepsilon)$-embedded into X and we write: $\ell_2^k \overset{1+\varepsilon}{\hookrightarrow} X$.

The example of $X = \ell_\infty^n$ shows that this result is best possible with respect to n (see [2] for the details). The approach of [15] is probabilistic in nature and provides

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that the vast majority of subspaces (in terms of the Haar probability measure on the Grassmannian manifold $G_{n,k}$) are $(1+\varepsilon)$-Euclidean, as long as $k \leq c(\varepsilon)k(X)$, where $k(X)$ is the *critical dimension* of X. Nowadays this is customary addressed as the randomized Dvoretzky theorem or random version of Dvoretzky's theorem. V. Milman in this work revealed the significance of the concentration of measure as a basic tool for the understanding of the high-dimensional structures. That was the starting point for many applications of the concentration of measure method in high-dimensional phenomena. The last five decades has been applied to various fields: quantum information, combinatorics, random matrices, compressed sensing, theoretical computer science, high-dimensional geometry of probability measures and more.

Another remarkable fact of V. Milman's approach is that the critical quantity $k(X)$ can be described in terms of the global parameters of the space. In particular, $k(X) := \mathbb{E}\|g\|^2/b^2(X)$ where g is standard gaussian random vector in X and $b(X) = \max_{\theta\in S^{n-1}} \|\theta\|_X$. Then, one can find a good position of the unit ball of X for which $k(X)$ is large enough with respect to n (see [16] for further details). It has been proved in [17] that this formulation is optimal with respect to the dimension $k(X)$ in the sense that the k-dimensional subspaces which are 4-Euclidean with probability greater than $\frac{n}{n+k}$ cannot exceed $Ck(X)$.

The proof of [15] gave the estimate $c(\varepsilon) \geq c\varepsilon^2/\log\frac{1}{\varepsilon}$ and this was improved to $c(\varepsilon) \geq c\varepsilon^2$ by Gordon in [9] and later, adopting the methods of V. Milman, by Schechtman in [20]. This dependence is known to be optimal (see the survey [23]). The recent works of Schectman in [22] and Tikhomirov in [26] established that the dependence on ε in the randomized Dvoretzky for ℓ_∞^n is of the order $\varepsilon/\log\frac{1}{\varepsilon}$ and this is best possible. Bounds on $c(\varepsilon)$ in the randomized Dvoretzky for ℓ_p^n, $1 \leq p \leq \infty$ have appeared in [18].

As far as the dependence on ε in the "existential version" of Dvoretzky's theorem is concerned, Schechtman proved in [21] that one can always $(1+\varepsilon)$-embed ℓ_2^k in any n-dimensional normed space X with $k \geq c\varepsilon\log n/(\log\frac{1}{\varepsilon})^2$. Tikhomirov in [26] proved that for 1-symmetric space X we may have $k \geq c\log n/\log\frac{1}{\varepsilon}$ and this was subsequently extended by Fresen in [8] for permutation invariant spaces. For more detailed information on the subject, explicit statements and historical remarks the reader is consulted in the recent monograph [2].

The purpose of this note is to study the dependence on ε in Dvoretzky's theorem for finite-dimensional subspaces of L_q, $2 < q < \infty$. The case of subspaces of L_p, $1 \leq p < \infty$ have been previously studied in the classical article [7] by Figiel, Lindenstrauss and V. Milman in 1977. The authors use V. Milman's techniques in randomized Dvoretzky from [15]. They select John's position for the unit ball of the underlying space and combine the *cotype* property with the classical Dvoretzky-Rogers Lemma, in order to $(1+\varepsilon)$-embed ℓ_2^k with $k \geq c(q)\varepsilon^2 n^{2/q}$, where $c(q) > 0$ depends only on q (see [7] for the details).

Let us recall that for $2 \leq q < \infty$ the q-cotype constant of a normed space X in n

vectors, denoted by $C_q(X,n)$ is defined as the smallest constant $C > 0$ which satisfies:

$$\left(\sum_{i=1}^{n} \|z_i\|_X^q\right)^{1/q} \leq C\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i z_i\right\|_X, \tag{1.1}$$

for any n vectors $z_1,\ldots,z_n \in X$. Then, the q-cotype constant of X is defined as $C_q(X) := \sup_n C_q(X,n)$. Following the terminology of G. Pisier, cotype is a *super-property* (that is it depends only on the finite dimensional subspaces of the space). It is also isomorphic invariant and the spaces L_p, $1 \leq p < \infty$ are of cotype $q = \max\{2,p\}$ with $C_q(L_q) = O(q^{1/2})$ (see [1] for a proof). Therefore, for any finite-dimensional X of L_q, $q > 2$ we have $C_q(X) \leq C\sqrt{q}$ and we may show for any finite dimensional normed space with cotype q, in John's position satisfies $k(X) \geq cC_q^{-2}(X)(\dim X)^{2/q}$. It follows that there exists an almost isometric linear embedding $\ell_2^k \overset{1+\varepsilon}{\hookrightarrow} X$ with $k \geq cq^{-1}\varepsilon^2(\dim X)^{2/q}$. Moreover, the above argument also provides $k(X) \geq cn$ for any n-dimensional subspace X of L_p with $1 \leq p < 2$, and thus ℓ_2^k can be $(1+\varepsilon)$-embedded into X with $k \geq c\varepsilon^2 n$ which is best possible. For the range $2 < p < \infty$ our approach is different and yields the following:

Theorem 1.1. *For any $p > 2$ there exists a constant $c(p) > 0$ with the following property: for any n-dimensional subspace X of L_p and for any $\varepsilon \in (0,1)$ there exists $k \geq c(p)\min\{\varepsilon^2 n, (\varepsilon n)^{2/p}\}$ so that ℓ_2^k can be $(1+\varepsilon)$-embedded into X.*

The core of the proof still lies on the concentration of measure phenomenon, but the main tool is a variant of an inequality due to Pisier from [19]. Our method depends on this gaussian functional analytic inequality rather than the spherical isoperimetric inequality that is used in the classical framework. The advantage of the argument is based on the fact that we may take into account the order of magnitude of the Euclidean norm of the gradient of the norm instead of the Lipschitz constant which is involved in the spherical concentration inequality. The idea of sufficiently estimating averages of the Euclidean norm of the gradient of a function in order to get sharp concentration results seems to be only recently applied and was also successfully exploited in [18]. Moreover, the selection of the position of the unit ball of the space is different. Instead of using John's position we use Lewis' position for the unit ball of finite-dimensional subspaces of L_p. This permits us to express the norm in an integral form, with respect to some isotropic measure on the sphere, and therefore to use the aforementioned inequality. Further information is also provided in this position. We show that the concentration result one obtains for this type of norms is best possible as the example of ℓ_p norms shows (see [18] for the exact formulation). As a result, the random version of Dvoretzky's theorem we prove for this position (or for this type of norms) is best possible in the sense that in the case of ℓ_p^n spaces the corresponding critical dimension is optimal. In other words the ℓ_p^n spaces occur as the extremal structure in this study.

The novelty of the result is not only observed in the techniques used but also in the content of the main theorem. It is clear that the dimension $k(\varepsilon,n,p) = \min\{\varepsilon^2 n, (\varepsilon n)^{2/p}\}$ one can find almost Euclidean subspaces is always better than the previously known $\varepsilon^2 n^{2/p}$. In addition, the new estimate of $k(\varepsilon, n, p)$ also yields "new dimensions" of almost Euclidean sections in the following sense: The previous setting was only permitting almost isometric embeddings of distortion $1+\varepsilon$ with $\varepsilon \gg n^{-1/p}$ in order to achieve non-trivial dimensions. Now this phenomenon admits a striking improvement and one can find $(1+\varepsilon)$-linear embeddings with $\varepsilon \gg n^{-1/2}$.

The rest of the paper is organized as follows: In Section 2 we introduce the notation, some background material on isotropic measures on the n-dimensional Euclidean sphere and finally we give the proof of the aforementioned Gaussian inequality. In Section 3 we prove concentration results for the family of the L_q-bodies associated to an isotropic measure on the n-dimensional Euclidean sphere. In Section 4 we provide the proof of our main result. Finally, in Section 5 we conclude with some further remarks.

2 Background material and auxiliary results

We work in $\mathbb{R}^n$ equipped with the standard Euclidean structure $\langle \cdot, \cdot \rangle$. The n-dimensional Euclidean sphere is defined as $S^{n-1} := \{x \in \mathbb{R}^n : \langle x, x \rangle = 1\}$. The ℓ_p norm is defined as: $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. We set $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ and let B_p^n its unit ball. More generally, for any centrally symmetric convex body K on $\mathbb{R}^n$ we write $\|\cdot\|_K$ for the norm induced by K. The n-dimensional Lebesgue measure (volume) of a body A is denoted by $|A|$. The space $L_p(\Omega, \mathcal{E}, \mu)$, $1 \le p < \infty$ consists of all $\mathcal{E}$-measurable functions $f : \Omega \to \mathbb{R}$ so that $\int_\Omega |f|^p \, d\mu < \infty$, equipped with the norm $\|f\|_{L_p(\mu)} := (\int_\Omega |f|^p \, d\mu)^{1/p}$.
The n-dimensional (standard) Gaussian measure is denoted by γ_n and its density is:

$$d\gamma_n(x) := (2\pi)^{-n/2} e^{-\|x\|_2^2/2} dx. \tag{2.1}$$

More generally, let $d\gamma_{n,\sigma}(x) := (2\pi\sigma^2)^{-n/2} e^{-\|x\|_2^2/(2\sigma^2)} dx$ for $\sigma > 0$. Random vectors distributed according to γ_n usually denoted by $X, Y, Z, \dots$ or $g = (g_1, \dots, g_n)$. The notation $\mathbb{E}(\cdot)$ is used for the expectation. The moments of norms with respect to γ_n whose unit ball is the body K are denoted by:

$$I_r(\gamma_n, K) := (\mathbb{E}\|X\|_K^r)^{1/r} = \left(\int_{\mathbb{R}^n} \|x\|_K^r \, d\gamma_n(x) \right)^{1/r} \tag{2.2}$$

and more generally, for an arbitrary probability measure ν as $I_r(\nu, K)$. Recall the pth moment σ_p of a standard gaussian random variable g_1:

$$\sigma_p^p := \mathbb{E}|g_1|^p = \frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) \sim \sqrt{2/e} \left(\frac{p+1}{e}\right)^{p/2}, \quad p \to \infty, \tag{2.3}$$

where $f \sim g$ means $f(t)/g(t) \to 1$ as $t \to \infty$. We write $f \lesssim g$ when there exists absolute constant $C > 0$ such that $f \le Cg$. We write $f \simeq g$ if $f \lesssim g$ and $g \lesssim f$, whereas the notation $f \lesssim_p g$ means that the involved constant depends only on p. The letters $C, c, C_1, c_0, \dots$ are frequently used throughout the text in order to denote absolute constants which may differ from line to line.

The formulation of Dvoretzky's theorem due to V. Milman from [15] (for the optimal dependence on ε see [9] and [20]) is given below:

Theorem 2.1. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be normed space. Define the critical dimension of X as the quantity:*

$$k(X) := \frac{\mathbb{E}\|g\|^2}{b^2(X)}, \tag{2.4}$$

where $b(X) := \max_{\theta\in S^{n-1}} \|\theta\|$. Then, for every $\varepsilon \in (0,1)$ and for any $k \leq c\varepsilon^2 k(X)$ there exists k-dimensional subspace of X which is $(1+\varepsilon)$-Euclidean.

2.1 Logarithmic Sobolev inequality

Let ν be a Borel probability measure on $\mathbb{R}^n$ which satisfies log-Sobolev inequality with constant $\rho > 0$:

$$\mathrm{Ent}_\nu(f^2) := \int f^2 \log f^2 \, d\nu - \int f^2 \, d\nu \log\left(\int f^2 \, d\nu\right) \leq \frac{2}{\rho}\int_{\mathbb{R}^n} \|\nabla f\|_2^2 \, d\nu, \tag{2.5}$$

for all smooth (or locally Lipschitz) functions $f : \mathbb{R}^n \to \mathbb{R}$. The n-dimensional Gaussian measure satisfies log-Sobolev inequality with $\rho = 1$ (see [12]).

Lemma 2.2. *Let ν be Borel probability measure on $\mathbb{R}^n$ which satisfies log-Sobolev inequality with constant ρ. Then, for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we have:*

$$\|f\|_{L_q(\nu)}^2 - \|f\|_{L_p(\nu)}^2 \leq \frac{1}{\rho}\int_p^q \left\| \|\nabla f\|_2 \right\|_{L_s(\nu)}^2 ds, \tag{2.6}$$

for all $2 \leq p \leq q$. Moreover, if f is Lipschitz continuous, then we have:

$$\|f\|_{L_q(\nu)}^2 - \|f\|_{L_p(\nu)}^2 \leq \frac{\|f\|_{\mathrm{Lip}}^2}{\rho}(q-p). \tag{2.7}$$

In particular, we obtain:

$$\frac{\|f\|_{L_q(\nu)}}{\|f\|_{L_2(\nu)}} \leq \sqrt{1 + \frac{q-2}{\rho k(f)}}, \tag{2.8}$$

for $q \geq 2$, where $k(f) := \|f\|_{L_2(\nu)}^2 / \|f\|_{\mathrm{Lip}}^2$.

Proof. For $p \geq 2$ we define $I(p) := \|f\|_{L_p}$. Differentiation with respect to p yields:

$$\frac{dI}{dp} = \frac{\mathrm{Ent}_\nu(|f|^p)}{p^2 I(p)^{p-1}}. \tag{2.9}$$

Applying log-Sobolev for $g = |f|^{p/2}$ we obtain:

$$\frac{dI}{dp} \leq \frac{1}{2\rho I(p)^{p-1}}\int_{\mathbb{R}^n} |f|^{p-2}\|\nabla f\|_2^2 \, d\nu \leq \frac{1}{2\rho I(p)^{p-1}} I(p)^{p-2} \left\| \|\nabla f\|_2 \right\|_{L_p(\nu)}^2, \tag{2.10}$$

by Hölder's inequality. This shows that $(I(p)^2)' \leq \frac{1}{\rho}\left\| \|\nabla f\|_2 \right\|_{L_p(\nu)}^2$, thus integration over the interval $[p,q]$ proves (2.6). □

2.2 Lewis' position

Given any finite Borel measure μ on S^{n-1} and any $1 \le p < \infty$ we can equip $\mathbb{R}^n$ with the norm

$$\|x\|_{\mu,p} := \left(\int_{S^{n-1}} |\langle x, \theta \rangle|^p \, d\mu(\theta) \right)^{1/p} \tag{2.11}$$

and then the space $X = (\mathbb{R}^n, \|\cdot\|)$ can be naturally embedded into $L_p(S^{n-1}, \mu)$ via the linear isometry $U : X \to L_p(S^{n-1}, \mu)$ with $Ux := \langle x, \cdot \rangle$.

The fundamental result of Lewis' from [13] states that essentially the converse is true under a suitable change of density (see also [24] for an alternative proof which extends to the whole range $0 < p < \infty$ and arises as a solution of an optimization problem). The formulation we use here follows the exposition from [14]:

Theorem 2.3 (Lewis). *Let $1 \le p < \infty$ and let X be n-dimensional subspace of L_p. Then, there exists even Borel measure μ on S^{n-1} which satisfies:*

$$\|x\|_2^2 = \int_{S^{n-1}} |\langle x, \theta \rangle|^2 d\mu(\theta), \tag{2.12}$$

for all $x \in \mathbb{R}^n$ and the normed space $(\mathbb{R}^n, \|\cdot\|_{\mu,p})$ is isometric to X.

Let us mention that property (2.12) is called *isotropic condition* and the measures satisfying it *isotropic measures*. It is also clear that taking into account this representation of any finite-dimensional subspace of L_p, the problem of embedding ℓ_2^k in subspaces of L_p is reduced to spaces $(\mathbb{R}^n, \|\cdot\|_{p,\mu})$ with μ isotropic on S^{n-1}. Hence, the next paragraph is devoted to the study of these measures.

2.3 Isotropic measures on the sphere

An even Borel measure μ on S^{n-1} it is said to be *isotropic* if it satisfies the condition:

$$\|x\|_2^2 = \int_{S^{n-1}} |\langle x, \theta \rangle|^2 \, d\mu(\theta), \tag{2.13}$$

for all $x \in \mathbb{R}^n$. Equivalently, for all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ we have:

$$\mathrm{trace}(T) = \int_{S^{n-1}} \langle \theta, T\theta \rangle \, d\mu(\theta). \tag{2.14}$$

For any such measure we may define the following family of centrally symmetric convex bodies $B_q(\mu)$ with associated norms:

$$x \mapsto \|x\|_{B_q(\mu)} := \|\langle x, \cdot \rangle\|_{L_q(\mu)} = \left(\int_{S^{n-1}} |\langle x, z \rangle|^q \, d\mu(z) \right)^{1/q}, \quad 1 \le q < \infty. \tag{2.15}$$

The corresponding spaces whose unit ball is $B_q(\mu)$ will be denoted by $X_q(\mu)$. Under this terminology and notation, Lewis' theorem reads as follows:

Theorem 2.4. *Let $1 \le p < \infty$ and let X be n-dimensional subspace of L_p. Then, there exists isotropic Borel measure μ on S^{n-1} and linear isometry $S : X_p(\mu) \to X$.*

Next simple lemma collects several properties for the bodies $B_p(\mu)$.

Lemma 2.5. *Let μ be Borel isotropic measure on S^{n-1}. Then, we have the following properties:*

i. $\mathbb{E}\|g\|_{B_q(\mu)}^q = \sigma_q^q \mu(S^{n-1})$.

ii. $\mu(S^{n-1}) = n$.

iii. *For $p \ge 2$ we have: $\|x\|_{B_p(\mu)} \le \|x\|_2$ and for $1 \le p < q < \infty$ we have: $\|x\|_{B_p(\mu)} \le n^{1/p-1/q}\|x\|_{B_q(\mu)}$, for all $x \in \mathbb{R}^n$.*

iv. *(K. Ball) For every $1 \le p < \infty$ we have: $|B_p(\mu)| \le |B_p^n|$.*

v. *For the body $B_q(\mu)$, $q \ge 1$ we have $k(B_q(\mu)) \ge cn^{\min\{1,2/q\}}$.*

vi. *There exists an absolute constant $c > 0$ such that for all $2 \le q \le c\log n$, one has: $(\mathbb{E}\|g\|_{B_q(\mu)}^2)^{1/2} \simeq q^{1/2}n^{1/q}$. In particular, for those q's one has: $k(B_q(\mu)) \ge cqn^{2/q}$.*

Proof. For the first assertion we use Fubini's theorem and the rotation invariance of the Gaussian measure to write:

$$\mathbb{E}\|g\|_{B_q(\mu)}^q = \int_{\mathbb{R}^n} \|x\|_{B_q(\mu)}^q \, d\gamma_n(x) = \int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle x, z\rangle|^q \, d\gamma_n(x) \, d\mu(z) = \sigma_q^q \mu(S^{n-1}).$$

The second assertion follows from the above formula applied for $q = 2$ and employing the isotropic condition. For the third one, note that for all $u \in S^{n-1}$ we have:

$$\|u\|_{B_p(\mu)}^p = \int_{S^{n-1}} |\langle u, z\rangle|^p \, d\mu(z) \le \int_{S^{n-1}} |\langle u, z\rangle|^2 \, d\mu(z) = 1, \tag{2.16}$$

while for the right-hand side estimate we apply Hölder's inequality:

$$\|x\|_{B_p(\mu)} = \left(\int_{S^{n-1}} |\langle x, z\rangle|^p \, d\mu(z) \right)^{1/p} \le \mu(S^{n-1})^{\frac{1}{p}-\frac{1}{q}} \left(\int_{S^{n-1}} |\langle x, z\rangle|^q \, d\mu(z) \right)^{1/q}.$$

iv. This result was essentially proved by K. Ball in [3]. A sketch of his very elegant proof is reproduced below for the sake of completeness: Without loss of generality we may assume that μ is discrete, i.e. $\mu = \sum_{i=1}^m c_i \delta_{u_i}$, for some vectors (u_i) in S^{n-1} and positive numbers (c_i) with $I = \sum_{i=1}^m c_i u_i \otimes u_i$. Now we use the formula, which holds true for any centrally symmetric convex body K on $\mathbb{R}^n$:

$$|K| = (\Gamma(1 + n/p))^{-1} \int_{\mathbb{R}^n} e^{-\|z\|_K^p} \, dz, \tag{2.17}$$

to get:

$$|B_p(\mu)| = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle z, u_i\rangle)^{c_i} \, dz, \tag{2.18}$$

where $f_i(t) = \exp(-|t|^p)$. The result follows by the Brascamp-Lieb inequality.
v. First consider the case $2 < q < \infty$. For the critical dimension of the space $X_q(\mu) := (\mathbb{R}^n, \|\cdot\|_{B_q(\mu)})$, note that $k(X_q(\mu)) = \mathbb{E}\|g\|^2_{B_q(\mu)}/b^2(B_q(\mu)) \geq n^{2/q}$ by the third assertion.
Now we turn in the range $1 \leq q \leq 2$. Using Hölder's inequality we may write:

$$\left(\mathbb{E}\|g\|^2_{B_q(\mu)}\right)^{1/2} \geq n^{1/2}\left(\frac{|B_2^n|}{|B_q(\mu)|}\right)^{1/n} \geq n^{1/2}\left(\frac{|B_2^n|}{|B_q^n|}\right)^{1/n} \simeq n^{1/q}, \tag{2.19}$$

where in the last step we have used Ball's volumetric estimate (iv). The result follows once we recall that $b(B_q(\mu)) \leq n^{1/q-1/2}$ for $1 \leq q \leq 2$.
vi. We define the parameter:

$$q_0 \equiv q_0(\mu) := \max\left\{q \in [2,n] : k(B_p(\mu)) \geq p,\ \forall p \in [2,q]\right\}. \tag{2.20}$$

By the continuity of the map $p \mapsto k(B_p(\mu))$ and the fact that $k(B_q(\mu)) \leq n$ for all $q \geq 2$, while $k(B_2(\mu)) = n$ we get: $q_0 = k(B_{q_0}(\mu))$. Lemma 2.2 shows that $\left(\mathbb{E}\|g\|^{q_0}_{B_{q_0}(\mu)}\right)^{1/q_0} \leq c_1\left(\mathbb{E}\|g\|^2_{B_{q_0}(\mu)}\right)^{1/2}$, so we may write:

$$q_0 = k(B_{q_0}(\mu)) = \frac{\mathbb{E}\|g\|^2_{B_{q_0}(\mu)}}{b^2(B_{q_0}(\mu))} \geq c_1^{-2}(\mathbb{E}\|g\|^{q_0}_{B_{q_0}(\mu)})^{2/q_0} = c_1^{-2}\sigma_{q_0}^2 n^{2/q_0} \Longrightarrow q_0 \geq c_2 \log n.$$

Therefore, by the definition of q_0 we have $k(B_q(\mu)) \geq q$ for all $2 \leq q \leq q_0$ and by Lemma 2.2 again, we get:

$$\sigma_q n^{1/q} = \left(\mathbb{E}\|g\|^q_{B_q(\mu)}\right)^{1/q} \leq c_1\left(\mathbb{E}\|g\|^2_{B_q(\mu)}\right)^{1/2}.$$

Moreover, we have:

$$k(B_q(\mu)) = \frac{\mathbb{E}\|g\|^2_{B_q(\mu)}}{b^2(B_q(\mu))} \geq c_1^{-2}(\mathbb{E}\|g\|^q_{B_q(\mu)})^{2/q} = c_1^{-2}\sigma_q^2 n^{2/q} \geq c_3 q n^{2/q}.$$

This can be interpreted as $k(B_q(\mu)) \geq ck(\ell_q^n)$, provided that $2 \leq q \leq c\log n$ for some absolute constant $c > 0$. □

Lemma 2.6. *Let μ be Borel isotropic measure on S^{n-1}. For $q \geq 2$ and for all $r \geq 1$ we have:*

$$I_{rq}(\gamma_n, B_q(\mu))/I_q(\gamma_n, B_q(\mu)) \leq \sqrt{1 + \frac{q(r-1)}{\sigma_q^2 n^{2/q}}} \leq \sqrt{1 + \frac{c(r-1)}{n^{2/q}}}, \tag{2.21}$$

where $c > 0$ is an absolute constant.

Proof. Note that Lemma 2.5 (iii) implies $\left|\|x\|_{B_q(\mu)} - \|y\|_{B_q(\mu)}\right| \leq \|x-y\|_2$ for all $x, y \in \mathbb{R}^n$. Hence, if we use Lemma 2.2 we obtain:

$$\left(\frac{I_{rq}}{I_q}\right)^2 \leq 1 + \frac{q(r-1)}{I_q^2} = 1 + \frac{q(r-1)}{\sigma_q^2 n^{2/q}}. \tag{2.22}$$

where the last estimate follows from Lemma 2.5. Finally, using the fact that $\sigma_q \simeq \sqrt{q}$ we conclude the second estimate. □

2.4 A Gaussian inequality

Next inequality is due to Pisier (for a proof see [19]).

Theorem 2.7. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex function and let $f : \mathbb{R}^n \to \mathbb{R}$ be C^1-smooth. Then, if X, Y are independent copies of a Gaussian random vector, then we have:*

$$\mathbb{E}\phi\left(f(X) - f(Y)\right) \leq \mathbb{E}\phi\left(\frac{\pi}{2}\langle \nabla f(X), Y\rangle\right). \tag{2.23}$$

Here we prove a generalization of this inequality in the context of Gaussian processes generated by the action of a random matrix with i.i.d standard gaussian entries on a fixed vector in S^{n-1}:

Theorem 2.8. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex function and let $f : \mathbb{R}^n \to \mathbb{R}$ be C^1-smooth. If $G = (g_{ij})_{i,j=1}^{n,k}$ is Gaussian matrix and $a, b \in S^{k-1}$, then we have:*

$$\mathbb{E}\phi\left(f(Ga) - f(Gb)\right) \leq \mathbb{E}\phi\left(\frac{\pi}{2}\|a-b\|_2\langle \nabla f(X), Y\rangle\right), \tag{2.24}$$

where X, Y are independent copies of a standard Gaussian n-dimensional random vector.

Proof. If $a = b$ then, there is nothing to prove. If $a = -b$ then, by setting $F(x) = f(x) - f(-x)$ we may write:

$$\mathbb{E}\phi\left(f(Ga) - f(Gb)\right) = \mathbb{E}\phi(F(X)) \leq \mathbb{E}\phi(F(X) - F(Y)), \tag{2.25}$$

for X, Y independent copies of a standard Gaussian random vector, where we have used the fact $\mathbb{E}F(X) = 0$ and Jensen's inequality. Then, a direct application of Theorem 2.7 yields:

$$\begin{aligned}
\mathbb{E}\phi(F(X) - F(Y)) &\leq \mathbb{E}\phi\left(\frac{\pi\langle \nabla f(X), Y\rangle + \pi\langle \nabla f(-X), Y\rangle}{2}\right) \\
&\leq \mathbb{E}\frac{\phi(\pi\langle \nabla f(X), Y\rangle) + \phi(\pi\langle \nabla f(-X), Y\rangle)}{2} \\
&= \mathbb{E}\phi(\pi\langle \nabla f(X), Y\rangle),
\end{aligned}$$

by the convexity of ϕ.

In the general case, fix $a, b \in S^{k-1}$ with $a \neq \pm b$ and define $p := \frac{a+b}{2}$. Note that since $\|a\|_2 = \|b\|_2$ we have that the vector $u := a - p$ is perpendicular to p. Set $X := G(u)$ and $Z := G(p)$ and note that X, Z are independent random vectors in $\mathbb{R}^n$ with $X \sim N(\mathbf{0}, \|u\|_2^2 I_n)$, $Z \sim N(\mathbf{0}, \|p\|_2^2 I_n)$ and $G(a) = Z + X$ while $G(b) = Z - X$. Thus, we may write:

$$\mathbb{E}\phi\left(f(Ga) - f(Gb)\right) = \mathbb{E}_Z\mathbb{E}_X\phi\left(f(Z+X) - f(Z-X)\right).$$

Define $F(x,z) := f(z+x) - f(z-x)$ and using this notation we may write:

$$\mathbb{E}\phi(f(Ga) - f(Gb)) = \iint \phi(F(x,z))\, d\gamma_{n,\sigma_1}(x)\, d\gamma_{n,\sigma_2}(z),$$

where $\sigma_1 = \|u\|_2 > 0$, $\sigma_2 = \|p\|_2 > 0$. For fixed z we may apply the Theorem 2.7 to the function $x \mapsto F(x,z)$ (note that $\int F(x,z)\, d\gamma_{n,\sigma_1}(x) = 0$) to get:

$$\begin{aligned}\int \phi(F(x,z))\, d\gamma_{n,\sigma_1}(x) &\le \iint \phi\left(\frac{\pi}{2}\langle \nabla_x F(x,z), y\rangle\right) d\gamma_{n,\sigma_1}(x)\, d\gamma_{n,\sigma_1}(y)\\ &\le \iint \frac{\phi(\pi\langle \nabla f(x+z), y\rangle) + \phi(\pi\langle \nabla f(z-x), y\rangle)}{2}\, d\gamma_{n,\sigma_1}(x)\, d\gamma_{n,\sigma_1}(y)\\ &= \iint \phi\left(\pi\langle \nabla f(x+z), y\rangle\right) d\gamma_{n,\sigma_1}(x)\, d\gamma_{n,\sigma_1}(y),\end{aligned}$$

by the convexity of ϕ. Integration with respect to γ_{n,σ_2} over z provides:

$$\begin{aligned}\iint \phi(F(x,z))\, d\gamma_{n,\sigma_1}(x) d\gamma_{n,\sigma_2}(z) &\le \int \left[\iint \phi\left(\pi\langle \nabla f(x+z), y\rangle\right) d\gamma_{n,\sigma_1}(x) d\gamma_{n,\sigma_2}(z)\right] d\gamma_{n,\sigma_1}(y)\\ &= \int \left[\int \phi\left(\pi\langle \nabla f(u), y\rangle\right) d(\gamma_{n,\sigma_1} * \gamma_{n,\sigma_2})(u)\right] d\gamma_{n,\sigma_1}(y)\\ &= \iint \phi\left(\pi\sigma_1\langle \nabla f(u), y\rangle\right) d\gamma_n(u)\, d\gamma_n(y),\end{aligned}$$

where we have used the fact that $\gamma_{n,\sigma_1} * \gamma_{n,\sigma_2} = \gamma_{n,\sigma_1^2+\sigma_2^2} \equiv \gamma_n$, since $\sigma_1^2 + \sigma_2^2 = \|a\|_2^2 = 1$. The result follows. □

Remark 2.9. 1. Applying this for $\phi(t) = |t|^r$, $r \ge 1$ and taking into account the invariance of Gaussian measure under orthogonal transformations we derive the next (r,r)-Poincaré inequalities:

$$(\mathbb{E}|f(Ga) - f(Gb)|^r)^{1/r} \le C\sqrt{r}\|a-b\|_2 \left(\mathbb{E}\|\nabla f(X)\|_2^r\right)^{1/r}, \tag{2.26}$$

for $a, b \in S^{k-1}$, where X is standard Gaussian random vector.
2. Assuming further that f is L-Lipschitz we may apply Theorem 2.8 for $\phi(t) = e^{\lambda t}$, $\lambda > 0$ to get:

(2.27)

$$\mathbb{E}\exp\left(\lambda\left(f(Ga) - f(Gb)\right)\right) \le \mathbb{E}\exp\left(\lambda^2\frac{\pi^2}{2}\|a-b\|_2^2\|\nabla f(X)\|_2^2\right) \le \exp\left(\lambda^2\frac{\pi^2}{2}\|a-b\|_2^2 L^2\right).$$

Then Markov's inequality yields Schechtman's distributional inequality from [20]:

$$\mathrm{Prob}\left(|f(Ga) - f(Gb)| > t\right) \le C\exp\left(-ct^2/(\|a-b\|^2 L^2)\right), \tag{2.28}$$

for all $t > 0$, where $a, b \in S^{k-1}$. Let us note that (2.27) for f being a norm, has also appeared in [25].
3. For $a, b \in S^{k-1}$ with $\langle a, b\rangle = 0$ the matrix G generates the vectors $X = Ga$ and $Y = Gb$ which are independent copies of a standard n-dimensional Gaussian random vector. For example, inequality (2.28) reduces to the classical concentration inequality:

$$\mathrm{Prob}\left(\left|f(X) - f(Y)\right| > t\right) \le C\exp\left(-ct^2/L^2\right), \tag{2.29}$$

for all $t > 0$.

3 Gaussian concentration for $B_p(\mu)$ bodies

A direct application of the Gaussian concentration inequality (2.29) for the bodies $B_p(\mu)$, $p > 2$ implies:

$$P\left(\left|\|X\|_{B_p(\mu)} - I_1\right| > tI_1\right) \leq C\exp(-ct^2 I_1^2) \leq Ce^{-ct^2 n^{2/p}}, \tag{3.1}$$

for all $t > 0$, where $I_1 \equiv I_1(\gamma_n, B_p(\mu))$. It is known (see [18]) that the large deviation estimate ($t \geq 1$) the inequality (2.29) provides is sharp (up to constants).

In this paragraph we prove that for $p > 2$ and μ isotropic Borel measure on S^{n-1}, the bodies $B_p(\mu)$ exhibit better concentration ($0 < t < 1$) than the one implied by the Gaussian concentration inequality on $\mathbb{R}^n$ in terms of the Lipschitz constant. Later, this will be used to prove the announced dependence on ε in Dvoretzky's theorem for any subspace of L_p. Our main tool is the probabilistic inequality proved in Theorem 2.8 and as was formulated further in Remark 2.9.1.

We apply inequality (2.26) for $f(x) = \|x\|^p = \int |\langle x, \theta\rangle|^p \, d\mu(\theta)$. To this end we have to compute the gradient. Note that:

$$\|\nabla f(x)\|_2^2 = p^2 \sum_{i=1}^n \left| \int_{S^{n-1}} \theta_i |\langle x, \theta\rangle|^{p-1} \operatorname{sgn}(\langle x, \theta\rangle) \, d\mu(\theta) \right|^2. \tag{3.2}$$

We also have the following:

Claim. For almost every $x \in \mathbb{R}^n$ we have:

$$\|\nabla f(x)\|_2^2 \leq p^2 \|x\|_{B_{2p-2}(\mu)}^{2p-2}. \tag{3.3}$$

Proof of Claim. Let $b_i \equiv b_i(x) := \int_{S^{n-1}} |\langle x, z\rangle|^{p-1} \operatorname{sgn}(\langle x, z\rangle) z_i \, d\mu(z)$. Using duality we may write:

$$\begin{aligned}
\sum_{i=1}^n \left(\int_{S^{n-1}} |\langle x, z\rangle|^{p-1} \operatorname{sgn}(\langle x, z\rangle) z_i \, d\mu(z) \right)^2 &= \max_{\theta \in S^{n-1}} \left| \sum_{i=1}^n b_i \theta_i \right|^2 \\
&= \max_{\theta \in S^{n-1}} \left| \int_{S^{n-1}} |\langle x, z\rangle|^{p-1} \operatorname{sgn}(\langle x, z\rangle) \langle z, \theta\rangle \, d\mu(z) \right|^2 \\
&\leq \int_{S^{n-1}} |\langle x, z\rangle|^{2p-2} \, d\mu(z),
\end{aligned}$$

where we have used Cauchy-Schwarz inequality and the isotropic condition. □

Therefore, using the Claim and the inequality (2.26) we get for every $a, b \in S^{k-1}$:

$$\left(\mathbb{E}|f(Ga) - f(Gb)|^r\right)^{1/r} \leq Cpr^{1/2} \|a - b\|_2 \left(\mathbb{E}\|X\|_{B_{2p-2}(\mu)}^{r(p-1)}\right)^{1/r}, \tag{3.4}$$

for all $r \geq 1$. By employing Lemma 2.6 we find:

$$(\mathbb{E}|f(Ga)-f(Gb)|^r)^{1/r} \leq Cpr^{1/2}\|a-b\|_2\left(\mathbb{E}\|X\|_{B_{2p-2}(\mu)}^{2p-2}\right)^{1/2}\left(1+\frac{(r-2)(p-1)}{\sigma_{2p-2}^2 n^{\frac{1}{p-1}}}\right)^{\frac{p-1}{2}}$$

$$< Cpr^{1/2}\|a-b\|_2\sigma_{2p-2}^{p-1}n^{1/2}\left(1+\frac{r(p-1)}{\sigma_{2p-2}^2 n^{\frac{1}{p-1}}}\right)^{\frac{p-1}{2}}$$

$$< Cp\|a-b\|_2\sigma_{2p-2}^{p-1}n^{1/2}2^{\frac{p-1}{2}}\max\left\{r^{1/2}, \frac{r^{p/2}(p-1)^{\frac{p-1}{2}}}{\sigma_{2p-2}^{p-1}n^{1/2}}\right\},$$

for all $r \geq 2$. We define

$$\alpha(n,p,r) := \max\left\{r^{1/2}, \frac{r^{p/2}(p-1)^{\frac{p-1}{2}}}{\sigma_{2p-2}^{p-1}n^{1/2}}\right\}, \quad r > 0 \tag{3.5}$$

and we summarize the above discussion to the following:

Proposition 3.1. *Let $2 < p < \infty$ and let μ be Borel isotropic measure on S^{n-1}. If $G = (g_{ij})_{i,j=1}^{n,k}$ is Gaussian matrix and $a, b \in S^{k-1}$, then we have:*

$$\left(\mathbb{E}\left|\|Ga\|_{B_p(\mu)}^p - \|Gb\|_{B_p(\mu)}^p\right|^r\right)^{1/r} \leq Cp\|a-b\|_2\sigma_{2p-2}^{p-1}n^{1/2}2^{\frac{p-1}{2}}\alpha(n,p,r), \tag{3.6}$$

for all $r \geq 2$, where $\alpha(n,p,\cdot)$ is defined in (3.5)

We are now ready to prove the main result of this Section:

Theorem 3.2. *Let $2 < p < \infty$ and let μ Borel isotropic measure on S^{n-1} with $n > e^p$. Then, we have*

$$P\left(\left|\|X\|_{B_p(\mu)} - (\mathbb{E}\|X\|_{B_p(\mu)}^p)^{1/p}\right| \geq \varepsilon(\mathbb{E}\|X\|_{B_p(\mu)}^p)^{1/p}\right) \leq C\exp(-c\psi(n,p,\varepsilon)), \tag{3.7}$$

for every $\varepsilon > 0$, where $\psi(n,p,\cdot)$ is defined as:

$$\psi(n,p,t) := \min\left\{\frac{t^2 n}{p4^p}, (tn)^{2/p}\right\}, \; t > 0 \tag{3.8}$$

and $C, c > 0$ are absolute constants.

Proof. Using Proposition 3.1 for $a, b \in S^{k-1}$ with $\langle a, b\rangle = 0$ and applying Jensen's inequality we obtain:

$$\left(\mathbb{E}\left|\|X\|_{B_p(\mu)}^p - \mathbb{E}\|X\|_{B_p(\mu)}^p\right|^r\right)^{1/r} \leq Cp\sigma_{2p-2}^{p-1}n^{1/2}2^{p/2}\alpha(n,p,r), \tag{3.9}$$

for all $r \geq 2$. Therefore Markov's inequality yields:

$$P\left(\left|\|X\|_{B_p(\mu)}^p - \mathbb{E}\|X\|_{B_p(\mu)}^p\right| > \varepsilon\right) \leq \left(\frac{Cp\sigma_{2p-2}^{p-1} n^{1/2} 2^{p/2} \alpha(n,p,r)}{\varepsilon}\right)^r. \tag{3.10}$$

note that:

$$\alpha^{-1}(n,p,s) = \min\left\{s^2, \frac{s^{2/p} n^{1/p} \sigma_{2p-2}^{\frac{2p-2}{p}}}{(p-1)^{\frac{p-1}{p}}}\right\}, \quad s > 0, \tag{3.11}$$

thus we may choose $r_\varepsilon \geq 2$ such that $\alpha(n,p,r_\varepsilon) = \frac{\varepsilon}{eCp\sigma_{2p-2}^{p-1} n^{1/2} 2^{p/2}}$, as long as the range of $\varepsilon > 0$ satisfies $\alpha(n,p,r_\varepsilon) \geq \alpha(n,p,2)$. Otherwise $\alpha(n,p,r_\varepsilon) < \alpha(n,p,2) \simeq \max\{1, (e^p/n)^{1/2}\} \simeq 1$ provided that n is large enough with respect to p. Thus, we get:

$$P\left(\left|\|X\|_{B_p(\mu)}^p - \mathbb{E}\|X\|_{B_p(\mu)}^p\right| > \varepsilon\right) \leq C_1 \exp\left(-\alpha^{-1}\left(n, p, \frac{\varepsilon}{eCp\sigma_{2p-2}^{p-1} n^{1/2} 2^{p/2}}\right)\right), \tag{3.12}$$

for all $\varepsilon > 0$. We may check that:

$$\alpha^{-1}\left(n, p, \frac{\varepsilon}{eCp\sigma_{2p-2}^{p-1} n^{1/2} 2^{p/2}}\right) \simeq \min\left\{\frac{\varepsilon^2}{np^2 2^p \sigma_{2p-2}^{2p-2}}, \frac{\varepsilon^{2/p}}{p}\right\}, \tag{3.13}$$

thus, we arrive at:

$$P\left(\left|\|X\|_{B_p(\mu)}^p - \mathbb{E}\|X\|_{B_p(\mu)}^p\right| > \varepsilon\right) \leq C_1 \exp\left(-c_1 \min\left\{\frac{\varepsilon^2}{np^2 2^p \sigma_{2p-2}^{2p-2}}, \frac{\varepsilon^{2/p}}{p}\right\}\right), \tag{3.14}$$

for every $\varepsilon > 0$. It follows that:

$$P\left(\left|\|X\|_{B_p(\mu)}^p - \mathbb{E}\|X\|_{B_p(\mu)}^p\right| > \varepsilon\mathbb{E}\|X\|_{B_p(\mu)}^p\right) \leq C_1 \exp\left(-c_1 \min\left\{\frac{\varepsilon^2 n \sigma_p^{2p}}{p^2 2^p \sigma_{2p-2}^{2p-2}}, \frac{(\varepsilon n)^{2/p} \sigma_p^2}{p}\right\}\right), \tag{3.15}$$

for every $\varepsilon > 0$. The asymptotic estimate (2.3) implies that $\sigma_p^{2p}/\sigma_{2p-2}^{2p-2} \simeq p2^{-p}$ and that $\sigma_p \simeq p^{1/2}$, thus we conclude:

$$P\left(\left|\|X\|_{B_p(\mu)}^p - \mathbb{E}\|X\|_{B_p(\mu)}^p\right| > \varepsilon\mathbb{E}\|X\|_{B_p(\mu)}^p\right) \leq C_1 \exp\left(-c_1' \min\left\{\frac{\varepsilon^2 n}{p4^p}, (\varepsilon n)^{2/p}\right\}\right), \tag{3.16}$$

for all $\varepsilon > 0$. This further implies that:

$$P\left(\left|\|X\|_{B_p(\mu)} - \left(\mathbb{E}\|X\|_{B_p(\mu)}^p\right)^{1/p}\right| > \varepsilon\left(\mathbb{E}\|X\|_{B_p(\mu)}^p\right)^{1/p}\right) \leq 2C_1 \exp\left(-c_1' \min\left\{\frac{\varepsilon^2 n}{p4^p}, (\varepsilon n)^{2/p}\right\}\right), \tag{3.17}$$

for all $\varepsilon > 0$. In order to verify that we write as follows:

$$P\left(\|X\|_{B_p(\mu)} > (1+\varepsilon)\left(\mathbb{E}\|X\|^p_{B_p(\mu)}\right)^{1/p}\right) \leq P\left(\|X\|^p_{B_p(\mu)} > (1+\varepsilon)\mathbb{E}\|X\|^p_{B_p(\mu)}\right)$$
$$\leq C_1 \exp\left(-c_1' \min\left\{\frac{\varepsilon^2 n}{p4^p},\ (\varepsilon n)^{2/p}\right\}\right),$$

for all $\varepsilon > 0$ by the estimate (3.16). We argue similarly for the other case. □

Remark. By the well known symmetrization argument for any random variable ξ:

$$P(|\xi - \mathrm{med}(\xi)| > t) \leq 4 \inf_{\alpha\in\mathbb{R}} P(|\xi - \alpha| > t/2), \quad t > 0, \tag{3.18}$$

we may replace $(\mathbb{E}\|X\|^p_{B_p(\mu)})^{1/p}$ by a median of $x \mapsto \|x\|_{B_p(\mu)}$ (or the expected value $\mathbb{E}\|X\|_{B_p(\mu)}$) with respect to the Gaussian measure γ_n (see also [16, Appendix V]).

4 Embedding ℓ_2^k in subspaces of L_p for $p > 2$

In this paragraph we prove the main result of the note:

Theorem 4.1. *Let $2 < p < \infty$. Then for every n-dimensional subspace X of L_p and any $0 < \varepsilon < 1$ there exists $k \geq c_p\psi(n, p, \varepsilon)$ and linear map $T : \ell_2^k \to X$ such that $\|x\|_2 \leq \|Tx\|_X \leq (1+\varepsilon)\|x\|_2$ for all $x \in \ell_2^k$, where $c_p > 0$ is constant depending only on p and $\psi(n, p, \cdot)$ is given by* (3.8).

We shall need the next variant of Theorem 3.2:

Theorem 4.2. *Let $2 < p < \infty$ and let μ be a Borel isotropic probability measure on S^{n-1} with $n > e^p$. If $(g_{ij})_{i,j=1}^{n,k}$ are i.i.d standard Gaussian random variables and $a, b \in S^{k-1}$, then:*

$$P\left(\left|\|Ga\|^p_{B_p(\mu)} - \|Gb\|^p_{B_p(\mu)}\right| > t\mathbb{E}\|X\|^p_{B_p(\mu)}\right) \leq C \exp\left(-c\psi\left(n, p, \frac{t}{\|a-b\|_2}\right)\right), \tag{4.1}$$

for all $t > 0$.

Proof. The proof is similar to the proof of Theorem 3.2. We omit the details. □

Now we turn in the proof of the main result:

Proof of Theorem 4.1. Let $2 < p < \infty$ and let X be n-dimensional normed space of L_p. Then, Theorem 2.4 yields the existence of an isotropic Borel measure μ on S^{n-1} and a linear isometry $S : X_p(\mu) \to X$. The next step is to establish an almost isometric embedding $G : \ell_2^k \to X_p(\mu)$ with k as large as possible. Let $\{g_{ij}(\omega)\}_{i,j=1}^{n,k}$ be i.i.d. standard normals in some probability space (Ω, P) and consider the random gaussian operator $G_\omega = (g_{ij}(\omega))_{i,j=1}^{n,k} : \ell_2^k \to X_p(\mu)$. We are going to show that with positive probability the operator G is $(1+\varepsilon)$-isomorphic embedding when k is sufficiently large. To this end, we employ Proposition 4.2 and a chaining argument from [20]: For each

$j = 1, 2, \ldots$ consider δ_j-nets $\mathcal{N}_j$ on S^{k-1} with cardinality $|\mathcal{N}_j| \leq (3/\delta_j)^k$ (see [16, Lemma 2.6]). Note that for any $\theta \in S^{k-1}$ and for all j there exist $u_j \in \mathcal{N}_j$ with $\|\theta - u_j\|_2 \leq \delta_j$ and by triangle inequality it follows that $\|u_j - u_{j-1}\|_2 \leq \delta_j + \delta_{j-1}$. Moreover, if we assume that $\delta_j \to 0$ as $j \to \infty$ and (t_j) is sequence of numbers with $t_j \geq 0$ and $\sum_j t_j \leq 1$ then, for any $\varepsilon > 0$ we have the next:

Claim. If we define the following sets:

$$A := \left\{\omega \mid \exists \theta \in S^{k-1} : \left| \|G_\omega(\theta)\|^p_{B_p(\mu)} - I_p^p \right| > \varepsilon I_p^p \right\}, \tag{4.2}$$

$$A_1 := \left\{\omega \mid \exists u_1 \in \mathcal{N}_1 : \left| \|G_\omega(u_1)\|^p_{B_p(\mu)} - I_p^p \right| > t_1 \varepsilon I_p^p \right\}$$

and for $j \geq 2$

$$A_j := \left\{\omega \mid \exists u_j \in \mathcal{N}_j, u_{j-1} \in \mathcal{N}_{j-1} : \left| \|G_\omega(u_j)\|^p_{B_p(\mu)} - \|G_\omega(u_{j-1})\|^p_{B_p(\mu)} \right| > t_j \varepsilon I_p^p \right\}, \tag{4.3}$$

where $I_p \equiv I_p(\gamma_n, B_p(\mu))$, then the following inclusion holds:

$$A \subseteq \bigcup_{j=1}^{\infty} A_j. \tag{4.4}$$

Proof of Claim. If $\omega \notin \bigcup_{j=1}^{\infty} A_j$ then for any j and any $u_j \in \mathcal{N}_j$ we have:

$$\left| \|G_\omega(u_1)\|^p_{B_p(\mu)} - I_p^p \right| \leq t_1 I_p^p \quad \text{and} \quad \left| \|G_\omega(u_j)\|^p_{B_p(\mu)} - \|G_\omega(u_{j-1})\|^p_{B_p(\mu)} \right| \leq t_j I_p^p, \quad j = 2, 3, \ldots.$$

For any θ there exist $\theta_j \in \mathcal{N}_j$ such that $\|\theta - \theta_j\|_2 < \delta_j$ for $j = 1, 2, \ldots$. Hence, for any $N \geq 2$ we may write:

$$\begin{aligned} \left| \|G_\omega(\theta)\|^p_{B_p(\mu)} - I_p^p \right| &\leq \left| I_p^p - \|G_\omega(\theta_1)\|^p_{B_p(\mu)} \right| + \sum_{j=2}^{N} \left| \|G_\omega(\theta_{j-1})\|^p_{B_p(\mu)} - \|G_\omega(\theta_j)\|^p_{B_p(\mu)} \right| + \\ &\quad + \left| \|G_\omega(\theta_N)\|^p_{B_p(\mu)} - \|G_\omega(\theta)\|^p_{B_p(\mu)} \right| \\ &\leq \sum_{j=1}^{N} \varepsilon t_j I_p^p + 2p \cdot \delta_N \cdot \|G_\omega\|^{p-1}_{2\to X}, \end{aligned}$$

which proves the assertion, since N is arbitrary.

Fix $0 < \varepsilon < 1$. Choose $\delta_j = e^{-j}$, $t_j = j^{p/2} e^{-j} / S_p$ with $S_p := \sum_{j=1}^{\infty} j^{p/2} e^{-j}$ (thus,

$\sum_j t_j \leq 1$). Then, according to the previous Claim we may write:

$$\begin{aligned}
P(A) &\leq C|\mathcal{N}_1| \exp(-c_1\psi(n,p,\varepsilon t_1)) + C\sum_{j=2}^{\infty} |\mathcal{N}_{j-1}|\cdot|\mathcal{N}_j| \exp(-c_1\psi(n,p,\varepsilon t_j e^j/4)) \\
&\leq C\sum_{j=1}^{\infty}(3e^j)^{2k}\exp\left(-c_1'\psi\left(n,p,\varepsilon j^{p/2}S_p^{-1}\right)\right) \\
&\leq C\sum_{j=1}^{\infty}\exp\left(c_2 jk - c_2' S_p^{-2}\psi\left(n,p,\varepsilon j^{p/2}\right)\right) \\
&\leq C\sum_{j=1}^{\infty}\exp\left(c_2 jk - c_2' S_p^{-2} j\psi(n,p,\varepsilon)\right) \\
&\leq C\sum_{j=1}^{\infty}\exp\left(-c_3(2e/p)^p j\psi(n,p,\varepsilon)\right) \leq C' \exp\left(-c_3(2e/p)^p\psi(n,p,\varepsilon)\right).
\end{aligned}$$

as long as $k \lesssim (2e/p)^p\psi(n,p,\varepsilon)$. Therefore, there exists $\omega \in \Omega$ with the following property:

$$(1-\varepsilon)^{1/p} I_p \leq \|G_\omega(\theta)\|_{B_p(\mu)} \leq (1+\varepsilon)^{1/p} I_p, \tag{4.5}$$

for all $\theta \in S^{k-1}$. Then, the mapping $T = T_\varepsilon : \ell_2^k \to X$ defined as $T := (1-\varepsilon)^{-1/p} I_p^{-1} S G_\omega$ satisfies:

$$\|x\|_2 \leq \|Tx\|_X \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/p}\|x\|_2, \tag{4.6}$$

for all $x \in \ell_2^k$, as required. □

Note. Let us note that if $n^{-\frac{p-2}{2(p-1)}} \lesssim_p \varepsilon < 1$ then we get $k \leq C(\varepsilon n)^{2/p}/p$ by taking into account the form of $\psi(n,p,\cdot)$.

5 Further remarks

1. Optimality of the result. If the isotropic measure μ on S^{n-1} is the one supported on $\pm e_i$'s i.e. $X_p(\mu) \equiv \ell_p^n$, then Theorem 3.2 is optimal (up to constants depending on p) as was proved in [18]. Moreover, Theorem 4.1 is optimal, in the sense that if the typical k-dimensional subspace of ℓ_p^n is $(1+\varepsilon)$-spherical, then $k \leq Cp(\varepsilon n)^{2/p}$ for some absolute constant $C > 0$ (see [18]). We should mention that it is known, that for concrete values of p one can embed ℓ_2^k into ℓ_p^n even isometrically (see [11] for details). However, this is not a typical subspace. Let us note that once we have established the concentration estimate of Theorem 3.2, then the standard net argument yields the result with an extra logarithmic on ε term. Section 4 serves exactly the purpose of removing this term: We utilize Theorem 2.8, to prove the distributional inequality of Theorem 4.2. Then we use this inequality along with the chaining method to conclude the logarithmic-free dependence on ε in our main result. This approach has been inspired by [20]. In probabilistic terms Theorem 4.2 says that the process $(\|G\theta\|_{B_p(\mu)}^p - I_p^p)_{\theta\in S^{k-1}}$ has two-level tail behavior described by $\psi(n,p,\cdot)$.

2. Selection of randomness. Embeddings of ℓ_2^k into L_q, $q>2$ under different randomness have appeared in the literature in [4]. The authors there consider large random matrices with independent Rademacher entries in order to $K(q)$-embed ℓ_2^k into ℓ_q^N with $N\simeq k^{q/2}$, where $K(q)>0$ depends only on q. Then, they use this result in order to prove that for any $1<p<2$ there exists uncomplemented subspace of L_p which is isomorphic to Hilbert space. It is worth mentioning, that one can prove a concentration result similar to that of Theorem 3.2 using other randomness than Gaussian. In particular, if ν is isotropic Borel probability measure on $\mathbb{R}^n$ which satisfies log-Sobolev inequality with constant $\rho>0$ then we may prove the following:

Theorem 5.1. *Let $2<p<\infty$ let μ be Borel isotropic measure on S^{n-1} and let ν be isotropic Borel probability measure on $\mathbb{R}^n$ which satisfies log-Sobolev inequality with constant $\rho>0$. Then, we have:*

$$\left(\iint \left|\|x\|_{B_p(\mu)}^p - \|y\|_{B_p(\mu)}^p\right|^r d\nu(x)d\nu(y)\right)^{1/r} \leq C(p,\rho) I_p^p(\nu, B_p(\mu)) \max\left\{\left(\frac{r}{n}\right)^{1/2}, \frac{r^{p/2}}{n}\right\}, \tag{5.1}$$

for all $r\geq 2$, where $C(p,\rho)>0$ is constant depending only on p and ρ.

Having proved Theorem 5.1, we apply Markov's inequality as in Section 3 to get the corresponding concentration inequality. For the proof of Theorem 5.1 we argue as follows: Consider the function $f(x)=\|x\|_{B_p(\mu)}^p$ and define $F=f-\mathbb{E}_\nu f$. Then, a direct application of Lemma 2.2 yields:

$$\|F\|_{L_r(\nu)}^2 \leq \|F\|_{L_2(\nu)}^2 + \frac{1}{\rho}\int_2^r \left\|\|\nabla f\|_2\right\|_{L_s(\nu)}^2 ds, \tag{5.2}$$

for all $r\geq 2$. Recall the known fact (e.g. see [12]) that if a measure ν satisfies log-Sobolev with ρ, also satisfies Poincaré with ρ:

$$\|h-\mathbb{E}_\nu h\|_{L_2(\nu)}^2 \leq \frac{1}{\rho}\int_{\mathbb{R}^n} \|\nabla h\|_2^2\, d\nu = \frac{1}{\rho}\left\|\|\nabla h\|_2\right\|_{L_2(\nu)}^2, \tag{5.3}$$

for any smooth function h. Therefore, (5.2) becomes:

$$\|F\|_{L_r(\nu)}^2 \leq \frac{2}{\rho}\int_2^r \left\|\|\nabla f\|_2\right\|_{L_s(\nu)}^2 ds \leq \frac{2r}{\rho}\left\|\|\nabla f\|_2\right\|_{L_r(\nu)}^2, \tag{5.4}$$

for all $r\geq 3$, where we have used the fact that $s\mapsto \|h\|_{L_s}$ is non-decreasing function. Taking into account the Claim in Section 4 we get:

$$\|F\|_{L_r(\nu)}^2 \leq \frac{2p^2 r}{\rho}\left(\int_{\mathbb{R}^n} \|x\|_{B_{2p-2}(\mu)}^{r(p-1)}\, d\nu(x)\right)^{2/r}, \quad r\geq 3. \tag{5.5}$$

Again, Lemma 2.2 implies that:

$$\left(\int_{\mathbb{R}^n} \|x\|_{B_{2p-2}(\mu)}^{r(p-1)}\, d\nu(x)\right)^{2/r} \leq I_{2p-2}^{2p-2}(\nu, B_{2p-2}(\mu))\left(1+\frac{r-2}{\rho I_{2p-2}^2(\nu, B_{2p-2}(\mu))}\right)^{p-1}, \tag{5.6}$$

for $r \geq 2$. Plug this back in (5.5) we obtain:

$$\|F\|_{L_r(\nu)} < \left(\frac{2p^2}{\rho}\right)^{1/2} r^{1/2} I_{2p-2}^{p-1}(\nu, B_{2p-2}(\mu))\left(1+\frac{r-2}{\rho I_{2p-2}^2(\nu, B_{2p-2}(\mu))}\right)^{\frac{p-1}{2}}, \tag{5.7}$$

for all $r \geq 3$. Note that the isotropicity and Lemma 2.2 for $x \mapsto \langle x, \theta\rangle$ imply

$$n \leq I_p^p(\nu, B_p(\mu)) \leq \left(1+\frac{p-2}{\rho}\right)^{p/2} n,$$

for all $p \geq 2$. Taking into account these estimates, we argue as in Section 3 to complete the proof. The details are left to the reader.

3. The variance in Lewis' position. Let us point out that our method also provides upper estimate for the variance of the norm of any finite dimensional subspace of L_p in Lewis' position. We should mention that the following estimate turns out to be optimal (up to the constant C) as the example of ℓ_p^n shows (see [18, Section 3] for details).

Theorem 5.2. *Let $1 \leq p < \infty$. Then, for any n-dimensional subspace X of L_p there exists position $\tilde{B}$ of the unit ball B_X of X such that:*

$$\mathrm{Var}\|g\|_{\tilde{B}} \leq C^p n^{\frac{2}{p}-1}, \tag{5.8}$$

where $C > 0$ is an absolute constant.

Sketch of Proof. It suffices to prove the assertion for $2 < p < \infty$. If $\tilde{B}$ is Lewis' position, we may identify X with $X_p(\mu)$ for some Borel isotropic measure μ on S^{n-1}. Then, we may write:

$$\mathrm{Var}\|g\|_{B_p(\mu)} = \mathbb{E}(\|g\|_{B_p(\mu)} - \|g'\|_{B_p(\mu)})^2 \leq \frac{1}{p^2}\mathbb{E}\left(\frac{\|g\|_{B_p(\mu)}^p - \|g'\|_{B_p(\mu)}^p}{\min\{\|g\|_{B_p(\mu)}^{p-1}, \|g'\|_{B_p(\mu)}^{p-1}\}}\right)^2, \tag{5.9}$$

where we have use the numerical inequality $|a^p - b^p| \geq p|a-b|\min\{a^{p-1}, b^{p-1}\}$ for $a, b > 0$ and $p > 1$. Thus, Cauchy-Schwarz inequality implies:

$$\sqrt{\mathrm{Var}\|g\|_{B_p(\mu)}} \leq \frac{2}{p}\frac{\left(\mathbb{E}\left|\|g\|_{B_p(\mu)}^p - \|g'\|_{B_p(\mu)}^p\right|^4\right)^{1/4}}{I_{-4(p-1)}^{p-1}(\gamma_n, B_p(\mu))}. \tag{5.10}$$

The numerator is estimated by Proposition 3.1 while for the denominator we employ the main result of [10] along with the fact that $k(B_p(\mu)) \geq c_1 p n^{2/p}$ for $n \geq e^{C_1 p}$. Putting them all together we arrive at the desired estimate. □

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