Estimates for the affine and dual affine quermassintegrals of convex bodies

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Abstract

We provide estimates for suitable normalizations of the affine and dual affine quermassintegrals of a convex body K in $\mathbb{R}^n$. These follow by a more general study of normalized p-means of projection and section functions of K.

1 Introduction

The starting point of this paper is an integral formula of Furstenberg and Tzkoni [5] about the volume of k-dimensional sections of ellipsoids: for every ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n$ one has

$$\int_{G_{n,k}} |\mathcal{E} \cap F|^n d\nu_{n,k}(F) = c_{n,k}|\mathcal{E}|^k, \tag{1.1}$$

where $\nu_{n,k}$ is the Haar measure on the Grassmannian $G_{n,k}$ and $c_{n,k}$ is a constant depending only on n and k; more precisely, $c_{n,k} = \Gamma\left(\frac{n}{2}+1\right)^k / \Gamma\left(\frac{k}{2}+1\right)^n$. It was proved by Miles [16] that this formula can be obtained in a simpler way as a consequence of classical formulas of Blaschke and Petkantschin.

Later, analogous quantities were considered by Lutwak and Grinberg in the setting of convex bodies. Lutwak introduced in [11] – for every convex body K in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$ – the quantities

$$\Phi_{n-k}(K) = \frac{\omega_n}{\omega_k} \left(\int_{G_{n,k}} |P_F(K)|^{-n} d\nu_{n,k}(F) \right)^{-1/n}, \tag{1.2}$$

where $P_F(K)$ is the orthogonal projection onto F and ω_k is the volume of the Euclidean unit ball in $\mathbb{R}^k$. For $k = 0$ and $k = n$ one sets $\Phi_0(K) = |K|$ and $\Phi_n(K) = \omega_n$ respectively. Grinberg [8] proved that these quantities are invariant under volume preserving affine transformations; this justifies the terminology "affine quermassintegrals" for $\Phi_{n-k}(K)$. From the definition of $\Phi_{n-k}(K)$ it is clear that

$$\Phi_{n-k}(K) \leqslant \frac{\omega_n}{\omega_k} \int_{G_{n,k}} |P_F(K)| \, d\nu_{n,k}(F) = W_{n-k}(K), \tag{1.3}$$

where $W_{n-k}(K) = V(K, [k]\, B_2^n, [n-k])$ are the Quermassintegrals of K. Lutwak conjectured in [12] that the affine quermassintegrals satisfy the inequalities

$$\omega_n^j \Phi_i^{n-j} \leqslant \omega_n^i \Phi_j(K)^{n-i} \tag{1.4}$$

for all $0 \leqslant i < j < n$. For example, Lutwak asks if

$$\Phi_{n-k}(K) \geqslant \omega_n^{(n-k)/n} |K|^{k/n} \tag{1.5}$$

with equality if and only if K is an ellipsoid; note that the weaker inequality $W_{n-k}(K) \geqslant \omega_n^{(n-k)/n}|K|^{k/n}$ holds true by the isoperimetric inequality. Most of these questions remain open (see [6, Chapter 9]); two cases of (1.5) follow from classical results: when $k = n-1$ this inequality is the Petty projection inequality and when $k = 1$ and K is symmetric then (1.5) is the Blaschke-Santaló inequality.

Lutwak proposed in [13] to study the dual affine quermassintegrals $\tilde{\Phi}_{n-k}(K)$. For every convex body K in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$ one defines

$$\tilde{\Phi}_{n-k}(K) = \frac{\omega_n}{\omega_k} \left(\int_{G_{n,k}} |K \cap F|^n d\nu_{n,k}(F) \right)^{1/n}. \tag{1.6}$$

For $k = 0$ and $k = n$ one sets $\tilde{\Phi}_0(K) = |K|$ and $\tilde{\Phi}_n(K) = \omega_n$ respectively. Grinberg proved in [8] that these quantities are also invariant under volume preserving linear transformations, and he established the inequality

$$\tilde{\Phi}_{n-k}(K) \leqslant \omega_n^{(n-k)/n} |K|^{k/n} \tag{1.7}$$

for all $1 \leqslant k \leqslant n-1$, with equality if and only if K is a centered ellipsoid. The case $k = n-1$ of this inequality is the Busemann intersection inequality (while the case $k = 1$ becomes an identity for symmetric convex bodies).

Being affinely invariant, affine and dual affine quermassintegrals appear to be useful in asymptotic convex geometry. So, one of the purposes of this work is to give upper and lower bounds for $\Phi_{n-k}(K)$ and $\tilde{\Phi}_{n-k}(K)$ in the remaining cases. We introduce a different notation and normalization which is better adapted to our needs. Nevertheless, the question we study is equivalent to e.g. [6, Problem 9.7].

Definition 1.1 (normalized affine quermassintegrals). For every convex body K in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$ we define

$$\Phi_{[k]}(K) = \left(\int_{G_{n,k}} |P_F(K)|^{-n} d\nu_{n,k}(F) \right)^{-\frac{1}{kn}}. \tag{1.8}$$

We also set $\Phi_{[n]}(K) = |K|^{1/n}$. Lutwak's conjectures about affine quermassintegrals can now be restated as follows:

(i) For every (symmetric) convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$,

$$\Phi_{[k]}(K) \geqslant \Phi_{[k]}(D_n), \tag{1.9}$$

where D_n is the Euclidean ball of volume 1.
(ii) For every convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$,

$$\Phi_{[k]}(K) \leqslant \Phi_{[k]}(S_n), \tag{1.10}$$

where S_n is the regular Simplex of volume 1.
In view of these conjectures, in the asymptotic setting it is reasonable to ask if the following holds true: There exist absolute constants $c_1, c_2 > 0$ such that for every convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$,

$$c_1\sqrt{n/k} \leqslant \Phi_{[k]}(K) \leqslant c_2\sqrt{n/k}. \tag{1.11}$$

For $k = 1$ the Blaschke-Santaló inequality shows that (1.9) holds true. Proving (1.10) for $k = 1$ corresponds to Malher's conjecture. Clearly, (1.11) for $k = 1$ follows from the Blaschke-Santaló and the reverse Santaló inequality of Bourgain-Milman [3].

Note that for $k = n-1$ we have

$$\Phi_{[n-1]}(K) = \left(\frac{|B_2^n|}{|\Pi^*(K)|}\right)^{\frac{1}{n(n-1)}}, \tag{1.12}$$

where $\Pi^*(K)$ is the polar projection body of K. Then, Hölder's inequality and the isoperimetric inequality show that (1.9) holds true. The same is true for (1.10): this follows from Zhang's inequality; see [30].

Definition 1.2 (normalized dual affine quermassintegrals). For every convex body K in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$ we define

$$\tilde{\Phi}_{[k]}(K) = \left(\int_{G_{n,k}} |K \cap F^{\perp}|^n d\nu_{n,k}(F)\right)^{\frac{1}{kn}}. \tag{1.13}$$

Grinberg's theorem about dual affine quermassintegrals states that if K has volume 1 then

$$\tilde{\Phi}_{[k]}(K) \leqslant \tilde{\Phi}_{[k]}(D_n) \leqslant c_2, \tag{1.14}$$

where $c_2 > 0$ is an absolute constant. As we will see, if the hyperplane conjecture has an affirmative answer then

$$\tilde{\Phi}_{[k]}(K) \geqslant c_1 \tag{1.15}$$

for every centered convex body of volume 1, where $c_1 > 0$ is an absolute constant. In view of the above, here one asks if the following holds true: There exist absolute constants $c_1, c_2 > 0$ such that for every centered convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$,

$$c_1 \leqslant \tilde{\Phi}_{[k]}(K) \leqslant c_2. \tag{1.16}$$

Our estimates on the normalized affine and dual affine quermassintegrals are summarized in the following:

Theorem 1.3. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \leqslant k \leqslant n-1$,*

$$\Phi_{[k]}(K) \leqslant c_1 \sqrt{n/k}\, \log n \tag{1.17}$$

and, if K is also centered,

$$\tilde{\Phi}_{[k]}(K) \geqslant \frac{c_2}{L_K}, \tag{1.18}$$

where L_K is the isotropic constant of K. In particular, assuming the hyperplane conjecture we have that $\tilde{\Phi}_{[k]}(K) \simeq 1$ for all $1 \leqslant k \leqslant n-1$. We also have the bounds

$$\Phi_{[k]}(K) \leqslant c_3 (n/k)^{3/2} \sqrt{\log (en/k)} \tag{1.19}$$

and

$$\tilde{\Phi}_{[k]}(K) \geqslant \frac{c_4}{\sqrt{n/k}\sqrt{\log(en/k)}}. \tag{1.20}$$

which are sharp when k is proportional to n.

For the proofs of these estimates, we attempt a more general study of normalized p-means of projection and section functions of K, which we introduce for every $1 \leqslant k \leqslant n-1$ and every $p \neq 0$ by setting

$$W_{[k,p]}(K) := \left(\int_{G_{n,k}} |P_F(K)|^p d\nu_{n,k}(F) \right)^{\frac{1}{kp}}. \tag{1.21}$$

and

$$\tilde{W}_{[k,p]}(K) = \left(\int_{G_{n,k}} |K \cap F^{\perp}|^p d\nu_{n,k}(F) \right)^{\frac{1}{kp}}. \tag{1.22}$$

respectively. The k-th normalized affine and dual affine quermassintegrals of K correspond to the cases $p = -n$ and $p = n$ respectively:

$$\Phi_{[k]}(K) = W_{[k,-n]}(K) \text{ and } \tilde{\Phi}_{[k]}(K) = \tilde{W}_{[k,n]}(K). \tag{1.23}$$

We list several properties of the p-means and prove some related inequalities.

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2 Notation and Preliminaries

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. We also write $\overline{A}$ for the homothetic image of volume 1 of a compact set $A\subseteq\mathbb{R}^n$ of positive volume, i.e. $\overline{A}:=|A|^{-\frac{1}{n}}A$. If A and B are compact sets in $\mathbb{R}^n$, then the covering number $N(A,B)$ of A by B is the smallest number of translates of B whose union covers A.

The letters c,c',c_1,c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a\simeq b$, we mean that there exist absolute constants $c_1,c_2>0$ such that $c_1a\leqslant b\leqslant c_2a$.

A star-shaped body C with respect to the origin is a compact set that satisfies $tC\subseteq C$ for all $t\in[0,1]$. We denote by $\|\cdot\|_C$ the gauge function of C:

$$\|x\|_C=\inf\{\lambda>0: x\in\lambda C\}. \tag{2.1}$$

A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x\in C$ implies that $-x\in C$. We say that C is centered if it has centre of mass at the origin: $\int_C\langle x,\theta\rangle\,dx=0$ for every $\theta\in S^{n-1}$. The support function $h_C:\mathbb{R}^n\to\mathbb{R}$ of C is defined by $h_C(x)=\max\{\langle x,y\rangle: y\in C\}$. The radius of C is the quantity $R(C)=\max\{\|x\|_2: x\in C\}$ and, if the origin is an interior point of C, the polar body C° of C is

$$C^\circ:=\{y\in\mathbb{R}^n:\langle x,y\rangle\leqslant 1 \text{ for all } x\in C\}. \tag{2.2}$$

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, the Blaschke–Santaló inequality and the Bourgain–Milman inequality imply that

$$|K^\circ|^{\frac{1}{n}}\simeq\frac{1}{n}. \tag{2.3}$$

Let K be a centered convex body in $\mathbb{R}^n$. For every $F\in G_{n,k}$, $1\leqslant k\leqslant n-1$, we have that $P_F(K^\circ)=(K\cap F)^\circ$, and hence,

$$|K\cap F|^{1/k}|P_FK^\circ|^{1/k}\simeq\frac{1}{k}. \tag{2.4}$$

The Rogers-Shephard inequality [26] states that

$$1\leqslant|P_FK|^{1/k}|K\cap F^\perp|^{1/k}\leqslant\binom{n}{k}^{1/k}\leqslant\frac{en}{k}. \tag{2.5}$$

We refer to the books [28], [21] and [25] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$ and $\theta \in S^{n-1}$ we define

$$h_{Z_q(K)}(\theta) := \left(\int_K |\langle x, \theta \rangle|^q dx \right)^{1/q}. \tag{2.6}$$

We define the L_q-centroid body $Z_q(K)$ of K to be the centrally symmetric convex set with support function $h_{Z_q(K)}$. L_q–centroid bodies were introduced in [14]. Here we follow the normalization (and notation) that appeared in [23].

It is easy to check that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \mathrm{conv}\{K, -K\}$. Note that if $T \in SL(n)$ then $Z_p(T(K)) = T(Z_p(K))$. Moreover, as a consequence of the Brunn–Minkowski inequality (see, for example, [23]), one can check that

$$Z_q(K) \subseteq c\frac{q}{p} Z_p(K) \tag{2.7}$$

for all $1 \leqslant p < q$, where $c \geqslant 1$ is an absolute constant, and

$$Z_q(K) \supseteq c\, K \tag{2.8}$$

for all $q \geqslant n$, where $c > 0$ is an absolute constant.

A centered convex body K of volume 1 in $\mathbb{R}^n$ is called isotropic if $Z_2(K)$ is a multiple of B_2^n. Then, we define the isotropic constant of K by

$$L_K := \left(\frac{|Z_2(K)|}{|B_2^n|} \right)^{1/n}. \tag{2.9}$$

It is known that $L_K \geqslant L_{B_2^n} \geqslant c > 0$ for every convex body K in $\mathbb{R}^n$. Bourgain proved in [2] that $L_K \leqslant c\sqrt[4]{n} \log n$ and, a few years ago, Klartag [9] obtained the estimate $L_K \leqslant c\sqrt[4]{n}$ (see also [10]). The hyperplane conjecture asks if $L_K \leqslant C$, where $C > 0$ is an absolute constant. We refer to [19], [7] and [23] for additional information on isotropic convex bodies.

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every star shaped body C in $\mathbb{R}^n$ and any $-n < p \leqslant \infty$, $p \neq 0$, we set

$$I_p(K, C) := \left(\int_K \|x\|_C^p dx \right)^{1/p}. \tag{2.10}$$

If $C = B_2^n$ we simply write $I_p(K)$ instead of $I_p(K, B_2^n)$.

3 p-mean projection functions and estimates for $\Phi_{[k]}(K)$

We first consider the question whether there exist absolute constants $c_1, c_2 > 0$ such that for every convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$,

$$c_1 \sqrt{n/k} \leqslant \Phi_{[k]}(K) \leqslant c_2 \sqrt{n/k}. \tag{3.1}$$

We can prove that the right-hand side inequality holds true up to a $\log n$ term.

Theorem 3.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \leqslant k \leqslant n-1$,*

$$\Phi_{[k]}(K) \leqslant c\sqrt{n/k}\,\log n. \tag{3.2}$$

For the proof of Theorem 3.1 we introduce a normalized version of the quermass-integrals of a convex body.

§*3.1 Normalized quermassintegrals.* Let K be a convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ we define the normalized k-quermassintegral of K by

$$W_{[k]}(K) := \left(\int_{G_{n,k}} |P_F(K)| d\nu_{n,k}(F)\right)^{1/k}. \tag{3.3}$$

We also set $W_{[n]}(K) = |K|^{1/n}$ and $W_{[0]}(K) = 1$. Note that

$$W_{[1]}(K) = \int_{S^{n-1}} [h_K(\theta) + h_K(-\theta)]\, d\sigma(\theta) = 2w(K). \tag{3.4}$$

From the definition and Kubota's formula (see [28]) it is clear that, for every $1 \leqslant k \leqslant n-1$ one has

$$W_{[k]}(K) = \left(\frac{\omega_k}{\omega_n} V(K,[k];B_2^n,[n-k])\right)^{1/k}. \tag{3.5}$$

Applying the Aleksandrov-Fenchel inequality (see [28, Chapter 6]) one can check the following:

(i) If K and L are convex bodies in $\mathbb{R}^n$, then, for all $1 \leqslant k \leqslant n$,

$$W_{[k]}(K+L) \geqslant W_{[k]}(K) + W_{[k]}(L). \tag{3.6}$$

(ii) For all $0 \leqslant k_1 < k_2 < k_3 \leqslant n$,

$$\frac{W_{[k_2]}(K)W_{[k_1]}(B_2^n)}{W_{[k_1]}(K)W_{[k_2]}(B_2^n)} \geqslant \left(\frac{W_{[k_3]}(K)W_{[k_1]}(B_2^n)}{W_{[k_1]}(K)W_{[k_3]}(B_2^n)}\right)^{\frac{(k_2-k_1)k_3}{k_2(k_3-k_1)}}. \tag{3.7}$$

(iii) For all $1 \leqslant k_1 \leqslant k_2 \leqslant n$,

$$\frac{W_{[k_2]}(K)}{W_{[k_2]}(B_2^n)} \leqslant \frac{W_{[k_1]}(K)}{W_{[k_1]}(B_2^n)}. \tag{3.8}$$

Proof of Theorem 3.1. Since $\Phi_{[k]}(K)$ is affine invariant we may assume that K is centered. It is well-known that Pisier's inequality (see [25, Chapter 2]) on the norm of the Rademacher projection implies that there exists $T \in SL(n)$ such that

$$W_{[1]}(T(K)) = 2w(T(K)) \leqslant c\sqrt{n}\log n. \tag{3.9}$$

More precisely, (3.9) follows from Pisier's inequality in the case where K is symmetric. However, it is not difficult to extend the inequality to the non necessarily symmetric case (see e.g. [22, Lemma3]). Then, using the affine invariance of $\Phi_{[k]}$ and the fact that $\Phi_{[k]}(K) \leqslant W_{[k]}(K)$, we write

$$\Phi_{[k]}(K) = \Phi_{[k]}(T(K)) \leqslant W_{[k]}(T(K)). \tag{3.10}$$

Since $W_{[k]}(B_2^n) = \omega_k^{1/k} \simeq \frac{1}{\sqrt{k}}$, it follows from (3.8) that

$$W_{[k]}(T(K)) \leqslant \frac{W_{[k]}(B_2^n)}{W_{[1]}(B_2^n)} W_{[1]}(T(K)) \leqslant c\sqrt{n/k}\,\log n. \tag{3.11}$$

This completes the proof. □

Next, we introduce the p-mean projection function $W_{[k,p]}(K)$ and the p-mean width $w_p(K)$ of a convex body K and prove a weak lower bound in the direction of the left hand side inequality of (3.1).

§3.2. p-mean projection function. Let K be a convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ and for every $p \neq 0$ we define the p-mean projection function $W_{[k,p]}(K)$ by

$$W_{[k,p]}(K) := \left(\int_{G_{n,k}} |P_F(K)|^p d\nu_{n,k}(F) \right)^{\frac{1}{kp}}. \tag{3.12}$$

We also set $W_{[n]}(K) := |K|^{1/n}$. Observe that the k-th normalized affine quermassintegral of K corresponds to the case $p = -n$:

$$\Phi_{[k]}(K) := W_{[k,-n]}(K). \tag{3.13}$$

It is clear that $W_{[k,p]}(K)$ is an increasing function of p, $W_{[s,p]}(\lambda K) = \lambda W_{[s,p]}(K)$ for every $\lambda > 0$ and $W_{[s,p]}(K) \leqslant W_{[s,p]}(L)$ whenever $K \subseteq L$. Moreover, for every $1 \leqslant k < m \leqslant n-1$ and every $p \neq 0$, one has

$$W_{[k,p]}(K) = \left(\int_{G_{n,m}} W_{[k,p]}^{kp}(P_E(K)) d\nu_{n,m}(E) \right)^{\frac{1}{kp}}. \tag{3.14}$$

In particular,

$$W_{[k,-m]}(K) = \left(\int_{G_{n,m}} \Phi_{[k]}^{-km}(P_E(K)) d\nu_{n,m}(E) \right)^{-\frac{1}{km}}. \tag{3.15}$$

§3.3. p-mean width. The p-mean width of K is defined for every $p \neq 0$ by

$$w_p(K) = \left(\int_{S^{n-1}} h_K^p(\theta) d\sigma(\theta) \right)^{1/p}. \tag{3.16}$$

It is clear that $w_p(K)$ is an increasing function of p, $w_p(\lambda K) = \lambda w_p(K)$ for every $\lambda > 0$ and $w_p(K) \leqslant w_p(L)$ whenever $K \subseteq L$. Note that, if K° is the polar body of K, then

$$w_{-n}(K) = \left(\frac{|B_2^n|}{|K^\circ|}\right)^{\frac{1}{n}}. \tag{3.17}$$

Also, for every $1 \leqslant k \leqslant n-1$,

$$w_p(K) = \left(\int_{G_{n,k}} w_p^p(P_E(K))d\nu_{n,k}(E)\right)^{1/p} \tag{3.18}$$

and, in particular,

$$w_{-k}(K) = \omega_k^{1/k}\left(\int_{G_{n,k}} |(P_E(K))^\circ| d\nu_{n,k}(E)\right)^{-1/k}. \tag{3.19}$$

Using the above we are able to prove that, in the symmetric case, $W_{[k,-q]}(K) \geqslant c\sqrt{n/k}$ as far as $q \leqslant n/k$; recall that $\Phi_{[k]}(K) = W_{[k,-n]}(K)$.

Theorem 3.2. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \leqslant k \leqslant n-1$,*

$$W_{[k,-n/k]}(K) \geqslant c\sqrt{n/k}. \tag{3.20}$$

Proof. Using Hölder's inequality, the Blaschke-Santaló and the reverse Santaló inequality, for every $p \geqslant 1$ we can write

$$\begin{aligned}
\left(\int_{G_{n,k}} |P_F(K)|^{-p} d\nu_{n,k}(F)\right)^{\frac{1}{kp}} &\simeq \left(\int_{G_{n,k}} \frac{|(P_F(K))^\circ|^p}{\omega_k^{2p}} d\nu_{n,k}(F)\right)^{\frac{1}{kp}} \\
&\simeq \sqrt{k}\left(\int_{G_{n,k}}\left(\int_{S_F}\frac{1}{h_K^k(\theta)}d\sigma_F(\theta)\right)^p d\nu_{n,k}(F)\right)^{\frac{1}{kp}} \\
&\leqslant c\sqrt{k}\left(\int_{G_{n,k}}\int_{S_F}\frac{1}{h_K^{kp}(\theta)}d\sigma_F(\theta)\,d\nu_{n,k}(F)\right)^{\frac{1}{kp}} \\
&= c\sqrt{k}\left(\int_{S^{n-1}}\frac{1}{h_K^{kp}(\theta)}d\sigma(\theta)\right)^{\frac{1}{kp}} \\
&= c\sqrt{k}\,w_{-kp}^{-1}(K).
\end{aligned}$$

We set $p := n/k \geqslant 1$. Then, from (3.17) we get

$$W_{[k,-n/k]}(K) \geqslant \frac{w_{-n}(K)}{c\sqrt{k}} \simeq \frac{1}{c\sqrt{k}}\frac{\omega_n^{1/n}}{|K^\circ|^{1/n}} \simeq \sqrt{n/k}. \tag{3.21}$$

This completes the proof. □

Note. What we have actually shown in the proof of Theorem 3.2 is that
(3.22)

$$W_{[k,-p]}(K) \simeq \sqrt{k}\left(\int_{G_{n,k}}\left(\int_{S_F}\frac{1}{h_K^k(\theta)}d\sigma_F(\theta)\right)^p d\nu_{n,k}(F)\right)^{-\frac{1}{kp}} \geqslant c\frac{w_{-kp}(K)}{\sqrt{k}}$$

for all $1 \leqslant k \leqslant n-1$ and $p \geqslant 1$.

4 p-mean section functions and estimates for $\tilde{\Phi}_{[k]}(K)$

Next, we consider the dual affine quermassintegrals. We first provide a lower bound which is sharp up to the isotropic constant of the body.

Theorem 4.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n-1$. Then,*

$$\tilde{\Phi}_{[k]}(K) \geqslant \frac{c}{L_K}. \tag{4.1}$$

Proof. By the linear invariance of $\tilde{\Phi}_{[k]}(K)$, we may assume that K is in the isotropic position. Let F be a k-dimensional subspace of $\mathbb{R}^n$. We denote by E the orthogonal subspace of F and for every $\phi \in F \setminus \{0\}$ we define $E^+(\phi) = \{x \in \mathrm{span}\{E,\phi\} : \langle x,\phi\rangle \geqslant 0\}$. K. Ball (see [1] and [19]) proved that, for every $q \geqslant 0$, the function

$$\phi \mapsto \|\phi\|_2^{1+\frac{q}{q+1}}\left(\int_{K\cap E^+(\phi)}\langle x,\phi\rangle^q dx\right)^{-\frac{1}{q+1}} \tag{4.2}$$

is the gauge function of a convex body $B_q(K,F)$ on F. We will make use of the fact that, if K is isotropic then

$$|K\cap F^\perp|^{1/k} \simeq \frac{L_{B_{k+1}(K,F)}}{L_K}. \tag{4.3}$$

See [19] and [23] for a proof. Therefore,

$$\tilde{\Phi}_{[k]}(K)L_K \simeq \left(\int_{G_{n,k}} L_{B_{k+1}(K,F)}^{kn} d\nu_{n,k}(F)\right)^{\frac{1}{kn}}. \tag{4.4}$$

Recall that the isotropic constant is uniformly bounded from below: we know that $L_{B_{k+1}(K,F)} \geqslant c$, where $c > 0$ is an absolute constant. It follows that

$$\tilde{\Phi}_k(K)L_K \simeq \left(\int_{G_{n,k}} L_{B_{k+1}(K,F)}^{kn} d\nu_{n,k}(F)\right)^{\frac{1}{kn}} \geqslant c, \tag{4.5}$$

and the result follows. $\square$

Note. Theorem 4.1 shows that if the hyperplane conjecture is correct then (if we also take into account Grinberg's theorem), for every centered convex body K of volume 1 in $\mathbb{R}^n$ and for every $1 \leqslant k \leqslant n-1$,

$$c_1 \leqslant \tilde{\Phi}_{[k]}(K) \leqslant c_2 \tag{4.6}$$

where $c_1, c_2 > 0$ are absolute constants. This would answer completely the asymptotic version of our original problems about the dual affine quermassintegrals.

The proof of Theorem 4.1 has some interesting consequences:

Corollary 4.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ we have*

$$\nu_{n,k}\left(\{F \in G_{n,k} : L_{B_{k+1}(K,F)} \geqslant cL_K\}\right) \leqslant e^{-kn}, \tag{4.7}$$

where $c > 0$ is an absolute constant.

Proof. From Grinberg's theorem – see (1.14) – we know that $\tilde{\Phi}_{[k]}(K) \leqslant \tilde{\Phi}_{[k]}(D_n) \leqslant c_2$, where $c_2 > 0$ is an absolute constant. From (4.5) we get

$$\left(\int_{G_{n,k}} L^{kn}_{B_{k+1}(K,F)} d\nu_{n,k}(F)\right)^{\frac{1}{kn}} \leqslant c_3 L_K, \tag{4.8}$$

and the result follows from Markov's inequality. $\square$

We complement Theorem 4.1 with a second lower bound for $\tilde{\Phi}_{[k]}(K)$, which is sharp when k is proportional to n.

Theorem 4.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ we have that*

$$\tilde{\Phi}_{[k]}(K) \geqslant \frac{c}{\sqrt{n/k}\sqrt{\log(en/k)}}. \tag{4.9}$$

For the proof of this bound, we introduce the p-mean section function $\tilde{W}_{[k,p]}(K)$ of a convex body K.

§4.1. p-mean section function. Let K be a convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ and for every $p \neq 0$ we define the p-mean $\tilde{W}_{[k,p]}(K)$ by

$$\tilde{W}_{[k,p]}(K) = \left(\int_{G_{n,k}} |K \cap F^{\perp}|^p d\nu_{n,k}(F)\right)^{\frac{1}{kp}}. \tag{4.10}$$

The *normalized dual k-quermassintegral* of K is $\tilde{W}_{[k]}(K) := \tilde{W}_{[k,1]}(K)$. Observe that the k-th normalized dual affine quermassintegral of K corresponds to the case $p = n$:

$$\tilde{\Phi}_{[k]}(K) = \tilde{W}_{[k,n]}(K). \tag{4.11}$$

Hölder's inequality implies that, for a fixed value of k, $\tilde{W}_{[k,p]}(K)$ is an increasing function of p.

The next Proposition shows that the normalized dual quermassintegrals $\tilde{W}_{[k]}(K)$ are strongly related to the quantities $I_p(K)$.

Proposition 4.4. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n-1$. Then,*

$$\tilde{W}_{[k]}(K) I_{-k}(K) = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} = \tilde{W}_{[k]}(D_n) I_{-k}(D_n). \tag{4.12}$$

Note. It is easy to check that $\left(\frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \simeq \sqrt{n}$.

Proof. We integrate in polar coordinates:

$$\begin{aligned} I_{-k}^{-k}(K) &= \frac{n\omega_n}{n-k} \int_{S^{n-1}} \frac{1}{\|x\|_K^{n-k}} d\sigma(x) \\ &= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} \omega_{n-k} \int_{S_F} \frac{1}{\|\theta\|_{K\cap F}^{n-k}} d\sigma(\theta) d\nu_{n,n-k}(F) \\ &= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} |K \cap F| d\nu_{n,n-k}(F) \\ &= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,k}} |K \cap F^{\perp}| d\nu_{n,k}(F). \end{aligned}$$

The definition of $\tilde{W}_{[k]}(K)$ completes the proof. □

Proposition 4.4 has the following consequence:

Proposition 4.5. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \leqslant s \leqslant m \leqslant n-1$,*

$$\tilde{W}_{[s]}(K) \leqslant \tilde{W}_{[s]}(D_n) \tag{4.13}$$

and

$$\frac{\tilde{W}_{[m]}(K)}{\tilde{W}_{[s]}(K)} \geqslant \frac{\tilde{W}_{[m]}(D_n)}{\tilde{W}_{[s]}(D_n)}. \tag{4.14}$$

Proof. It is known (see [24]) that for any $q \geqslant p \geqslant -n$ we have

$$I_p(K) \geqslant I_p(D_n) \tag{4.15}$$

and

$$\frac{I_q(K)}{I_p(K)} \geqslant \frac{I_q(D_n)}{I_p(D_n)}. \tag{4.16}$$

Then, the result follows from Proposition 4.4. $\square$

Note. It is easy to check that

$$\tilde{W}_{[k]}(D_n) = \tilde{W}_{[k,p]}(D_n) = \tilde{\Phi}_{[k]}(D_n) \simeq 1. \tag{4.17}$$

Proof of Theorem 4.3. Hölder's inequality and Proposition 4.4 imply that

$$\tilde{\Phi}_{[k]}(K) \geqslant \tilde{W}_{[k]}(K) \geqslant \frac{c\sqrt{n}}{I_{-k}(K)}. \tag{4.18}$$

Now, we use the fact (see Theorem 5.2 and Lemma 5.6 in [4]) that there exists $T \in SL(n)$ such that

$$I_{-k}(T(K)) \leqslant c\sqrt{n}\sqrt{n/k}\sqrt{\log en/k}. \tag{4.19}$$

By the affine invariance of $\tilde{\Phi}_{[k]}(K)$ we have

$$\tilde{\Phi}_{[k]}(K) = \tilde{\Phi}_{[k]}(T(K)) \geqslant \frac{c\sqrt{n}}{I_{-k}(T(K))}, \tag{4.20}$$

and this completes the proof. $\square$

5 Duality relations

In this Section we prove some inequalities involving the p-means of projection and section functions of a convex body. In particular, we obtain duality relations between $\Phi_{[n/2]}(K)$ and $\tilde{\Phi}_{[n/2]}(K^\circ)$. These will allow us to obtain a second upper bound for $\Phi_{[k]}(K)$ which is sharp when k is proportional to n.

One source of such inequalities is the following "L_q–version of the Rogers-Shephard inequality" which was proved in [24].

Lemma 5.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \leqslant k \leqslant n-1$ and every $F \in G_{n,k}$ we have that*

$$c_1 \leqslant |K \cap F^\perp|^{1/k} |P_F(Z_k(K))|^{1/k} \leqslant c_2, \tag{5.1}$$

where $c_1, c_2 > 0$ are universal constants.

A direct application of Lemma 5.1 leads to the following:

Proposition 5.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ and $p \neq 0$ we have that*

(i) $c_1 \leqslant \tilde{W}_{[k,p]}(K) W_{[k,-p]}(Z_k(K)) \leqslant c_2$,

(ii) $c_3 \leqslant \tilde{\Phi}_{[k]}(K) \Phi_{[k]}(Z_k(K)) \leqslant c_4$,

(iii) $c_5 \leqslant \tilde{\Phi}_{[k]}(K) \Phi_{[k]}(K) \leqslant c_6 n/k$, *where $c_i > 0$, $i = 1, \dots, 6$ are absolute constants.*

Proof. From the definitions and (5.1) we readily see that

$$\begin{aligned}\tilde{W}_{[k,p]}(K) &= \left(\int_{G_{n,k}} |K \cap F^{\perp}|^p d\nu_{n,k}(F)\right)^{1/(kp)} \\ &\simeq \left(\int_{G_{n,k}} |P_F(Z_k(K))|^{-p} d\nu_{n,k}(F)\right)^{1/(kp)} \\ &= W_{[k,-p]}^{-1}(Z_k(K)).\end{aligned}$$

This proves (i). Then, (ii) corresponds to the special case $p = n$. Since $K \subseteq \frac{cn}{k} Z_k(K)$, (iii) follows. $\square$

A second source of inequalities is the Blaschke-Santaló and the reverse Santaló inequality. Since $(K \cap F^{\perp})^{\circ} = P_{F^{\perp}}(K^{\circ})$, for every $1 \leqslant k \leqslant n-1$ and $F \in G_{n,k}$ we have

$$c^{n-k}\omega_{n-k}^2 \leqslant |P_{F^{\perp}}(K^{\circ})|\,|K \cap F^{\perp}| \leqslant \omega_{n-k}^2. \tag{5.2}$$

Therefore,

$$\begin{aligned}\tilde{W}_{[k,p]}(K) &= \left(\int_{G_{n,k}} |K \cap F^{\perp}|^p d\nu_{n,k}(F)\right)^{1/(kp)} \\ &\leqslant \omega_{n-k}^{2/k}\left(\int_{G_{n,k}} |P_{F^{\perp}}(K^{\circ})|^{-p} d\nu_{n,k}(F)\right)^{1/(kp)} \\ &= \omega_{n-k}^{2/k}\left(\int_{G_{n,n-k}} |P_F(K^{\circ})|^{-p} d\nu_{n,n-k}(F)\right)^{1/(kp)} \\ &= \omega_{n-k}^{2/k} W_{[n-k,p]}^{-(n-k)/k}(K^{\circ}).\end{aligned}$$

Working in the same way we check that

$$\tilde{W}_{[k,p]}(K) W_{[k,p]}^{(n-k)/k}(K^{\circ}) \geqslant c^{(n-k)/k}\omega_{n-k}^{2/k}. \tag{5.3}$$

We summarize in the following Proposition.

Proposition 5.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ and $p \neq 0$ we have:*

(i) $c^{(n-k)/k}\omega_{n-k}^{2/k} \leqslant \tilde{W}_{[k,p]}(K) W_{[k,p]}^{(n-k)/k}(K^{\circ}) \leqslant \omega_{n-k}^{2/k}$.

(ii) *If n is even, then* $\tilde{W}_{[n/2,p]}(K) W_{[n/2,p]}(K^{\circ}) \simeq \frac{1}{n}$.

(iii) *If n is even, then* $\tilde{\Phi}_{[n/2]}(K)\Phi_{[n/2]}(\overline{K^{\circ}}) \simeq 1$.

Taking into account Proposition 5.2(iii) we have the following:

Corollary 5.4. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\tilde{\Phi}_{[n/2]}(K) \simeq \tilde{\Phi}_{[n/2]}(\overline{K^\circ}) \text{ and } \Phi_{[n/2]}(K) \simeq \Phi_{[n/2]}(\overline{K^\circ}). \tag{5.4}$$

We can get more precise information if we use the M-ellipsoid of K. Let K be a convex body of volume 1 in $\mathbb{R}^n$. Milman (see [17], [18] and also [20] for the not necessarily symmetric case) proved that there exists an ellipsoid $\mathcal{E}$ with $|\mathcal{E}| = 1$, such that

$$\log N(K, \mathcal{E}) \leqslant \nu n, \tag{5.5}$$

where $\nu > 0$ is an absolute constant. In other words, for any centered convex body K of volume 1 in $\mathbb{R}^n$ there exists $T \in SL(n)$ such that

$$N(T(K), D_n) \leqslant e^{\nu n}. \tag{5.6}$$

Theorem 5.5. *Let n be even and let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$c_1 \leqslant \tilde{\Phi}_{[n/2]}(K) \leqslant c_2, \tag{5.7}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. We will use the following inequality of Rogers and Shephard [27]. If K is a centered convex body of volume 1 in $\mathbb{R}^n$ then

$$|K - K| \leqslant 4^n. \tag{5.8}$$

We choose $T \in SL(n)$ so that

$$N(T(\overline{K - K}), D_n) \leqslant e^{\nu n} \tag{5.9}$$

Then, for any $F \in G_{n,\frac{n}{2}}$,

$$|P_F(T(\overline{K - K}))| \leqslant N\left(T(\overline{K - K})), D_n\right) |P_F(D_n)| \leqslant e^{\nu n} c^n. \tag{5.10}$$

Moreover, using (5.8) we have that

$$|P_F(Z_{\frac{n}{2}}(T(K)))| \leqslant |P_F(\mathrm{conv}(T(K), -T(K)))| \leqslant |P_F(T(K-K))| \leqslant 4^n |P_F(T(\overline{K - K}))|.$$

Combining the above with (5.10) and (5.1) we have that

$$|T(K) \cap F^\perp| \geqslant \frac{c_0^{\frac{n}{2}}}{|P_F(Z_{\frac{n}{2}}(T(K)))|} \geqslant \frac{c_0^{\frac{n}{2}}}{e^{\nu n} c^n} =: c_1^{\frac{n}{2}}. \tag{5.11}$$

So, we have shown that for any $F \in G_{n,\frac{n}{2}}$,

$$|T(K) \cap F| \geqslant c_1^{\frac{n}{2}}. \tag{5.12}$$

This implies that

$$\tilde{\Phi}_{[\frac{n}{2}]}(K) = \tilde{\Phi}_{[\frac{n}{2}]}(T(K)) \geqslant \min_{F\in G_{n,\frac{n}{2}}} |T(K)\cap F|^{\frac{2}{n}} \geqslant c_2. \tag{5.13}$$

This shows the left hand side inequality in (5.7). The right hand side inequality follows from (1.14). □

Combining Theorem 5.5 with Proposition 5.3 and Corollary 5.4 we conclude the following:

Corollary 5.6. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\tilde{\Phi}_{[n/2]}(K) \simeq \tilde{\Phi}_{[n/2]}(\overline{K^\circ}) \simeq \Phi_{[n/2]}(K) \simeq \Phi_{[n/2]}(\overline{K^\circ}) \simeq 1. \tag{5.14}$$

Note. In view of Corollary 5.6, if n is even and $k = n/2$, then (4.4) becomes a formula:

Corollary 5.7. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$L_K \simeq \left(\int_{G_{n,n/2}} L^{n^2/2}_{B_{\frac{n}{2}+1}(K,F)} d\nu_{n,n/2}(F)\right)^{2/n^2}. \tag{5.15}$$

In particular, there exists $F\in G_{n,n/2}$ such that

$$L_K \leqslant cL_{B_{\frac{n}{2}+1}(K,F)}. \tag{5.16}$$

Making use of Theorem 4.3 and of Proposition 5.2 we can now give a second upper bound for $\Phi_{[k]}(K)$, which sharpens the estimate in Theorem 3.1 when k is proportional to n.

Theorem 5.8. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1\leqslant k\leqslant n-1$. Then,*

$$\Phi_{[k]}(K) \leqslant c(n/k)^{3/2}\sqrt{\log en/k}. \tag{5.17}$$

Proof. We may assume that K is also centered. By Proposition 5.2 we have that

$$\Phi_{[k]}(K) = \frac{\Phi_{[k]}(K)\tilde{\Phi}_{[k]}(K)}{\tilde{\Phi}_{[k]}(K)} \leqslant \frac{cn/k}{\tilde{\Phi}_{[k]}(K)}. \tag{5.18}$$

Then, we use the lower bound of Theorem 4.3 for $\tilde{\Phi}_{[k]}(K)$. □

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