On the intersection of random rotations of a symmetric convex body

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Abstract

Let C be a symmetric convex body of volume 1 in $\mathbb{R}^n$. We provide general estimates for the volume and the radius of $C \cap U(C)$ where U is a random orthogonal transformation of $\mathbb{R}^n$. In particular, we consider the case where C is in the isotropic position or C is the volume normalized L_q-centroid body $Z_q(\mu)$ of an isotropic log-concave measure μ on $\mathbb{R}^n$.

1 Introduction

A well-known principle in the asymptotic theory of convex bodies asserts that local statements describing the structure of lower dimensional sections and projections of a symmetric convex body C in $\mathbb{R}^n$ can be “translated” to global statements about properties of C and its orthogonal images $U(C)$. A number of results, including the global form of Dvoretzky theorem proved by V. Milman and Schechtman in [26], illustrate this point of view. The volume ratio theorem is another classical example of the parallelism between the global and the local asymptotic theory. Szarek and Tomczak-Jaegermann [32], generalizing previous work of Kashin [12] for the unit ball of ℓ_1^n, proved that if C is a symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \subseteq C$ and $|C| = \alpha^n |B_2^n|$ for some $\alpha > 1$ then, for every $1 \leqslant k \leqslant n$, a random subspace $F \in G_{n,k}$ satisfies with probability greater than $1 - e^{-n}$

$$B_2^n \cap F \subseteq C \cap F \subseteq (c\alpha)^{\frac{n}{n-k}} B_2^n \cap F,$$

where $c > 0$ is an absolute constant. The global analogue of this statement is that, under the same hypothesis, there exists $U \in O(n)$ with the property

$$B_2^n \subset C \cap U(C) \subset c\alpha^2 B_2^n,$$

where $c > 0$ is an absolute constant. In a few words, the fact that most of the $n/2$-dimensional sections of C are α^2-equivalent to a Euclidean ball can be translated to the global statement that intersecting C with a random rotation $U(C)$ we obtain a convex body which is α^2-equivalent to B_2^n.

In this note we consider the intersection of a symmetric convex body C with $U(C)$, where $U \in O(n)$ is a random orthogonal transformation of $\mathbb{R}^n$, and we are mainly interested in the expectation of the volume and the radius $R(C\cap U(C)) := \max\{\|x\|_2 : x\in C\cap U(C)\}$ of $C\cap U(C)$. One motivation for this work was to understand the way these two quantities depend on classical parameters of the body C; a second motivation was to understand better the regularity properties of the L_q-centroid bodies of isotropic log-concave measures. Naturally, our results are directed to these two questions. We write $\|\cdot\|_C$ for the norm induced by C on $\mathbb{R}^n$ and we denote by $M := M(C)$ and by $w(C)$ the expectation of this norm on the unit sphere and the mean width of C respectively.

Starting with the volume, it is clear that $|C\cap U(C)|\leqslant 1$ for all U, and the example of the Euclidean ball $\overline{B_2^n}$ of volume 1 shows that, in full generality, one cannot expect anything better than this trivial upper bound. However, we will see that, under some natural condition on C, one can provide subexponential upper bounds for the expectation

$$\mathbb{E}_U|C\cap U(C)| = \int_{O(n)}|C\cap U(C)|\,d\nu(U)$$

where ν is the Haar measure on $O(n)$. Our starting point is a simple formula for this expectation; one has

$$\int_{O(n)}|C\cap U(C)|\,d\nu(U) = \int_C \sigma\Big(S^{n-1}\cap \frac{1}{\|x\|_2}C\Big)\,dx, \tag{1.1}$$

where σ is the rotationally invariant probability measure on the unit sphere S^{n-1}. Therefore, one has to understand the behaviour of $\sigma(S^{n-1}\cap tC)$ for small values of t or, equivalently, the behaviour of $\gamma_n(\alpha C)$ for $\alpha\simeq 1$ (see Lemma 4.1 below). A parameter which plays a key role in small ball probability estimates and is very much related to this question was introduced by Klartag and Vershynin in [18]: they defined $d(C)$ as follows:

$$d(C) := \min\left\{-\log\sigma\left(\left\{x\in S^{n-1} : \|x\|_C\leqslant \frac{M(C)}{2}\right\}\right), n\right\}.$$

Using the B-theorem of Cordero-Erausquin, Fradelizi and Maurey [5], in Section 4 we obtain the following estimate.

Theorem 1.1. *There exists an absolute constant $B_0>0$ such that if C is a symmetric convex body of volume 1 in $\mathbb{R}^n$ with $\sqrt{n}M(C)\geqslant B_0$ then*

$$\int_{O(n)}|C\cap U(C)|\,d\nu(U)\leqslant e^{-cd(C)}, \tag{1.2}$$

where $c>0$ is an absolute constant.

The condition $\sqrt{n}M(C)\geqslant B_0$ in Theorem 1.1 is rather natural; observe that if we express the volume of C as an integral in polar coordinates and use Hölder's inequality then we get

$$\mathrm{vrad}(C)M(C)\geqslant 1 \tag{1.3}$$

with equality if C is a Euclidean ball. If $|C| = 1$ then $\mathrm{vrad}(C) := (|C|/|B_2^n|)^{1/n} \simeq \sqrt{n}$ and hence the bodies for which $\mathrm{vrad}(C)M(C) \leqslant B_0$ form a rather restricted class.

The proof of Theorem 1.1 is given in Section 4. The general upper bound in (1.2) depends on the order of $d(C)$. We give some concrete applications in the case where C is a (normalized) ℓ_p^n-ball. We also discuss some classical positions of the body C from this point of view. A case of interest is when the body is in the isotropic position (see Section 2 for the definition and background information). In this case, using the thin shell estimate (see e.g. [11]) we obtain an alternative bound.

Theorem 1.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, either $L_K \leqslant 1$ or*

$$\int_{O(n)} |K \cap U(K)| \, d\nu(U) \leqslant c_1 e^{-c_2\sqrt{n}}, \tag{1.4}$$

where $c_1, c_2 > 0$ are absolute constants.

In fact, one can obtain a similar sub-exponential estimate in Theorem 1.2 under the assumption $L_K \geqslant t$, for any $t > \sqrt{2/\pi}$; this would only affect the constant c_2 (see Proposition 4.8 for a precise statement). Note also that the condition that is used in Theorem 1.2 is different from the one in Theorem 1.1; here, we require that the isotropic constant L_K of K is large enough: $L_K \geqslant 1$. It is a major open problem whether there exists an absolute constant $c_0 > 0$ such that $L_K \leqslant c_0$ for all isotropic convex bodies in any dimension; if this is true, and in particular if $c_0 < 1$, then the statement of Theorem 1.2 does not provide significant information.

In order to give lower bounds for $|C \cap U(C)|$ we use simple entropy estimates. In fact, our bounds are valid for every $U \in O(n)$. In Section 3 we show that, for every $\varrho > 0$ and any $U \in O(n)$, one has

$$|C \cap U(C)| \geqslant \left[\min\{(4\varrho)^{n/2} N(C, \varrho\overline{B_2^n}), (4/\varrho)^{n/2} N(\varrho\overline{B_2^n}, C)\} \right]^{-2}, \tag{1.5}$$

where $N(A, B)$ is the covering number of A by B, i.e. the least number of translates of B needed to cover A. Then, using known results on covering numbers, we obtain the following.

Theorem 1.3. *Let C be a symmetric convex body of volume 1 in $\mathbb{R}^n$. For any $U \in O(n)$ we have*

$$|C \cap U(C)| \geqslant e^{-cn \min\{w^2(C)/n, nM^2(C)\}}, \tag{1.6}$$

where $c > 0$ is an absolute constant. In particular, for any $1 \leqslant p \leqslant \infty$ and any $U \in O(n)$ we have

$$|\overline{B_p^n} \cap U(\overline{B_p^n})| \geqslant e^{-cn}, \tag{1.7}$$

where $c > 0$ is an absolute constant and $\overline{B_p^n}$ is the "normalized" ℓ_p^n-ball.

The dependence on $M(C)$ and $w(C)$ in Theorem 1.3 indicates that in order to obtain some non trivial information, we should consider some "good position" of the body C. We provide a number of results of this type: If C is in M-position with constant β then, for any $U\in O(n)$ we have $|C\cap U(C)|\geqslant e^{-2(\beta+1)n}$. Similarly, if K is an isotropic symmetric convex body in $\mathbb{R}^n$ then

$$|K\cap U(K)|\geqslant (cL_K)^{-n} \tag{1.8}$$

for every $U\in O(n)$, where $c\geqslant 4$ is an absolute constant.

In Section 5 we recall some known results from the local theory of normed spaces which lead to upper bounds for the radius of $C\cap U(C)$. It is well understood that if one has an upper bound for the radius of a random k-dimensional section $C\cap F$ of C where $k\geqslant (1-c_0)n$ (for some small absolute constant $c_0\in(0,1)$) then the same bound holds true for the radius of a random intersection $C\cap U(C)$. There are several versions of this statement; we review the strongest and most recent ones (see [8], [33], [20]). In particular, combining these results with the low M^*-estimate, one gets the next very general fact, in the spirit of [26, Theorem 2.2] and most probably known to experts: a random $U\in O(n)$ satisfies

$$R(C\cap U(C))\leqslant cw(C)$$

with probability greater than $1-e^{-n}$, where $c>0$ is an absolute constant.

In the last section of this article we apply the previous results to the L_q-centroid bodies $Z_q(\mu)$ of an isotropic log-concave measure μ on $\mathbb{R}^n$. Recall that, if μ is a log-concave probability measure on $\mathbb{R}^n$ and $q\geqslant 1$ then the L_q-centroid body $Z_q(\mu)$ of μ is the symmetric convex body with support function

$$h_{Z_q(\mu)}(y):=\left(\int_{\mathbb{R}^n}|\langle x,y\rangle|^q d\mu(x)\right)^{1/q}. \tag{1.9}$$

The study of random rotations of $Z_q(\mu)$ proved to be useful in recent works on the thin shell conjecture. The following fact plays a key role in the article of Klartag and E. Milman [17] which introduces a regularization step for the study of this problem and strengthens the small ball estimates of [11]: if $2\leqslant q\leqslant\sqrt{n}$ then a random $U\in O(n)$ satisfies

$$Z_q(\mu)+U(Z_q(\mu))\supseteq c\sqrt{q}\,B_2^n$$

with probability greater than $1-e^{-cn}$. Using the results of Section 5 and the estimates of [9] on the inradius of random proportional projections of $Z_q(\mu)$ we provide a second proof. In a similar way one can prove an analogous inner regularization result for the polar body $Z_q^{\circ}(\mu)$; this is actually simpler. The precise statement, in the spirit of this note, is as follows.

Theorem 1.4. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For any $q\leqslant\sqrt{n}$, a random $U\in O(n)$ satisfies*

$$\overline{Z_q^{\circ}}(\mu)\cap U(\overline{Z_q^{\circ}}(\mu))\subseteq c\sqrt{n}B_2^n\quad \textit{and}\quad \overline{Z_q}(\mu)\cap U(\overline{Z_q}(\mu))\subseteq c\sqrt{n}B_2^n$$

with probability greater than $1-2e^{-n}$.

In the case of $Z_q(\mu)$, Theorem 1.1 leads to the following estimate: Let μ be an isotropic log-concave measure on $\mathbb{R}^n$ and let $2 \leqslant q \leqslant \sqrt{n}$. If $\sqrt{q}M(Z_q(\mu)) \geqslant B_1$ then

$$\int_{O(n)} |\overline{Z_q}(\mu) \cap U(\overline{Z_q}(\mu))| \, d\nu(U) \leqslant e^{-c_1 n/q}, \tag{1.10}$$

where $B_1, c_1 > 0$ are absolute constants. The question to give an upper bound for $M(Z_q(\mu))$ is naturally related to the necessary condition for (1.10). This was one of the main objects of study in [9], where a partial non-trivial upper bound was obtained: for every $1 \leqslant q \leqslant n^{3/7}$ one has $M\big(Z_q(\mu)\big) \leqslant C(\log q)^{5/6}/\sqrt[6]{q}$.

2 Notation and background material

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote the corresponding Euclidean norm by $\|\cdot\|_2$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. The volume ω_n of B_2^n is equal to $\pi^{n/2}/\Gamma\left(\frac{n}{2}+1\right)$; from Stirling's formula we see that $\omega_n^{1/n} \simeq 1/\sqrt{n}$. We write σ for the rotationally invariant probability measure on S^{n-1} and denote the Haar measure on $O(n)$ by ν. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $1 \leqslant k \leqslant n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F := B_2^n \cap F$ and $S_F := S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$.

Basic references for the theory of convex bodies and the asymptotic theory of finite dimensional normed spaces are the classical books of Schneider [31], Milman and Schechtman [25] and Pisier [30].

Symmetric convex bodies. A convex body in $\mathbb{R}^n$ is a compact convex set $C \subset \mathbb{R}^n$ with non-empty interior. In this article we discuss symmetric convex bodies, namely convex bodies C with the property that $x \in C$ if and only if $-x \in C$. The volume radius of C is the quantity $\mathrm{vrad}(C) = (|C|/|B_2^n|)^{1/n}$. The support function of C is defined by $h_C(y) := \max\big\{\langle x, y\rangle : x \in C\big\}$, and the mean width of C is the average

$$w(C) := \int_{S^{n-1}} h_C(\theta)\, d\sigma(\theta) \tag{2.1}$$

of h_C on S^{n-1}. The radius $R(C)$ of C is the smallest $R > 0$ such that $C \subseteq RB_2^n$ and the inradius $r(C)$ of C is the largest $r > 0$ for which $rB_2^n \subseteq C$.

The polar body C° of a symmetric convex body C in $\mathbb{R}^n$ is defined by

$$C^\circ := \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\big\}. \tag{2.2}$$

The Blaschke-Santaló inequality states that $|C||C^\circ| \leqslant \omega_n^2$, with equality if and only if C is an ellipsoid. The reverse Santaló inequality of Bourgain and V. Milman [3] states that there exists an absolute constant $c>0$ such that, conversely,

$$(|C||C^\circ|)^{1/n} \geqslant c/n, \tag{2.3}$$

where $c>0$ is an absolute constant.

Every symmetric convex body $C\subset \mathbb{R}^n$ induces a norm to $\mathbb{R}^n$, given by $\|x\|_C = \min\{t\geqslant 0 : x\in tC\}$. Then, the dual norm $\|\cdot\|_*$ of $\|\cdot\|$, defined by

$$\|y\|_* = \max\{|\langle x,y\rangle| : \|x\|_C\leqslant 1\},$$

is the norm induced by C° to $\mathbb{R}^n$: we have

$$h_C(x) = \|x\|_{C^\circ} = \|x\|_*$$

for all $x\in\mathbb{R}^n$. We also write $b:=b(C)$ for the smallest positive constant for which $\|x\|_C\leqslant b\|x\|_2$ for all $x\in\mathbb{R}^n$. Note that $b(C)=r(C)^{-1}$.

We will use some basic facts from the asymptotic theory of finite dimensional normed spaces. A parameter that plays a central role in the theory, and in this article, is the average

$$M(C) := \int_{S^{n-1}} \|\theta\|_C d\sigma(\theta) \tag{2.4}$$

of $\|\cdot\|_C$ on S^{n-1}. Note that $M(C)=w(C^\circ)$ and that

$$M(C)^{-1}\leqslant \mathrm{vrad}(C)\leqslant w(C) = M(C^\circ); \tag{2.5}$$

the left hand side inequality is easily checked if we express the volume of C as an integral in polar coordinates and use Hölder's inequality, while the right hand side inequality is the classical Urysohn's inequality.

The critical dimension $k(C)$ of a symmetric convex body C in $\mathbb{R}^n$ is the largest positive integer $k\leqslant n$ with the property that the measure $\nu_{n,k}$ of $F\in G_{n,k}$ for which we have $\frac{1}{2M(C)}B_F\subseteq C\cap F\subseteq \frac{2}{M(C)}B_F$ is greater than $\frac{n}{n+k}$. This parameter was studied in [26] where it is shown that it is completely determined by the dimension, the parameter $M(C)$ and the inradius of C: one always has

$$c_1 n\frac{M(C)^2}{b(C)^2}\leqslant k(C)\leqslant c_2 n\frac{M(C)^2}{b(C)^2}, \tag{2.6}$$

where $c_1,c_2>0$ are absolute constants. We also define $k_*(C)=k(C^\circ)$. Note that $k_*(C)\simeq n\frac{w(C)^2}{R(C)^2}$. Generalizing the definition of $d(C)$ which was given in the introduction, for every $r>1$ we define

$$d_r(C) = \min\left\{-\log\sigma\left(\left\{x\in S^{n-1} : \|x\|_C\leqslant \frac{M(C)}{r}\right\}\right), n\right\}.$$

Note that $d(C)=d_2(C)$. One can check (see [18]) that $d(C)\geqslant ck(C)$, where $c>0$ is an absolute constant.

Finally, we will need Milman's low M^*-estimate which states that there exists a function $\lambda : (1,\infty) \to \mathbb{R}^+$ such that if C is a symmetric convex body in $\mathbb{R}^n$, then a subspace $F \in G_{n,k}$ satisfies

$$R(C \cap F) \leqslant \lambda\left(\tfrac{n}{n-k}\right) w(C) \tag{2.7}$$

with probability greater than $1 - \exp(-c_2(n-k))$, where $c_1, c_2 > 0$ are absolute constants. Milman's first proof of (2.7) appears in [21], and a second proof from [22] establishes (2.7) with $\lambda(t) = ct$. Pajor and Tomczak-Jaegermann proved in [27] that the same statement holds true with $\lambda(t) = c\sqrt{t}$, which is the asymptotically best possible behavior. Finally, Gordon [10] proved a sharp form of the latter result; in particular, he showed that the value of the constant c can be assumed asymptotically equal to 1.

Given two convex bodies $C, L \subseteq \mathbb{R}^n$, we will write $C \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 C \subseteq L \subseteq c_2 C$. For notational convenience we write $\overline{C}$ for the homothetic image of volume 1 of a convex body $C \subseteq \mathbb{R}^n$, i.e. $\overline{C} := |C|^{-1/n} C$.

Log-concave probability measures. We denote by $\mathcal{P}_n$ the class of all Borel probability measures on $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. The density of $\mu \in \mathcal{P}_n$ is denoted by f_μ. We say that $\mu \in \mathcal{P}_n$ is centered and we write $\mathrm{bar}(\mu) = 0$ if, for all $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} \langle x, \theta\rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \theta\rangle f_\mu(x) dx = 0. \tag{2.8}$$

A measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for all compact subsets A and B of $\mathbb{R}^n$ and all $\lambda \in (0,1)$. A function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set and the restriction of $\log f$ to it is concave. Borell has proved in [1] that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H, then $\mu \in \mathcal{P}_n$ and its density f_μ is log-concave. Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_K$ is the density of a log-concave measure.

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x\in\mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x) dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}, \tag{2.9}$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} \frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx}. \tag{2.10}$$

Note that L_μ is an affine invariant of μ and does not depend on the choice of the Euclidean structure. We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if $\mathrm{bar}(\mu) = 0$ and $\mathrm{Cov}(\mu)$ is the identity matrix.

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and if its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{2.11}$$

for every θ in the Euclidean unit sphere S^{n-1}. Note that a centered convex body K of volume 1 in $\mathbb{R}^n$ is isotropic, i.e. it satisfies (2.11), if and only if the log-concave probability measure μ_K with density $x \mapsto L_K^n \mathbf{1}_{K/L_K}(x)$ is isotropic. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C \tag{2.12}$$

for all $n \geqslant 1$. Bourgain proved in [2] that $L_n \leqslant c\sqrt[4]{n}\log n$, while Klartag [13] obtained the bound $L_n \leqslant c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [16].

Let $\mu \in \mathcal{P}_n$. For every $1 \leqslant k \leqslant n-1$ and every $E \in G_{n,k}$, the marginal $\pi_E\mu$ of μ with respect to E is the probability measure with density

$$f_{\pi_E\mu}(x) = \int_{x+E^{\perp}} f_\mu(y)dy. \tag{2.13}$$

It is easily checked that if μ is centered, isotropic or log-concave, then $\pi_E\mu$ is also centered, isotropic or log-concave, respectively.

Recall that, if μ is a log-concave probability measure on $\mathbb{R}^n$ and $q \geqslant 1$ then the L_q-centroid body $Z_q(\mu)$ of μ is the symmetric convex body with support function

$$h_{Z_q(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^q d\mu(x)\right)^{1/q}. \tag{2.14}$$

Observe that μ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$. From Hölder's inequality it follows that $Z_1(\mu) \subseteq Z_p(\mu) \subseteq Z_q(\mu)$ for all $1 \leqslant p \leqslant q < \infty$. Conversely, using Borell's lemma (see [25, Appendix III]), one can check that

$$Z_q(\mu) \subseteq c\frac{q}{p} Z_p(\mu) \tag{2.15}$$

for all $1 \leqslant p < q$. In particular, if μ is isotropic, then $R(Z_q(\mu)) \leqslant cq$. From [28] and [29] one knows that the "q-moments"

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q dx\right)^{1/q}, \quad q \in (-n, +\infty) \setminus \{0\}, \tag{2.16}$$

of the Euclidean norm with respect to an isotropic log-concave probability measure μ on $\mathbb{R}^n$ are equivalent to $I_2(\mu) = \sqrt{n}$ as long as $|q| \leqslant \sqrt{n}$. Also, Paouris has proved in [28] that

$$w(Z_q(\mu)) \simeq \sqrt{q} \tag{2.17}$$

for all $q \leqslant \sqrt{n}$ and

$$|Z_q(\mu)|^{1/n} \leqslant c_2\sqrt{q/n} \tag{2.18}$$

for all $2 \leqslant q \leqslant n$. On the other hand, in [16] Klartag and Milman prove that

$$\big|Z_q(\mu)\big|^{1/n} \geqslant c_3\sqrt{q/n} \tag{2.19}$$

for all $q \leqslant \sqrt{n}$, where $c_3 > 0$ is an absolute constant. This determines the volume radius of $Z_q(\mu)$ for all $q \leqslant \sqrt{n}$.

Finally, let us recall the thin-shell estimate, first obtained by Klartag in [14] and [15] (see also [7] and [6]). The currently best known result is due to Guédon and E. Milman [11]: If μ is an isotropic log-concave measure on $\mathbb{R}^n$ then

$$\mu\big(\{x \in \mathbb{R}^n : |\,\|x\|_2 - \sqrt{n}\,| \geqslant t\sqrt{n}\}\big) \leqslant c_1 \exp(-c_2\sqrt{n}\min(t^3, t)) \tag{2.20}$$

for all $t \geqslant 0$, and (see [17])

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 \leqslant \varepsilon\sqrt{n}\}) \leqslant (C\varepsilon)^{c_2\sqrt{n}} \tag{2.21}$$

for all $0 \leqslant \varepsilon \leqslant 1/c_1$, where $c_1, c_2 > 0$ are absolute constants.

3 Lower bounds for the volume

For the lower bound we use a simple argument which is based on entropy estimates. Recall that the covering number $N(A, B)$ of a body A by a second body B is the least integer N for which there exist N translates of B whose union covers A. We need some standard estimates on covering numbers, that can be found e.g. in Pisier's book [30, Chapter 7]:

Fact 3.1. (i) If C is a convex body and L is a symmetric convex body in $\mathbb{R}^n$, then

$$2^{-n}\frac{|C + L|}{|L|} \leq N(C, L) \leq 2^n\frac{|C + L|}{|L|}. \tag{3.1}$$

(ii) If both C and L are symmetric, then

$$|C| \leq N(C, L)|C \cap L|. \tag{3.2}$$

Let C be a symmetric convex body in $\mathbb{R}^n$. Using Fact 3.1 we can give a lower bound for $|C \cap U(C)|$ which is actually valid for every $U \in O(n)$. From (3.2) it follows that

$$1 = |C| \leqslant N(C, U(C))\,|C \cap U(C)|. \tag{3.3}$$

In order to estimate $N(C, U(C))$, for every $\varrho > 0$ we write

$$N(C, U(C)) \leqslant N(C, \varrho\overline{B_2^n})\,N(\varrho\overline{B_2^n}, U(C)) = N(C, \varrho\overline{B_2^n})\,N(\varrho\overline{B_2^n}, C). \tag{3.4}$$

On the other hand,

$$N(\varrho\overline{B_2^n}, C) \leqslant 2^n|\varrho\overline{B_2^n} + C| \leqslant (4\varrho)^n N(C, \varrho\overline{B_2^n}), \tag{3.5}$$

using (3.1) and the fact that $|C| = |\overline{B_2^n}| = 1$. It follows that

$$N(C, U(C)) \leqslant (4\varrho)^n \big[N(C, \varrho\overline{B_2^n})\big]^2, \tag{3.6}$$

and in a similar way we check that

$$N(C, U(C)) \leqslant (4/\varrho)^n \big[N(\varrho\overline{B_2^n}, C)\big]^2, \tag{3.7}$$

Putting these estimates together, we get:

Lemma 3.2. *Let C be a symmetric convex body of volume 1 in $\mathbb{R}^n$. For every $\varrho > 0$ and any $U \in O(n)$ one has*

$$|C \cap U(C)| \geqslant \big[\min\{(4\varrho)^{n/2}N(C, \varrho\overline{B_2^n}), (4/\varrho)^{n/2}N(\varrho\overline{B_2^n}, C)\}\big]^{-2}. \tag{3.8}$$

One can estimate the covering numbers $N(C, \varrho\overline{B_2^n})$ and $N(\varrho\overline{B_2^n}, C)$ using Sudakov's inequality and its dual (see e.g. [30]). Recall that $\overline{B_2^n} \simeq \sqrt{n}B_2^n$ and hence

$$N(C, \varrho\overline{B_2^n}) \leqslant \exp(c_1 w^2(C)/\varrho^2) \qquad \text{and} \qquad N(\varrho\overline{B_2^n}, C) \leqslant \exp(c_1\varrho^2 n^2 M^2(C)),$$

where $c_1 > 0$ is an absolute constant. Choosing $\varrho = 1$ in Lemma 3.2, we get

$$|C \cap U(C)| \geqslant \frac{1}{4^n \exp(\min\{2c_2 w^2(C), 2c_1 n^2 M^2(C)\})}. \tag{3.9}$$

Taking into account the fact that $\min\{w(C)/\sqrt{n}, \sqrt{n}M(C)\} \geqslant c_3$ (which implies that $4^n \leqslant \exp(c_4 n \min\{w^2(C)/n, nM^2(C)\})$, we conclude the following.

Theorem 3.3. *Let C be a symmetric convex body of volume 1 in $\mathbb{R}^n$. For any $U \in O(n)$ we have*

$$|C \cap U(C)| \geqslant e^{-cn\min\{w^2(C)/n, nM^2(C)\}}, \tag{3.10}$$

where $c > 0$ is an absolute constant.

We may apply Theorem 3.3 to the normalized balls $\overline{B_p^n}$, $1 \leqslant p \leqslant \infty$. The known estimates for $|B_p^n|$ imply that if $1 \leqslant p \leqslant 2$ then $\overline{B_p^n} \simeq n^{1/p}B_p^n$. On the other hand, $\|x\|_p \leqslant n^{\frac{1}{p}-\frac{1}{2}}\|x\|_2$ and hence $M(B_p^n) \leqslant n^{\frac{1}{p}-\frac{1}{2}}$. Therefore,

$$M(\overline{B_p^n}) \leqslant cn^{-\frac{1}{p}}M(B_p^n) \leqslant c/\sqrt{n}.$$

Moreover, if $2 \leqslant p \leqslant \infty$ and if q is the conjugate exponent of p, then $\overline{B_p^n} \simeq n^{1/p}B_p^n$ and hence

$$w(\overline{B_p^n}) \leqslant cn^{\frac{1}{p}}w(B_p^n) = cn^{\frac{1}{p}}M(B_q^n) \leqslant cn^{\frac{1}{p}}n^{\frac{1}{q}-\frac{1}{2}} = c\sqrt{n}.$$

Combining the above, we see that

$$\min\{w^2(\overline{B_p^n})/n, nM^2(\overline{B_p^n})\} \leqslant c$$

for all $1 \leqslant p \leqslant \infty$, where $c > 0$ is an absolute constant. Theorem 3.3 gives:

Proposition 3.4. *For any $1 \leqslant p \leqslant \infty$ and any $U \in O(n)$ we have*

$$|\overline{B_p^n} \cap U(\overline{B_p^n})| \geqslant e^{-cn}, \tag{3.11}$$

where $c > 0$ is an absolute constant.

A second application of Lemma 3.2 can be given in the case where C is in M-position. Milman (see e.g. [23]) proved that there exists an absolute constant $\beta > 0$ such that every symmetric convex body C in $\mathbb{R}^n$ has a linear image $\tilde{C}$ of volume 1 which satisfies

$$\max\big\{N(\tilde{C}, \overline{B_2^n}), N(\overline{B_2^n}, \tilde{C})\big\} \leqslant \exp(\beta n). \tag{3.12}$$

We say that a convex body C which satisfies this estimate is in M-position with constant β. Applying Lemma 3.2 we get:

Proposition 3.5. *Let C be a symmetric convex body of volume 1 in $\mathbb{R}^n$. If C is in M-position with constant β then, for any $U \in O(n)$ we have*

$$|C \cap U(C)| \geqslant e^{-2(\beta+1)n}. \tag{3.13}$$

Next, assume that K is in the isotropic position. We use the following lemma (see [4, Section 3.2]): If K is an isotropic convex body in $\mathbb{R}^n$ then, for every $t > 0$,

$$N(K, t\overline{B_2^n}) \leqslant \exp\left(\frac{cnL_K}{t}\right), \tag{3.14}$$

where $c > 0$ is an absolute constant. In particular,

$$N(K, L_K\, \overline{B}_2^n) \leqslant e^{cn}. \tag{3.15}$$

But then, from (3.6) we have

$$N(K, U(K)) \leqslant (4L_K e^{2c})^n \tag{3.16}$$

and Lemma 3.2 implies the following:

Proposition 3.6. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. Then,*

$$|K \cap U(K)| \geqslant (c_1 L_K)^{-n} \tag{3.17}$$

for every $U \in O(n)$, where $c_1 \geqslant 4$ is an absolute constant.

4 Upper bounds for the volume

Our upper bounds for $\mathbb{E}_U|C \cap U(C)|$ will be based on Lemma 4.2. This follows from Fubini's theorem and the next fact (see e.g. [18]; in fact, both statements hold true for any star body C in $\mathbb{R}^n$).

Lemma 4.1. *If A is a symmetric convex body in $\mathbb{R}^n$, then*

$$\tfrac{1}{2}\,\sigma\big(S^{n-1}\cap \tfrac{1}{2}A\big)\leqslant \gamma_n(\sqrt{n}A)\leqslant \sigma(S^{n-1}\cap 2A)+e^{-cn}.$$

Lemma 4.2. *Let C be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$\int_{O(n)}|C\cap U(C)|\,d\nu(U)\leqslant 2\int_C \gamma_n\left(\frac{2\sqrt{n}}{\|x\|_2}C\right)\,dx. \tag{4.1}$$

Proof. Using basic properties of the Haar measure ν on $O(n)$ we may express the expectation of $|C\cap U(C)|$ as follows:

$$\begin{aligned}\int_{O(n)}|C\cap U(C)|\,d\nu(U) &= \int_{O(n)}\int_{\mathbb{R}^n}\chi_C(x)\chi_C(Ux)\,dx\,d\nu(U)\\ &= \int_{\mathbb{R}^n}\chi_C(x)\int_{O(n)}\chi_C(Ux)\,d\nu(U)\,dx\\ &= \int_C \nu(\{U\in O(n): Ux\in C\})\,dx\\ &= \int_C \nu(\{U\in O(n): \|x\|_2U(x/\|x\|_2)\in C\})\,dx\\ &= \int_C \sigma\left(\left\{\theta\in S^{n-1}:\theta\in \tfrac{1}{\|x\|_2}C\right\}\right)\,dx\\ &= \int_C \sigma\left(S^{n-1}\cap \tfrac{1}{\|x\|_2}C\right)\,dx.\end{aligned} \tag{3.8}$$

From Lemma 4.1 we get

$$\sigma\Big(S^{n-1}\cap \tfrac{1}{\|x\|_2}C\Big)\leqslant 2\gamma_n\left(\frac{2\sqrt{n}}{\|x\|_2}C\right) \tag{4.2}$$

and the lemma follows. □

A simple consequence of Lemma 4.2 is the next fact.

Proposition 4.3. *There exists an absolute constant $\alpha_0>0$ such that if C is a symmetric convex body of volume 1 in $\mathbb{R}^n$ then*

$$\int_{O(n)}|C\cap U(C)|\,d\nu(U)\leqslant \gamma_n(\alpha_0 C)+e^{-n}. \tag{4.3}$$

Proof. Let $\rho_n=e^{-1}\omega_n^{-1/n}$. Then, we have

$$\int_{C\cap\rho_nB_2^n}\gamma_n\left(\frac{2\sqrt{n}}{\|x\|_2}C\right)\,dx\leqslant |C\cap\rho_nB_2^n|\leqslant|\rho_nB_2^n|=e^{-n}.$$

On the other hand, if $x\in C\setminus\rho_nB_2^n$ then

$$\frac{2\sqrt{n}}{\|x\|_2}\leqslant\frac{2\sqrt{n}}{\rho_n}=2e\sqrt{n}\omega_n^{1/n}.$$

Therefore,

$$\int_{C\setminus\rho_n B_2^n} \gamma_n\left(\frac{2\sqrt{n}}{\|x\|_2}C\right)\,dx \leqslant \gamma_n(\alpha_0 C),$$

where $\alpha_0 = \sup_n 2e\sqrt{n}\omega_n^{1/n} \sim 2e\sqrt{2\pi e}$. □

In view of Proposition 4.3 we need to control $\gamma_n(tC)$, $t>0$. One way is to use the parameter $d_r(C)$. Recall that, for any $r>1$, we set

$$d_r(C) = \min\left\{-\log\sigma\left(\left\{x\in S^{n-1}:\|x\|_C \leqslant \frac{M(C)}{r}\right\}\right), n\right\}.$$

One of the main results in [18] is the following small ball probability estimate:

Theorem 4.4. *For every $r>1$ and every $0<\varepsilon<\frac{1}{32r^2}$ we have*

$$\gamma_n(\varepsilon\sqrt{n}M(C)C) \leqslant (c_1\varepsilon)^{c_2(r)d_r(C)} \leqslant (c_1\varepsilon)^{c_3(r)k(C)}, \tag{4.4}$$

where $c_1>0$ is an absolute constant and $c_2(r), c_3(r) \simeq \frac{1}{\log(8r)}$.

For completeness we sketch the proof of Theorem 4.4. The main tool is the B-theorem of Cordero-Erausquin, Fradelizi and Maurey [5]: if C is a symmetric convex body in $\mathbb{R}^n$ then the function

$$t \mapsto \gamma_n(e^t C)$$

is log-concave on $\mathbb{R}$. This implies that $\gamma_n(a^\lambda b^{1-\lambda}C) \geqslant \gamma_n(aC)^\lambda \gamma_n(bC)^{1-\lambda}$ for all $a,b>0$ and $\lambda\in(0,1)$. We use this fact in the following way. Let $m = \mathrm{med}(\|\cdot\|_C)$ denote the median (or Lévy mean) of $\|\cdot\|_C$ on S^{n-1}. Markov's inequality shows that

$$\frac{m}{2} \leqslant \int_{\{\theta:\|\theta\|_C\geqslant m\}} \|\theta\|_C\,d\sigma(\theta) \leqslant M(C).$$

It is also known that, conversely, $M(C)\leqslant c_0 m$ for some absolute constant $c_0>0$, a fact that will be used in the end of the proof.

We set $D = m\sqrt{n}C$. According to Lemma 4.1 we have

$$\gamma_n(2D) \geqslant \frac{1}{2}\sigma(S^{n-1}\cap mC) \geqslant \frac{1}{4}, \tag{4.5}$$

by the definition of the median. On the other hand, using Lemma 4.1 again, we have

$$\begin{aligned}\gamma_n(\tfrac{1}{4r}D) &\leqslant \sigma\left(S^{n-1}\cap\frac{m}{2r}C\right) + e^{-cn}\\ &= \sigma\left(\{\theta\in S^{n-1}:\|\theta\|_C\leqslant\frac{m}{2r}\}\right) + e^{-cn}\\ &\leqslant \sigma\left(\{\theta\in S^{n-1}:\|\theta\|_C\leqslant\frac{M(C)}{r}\}\right) + e^{-cn}\\ &\leqslant 2e^{-c_1 d_r(C)},\end{aligned} \tag{4.6}$$

where $c_1 > 0$ is a suitable absolute constant. We may assume that $0 < \varepsilon < \frac{1}{32r^2}$ and then we apply the B-theorem for the body D, with $a = \varepsilon$, $b = 2$ and $\lambda = \log(8r)/\log\frac{2}{\varepsilon}$. This gives

$$\gamma_n(\varepsilon D)^{\frac{\log(8r)}{\log(2/\varepsilon)}}\gamma_n(2D)^{1-\frac{\log(8r)}{\log(2/\varepsilon)}} \leqslant \gamma_n(\tfrac{1}{4r}D). \tag{4.7}$$

Note that $\frac{\log(8r)}{\log(2/\varepsilon)} < \frac{1}{2}$, and hence $\gamma_n(2D)^{1-\frac{\log(8r)}{\log(2/\varepsilon)}} \geqslant \frac{1}{2}$. Combining (4.5), (4.6) and (4.7) we see that

$$\gamma_n(\varepsilon D) \leqslant \Big(4e^{-c_1 d_r(C)}\Big)^{\frac{\log(2/\varepsilon)}{\log(8r)}} \leqslant (c_2\varepsilon)^{c_3(r)d_r(C)}, \tag{4.8}$$

where $c_3(r) = \frac{c_3}{\log(8r)}$ for some absolute constant $c_3 > 0$. This proves (4.4). □

Next, we combine Proposition 4.3 with Theorem 4.4.

Proposition 4.5. *There exists an absolute constant $B_0 > 0$ such that if $r > 1$ and C is a symmetric convex body of volume 1 in $\mathbb{R}^n$ with $\sqrt{n}M(C) \geqslant B_0 r^2$ then*

$$\int_{O(n)} |C \cap U(C)|\, d\nu(U) \leqslant e^{-\frac{c}{\log(8r)}d_r(C)} \tag{4.9}$$

for all $r > 1$, where $c > 0$ is an absolute constant.

Proof. From Theorem 4.4 we know that if $0 < \varepsilon < \frac{1}{c_2 r^2}$ then

$$\gamma_n(\varepsilon M(C)\sqrt{n}C) \leqslant (c_1\varepsilon)^{c_2(r)d_r(C)}.$$

Let $\varepsilon = \frac{\alpha_0}{\sqrt{n}M(C)}$. If $\sqrt{n}M(C) \geqslant \max\{c_1 e, c_2 r^2\}\alpha_0$ then we get

$$\gamma_n(\alpha_0 C) \leqslant e^{-\frac{c_3}{\log(8r)}d_r(C)}.$$

The result follows from Proposition 4.3. □

Remark 4.6. An alternative estimate can be given in terms of the inradius r

$$r(C) = \sup\{r > 0 : rB_2^n \subseteq C\}$$

of the body C. One can use the next small ball probability estimate which is due to Latała and Oleszkiewicz [19] (the proof is based again on the B-theorem): *Let A be a symmetric convex body in $\mathbb{R}^n$ with inradius $r = r(A)$ and $\gamma_n(A) \leqslant 1/2$. For any $0 \leqslant \varepsilon \leqslant 1$ we have*

$$\gamma_n(\varepsilon A) \leqslant (2\varepsilon)^{r(A)^2/4}\gamma_n(D). \tag{4.10}$$

We use this result as follows: assume that C is a symmetric convex body of volume 1 in $\mathbb{R}^n$ with $\sqrt{n}M(C) \geqslant B_0$ and $d(C) \geqslant B_0$. We check that $A = \frac{m\sqrt{n}}{8}C$ satisfies $\gamma_n(A) \leqslant \frac{1}{2}$, and hence, applying (4.10) for A, we see that if $0 < \varepsilon < \frac{1}{2e}$ then

$\gamma_n(\varepsilon A) \leqslant 2e^{-r^2(A)/4}$. Note that $\frac{\varepsilon m\sqrt{n}}{8} = \alpha_0$ if we choose $\varepsilon = \frac{8\alpha_0}{\sqrt{n}m}$. If $\sqrt{n}m \geqslant 16e\alpha_0$ then we get

$$\gamma_n(\alpha_0 C) \leqslant 2e^{-r^2(A)/4} = 2e^{-nm^2r^2(C)}.$$

From Proposition 4.3 we get $\mathbb{E}_U|C\cap U(C)| \leqslant \exp(-cnm^2(C)r^2(C))$. However, note that $nm^2(C)r^2(C) \leqslant 4nM^2(C)r^2(C) \simeq k(C) \leqslant c'd(C)$.

Remark 4.7. Recall that for every symmetric convex body C of volume 1 in $\mathbb{R}^n$ one has

$$\sqrt{n}M(C) \geqslant \sqrt{n}\left(\frac{|B_2^n|}{|C|}\right)^{1/n} = \sqrt{n}\omega_n^{1/n} \sim \sqrt{2\pi e}.$$

So, the condition $\sqrt{n}M(C) \geqslant B_0$ is not satisfied by those bodies for which

$$M(C)\mathrm{vrad}(C) \simeq 1.$$

An example is given by the Euclidean ball $\overline{B_2^n}$ of volume 1. However, in this case one has $|\overline{B_2^n} \cap U(\overline{B_2^n})| = 1$ for all $U \in O(n)$. In other words, if one asks for a non-trivial (exponentially small) upper bound for the expectation of $|C \cap U(C)|$ then some condition is required. Thus, the condition $\sqrt{n}M(C) \geqslant B_0$ seems very natural.

In the example of the cube $Q_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ one has $\sqrt{n}M(Q_n) \simeq \sqrt{\log n}$ and hence Proposition 4.5 applies. However, it is easier to compute $\gamma_n(\alpha_0 Q_n)$ directly and then to apply Proposition 4.3. If $\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t e^{-s^2/2}ds$ is the distribution function of a standard normal random variable then one has

$$\gamma_n(\alpha_0 Q_n) = (2\Phi(\alpha_0/2) - 1)^n = e^{-\delta_0 n},$$

where $\delta_0(\alpha_0) > 0$ is defined by the equation $2\Phi(\alpha_0/2) - 1 = e^{-\delta_0}$. In [18] it is checked that $c_1 n^{1-c_1 r^{-2}} \leqslant d_r(Q_n) \leqslant c_2 n^{1-c_2 r^{-2}}$ for all $r > 1$. Therefore, using Proposition 4.5 with $r \simeq \sqrt[4]{\log n}$, one would obtain the estimate

$$\mathbb{E}_U(Q_n \cap U(Q_n)) \leqslant e^{-cn^{1-\delta}}$$

for any $\delta > 0$ and any $n \geqslant n_0(\delta)$. An analogous situation appears for any $2 < q < \infty$; one has $d_{c_q}(B_q^n) \geqslant C_q n$ for some constants $c_q, C_q > 0$ depending only on q. This leads to an upper bound for $\mathbb{E}_U|\overline{B_q^n} \cap U(\overline{B_q^n})|$ of the form $\exp(-n^{1-\delta})$ for any $0 < \delta < 1$, at least when q and n are large enough. These bounds should be compared with the lower bound from Fact 3.4.

In the last section of this article we apply Proposition 4.5 to the centroid bodies $Z_q(\mu)$ of an isotropic log-concave measure μ on $\mathbb{R}^n$. The next proposition provides an alternative argument leading to an upper bound for $\mathbb{E}_U|K \cap U(K)|$ in the case where K is an isotropic convex body in $\mathbb{R}^n$.

Proposition 4.8. *Let K be an isotropic convex body in $\mathbb{R}^n$. Assume that $L_K = (1+\delta)\sqrt{2/\pi}$ for some $\delta > 0$. Then,*

$$\int_{O(n)} |K \cap U(K)|\, d\nu(U) \leqslant c_1 e^{-c_2(\delta)\sqrt{n}}, \tag{4.11}$$

where $c_1 > 0$ *is an absolute constant and* $c_2(\delta) \simeq \min\{1, \delta^3\}$.

Proof. Assume that $L_K > \sqrt{2/\pi}$, and write $L_K = (1+\delta)\sqrt{2/\pi}$ for some $\delta > 0$. Let $\varepsilon \in (0,1)$ that will be suitably chosen depending on δ. From the thin-shell estimate (2.20) we know that if

$$A := \left\{ x \in K : \left| \, \|x\|_2 - \sqrt{n} L_K \, \right| \leqslant \varepsilon \sqrt{n} L_K \right\}, \tag{4.12}$$

then

$$|A| \geqslant 1 - C_1 \exp(-c_2 \varepsilon^3 \sqrt{n}), \tag{4.13}$$

provided n is large enough. Let $\rho = \varepsilon \sqrt{n} L_K$. If $K_\rho = K \cap \rho B_2^n$, from (4.13) we know that $|K_\rho| \leqslant C_1 \exp(-c_2 \varepsilon^3 \sqrt{n})$. Then,

$$\int_{K_\rho} \gamma_n \left(\frac{2\sqrt{n}}{\|x\|_2} K \right) dx \leqslant |K_\rho| \leqslant C_1 \exp(-c_2 \varepsilon^3 \sqrt{n}). \tag{4.14}$$

On the other hand,

$$\int_{K \setminus K_\rho} \gamma_n \left(\frac{2\sqrt{n}}{\|x\|_2} K \right) dx \leqslant |K \setminus K_\rho| \, \gamma_n \left(\frac{2}{(1-\varepsilon) L_K} K \right), \tag{4.15}$$

and

$$\gamma_n(aK) \leqslant \left(\frac{a}{\sqrt{2\pi}} \right)^n |K| \tag{4.16}$$

for every $a > 0$, so

$$\begin{aligned} \int_{K \setminus K_\rho} \gamma_n \left(\frac{2\sqrt{n}}{\|x\|_2} K \right) dx &\leqslant \left(\frac{2}{(1-\varepsilon)\sqrt{2\pi} L_K} \right)^n \\ &= \left(\frac{1}{(1-\varepsilon)(1+\delta)} \right)^n \leqslant C_2 e^{-c_3 \min\{1,\delta\} n}, \end{aligned} \tag{4.17}$$

if we choose $\varepsilon < \min\{1, \delta\}/3$. It follows that

$$\int_{O(n)} |K \cap U(K)| \, d\nu(U) \leqslant c_1 e^{-c_4(\delta)\sqrt{n}} \tag{4.18}$$

with $c_4(\delta) \simeq [\min\{1, \delta\}]^3$. □

5 Upper bounds for the radius

Let C be a symmetric convex body of volume 1 in $\mathbb{R}^n$. In this Section we briefly recall known arguments leading to an upper bound for the radius $R(C \cap U(C))$ of the intersection of C with its random rotations $U(C)$.

Proposition 5.1. *If $R(C\cap E)\leqslant r$ for all E in a subset of $G_{n,n/2}$ of measure greater than $1/2$ then there exists $U\in O(n)$ such that $R(C\cap U(C))\leqslant \sqrt{2}\,r$.*

Proof. We use a standard argument which goes back to Krivine (see [30] or [24]). From the assumption we know that there exists $E\in G_{n,n/2}$ such that

$$\|y\|_C\geqslant \frac{1}{r}\|y\|_2 \tag{5.1}$$

for all $y\in E$ and all $y\in E^{\perp}$. We write $P_1=P_E$ and $P_2=P_{E^{\perp}}$. Then, we write $I=P_1+P_2$ and we define $U=P_1-P_2\in O(n)$. Let $x\in\mathbb{R}^n$. We write $x=x_1+x_2$, where $x_1=P_1(x)$ and $x_2=P_2(x)$. Then,

$$\begin{aligned}\|x_1+x_2\|_C+\|x_1-x_2\|_C &\geqslant 2\max\{\|x_1\|_C,\|x_2\|_C\}\geqslant \frac{2}{r}\max\{\|x_1\|_2,\|x_2\|_2\}\\ &\geqslant \frac{\sqrt{2}}{r}\sqrt{\|x_1\|_2^2+\|x_2\|_2^2}=\frac{\sqrt{2}}{r}\|x\|_2.\end{aligned}$$

This means that

$$\|x\|_C+\|x\|_{U^{-1}(C)}\geqslant \frac{\sqrt{2}}{r}\|x\|_2, \tag{5.2}$$

or equivalently, since $U=U^*$,

$$2\text{conv}(C^{\circ}\cup U(C^{\circ}))\supseteq C^{\circ}+U(C^{\circ})\supseteq \frac{\sqrt{2}}{r}B_2^n. \tag{5.3}$$

Taking polars we conclude the proof. □

The next observation is that the existence of one e.g. $3n/4$-dimensional section with radius r implies that random $n/2$-dimensional sections have radius of the same order. Then, we may apply Proposition 5.1 to find $U\in O(n)$ with $R(C\cap U(C))\leqslant c_3r$.

Theorem 5.2. *If $R(C\cap F)\leqslant r$ for some $F\in G_{n,3n/4}$ then a random subspace $E\in G_{n,n/2}$ satisfies*

$$R(C\cap E)\leqslant c_1r$$

with probability greater than $1-e^{-c_2n}$.

Proof. This fact has been observed in [8], [33] and soon after, in a sharper form, in [20] where it was proved that if C is a symmetric convex body in $\mathbb{R}^n$, and if $1\leqslant k<m<n$ and $\mu=\frac{n-k}{n-m}$, then assuming that $R(C\cap F)\leqslant r$ for some $F\in G_{n,m}$ we have that a random subspace $E\in G_{n,k}$ satisfies

$$R(C\cap E)\leqslant r\left(c_2\sqrt{\tfrac{n}{n-m}}\right)^{\frac{\mu}{\mu-1}}$$

with probability greater than $1-2e^{-(n-k)/2}$, where $c_2>0$ is an absolute constant.

Assume that $R(C\cap F)\leqslant r$ for some $F\in G_{n,m}$, where $m=3n/4$. Applying the above with $k=n/2$ (and $\mu=2$) we conclude the proof. $\square$

We can actually prove an analogue of Proposition 5.1 for a random $U\in O(n)$ using the next result of Vershynin and Rudelson [33]: There exist absolute constants $c_0,c_1>0$ with the following property: if C and D are two symmetric convex bodies in $\mathbb{R}^n$ which have sections of dimensions at least k and $n-c_0k$ whose radius is bounded by 1, then a random $U\in O(n)$ satisfies $R(C\cap U(D))\leqslant c_1^{n/k}$ with probability greater than $1-e^{-n}$. We set $D=C$ and $k=n/2$ to get the following.

Theorem 5.3. *If*

$$r_C:=\min\{R(C\cap F):\dim(F)=\lceil(1-c_0/2)n\rceil\}$$

then $R(C\cap U(C))\leqslant c_2r_C$ with probability greater than $1-e^{-n}$ with respect to $U\in O(n)$.

An immediate application of Theorem 5.3 is an estimate for $R(C\cap U(C))$ in terms of the mean width $w(C)$. By the low M^*-estimate (2.7) we know that $r_C\leqslant c_3w(C)$. Thus, we have:

Proposition 5.4. *Let C be a symmetric convex body in $\mathbb{R}^n$. A random $U\in O(n)$ satisfies*

$$R(C\cap U(C))\leqslant cw(C)$$

with probability greater than $1-e^{-n}$, where $c>0$ is an absolute constant.

6 Applications to centroid bodies of log-concave measures

As an application of the results of the previous sections, we discuss the case of the centroid bodies $Z_q(\mu)$ of an isotropic log-concave measure μ on $\mathbb{R}^n$.

Starting with the volume, and in view of Proposition 4.5, we need a lower bound for $d(Z_q(\mu))$. We will use the fact that

$$d(Z_q(\mu))\geqslant c_1k(Z_q(\mu))=c_1k_*(Z_q^{\circ}(\mu)).$$

Assuming that $2\leqslant q\leqslant q_*(\mu)$ we have

$$w(Z_q^{\circ}(\mu))=M(Z_q(\mu))\geqslant\left(\frac{|B_2^n|}{|Z_q(\mu)|}\right)^{1/n}\geqslant\frac{c_2}{\sqrt{q}},$$

while the inclusion $B_2^n=Z_2(\mu)\subseteq Z_q(\mu)$ implies that $R\big(Z_q^{\circ}(\mu)\big)\leqslant 1$. It follows that

$$d(Z_q(\mu))\geqslant c_1k_*\big(Z_q^{\circ}(\mu)\big)\geqslant c_4n\frac{w^2\big(Z_q^{\circ}(\mu)\big)}{R^2\big(Z_q^{\circ}(\mu)\big)}\geqslant\frac{c_5n}{q}. \tag{6.1}$$

It is convenient to normalize the volume, and consider $\overline{Z_q}(\mu)$ instead of $Z_q(\mu)$. Recall from Section 2 that if $q \leqslant \sqrt{n}$ then $|Z_q(\mu)|^{1/n} \simeq \sqrt{q/n}$, and hence

$$\overline{Z_q}(\mu) \simeq \sqrt{n/q}\, Z_q(\mu).$$

Then,

$$M(\overline{Z_q}(\mu)) \simeq \sqrt{q/n} M(Z_q(\mu)). \tag{6.2}$$

We can now apply Proposition 4.5.

Proposition 6.1. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$ and let $2 \leqslant q \leqslant \sqrt{n}$. If $\sqrt{q} M(Z_q(\mu)) \geqslant B_1$ then*

$$\int_{O(n)} |\overline{Z_q}(\mu) \cap U(\overline{Z_q}(\mu))| \, d\nu(U) \leqslant e^{-c_1 n/q}, \tag{6.3}$$

where $B_1, c_1 > 0$ are absolute constants. □

On the other hand, from 2.17 we see that $w(\overline{Z_q}(\mu)) \simeq \sqrt{n/q} w(Z_q(\mu)) \simeq \sqrt{n}$. Therefore, Theorem 3.3 gives:

Proposition 6.2. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$ and let $2 \leqslant q \leqslant \sqrt{n}$. For any $U \in O(n)$ one has*

$$|\overline{Z_q}(\mu) \cap U(\overline{Z_q}(\mu))| \geqslant e^{-c_2 n}, \tag{6.4}$$

where $c_2 > 0$ is an absolute constant. □

Next, we discuss the radius of $\overline{Z_q}(\mu) \cap U(\overline{Z_q}(\mu))$. Our main tool is a (simplified version of a) result from [9] about proportional projections of the centroid bodies.

Theorem 6.3. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $0 < \varepsilon < 1$ and any $q \leqslant \sqrt{\varepsilon n}$ we may find $k \geqslant (1-\varepsilon)n$ and $F \in G_{n,k}$ such that*

$$P_F\big(Z_q(\mu)\big) \supseteq c_1 \varepsilon^2 \sqrt{q}\, B_F, \tag{6.5}$$

where $c_1 > 0$ is an absolute constant.

Now, we can use Theorem 5.3 to give a lower bound for the radius of $Z_q(\mu) \cap U(Z_q(\mu))$ or $Z_q^\circ(\mu) \cap U(Z_q^\circ(\mu))$ for a random $U \in O(n)$. Since the mean width of $Z_q(\mu)$, $2 \leqslant q \leqslant \sqrt{n}$, is known to be of the order of $\sqrt{q}$, we can use the low M^*-estimate to get that if $\varepsilon \in (0,1)$ and $k = (1-\varepsilon)n$, then a subspace $F \in G_{n,k}$ satisfies

$$R(Z_q(\mu) \cap F) \leqslant \frac{c_2 \sqrt{q}}{\sqrt{\varepsilon}} \tag{6.6}$$

with probability greater than $1 - \exp(-c_2 \varepsilon n)$, where $c_1, c_2 > 0$ are absolute constants. Applying this fact with $k = n/2$ we see that the bodies $C = D = \frac{c_3}{\sqrt{q}} Z_q(\mu)$

have sections of dimensions at least $n/2$ and $(1-c_0/2)n$ whose radius is bounded by 1 (it suffices to choose $c_3>0$ small enough). Then, from Theorem 5.3 we get $R(Z_q(\mu)\cap U(Z_q(\mu)))\leqslant c_4\sqrt{q}$ with probability greater than $1-e^{-n}$.

Similarly, from Theorem 6.3 we know that

$$R(Z_q^\circ(\mu)\cap F)\leqslant \frac{c_2}{\varepsilon^2\sqrt{q}}$$

for a random $F\in G_{n,(1-\varepsilon)n}$. Applying this fact with $k=n/2$ we see that the bodies $C=D=c_3\sqrt{q}Z_q^\circ(\mu)$ have sections of dimensions at least $n/2$ and $(1-c_0/2)n$ whose radius is bounded by 1 (it suffices to choose $c_3>0$ small enough). Then, from Theorem 5.3 we get $R(Z_q^\circ(\mu)\cap U(Z_q^\circ(\mu)))\leqslant c_4/\sqrt{q}$ with probability greater than $1-e^{-n}$.

We summarize in the next theorem.

Theorem 6.4. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For any $2\leqslant q\leqslant\sqrt{n}$, a random $U\in O(n)$ satisfies*

$$Z_q(\mu)+U(Z_q(\mu))\supseteq c_1\sqrt{q}B_2^n \quad \textit{and} \quad Z_q^\circ(\mu)+U(Z_q^\circ(\mu))\supseteq \frac{c_1}{\sqrt{q}}B_2^n,$$

or equivalently,

$$\overline{Z_q^\circ}(\mu)\cap U(\overline{Z_q^\circ}(\mu))\subseteq c_2\sqrt{n}B_2^n \quad \textit{and} \quad \overline{Z_q}(\mu)\cap U(\overline{Z_q}(\mu))\subseteq c_2\sqrt{n}B_2^n$$

with probability greater than $1-2e^{-n}$.

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