Banach spaces with the Lebesgue property of Riemann integrability

Bünyamin Sarı
University of North Texas

with Harrison Gaebler (UNT) and Pavlos Motakis (York)

Mathematical Analysis in honor of Spiros Argyros, 7th Summer school, Athens, 2024

Riemann integral

Recall from Calculus

$f : [0, 1] \to \mathbb{R}$ is **Riemann integrable** with the integral equal to a number x_f if for all $\varepsilon > 0$ there exists $\delta > 0$ such that all Riemann sums satisfy

$$\Big| \sum_{i=1}^{N} f(s_i)(t_i - t_{i-1}) - x_f \Big| < \varepsilon$$

where $0 = t_0 < t_1 < \ldots < t_N = 1$ is a **partition** of $[0, 1]$ with **tags** $s_i \in [t_{i-1}, t_i]$ and $\max_{1 \le i \le N}(t_i - t_{i-1}) < \delta$.

Riemann integral in Banach spaces

Let X be a Banach space.

$f : [0,1] \to X$ is **Riemann integrable** with the integral equal to a vector x_f if for all $\varepsilon > 0$ there exists $\delta > 0$ such that all Riemann sums satisfy

$$\Big\| \sum_{i=1}^{N} f(s_i)(t_i - t_{i-1}) - x_f \Big\| < \varepsilon$$

where $0 = t_0 < t_1 < \ldots < t_N = 1$ is a **partition** of $[0,1]$ with **tags** $s_i \in [t_{i-1}, t_i]$ and $\max_{1 \le i \le N}(t_i - t_{i-1}) < \delta$.

Equivalent definition via dyadic partitions

$f : [0, 1] \to X$ is **Riemann integrable** with the integral equal to a vector x_f if for all $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that for all $m \geq n$ the Riemann sums over dyadic partitions with interior tags satisfy

$$\left\| \frac{1}{2^m} \sum_{i=1}^{2^m} f(s_i) - x_f \right\| < \varepsilon$$

where $s_i \in (\frac{i-1}{2^m}, \frac{i}{2^m})$.

Lebesgue criterion for Riemann integrability

Recall from intro. Analysis class

A bounded $f : [0, 1] \to \mathbb{R}$ is Riemann integrable iff the set of discontinuities of f has Lebesgue measure zero.

Lebesgue criterion fails for Banach spaces

Example

Let (e_i) be the unit vector basis of c_0 or of ℓ_p, $1 < p < \infty$. Consider the Dirichlet function $f : [0, 1] \to c_0, \ell_p$ defined by

$$f(t) = \begin{cases} e_i & t = r_i \\ 0 & t \neq r_i \end{cases}$$

where (r_i) is the set of rationals. The function is discontinuous everywhere but it is integrable with integral is zero: The Riemann sums

$$\frac{1}{2^m}\Big\| \sum_{i=1}^{2^m} f(s_i) \Big\| \leq \frac{1}{2^m} \text{ (in } c_0\text{) and } \frac{(2^m)^{1/p}}{2^m} \text{ (in } \ell_p\text{)} \to 0.$$

Checking Lebesgue property

Haydon-Odell, unpublished, 83. If X does not have the Lebesgue property then there exists a normalized basic (block) sequence (x_j) in X so that the Dirichlet function

$$f(t) = \begin{cases} x_j & t = d_j \\ 0 & t \neq d_j \end{cases}$$

where (d_i) is the set of dyadic rationals (ordered in natural way) is Riemann integrable, that is,

$$\frac{1}{2^n} \sup_{t_j \in I_j^n} \Big\| \sum_{j=0}^{2^n-1} f(t_j) \Big\| \to 0$$

where $I_j^n = (\frac{j}{2^n}, \frac{j+1}{2^n})$ is the j'th dyadic interval at level n.

Examples

- In ℓ_1, Tsirelson space T or more generally in asymptotic-ℓ_1 spaces, the above expression is > 0 for every normalized block sequence. Thus they satisfy the Lebesgue property.

Examples

- In ℓ_1, Tsirelson space T or more generally in asymptotic-ℓ_1 spaces, the above expression is > 0 for every normalized block sequence. Thus they satisfy the Lebesgue property.
- **Pelczynski, da Roch Filho (80's, unpublished)** Every spreading model of X with the Lebesgue property is isomorphic to ℓ_1.

Examples

- In ℓ_1, Tsirelson space T or more generally in asymptotic-ℓ_1 spaces, the above expression is > 0 for every normalized block sequence. Thus they satisfy the Lebesgue property.
- **Pelczynski, da Roch Filho (80's, unpublished)** Every spreading model of X with the Lebesgue property is isomorphic to ℓ_1.
- **Pelczynski, da Roch Filho.** For $X \subset L_1$, the following are equivalent:
  i) all spreading models of X are isomorphic to ℓ_1
  ii) X has Schur property
  iii) X has the Lebesgue property.

Examples

- In ℓ_1, Tsirelson space T or more generally in asymptotic-ℓ_1 spaces, the above expression is > 0 for every normalized block sequence. Thus they satisfy the Lebesgue property.
- **Pelczynski, da Roch Filho (80's, unpublished)** Every spreading model of X with the Lebesgue property is isomorphic to ℓ_1.
- **Pelczynski, da Roch Filho.** For $X \subset L_1$, the following are equivalent:
  i) all spreading models of X are isomorphic to ℓ_1
  ii) X has Schur property
  iii) X has the Lebesgue property.
- **Haydon, 83.** Replace L_1 in above by stable X with uniformly separable types.

Examples

- In ℓ_1, Tsirelson space T or more generally in asymptotic-ℓ_1 spaces, the above expression is > 0 for every normalized block sequence. Thus they satisfy the Lebesgue property.
- **Pelczynski, da Roch Filho (80's, unpublished)** Every spreading model of X with the Lebesgue property is isomorphic to ℓ_1.
- **Pelczynski, da Roch Filho.** For $X \subset L_1$, the following are equivalent:
  i) all spreading models of X are isomorphic to ℓ_1
  ii) X has Schur property
  iii) X has the Lebesgue property.
- **Haydon, 83.** Replace L_1 in above by stable X with uniformly separable types.
- **Haydon, 83, Naralenkov, 08 (Talagrand's example, 84)** There are examples of stable spaces X with Schur property but fail the Lebesgue property.

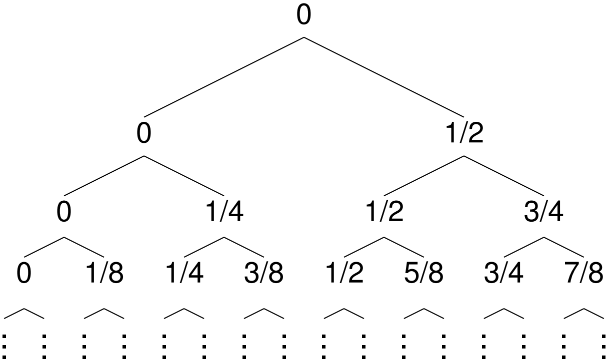
So Banach spaces with the Lebesgue property have some proximity to ℓ_1. We like to pinpoint exactly what it is.

So Banach spaces with the Lebesgue property have some proximity to ℓ_1. We like to pinpoint exactly what it is.

Characterization of the Lebesgue property is based on partitions of $\mathbb{N}$ that mimic the structure of the dyadic rationals in $[0, 1)$

Haar systems of partitions of $\mathbb{N}$

The dyadic rationals as the dyadic tree $\mathcal{D}$

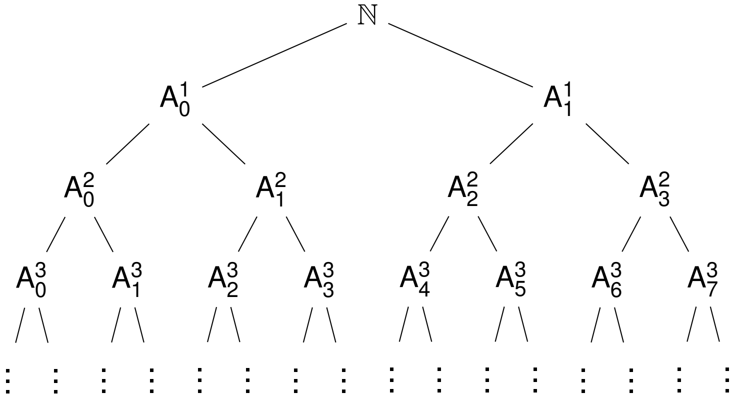


Haar systems of partitions of $\mathbb{N}$

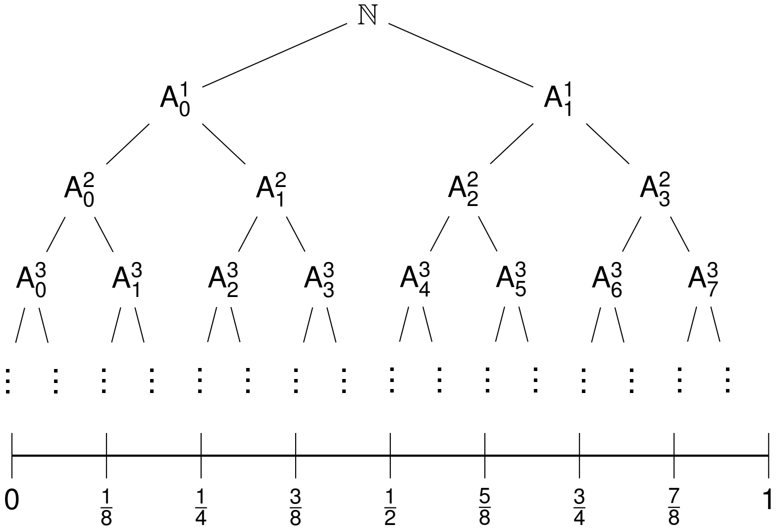
A collection $(A_j^n)_{j=0,n\in\mathbb{N}}^{2^n-1}$ of infinite subsets of $\mathbb{N}$ is said to be a Haar system of partitions of $\mathbb{N}$ if:

1. for every $n \in \mathbb{N}$, $\bigcup_{j=0}^{2^n-1} A_j^n = \mathbb{N}$ and $A_j^n \cap A_{j'}^n = \emptyset$ if $j \neq j'$
2. for every $n \in \mathbb{N}$ and $0 \leq j < 2^n - 1$, $A_j^n = A_{2j}^{n+1} \cup A_{2j+1}^{n+1}$

Haar systems of partitions of $\mathbb{N}$



Haar systems of partitions of $\mathbb{N}$



define embedding σ that takes d_j^n to the $(2^n)^{\text{th}}$ member of A_j^n

Characterization of Lebesgue property

Definition

We say that a normalized basic sequence (e_i) in a Banach space X is **Haar-ℓ_1^+ sequence** if for every Haar system of partitions $(A_j^n)_{j=0,n\in\mathbb{N}}^{2^n-1}$ of $\mathbb{N}$, there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \leq \lim_{n\to\infty} \sup \left\{ \frac{1}{2^m} \left\| \sum_{j=0}^{2^m-1} e_{i_j} \right\| : \; m \geq n \text{ and } 2^m \leq i_j \in A_j^m \right\}.$$

Characterization of Lebesgue property

Definition

We say that a normalized basic sequence (e_i) in a Banach space X is Haar-ℓ_1^+ sequence if for every Haar system of partitions $(A_j^n)_{j=0,n\in\mathbb{N}}^{2^n-1}$ of $\mathbb{N}$, there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \leq \lim_{n\to\infty} \sup \left\{ \frac{1}{2^m} \left\| \sum_{j=0}^{2^m-1} e_{i_j} \right\| : \ m \geq n \text{ and } 2^m \leq i_j \in A_j^m \right\}.$$

Theorem

A Banach space X has the Lebesgue property if and only if every normalized basic sequence (e_i) in X is a Haar-ℓ_1^+ sequence.

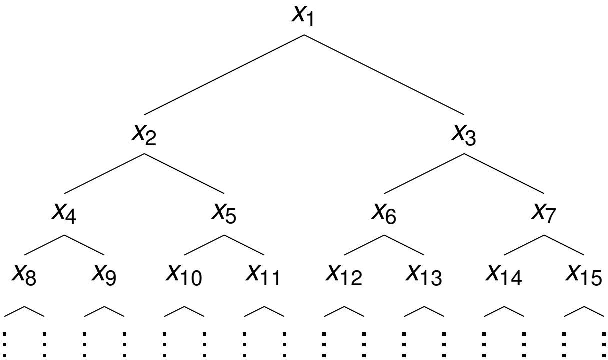
Sequential asymptotic property

The Haar-ℓ_1^+ condition is a sequential asymptotic property.

Equivalent formulation

For every collection $(x_\lambda)_{\lambda\in\mathcal{D}}$ of norm-one vectors in X, there exists a constant $\theta > 0$ such that, for every $n \in \mathbb{N}$ and for every $\lambda \in \{0,1\}^n$, there exist some collection of nodes $(\mu_\lambda)_{\lambda\in\{0,1\}^n}$ with $\mu_\lambda \geq \lambda$ so that

$$\frac{1}{2^n}\Big\|\sum_{\lambda\in\{0,1\}^n} x_{\mu_\lambda}\Big\| \geq \theta.$$



x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8
x_9
x_{10}
x_{11}
x_{12}
x_{13}
x_{14}
x_{15}

Comparing with other sequential asymptotic notions

1. If every normalized basic sequence in X is Haar-ℓ_1^+, then every **spreading model** of X is (not necessarily uniformly) equivalent to the unit vector basis ℓ_1.

Comparing with other sequential asymptotic notions

1. If every normalized basic sequence in X is Haar-ℓ_1^+, then every **spreading model** of X is (not necessarily uniformly) equivalent to the unit vector basis ℓ_1.
2. Let (e_i) be a normalized basic sequence in X. If every **asymptotic model** generated by an array of normalized block bases of (e_i) is equivalent to the unit vector basis of ℓ_1, then (e_i) is a Haar-ℓ_1^+ sequence. In particular, X has the Lebesgue property if every asymptotic model of X is equivalent to the unit vector basis of ℓ_1.

Comparing with other sequential asymptotic notions

1. If every normalized basic sequence in X is Haar-ℓ_1^+, then every **spreading model** of X is (not necessarily uniformly) equivalent to the unit vector basis ℓ_1.
2. Let (e_i) be a normalized basic sequence in X. If every **asymptotic model** generated by an array of normalized block bases of (e_i) is equivalent to the unit vector basis of ℓ_1, then (e_i) is a Haar-ℓ_1^+ sequence. In particular, X has the Lebesgue property if every asymptotic model of X is equivalent to the unit vector basis of ℓ_1.
3. Converses to both (1) and (2) are false.

Comparing with other sequential asymptotic notions

1. If every normalized basic sequence in X is Haar-ℓ_1^+, then every **spreading model** of X is (not necessarily uniformly) equivalent to the unit vector basis ℓ_1.
2. Let (e_i) be a normalized basic sequence in X. If every **asymptotic model** generated by an array of normalized block bases of (e_i) is equivalent to the unit vector basis of ℓ_1, then (e_i) is a Haar-ℓ_1^+ sequence. In particular, X has the Lebesgue property if every asymptotic model of X is equivalent to the unit vector basis of ℓ_1.
3. Converses to both (1) and (2) are false.
4. (**Tsirelson sum**) Let $\{X_i\}_{i=1}^{\infty}$ be a collection of Banach spaces such that X_i has the Lebesgue property for each $i \in \mathbb{N}$. Then, $X = (\sum_{i=1}^{\infty} X_i)_T$ has the Lebesgue property.
   Not all asymptotic models of $(T \oplus T \oplus \ldots)_T$ is isomorphic to ℓ_1.

Complete separation of sequential asymptotic structures

So we have the picture

$$\begin{gathered}\text{all asymptotic models isomorphic to } \ell_1 \\ \Downarrow \not\Uparrow \\ \text{Lebesgue property} \\ \Downarrow \not\Uparrow \\ \text{all spreading models isomorphic to } \ell_1\end{gathered}$$

Complete separation of sequential asymptotic structures

So we have the picture

$$\text{all asymptotic models isomorphic to } \ell_1$$

$$\Downarrow \not\Uparrow$$

$$\text{Lebesgue property}$$

$$\Downarrow \not\Uparrow$$

$$\text{all spreading models isomorphic to } \ell_1$$

Can we get the reverse implications on a subspace?

Argyros-Motakis space X_{iw}

Passing to subspaces doesn't necessarily improve the Lebesgue property.

1. Argyros-Motakis space X_{iw} has remarkable property that while all spreading models are **uniformly** equivalent to the unit vector basis of ℓ_1, every subspace admits c_0 asymptotic model!

Argyros-Motakis space X_{iw}

Passing to subspaces doesn't necessarily improve the Lebesgue property.

1. Argyros-Motakis space X_{iw} has remarkable property that while all spreading models are **uniformly** equivalent to the unit vector basis of ℓ_1, every subspace admits c_0 asymptotic model!
2. X_{iw} has the Lebesgue property.

The space $X_{\mathcal{D}}$

Theorem (Gaebler, Motakis, S., '24)

There exists a reflexive Banach space $X_{\mathcal{D}}$ with an unconditional basis such that all spreading models of $X_{\mathcal{D}}$ are uniformly equivalent to the unit vector basis of ℓ_1, yet every infinite-dimensional closed subspace of $X_{\mathcal{D}}$ fails the Lebesgue property.

The space $X_{\mathcal{D}}$

Theorem (Gaebler, Motakis, S., '24)

There exists a reflexive Banach space $X_{\mathcal{D}}$ with an unconditional basis such that all spreading models of $X_{\mathcal{D}}$ are uniformly equivalent to the unit vector basis of ℓ_1, yet every infinite-dimensional closed subspace of $X_{\mathcal{D}}$ fails the Lebesgue property.

The construction is based on X_{iw} space construction with additional metric constraints.

Thank you!