On some random convex sets generated by isotropic log-concave random vectors

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Abstract

For any $\mathbf{x} = (x_1, \ldots, x_N) \in \oplus_{i=1}^N \mathbb{R}^n$ we denote by $T_{\mathbf{x}} = [x_1 \cdots x_N]$ the $n \times N$ matrix whose columns are the vectors x_i. Paouris and Pivovarov showed that if $N \geqslant n$ and $f_1, \ldots, f_N$ are probability densities on $\mathbb{R}^n$ with $\|f_i\|_\infty \leqslant 1$ then, for any centrally symmetric convex body K in $\mathbb{R}^N$, the expected volume

$$\mathcal{F}_K(f_1, \ldots, f_N) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K)) \right) \prod_{i=1}^N f_i(x_i) \, dx_N \cdots dx_1$$

of $T_{\mathbf{x}}(K)$ is minimized when each f_i is the indicator function of the Euclidean ball D_n of volume 1 in $\mathbb{R}^n$. We discuss upper and lower bounds for $\mathcal{F}_K(f_1, \ldots, f_N)$ in the case where f_i are isotropic densities. In the second part of this note, given $N, n \geqslant 1$ and $r > 0$, we discuss upper and lower bounds for the expected volume $\mathbb{E} \left[\mathrm{vol}_n \left(\cap_{i=1}^N B(x_i, r) \right) \right]$ of random ball polyhedra defined by an N-tuple of i.i.d. random points $x_1, \ldots, x_N$ in $\mathbb{R}^n$ whose density f satisfies $\|f\|_\infty \leqslant 1$.

1 Introduction

The purpose of this note is to provide estimates on the expected volume of two classes of random convex sets. Both of them were studied by Paouris and Pivovarov in [18] and [20].

Let K be a centrally symmetric convex body in $\mathbb{R}^N$. For any $N \geqslant n$ and $\mathbf{x} = (x_1, \ldots, x_N) \in \oplus_{i=1}^N \mathbb{R}^n$ we denote by $T_{\mathbf{x}} = [x_1 \cdots x_N]$ the $n \times N$ matrix whose columns are the vectors x_i. Then, we consider the convex set

$$T_{\mathbf{x}}(K) = \left\{ \sum_{i=1}^N t_i x_i : \mathbf{t} = (t_1, \ldots, t_N) \in K \right\}.$$

Two examples of obvious geometric interest are obtained if we choose $K = B_1^N$ or $K = B_\infty^N$. Note that $T_{\mathbf{x}}(B_1^N) = \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\}$ is the absolute convex hull of $x_1, \ldots, x_N$, and $T_{\mathbf{x}}(B_\infty^N) = \sum_{i=1}^N [-x_i, x_i]$ is the zonotope defined as the Minkowski sum of the line segments $[-x_i, x_i]$. Now, let $\mu_1, \ldots, \mu_N$ be probability measures on $\mathbb{R}^n$ with densities $f_1, \ldots, f_N$, respectively. Consider the random convex set $T_{\mathbf{x}}(K)$, where x_i has distribution μ_i for $1 \leqslant i \leqslant N$. The next theorem from [18] asserts that if $\|f_i\|_\infty \leqslant 1$ then the expected volume of $T_{\mathbf{x}}(K)$ is minimized when each μ_i is the uniform measure on the Euclidean ball D_n of volume 1 in $\mathbb{R}^n$.

Theorem 1.1 (Paouris-Pivovarov). *Let $p > 0$, $N \geqslant n$ and $\mu_1, \ldots, \mu_N$ be probability measures on $\mathbb{R}^n$ with densities $f_1, \ldots, f_N$, respectively, with respect to the Lebesgue measure on $\mathbb{R}^n$, that satisfy $\|f_i\|_\infty \leqslant 1$. Consider a centrally symmetric convex body K in $\mathbb{R}^N$ and define*

$$\mathcal{F}_K(f_1, \ldots, f_N) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K)) \right)^p d\mu_N(x_N) \cdots d\mu_1(x_1).$$

Then,

$$\mathcal{F}_K(f_1, \ldots, f_N) \geqslant \mathcal{F}_K(\mathbf{1}_{D_n}, \ldots, \mathbf{1}_{D_n}).$$

In the first part of this note, our aim is to obtain upper and lower bounds for the expected volume of the random convex set $T_{\mathbf{x}}(K)$ under the assumption that $\mu_1 = \cdots = \mu_N = \mu$ is an isotropic log-concave probability measure in $\mathbb{R}^n$. We say that μ is isotropic if the barycenter of μ is at the origin, the density f of μ satisfies $\|f\|_\infty = 1$, and the covariance matrix of μ is $\mathrm{Cov}(\mu) = L_\mu^2 I_n$, where L_μ is the isotropic constant of μ. Our starting point is the formula

$$\mathrm{vol}_n(T_{\mathbf{x}}(K)) = \sqrt{\det(T_{\mathbf{x}} T_{\mathbf{x}}^*)}\, \mathrm{vol}_n(P_{E_{\mathbf{x}}}(K)),$$

where $E_{\mathbf{x}} = \ker(T_{\mathbf{x}})^\perp = \mathrm{Range}(T_{\mathbf{x}}^*)$, and A^* is the transpose of a matrix A. First we show that if $x_1, \ldots, x_N$ are independent random vectors distributed according to an isotropic log-concave probability measure μ on $\mathbb{R}^n$, then

$$c_1 L_\mu \sqrt{N} \leqslant \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\det(T_{\mathbf{x}} T_{\mathbf{x}}^*) \right)^{\frac{1}{2n}} d\mu^N(\mathbf{x}) \leqslant L_\mu \sqrt{N},$$

where $c_1 > 0$ is an absolute constant. Using this result we can give lower and upper bounds for the expected value

$$\int_{O(N)} \left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\mathrm{vol}_n(T_{\mathbf{x}}(U(K))) \right) d\mu^N(\mathbf{x}) \right) d\nu_N(U)$$

with respect to $U \in O(N)$, which indicates what might be a "good estimate" for the volume of the random convex set $T_{\mathbf{x}}(K)$: If μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ then for every $N \geqslant n$ and every centrally symmetric convex body K in $\mathbb{R}^N$ we have that

$$c_1 L_\mu \sqrt{N/n}\, \mathrm{vrad}(K) \leqslant \left(\int_{O(N)} \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K)) \, d\nu_N(U) \right) \right)^{\frac{1}{n}} \leqslant c_2 L_\mu \sqrt{N/n}\, w(K)$$

where $c_1, c_2 > 0$ are absolute constants and $\mathrm{vrad}(K) := (\mathrm{vol}_N(K)/\mathrm{vol}_N(B_2^N))^{1/N}$ is the volume radius of K.

Then, we study the basic examples $K = B_1^N$ and $K = B_\infty^N$. Using, additionally, known results of Bobkov and Nazarov which describe the geometry of an isotropic unconditional convex body in $\mathbb{R}^N$ we obtain estimates for the problem in this case. For example, in the range $n \leqslant N \leqslant \exp(\sqrt{n})$ we have:

Theorem 1.2. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $n \leqslant N \leqslant \exp(\sqrt{n})$ and any unconditional isotropic convex body K in $\mathbb{R}^N$ we have*

$$c_1 \sqrt{N/n}\, \mathrm{vrad}(K) \leqslant \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K))^{\frac{1}{n}} \right) \leqslant c_2 L_\mu \sqrt{N/n}\, (\log n)^2 \mathrm{vrad}(K),$$

where $c_1, c_2 > 0$ are absolute constants.

In the case $K = \overline{B}_q^N$, $2 \leqslant q \leqslant \infty$, we can give more precise asymptotic estimates for the expected value of the volume of $T_{\mathbf{x}}(\overline{B}_q^N)$ (see Theorem 3.11). For every $N \geqslant n$ and every $2 \leqslant q \leqslant \infty$ we have

$$c_1 \sqrt{N/n}\, \mathrm{vrad}(\overline{B}_q^N) \leqslant \left(\mathbb{E}_{\mu^N} \mathrm{vol}_n(T_{\mathbf{x}}(\overline{B}_q^N)) \right)^{1/n} \leqslant c_2 L_\mu \sqrt{N/n}\, \mathrm{vrad}(\overline{B}_q^N),$$

where $c_1, c_2 > 0$ are absolute constants.

We also provide a general upper bound under the assumption that both μ and K are isotropic.

Theorem 1.3. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $N \geqslant n$ and any isotropic convex body K in $\mathbb{R}^N$ we have*

$$\frac{c_2 L_\mu}{L_n} \mathrm{vrad}(K) \leqslant \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K))^{\frac{1}{n}} \right) \leqslant \frac{c_1 L_\mu N}{n} L_K \mathrm{vrad}(K),$$

where $c_1, c_2 > 0$ are absolute constants.

In the statement above, $L_n := \max\{L_C : C \text{ is an isotropic convex body in } \mathbb{R}^n\}$ (see the next section for more information and the known upper bounds for L_n).

In the second part of this note we provide estimates for the expected volume of random ball-polyhedra. Let f be a probability density on $\mathbb{R}^n$ with $\|f\|_\infty \leqslant 1$, fix $N \geqslant 1$ and an N-tuple $\mathbf{r} = (r_1, \ldots, r_N)$ of positive real numbers. Consider a sequence $x_1, \ldots, x_N$ of independent random points in $\mathbb{R}^n$ distributed according to f, and define the random ball-polyhedron

$$B(\mathbf{x}, \mathbf{r}) := \bigcap_{i=1}^N B(x_i, r_i)$$

which is the intersection of the Euclidean balls $B(x_i, r_i)$. Paouris and Pivovarov proved in [20] that the expected volume of this random ball polyhedron is maximized when $f = \mathbf{1}_{D_n}$, the density of the uniform measure on D_n.

Theorem 1.4 (Paouris-Pivovarov). *Let $N, n \geqslant 1$ and $r_1, \ldots, r_N \in (0, \infty)$. Consider independent random points $x_1, \ldots, x_N$ and $x_1^*, \ldots, x_N^*$ so that x_i has density f_i with $\|f_i\|_\infty \leqslant 1$, and x_i^* has density $\mathbf{1}_{D_n}$, $i = 1, \ldots, N$. Then, for any $r_1, \ldots, r_N > 0$,*

$$\mathbb{E}_{\mu_1 \otimes \cdots \otimes \mu_N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^N B(x_i, r_i) \right) \right) \leqslant \mathbb{E}_{\mu_{D_n}^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^N B(x_i^*, r_i) \right) \right).$$

Let K be a centrally symmetric convex body of volume 1 in $\mathbb{R}^n$. Our first observation is that in the case $r_1 = \cdots = r_N = r$ one has a very simple formula for the expectation

$$\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^N B(x_i, r) \right) \right).$$

Namely,

$$\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^N B(x_i, r) \right) \right) = \int_{K+rB_2^n} \mathrm{vol}_n ((K - y) \cap rB_2^n)^N \, dy.$$

In fact, one may replace Euclidean balls by r-homethets of any centrally symmetric convex body C in $\mathbb{R}^n$; the corresponding formula is

$$\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^N (x_i + rC) \right) \right) = \int_{K+rC} \mathrm{vol}_n ((K - y) \cap rC)^N \, dy.$$

Using an argument, based on the Brunn-Minkowski inequality, that goes back to Rogers and Shephard, we obtain the next lower bound, which is valid for all $r > 0$.

Theorem 1.5. *Let K be a centrally symmetric convex body of volume 1 in $\mathbb{R}^n$ and $x_1, \ldots, x_N$ be independent random points uniformly distributed in K. Then, for any centrally symmetric convex body C in $\mathbb{R}^n$ we have that*

$$\binom{nN + n}{n}^{-1} \mathrm{vol}_n(K \cap rC)^N \mathrm{vol}_n(K + rC) \leqslant \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^N (x_i + rC) \right) \right) \leqslant \mathrm{vol}_n(K \cap rC)^N \mathrm{vol}_n(K + rC).$$

An interesting question is to determine the best constants in the inequality of Theorem 1.5. Note that the behavior of $\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\cap_{i=1}^N (x_i + rC) \right) \right)$ is different for small and large values of r. One has

$$\lim_{r \to \infty} \frac{1}{\mathrm{vol}_n(rC)} \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^N (x_i + rC) \right) \right) = 1 \quad \text{and} \quad \lim_{r \to 0^+} \frac{1}{\mathrm{vol}_n(rC)^N} \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^N (x_i + rC) \right) \right) = 1.$$

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2 Notation and backgound information

In this section we introduce notation and terminology that we use throughout this work, and provide background information on isotropic convex bodies and log-concave probability measures in $\mathbb{R}^n$. We write $\langle \cdot, \cdot \rangle$ for the standard inner product in $\mathbb{R}^n$ and denote the Euclidean norm by $\| \cdot \|_2$. In what follows, B_2^n is the Euclidean unit ball and S^{n-1} is the unit sphere in $\mathbb{R}^n$, and σ is the unique rotationally invariant probability measure on S^{n-1}. The Lebesgue measure in $\mathbb{R}^n$ is denoted by vol_n. The letters c, c', c_j, c_j' etc. denote absolute positive constants whose value may change from line to line. Sometimes we relax our notation: $a \approx b$ will mean "$c_1 a \leqslant b \leqslant c_2 a$" for some absolute constants $c_i > 0$. We write ω_n for the volume of B_2^n; direct computation shows that $\omega_n^{\frac{1}{n}} \approx 1/\sqrt{n}$.

A convex body in $\mathbb{R}^n$ is a compact convex set $C \subset \mathbb{R}^n$ with non-empty interior. For notational convenience we write $\overline{C}$ for the homothetic image of volume 1 of a convex body $C \subseteq \mathbb{R}^n$, i.e. $\overline{C} := \mathrm{vol}_n(C)^{-1/n} C$. We say that C is centrally symmetric if $-C = C$. We say that C is unconditional with respect to the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ if $x = (x_1, \ldots, x_n) \in C$ implies that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in C$ for any choice of signs $\epsilon_j \in \{-1, 1\}$, $j = 1, \ldots, n$. The volume radius of C is the quantity $\mathrm{vrad}(C) = (\mathrm{vol}_n(C)/\mathrm{vol}_n(B_2^n))^{1/n}$. The support function of C is defined by $h_C(y) := \max\{\langle x, y \rangle : x \in C\}$, and the mean width of C is the average

$$w(C) := \int_{S^{n-1}} h_C(\xi) \, d\sigma(\xi)$$

of h_C on S^{n-1}.

A convex body C in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_C > 0$ such that

$$\|\langle \cdot, \xi \rangle\|_{L_2(C)}^2 := \int_C \langle x, \xi \rangle^2 dx = L_C^2$$

for all $\xi \in S^{n-1}$. The hyperplane conjecture asks whether there exists an absolute constant $A > 0$ such that

$$L_n := \max\{L_C : C \text{ is an isotropic convex body in } \mathbb{R}^n\} \leqslant A$$

for all $n \geqslant 1$. Bourgain proved in [4] that $L_n \leqslant c \sqrt[4]{n} \log n$; later, Klartag, in [11], improved this bound to $L_n \leqslant c \sqrt[4]{n}$. Very recently, in a breakthrough work, Chen [7] proved that for any $\epsilon > 0$ there exists $n_0(\epsilon) \in \mathbb{N}$ such that $L_n \leqslant n^\epsilon$ for every $n \geqslant n_0(\epsilon)$.

A Borel measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A + (1 - \lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0, 1)$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set in $\mathbb{R}^n$ and the restriction of $\log f$ to it is concave. It is known that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H in $\mathbb{R}^n$, then μ has a log-concave density f_μ. Note that if C is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_C$ is the density of a log-concave measure, the uniform measure on C.

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x \in \mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x) dx} \right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x) \, dx}{\int_{\mathbb{R}^n} f_\mu(x) \, dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x) \, dx}{\int_{\mathbb{R}^n} f_\mu(x) \, dx} \frac{\int_{\mathbb{R}^n} x_j f_\mu(x) \, dx}{\int_{\mathbb{R}^n} f_\mu(x) \, dx}.$$

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered, i.e. if

$$\int_{\mathbb{R}^n} \langle x, \xi \rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \xi \rangle f_\mu(x) dx = 0$$

for all $\xi \in S^{n-1}$, $\|f_\mu\|_\infty = 1$ and $\mathrm{Cov}(\mu) = L_\mu^2 I_n$, where I_n is the identity $n \times n$ matrix.

For every $q \geqslant 1$ and every $y \in \mathbb{R}^n$ we set

$$h_{Z_q(\mu)}(y) = \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^q d\mu(x) \right)^{1/q}.$$

The L_q-centroid body $Z_q(\mu)$ of μ is the centrally symmetric convex body with support function $h_{Z_q(\mu)}$. Note that μ is isotropic if and only if it is centered and $Z_2(\mu) = L_\mu B_2^n$. It was shown by Paouris [17] that if $1 \leqslant q \leqslant \sqrt{n}$ then $w(Z_q(\mu)) \simeq \sqrt{q} L_\mu$, and that for all $1 \leqslant q \leqslant n$ one has $\mathrm{vrad}(Z_q(\mu)) \leqslant c_1 \sqrt{q} L_\mu$. Conversely, it was shown by B. Klartag and E. Milman in [12] that if $1 \leqslant q \leqslant \sqrt{n}$ then $\mathrm{vrad}(Z_q(\mu)) \geqslant c_2 \sqrt{q} L_\mu$. This determines the volume radius of $Z_q(\mu)$ for all $1 \leqslant q \leqslant \sqrt{n}$. For larger values of q one can still use the lower bound: $\mathrm{vrad}(Z_q(\mu)) \geqslant c_2 \sqrt{q}$, obtained by Lutwak, Yang and Zhang in [14] for convex bodies and extended by Paouris and Pivovarov in [18] to the class of log-concave probability measures.

For every $1 \leqslant k \leqslant n-1$ and every $E \in G_{n,k}$, the marginal of the measure μ with respect to E is the probability measure $\pi_E(\mu)$ on E, with density

$$f_{\pi_E(\mu)}(x) = \int_{x+E^\perp} f_\mu(y) dy.$$

It is easily checked that if μ is centered, isotropic or log-concave, then $\pi_E(\mu)$ is also centered, isotropic or log-concave, respectively.

We refer the reader to the book [6] for an updated exposition of isotropic log-concave measures and more information on the hyperplane conjecture.

We close this section with a rough description of the main ideas behind the proof of Theorem 1.1 and Theorem 1.4. The approach of Paouris and Pivovarov is based on rearrangement inequalities; in particular, on the Brascamp-Lieb-Luttinger inequality. Let $H : \oplus_{i=1}^N \mathbb{R}^n \to \mathbb{R}^+$ be a non-negative measurable function and consider the multilinear operator $\mathcal{F}_H$ defined by

$$\mathcal{F}_H(f_1, \ldots, f_N) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} H(x_1, \ldots, x_N) f_1(x_1) \cdots f_N(x_N) \, dx_1 \cdots dx_N,$$

where $f_1, \ldots, f_N : \mathbb{R}^n \to \mathbb{R}^+$ are integrable functions. Assume that the function $H : \otimes_{i=1}^N \mathbb{R}^n \to \mathbb{R}^+$ has the following property: for any $z \in S^{n-1}$ and any $Y = \{y_1, \ldots, y_N\} \subset z^\perp$, the function $H_Y : \mathbb{R}^N \to \mathbb{R}^+$ which is defined by

$$H_Y(t) = H(y_1 + t_1 z, \ldots, y_N + t_N z)$$

is even and quasi-convex. Then,

$$\mathcal{F}_H(f_1, \ldots, f_N) \geqslant \mathcal{F}_H(f_1^*, \ldots, f_N^*),$$

where f^* is the symmetric decreasing rearrangement of f. Moreover, if $\|f_i\|_\infty \leqslant 1$ for all $i = 1, \ldots, N$, then

$$\mathcal{F}_H(f_1, \ldots, f_N) \geqslant \mathcal{F}_H(f_1^*, \ldots, f_N^*) \geqslant \mathcal{F}_H(\mathbf{1}_{D_n}, \ldots, \mathbf{1}_{D_n}),$$

where D_n is the Euclidean ball of volume 1 in $\mathbb{R}^n$. On the other hand, if for every $z \in S^{n-1}$ and any $Y = \{y_1, \ldots, y_N\} \subset z^\perp$ the function H_Y is even and quasi-concave then the above inequalities are reversed.

Theorem 1.1 is a consequence of this general result. Define

$$H(x_1, \ldots, x_N) = \left(\mathrm{vol}_n(T_\mathbf{x}(K)) \right)^p = \left(\mathrm{vol}_n([x_1 \cdots x_N]K) \right)^p.$$

One can show that for any $\xi \in S^{n-1}$ and $y_1, \ldots, y_N \in z^\perp$, if we set $Y = \{y_1, \ldots, y_N\}$ and define $T_Y(\mathbf{t}) := [y_1 + t_1 \xi, \ldots, y_N + t_N \xi]$ then the function $H_Y : \mathbb{R}^N \to \mathbb{R}^+$ defined by $H_Y(t) = \mathrm{vol}_n(T_{Y(t)}(K))^p$ is even and quasi-convex. Theorem 1.4 is again a consequence of this approach. Given $r_1, \ldots, r_N > 0$, define

$$H(x_1, \ldots, x_N) = \mathrm{vol}_n \left(\bigcap_{i=1}^N B(x_i, r_i) \right).$$

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Then, H is even and quasi-concave on its support. Moreover, for any $z \in S^{n-1}$ and $y_1, \ldots, y_N \in z^\perp$ the function $H_{z,Y} : \mathbb{R}^N \to [0, \infty)$ defined by $H_{z,Y}(t) = \mathrm{vol}_n \left(\cap_{i=1}^N B(y_i + t_i z, r_i) \right)$ is even and quasi-concave on its support. The reader will find more information in the survey article [21] of Paouris and Pivovarov.

3   Estimates for the expected volume of $T_{\mathbf{x}}(K)$

Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $N \geqslant n$ and any centered convex body K of volume 1 in $\mathbb{R}^N$ we want to give upper and lower bounds for the quantity

$$\mathbb{E}_{\mu^N} \left(\left(\mathrm{vol}_n(T_{\mathbf{x}}(K)) \right)^{\frac{1}{n}} \right) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K)) \right)^{\frac{1}{n}} d\mu^N(\mathbf{x}),$$

where $T_{\mathbf{x}}$ is the random $n \times N$ matrix with columns N independent random vectors $x_1, \ldots, x_N$ distributed according to μ. Our starting point is the formula (see for example [19, Proposition 2.1])

$$(3.1) \qquad \mathrm{vol}_n(T_{\mathbf{x}}(K)) = \sqrt{\det(T_{\mathbf{x}} T_{\mathbf{x}}^*)} \, \mathrm{vol}_n(P_{E_{\mathbf{x}}}(K)),$$

where $E_{\mathbf{x}} = \ker(T_{\mathbf{x}})^\perp = \mathrm{Range}(T_{\mathbf{x}}^*)$, and A^* denotes the transpose of a matrix A. We start with some preliminary observations regarding the expectation of $\sqrt{\det(T_{\mathbf{x}} T_{\mathbf{x}}^*)}$.

3.1   Preliminary estimates

It is known that $\sqrt{\det(T_{\mathbf{x}} T_{\mathbf{x}}^*)}$ is equal to the volume of the n-dimensional parallelotope spanned in $\mathbb{R}^N$ by the rows $y_1, \ldots, y_n$ of $T_{\mathbf{x}}$. The next lemma provides some estimates for $\mathbb{E}_{\mu^N} \left(\det(T_{\mathbf{x}} T_{\mathbf{x}}^*)^{\frac{1}{2n}} \right)$. Note that the assumption that μ is log-concave is needed only for the lower bound.

Lemma 3.1. *Let $x_1, \ldots, x_N$ be independent random points which are distributed according to an isotropic log-concave probability measure μ on $\mathbb{R}^n$. Then,*

$$(3.2) \qquad c_1 L_\mu \sqrt{N} \leqslant \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\det(T_{\mathbf{x}} T_{\mathbf{x}}^*) \right)^{\frac{1}{2n}} d\mu^N(\mathbf{x}) \leqslant L_\mu \sqrt{N},$$

where $c_1 > 0$ is an absolute constant.

Proof. We use the Cauchy-Binet formula: For any $S = \{i_1, \ldots, i_n\} \subseteq [N]$ with $|S| = n$ we denote by $T_{\mathbf{x}}|_S$ the $n \times n$ matrix whose columns are $x_{i_1}, \ldots, x_{i_n}$. Then,

$$(3.3) \qquad \det(T_{\mathbf{x}} T_{\mathbf{x}}^*) = \sum_{|S|=n} \det((T_{\mathbf{x}}|_S)(T_{\mathbf{x}}|_S)^*).$$

From a well-known formula that goes back to Blaschke (see [6, Proposition 3.5.5] for a proof) we see that

$$(3.4) \qquad \mathbb{E}_{\mu^S} \left(\det((T_{\mathbf{x}}|_S)(T_{\mathbf{x}}|_S)^*) \right) = n! \det(\mathrm{Cov}(\mu)),$$

where $\mu^S := \otimes_{i \in S} \mu$. Note that this identity holds true for any centered probability measure μ on $\mathbb{R}^n$. Assuming that μ is isotropic, we have $\det(\mathrm{Cov}(\mu)) = L_\mu^{2n}$ and it follows that

$$(3.5) \qquad \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \det(T_{\mathbf{x}} T_{\mathbf{x}}^*) \, d\mu^N(\mathbf{x}) = \binom{N}{n} n! \det(\mathrm{Cov}(\mu)) \leqslant N^n \det(\mathrm{Cov}(\mu)) = N^n L_\mu^{2n}.$$

Applying Hölder's inequality we obtain the upper bound in (3.2).

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For the lower bound, using first the concavity of the function $x \mapsto x^p$ for $p \in (0,1)$, we write

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\det(T_\mathbf{x} T_\mathbf{x}^*) \right)^{\frac{1}{2n}} d\mu^N(\mathbf{x}) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\sum_{|S|=n} \det((T_\mathbf{x}|_S)(T_\mathbf{x}|_S)^*) \right)^{\frac{1}{2n}} d\mu^N(\mathbf{x})$$

$$\geqslant \binom{N}{n}^{\frac{1}{2n}} \frac{1}{\binom{N}{n}} \sum_{|S|=n} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \det((T_\mathbf{x}|_S)(T_\mathbf{x}|_S)^*)^{\frac{1}{2n}} d\mu^N(\mathbf{x}).$$

From [22, Corollary 1] (see also [16, Section 3.7]) we see that, for any $S \subset [N]$ with $|S| = n$, one has $\det((T_\mathbf{x}|_S)(T_\mathbf{x}|_S)^*) \geqslant (c_2 n)^n L_\mu^{2n}$ for some absolute constant $c_2 > 0$, with probability greater than $1 - e^{-n}$. It follows that

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \det((T_\mathbf{x}|_S)(T_\mathbf{x}|_S)^*)^{\frac{1}{2n}} d\mu^N(\mathbf{x}) \geqslant c_3 L_\mu \sqrt{n}$$

for some absolute constant $c_3 > 0$. Therefore,

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \det(T_\mathbf{x} T_\mathbf{x}^*)^{\frac{1}{2n}} d\mu^N(\mathbf{x}) \geqslant c_3 L_\mu \sqrt{n} \binom{N}{n}^{\frac{1}{2n}} \geqslant c_1 L_\mu \sqrt{N}$$

for some absolute constant $c_1 > 0$. $\qquad\square$

Remark 3.2. From the proof of Lemma 3.1 one may easily check that, for any isotropic log-concave probability measure μ on $\mathbb{R}^n$ and any $N \geqslant n$, the estimate

(3.6) $$c_1 L_\mu \sqrt{N} \leqslant \left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left(\det(T_\mathbf{x} T_\mathbf{x}^*) \right)^p d\mu^N(\mathbf{x}) \right)^{\frac{1}{2pn}} \leqslant L_\mu \sqrt{N}$$

holds true for all $p \in [e^{-n}, 1]$.

The next proposition gives an upper and a lower bound for the average

$$\left(\int_{O(N)} \left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathrm{vol}_n(T_\mathbf{x}(U(K))) \, d\mu^N(\mathbf{x}) \right) d\nu_N(U) \right)^{\frac{1}{n}}$$

over all $U \in O(N)$ in terms of the mean width and the volume radius of K respectively, and shows what one should expect as a reasonable estimate for the expected volume radius of $T_\mathbf{x}(K)$.

Proposition 3.3. *Let μ be an log-concave isotropic probability measure on $\mathbb{R}^n$. For any $N \geqslant n$ and any centrally symmetric convex body K in $\mathbb{R}^N$ we have*

$$c_1 L_\mu \sqrt{N/n} \, \mathrm{vrad}(K) \leqslant \left(\int_{O(N)} \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_\mathbf{x}(K)) \, d\nu_N(U) \right)^{\frac{1}{n}} \leqslant c_2 L_\mu \sqrt{N/n} \, w(K) \right.$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Our starting point is (3.1). Let $U \in O(N)$ be independent from $\mathbf{x}$ and distributed according to the Haar probability measure ν_N on $O(N)$. Since $\det((T_\mathbf{x} U)(U^* T_\mathbf{x}^*)) = \det(T_\mathbf{x} T_\mathbf{x}^*)$ and $P_{E_\mathbf{x}} \circ U = P_{U^*(E_\mathbf{x})}$, we see that

$$\mathrm{vol}_n(T_\mathbf{x} U(K)) = \sqrt{\det(T_\mathbf{x} T_\mathbf{x}^*)} \, \mathrm{vol}_n(P_{U^* E_\mathbf{x}}(K)),$$

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where $E_\mathbf{x} = \ker(T_\mathbf{x})^\perp = \text{Range}(T_\mathbf{x}^*)$. Note that $E_\mathbf{x}$ is n-dimensional with probability 1, therefore the distribution of $U^*(E_\mathbf{x})$ is the Haar probability measure $\nu_{N,n}$ on $G_{N,n}$ for almost all $\mathbf{x}$. It follows that

$$(3.7) \qquad \int_{O(N)} \mathbb{E}_{\mu^N}\big(\text{vol}_n(T_\mathbf{x}(U(K)))\big)\,d\nu_N(U)$$

$$= \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\Big(\int_{O(N)}\text{vol}_n(T_\mathbf{x}(U(K)))\,d\nu_N(U)\Big)\,d\mu^N(\mathbf{x})$$

$$= \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\Big(\det(T_\mathbf{x}T_\mathbf{x}^*)^{1/2}\int_{O(n)}\text{vol}_n(P_{U^*E_\mathbf{x}}(K)d\nu^N(U))\Big)d\mu^N(\mathbf{x})$$

$$= \Big(\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\det(T_\mathbf{x}T_\mathbf{x}^*)^{1/2}d\mu^N(\mathbf{x})\Big)\Big(\int_{G_{N,n}}\text{vol}_n(P_E(K))\,d\nu_{N,n}(E)\Big).$$

From Lemma 3.1 we get

$$(cL_\mu)^n N^{\frac{n}{2}}\int_{G_{N,n}}\text{vol}_n(P_E(K))\,d\nu_{N,n}(E) \leqslant \int_{O(N)}\mathbb{E}_{\mu^N}\big(\text{vol}_n(T_\mathbf{x}(U(K)))\big)\,d\nu_N(U)$$

$$\leqslant L_\mu^n N^{\frac{n}{2}}\int_{G_{N,n}}\text{vol}_n(P_E(K))\,d\nu_{N,n}(E)$$

for an absolute constant $c > 0$. From Aleksandrov's inequalities we know that

$$\text{vrad}(K) \leqslant \Big(\frac{1}{\omega_n}\int_{G_{N,n}}\text{vol}_n(P_E(K))\,d\nu_{N,n}(E)\Big)^{\frac{1}{n}} \leqslant \frac{1}{\omega_1}\int_{G_{N,1}}\text{vol}_1(P_E(K))\,d\nu_{N,1}(E) = w(K),$$

and hence,

$$cL_\mu\sqrt{N}\omega_n^{1/n}\text{vrad}(K) \leqslant \Big(\int_{O(N)}\mathbb{E}_{\mu^N}\big(\text{vol}_n(T_\mathbf{x}(U(K)))\big)\,d\nu_N(U)\Big)^{\frac{1}{n}} \leqslant L_\mu\sqrt{N}\omega_n^{1/n}w(K).$$

Taking into account the fact that $\omega_n^{1/n} \approx 1/\sqrt{n}$, we obtain the result. $\qquad\square$

3.2 Two basic examples

There are two main examples of convex bodies K for which the expected volume of $T_\mathbf{x}(K)$ is well studied. The first one is $K = B_\infty^N$; then, $T_\mathbf{x}(B_\infty^N) = \sum_{i=1}^N[-x_i, x_i]$ is the zonotope defined as the Minkowski sum of the line segments $[-x_i, x_i]$.

Proposition 3.4. *Let $\overline{B}_\infty^N$ denote the cube of volume 1 in $\mathbb{R}^N$. Then,*

$$\mathbb{E}_{\mu_{D_n}^N}\big(\text{vol}_n(T_\mathbf{x}(\overline{B}_\infty^N))^{\frac{1}{n}}\big) \approx \sqrt{N/n}\,\text{vrad}(\overline{B}_\infty^N).$$

Proof. Let

$$I_p(D_n; m) := \int_{D_n}\cdots\int_{D_n}\text{vol}_n\Big(\sum_{i=1}^m[-x_i, x_i]\Big)^p dx_m\cdots dx_1.$$

Note that

$$I_{1/n}(D_n; N) = \mathbb{E}_{\mu_{D_n}^N}\big(\text{vol}_n(T_\mathbf{x}(\overline{B}_\infty^N))^{\frac{1}{n}}\big).$$

A direct computation based on the Blaschke-Petkantschin formula (see [24, Theorem 8.2.2]) shows that

$$\frac{1}{\text{vol}_n(B_2^n)^n}\int_{B_2^n}\cdots\int_{B_2^n}\text{vol}_n\Big(\sum_{i=1}^n[0, x_i]\Big)^p dx_n\cdots dx_1 = \frac{\omega_{n+p}^n}{\omega_n^n}\prod_{j=0}^{n-1}\frac{(n-j)\omega_{n-j}}{(n+p-j)\omega_{n+p-j}},$$

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where $\omega_k = \mathrm{vol}_k(B_2^k)$. It follows that

$$I_p(D_n;n) := \int_{D_n} \cdots \int_{D_n} \mathrm{vol}_n\Big(\sum_{i=1}^{n}[0,x_i]\Big)^p dx_n \cdots dx_1 = \frac{\omega_{n+p}^n}{\omega_n^{n+p}} \prod_{j=0}^{n-1} \frac{(n-j)\omega_{n-j}}{(n+p-j)\omega_{n+p-j}}.$$

Choosing $p = 1/n$ one may check that

$$c_1\sqrt{n} \leqslant I_{1/n}(D_n;n) \leqslant c_2\sqrt{n}$$

where $c_1, c_2 > 0$ are absolute constants. Note that

$$\mathrm{vol}_n\Big(\sum_{i=1}^{N}[-x_i,x_i]\Big) = 2^n \sum_{I \subset [N], |I|=n} \mathrm{vol}_n\Big(\sum_{j\in I}[0,x_j]\Big).$$

Using the inequalities

$$\binom{N}{n}^p \frac{1}{\binom{N}{n}} \sum_{I\subset[N],|I|=n} t_I^p \leqslant \Big(\sum_{I\subset[N],|I|=n} t_I\Big)^p \leqslant \sum_{I\subset[N],|I|=n} t_I^p$$

with $t_I = \mathbb{E}_{\mu_{D_n}^n}\Big(\mathrm{vol}_n\big(\sum_{j\in I}[0,x_j]\big)\Big)$ we see that

$$c_1\sqrt{n}\binom{N}{n}^{1/n} \leqslant \binom{N}{n}^{1/n} I_{1/n}(D_n;n) \leqslant \frac{1}{2}I_{1/n}(D_n;N) \leqslant \binom{N}{n}^{1/n} I_{1/n}(D_n;n) \leqslant c_2\sqrt{n}\binom{N}{n}^{1/n}.$$

Since $\binom{N}{n}^{1/n} \approx \frac{N}{n}$ and $\mathrm{vrad}(B_\infty^N) \approx \sqrt{N}$, we obtain the result. $\qquad\square$

As an immediate corollary of Theorem 1.1 we have the following.

Proposition 3.5. *Let $N \geqslant n$ and $\mu_1, \ldots, \mu_N$ be probability measures on $\mathbb{R}^n$ with densities f_i, respectively, with respect to the Lebesgue measure, that satisfy $\|f_i\|_\infty \leqslant 1$. Then,*

$$\mathbb{E}_{\otimes_{i=1}^N \mu_i}\big(\mathrm{vol}_n(T_\mathbf{x}(\overline{B}_\infty^N))^{\frac{1}{n}}\big) \geqslant c\sqrt{N/n}\,\mathrm{vrad}(\overline{B}_\infty^N),$$

where $c > 0$ is an absolute constant.

The second well-studied example is when $K = \overline{B}_1^N$. Note that $T_\mathbf{x}(B_1^N) = \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\}$ for all $\mathbf{x} = (x_1, \ldots, x_N)$.

Proposition 3.6. *Let $\overline{B}_1^N$ denote the multiple of the cross-polytope B_1^N of volume 1 in $\mathbb{R}^N$. Then, for any isotropic log-concave probability measure μ on $\mathbb{R}^n$ we have that*

$$(3.8) \qquad c_1 L_\mu \sqrt{N/n}\,\sqrt{\log(2N/n)}\mathrm{vrad}(\overline{B}_1^N) \leqslant \mathbb{E}_{\mu^N}\big(\mathrm{vol}_n(T_\mathbf{x}(\overline{B}_1^N))^{\frac{1}{n}}\big)$$

$$\leqslant c_2 L_\mu \sqrt{N/n}\,\sqrt{\log N}\mathrm{vrad}(\overline{B}_1^N)$$

if $n \leqslant N \leqslant \exp(\sqrt{n})$, and

$$(3.9) \qquad c_1\sqrt{N/n}\,\sqrt{\log(2N/n)}\mathrm{vrad}(\overline{B}_1^N) \leqslant \mathbb{E}_{\mu^N}\big(\mathrm{vol}_n(T_\mathbf{x}(\overline{B}_1^N))^{\frac{1}{n}}\big)$$

$$\leqslant c_2 L_\mu \sqrt{N/n}\,\sqrt{\log N}(\log\log N)^2\mathrm{vrad}(\overline{B}_1^N)$$

if $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$.

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Proof. Observe that $\overline{B}_1^N \approx N B_1^N$, which implies that $T_{\mathbf{x}}(\overline{B}_1^N) \approx N \operatorname{conv}\{\pm x_1, \ldots, \pm x_N\}$. Therefore,

$$\mathbb{E}_{\mu^N}\left(\operatorname{vol}_n(T_{\mathbf{x}}(\overline{B}_1^N))^{\frac{1}{n}}\right) \approx N \mathbb{E}_{\mu^N}\left(\operatorname{vol}_n(\operatorname{conv}\{\pm x_1, \ldots, \pm x_N\})^{\frac{1}{n}}\right).$$

It is proved in [8] that

$$\mathbb{E}_{\mu^N}\left(\operatorname{vol}_n(\operatorname{conv}\{\pm x_1, \ldots, \pm x_N\})\right)^{\frac{1}{n}} \leqslant \frac{c_1 w(Z_{\log N}(\mu))}{\sqrt{n}}$$

for all $N \leqslant e^n$, where $Z_q(\mu)$ is the L_q-centroid body of μ. Since $\operatorname{vrad}(\overline{B}_1^N) \approx \sqrt{N}$, this implies that

$$\mathbb{E}_{\mu^N}\left(\operatorname{vol}_n(T_{\mathbf{x}}(\overline{B}_1^N))^{\frac{1}{n}}\right) \leqslant c_2 \sqrt{N/n}\, \operatorname{vrad}(\overline{B}_1^N)\, w(Z_{\log N}(\mu)).$$

Then, the upper bounds in (3.8) and (3.9) follow from the known upper bounds for $w(Z_q(\mu))$, where μ is an isotropic log-concave probability measure on $\mathbb{R}^n$. Recall that if $1 \leqslant q \leqslant \sqrt{n}$ then $w(Z_q(\mu)) \leqslant c\sqrt{q}L_\mu$. On the other hand, E. Milman has proved in [15] that for all $\sqrt{n} \leqslant q \leqslant n$,

$$w(Z_q(\mu)) \leqslant c L_\mu \log(1 + \min\{q, n\}) \max\left\{\frac{q\log(1+q)}{\sqrt{n}}, \sqrt{q}\right\}$$

for some absolute constant $c > 0$. Note that this quantity is always bounded by $cL_\mu\sqrt{n}(\log n)^2$.

For the lower bound we use the fact, proved in [8] that if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ and if $x_1, \ldots, x_N$ are independent random points which are distributed according to μ, then for $n \leqslant N \leqslant e^{\sqrt{n}}$ one has

$$(3.10) \qquad\qquad \operatorname{vol}_n(\operatorname{conv}\{\pm x_1, \ldots, \pm x_N\})^{1/n} \geqslant c_1 L_\mu \frac{\sqrt{\log(2N/n)}}{\sqrt{n}}$$

with probability greater than $1 - \exp(-c_2\sqrt{N})$, while in the range $e^{\sqrt{n}} \leqslant N \leqslant e^n$ one has

$$(3.11) \qquad\qquad \operatorname{vol}_n(\operatorname{conv}\{\pm x_1, \ldots, \pm x_N\})^{1/n} \geqslant c_1 \frac{\sqrt{\log(2N/n)}}{\sqrt{n}},$$

again with probability exponentially close to 1. This shows that

$$\mathbb{E}_{\mu^N}\left(\operatorname{vol}_n(T_{\mathbf{x}}(\overline{B}_1^N))^{\frac{1}{n}}\right) \geqslant c\sqrt{N/n}L_\mu\sqrt{\log(2N/n)}\operatorname{vrad}(\overline{B}_1^N)$$

in the range $n \leqslant N \leqslant \exp(\sqrt{n})$, and the lower bound of (3.9) follows in the same way from (3.11). $\qquad\square$

3.3 Some general estimates

We can give some general estimates using the following bounds for the volume radius of an n-dimensional projection of a convex body in $\mathbb{R}^N$.

Lemma 3.7. *Let K be a centrally symmetric convex body in $\mathbb{R}^N$. For any $1 \leqslant n < N$ and any $E \in G_{N,n}$ we have that*

$$c_1 \sqrt{n/N} \frac{1}{\sqrt{n}M(K)} \leqslant \operatorname{vol}_n(P_E(K))^{1/n} \leqslant c_2 \sqrt{N/n} \frac{w(K)}{\sqrt{n}},$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Let $N(A, B)$ denote the covering number of A by B, i.e. the least number of translates of B whose union covers A. The classical Sudakov inequality (see [1, Chapter 4]) states that $N(K, tB_2^N) \leqslant \exp(c_3 N w^2(K)/t^2)$ for all $t > 0$. Since $N(P_E(K), tP_E(B_2^N)) \leqslant N(K, tB_2^N)$ for all $E \in G_{N,n}$, it follows that

$$\mathrm{vol}_n(P_E(K))^{1/n} \leqslant \exp(c_3 N w^2(K)/(t^2 n)) \mathrm{vol}_n(tP_E(B_2^N))^{1/n}$$

for all $t > 0$, and choosing $t = \sqrt{N/n} w(K)$ we get

$$\mathrm{vol}_n(P_E(K))^{1/n} \leqslant c_4 \sqrt{N/n} w(K) \mathrm{vol}_n(B_E)^{1/n},$$

where $B_E = P_E(B_2^N) = B_2^N \cap E$, and hence $\mathrm{vol}_n(B_E)^{1/n} \approx 1/\sqrt{n}$. This proves the right hand side inequality. For the lower bound we use a similar argument, this time employing the dual Sudakov inequality (see [1, Chapter 4]) $N(B_2^N, tK) \leqslant \exp(c_3 N M^2(K)/t^2)$, which implies that

$$\mathrm{vol}_n(P_E(B_2^N))^{1/n} \leqslant \exp(c_3 N M^2(K)/(t^2 n)) \mathrm{vol}_n(tP_E(K))^{1/n}$$

for all $t > 0$, and then choose $t = \sqrt{N/n} M(K)$. $\qquad\square$

Taking into account Lemma 3.1 and Lemma 3.7 we have the next general estimates.

Theorem 3.8. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $N \geqslant n$ and any centrally symmetric convex body K in $\mathbb{R}^N$ we have*

$$\frac{c_1 L_\mu}{M(K)} \leqslant \left(\mathbb{E}_{\mu^N} \, \mathrm{vol}_n(T_{\mathbf{x}}(K))^{\frac{1}{n}} \right) \leqslant \frac{c_2 L_\mu N}{n} \, w(K),$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. We may write

$$\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K)) \right) = \mathbb{E}_{\mu^N} \left(\sqrt{\det(T_{\mathbf{x}} T_{\mathbf{x}}^*)} \, \mathrm{vol}_n(P_{E_{\mathbf{x}}}(K)) \right)$$

$$\leqslant L_\mu^n N^{n/2} \max_{E \in G_{N,n}} \mathrm{vol}_n(P_E(K)),$$

by the proof of Lemma 3.1, and then the upper bound from Proposition 3.7 implies that

$$\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K))^{\frac{1}{n}} \right) \leqslant L_\mu \sqrt{N} \cdot c_2 \sqrt{N/n} \frac{w(K)}{\sqrt{n}} = \frac{c_2 L_\mu N}{n} w(K).$$

On the other hand, a similar argument shows that

$$\mathbb{E}_{\mu^N} \left((\mathrm{vol}_n(T_{\mathbf{x}}(K)))^{\frac{1}{n}} \right) = \mathbb{E}_{\mathbf{x}} \left((\det(T_{\mathbf{x}} T_{\mathbf{x}}^*))^{\frac{1}{2n}} \, \mathrm{vol}_n(P_{E_{\mathbf{x}}}(K))^{\frac{1}{n}} \right)$$

$$\geqslant \min_{E \in G_{N,n}} \mathrm{vol}_n(P_E(K))^{\frac{1}{n}} \, \mathbb{E}_{\mu^N} \left((\det(T_{\mathbf{x}} T_{\mathbf{x}}^*))^{\frac{1}{2n}} \right),$$

and combining the lower bounds from Lemma 3.1 and Lemma 3.7 we get

$$\mathbb{E}_{\mu^N} \left((\mathrm{vol}_n(T_{\mathbf{x}}(K)))^{\frac{1}{n}} \right) \geqslant c_3 L_\mu \sqrt{N} \cdot c_4 \sqrt{n/N} \frac{1}{\sqrt{n} M(K)} = \frac{c_5 L_\mu}{M(K)},$$

as claimed. $\qquad\square$

Our next result gives a general upper bound under the assumption that both μ and K are isotropic.

Theorem 3.9. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $N \geqslant n$ and any isotropic convex body K in $\mathbb{R}^N$ we have*

$$\frac{c_2 L_\mu}{L_n} \mathrm{vrad}(K) \leqslant \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_\mathbf{x}(K))^{\frac{1}{n}} \right) \leqslant \frac{c_1 L_\mu N}{n} L_K \mathrm{vrad}(K),$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Starting from (3.1) and using the Cauchy-Schwarz inequality we get

$$\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_\mathbf{x}(K))^{\frac{1}{n}} \right) \leqslant \left(\mathbb{E}_{\mu^N} \left(\det(T_\mathbf{x} T_\mathbf{x}^*)^{\frac{1}{n}} \right) \right)^{\frac{1}{2}} \left(\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(P_{E_\mathbf{x}}(K))^{\frac{2}{n}} \right) \right)^{\frac{1}{2}}$$

$$\leqslant L_\mu \sqrt{N} \left(\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(P_{E_\mathbf{x}}(K))^{\frac{2}{n}} \right) \right)^{\frac{1}{2}}$$

taking into account Lemma 3.1. From a classical inequality of Rogers and Shephard (see [1, Lemma 1.5.6]) we also know that

$$\mathrm{vol}_n(K \cap E_\mathbf{x}^\perp)^{-1} \leqslant \mathrm{vol}_n(P_{E_\mathbf{x}}(K)) \leqslant \binom{N}{n} \mathrm{vol}_n(K \cap E_\mathbf{x}^\perp)^{-1}$$

for all $\mathbf{x}$. Assuming that K is also isotropic, we have that

$$\mathrm{vol}_n(K \cap E_\mathbf{x}^\perp)^{1/n} \approx \frac{L_{K_{n+1}(\pi_{E_\mathbf{x}}(\mu_K))}}{L_K} \geqslant \frac{c_2}{L_K}$$

where $\pi_{E_\mathbf{x}}(\mu_K)$ is the marginal of K with respect to $E_\mathbf{x}$ (see [6] for the definition of the family of convex bodies $\{K_p(\nu)\}_{p>0}$ associated with a log-concave probability measure ν and, in particular, [6, Proposition 5.1.15] for this statement). Combining the above, we finally get

$$\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_\mathbf{x}(K))^{\frac{1}{n}} \right) \leqslant L_\mu \sqrt{N} \binom{N}{n}^{\frac{1}{n}} \cdot \frac{1}{c_2} L_K,$$

and the result follows from the fact that $\mathrm{vol}_N(K) = 1$ and hence $\mathrm{vrad}(K) \approx \sqrt{N}$. For the lower bound we recall that

$$\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_\mathbf{x}(K))^{\frac{1}{n}} \right) = \mathbb{E}_{\mu^N} \left(\det(T_\mathbf{x} T_\mathbf{x}^*)^{\frac{1}{2n}} \mathrm{vol}_n(P_{E_\mathbf{x}}(K))^{\frac{1}{n}} \right)$$

by (3.1). Then, we observe that

$$\mathrm{vol}_n(P_{E_\mathbf{x}}(K))^{1/n} \geqslant \mathrm{vol}_n(K \cap E_\mathbf{x}^\perp)^{-1/n} \approx \frac{L_K}{L_{K_{n+1}(\pi_{E_\mathbf{x}}(\mu_K))}} \geqslant \frac{c_1}{L_n}$$

and conclude that

$$\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_\mathbf{x}(K))^{\frac{1}{n}} \right) \geqslant \frac{c_1}{L_n} \mathbb{E}_{\mu^N} \left(\det(T_\mathbf{x} T_\mathbf{x}^*)^{\frac{1}{2n}} \right) \geqslant \frac{c_2 L_\mu}{L_n} \mathrm{vrad}(K),$$

where the last inequality follows from Lemma 3.1. $\square$

In the next theorem we assume that K is an unconditional isotropic convex body in $\mathbb{R}^N$ and using Theorem 3.8 and Theorem 3.9 we obtain a better estimate.

Theorem 3.10. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $n \leqslant N \leqslant \exp(\sqrt{n})$ and any unconditional isotropic convex body K in $\mathbb{R}^N$ we have*

$$c_1 \sqrt{N/n} \,\mathrm{vrad}(K) \leqslant \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_\mathbf{x}(K))^{\frac{1}{n}} \right) \leqslant c_2 L_\mu \sqrt{N/n} \,(\log n)^2 \mathrm{vrad}(K),$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. By a result of Bobkov and Nazarov from [3] we know that $c_1 \overline{B}_\infty^N \subseteq K \subseteq c_2 \overline{B}_1^N$ for some absolute constants $c_1, c_2 > 0$. It follows that $T_{\mathbf{x}}(K) \subseteq c_2 T_{\mathbf{x}}(\overline{B}_1^N)$ for any $\mathbf{x} = (x_1, \ldots, x_N)$, and hence

$$\mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K))^{\frac{1}{n}} \right) \leqslant c_2 \mathbb{E}_{\mu^N} \left[\mathrm{vol}_n(T_{\mathbf{x}}(\overline{B}_1^N))^{\frac{1}{n}} \right].$$

Using Proposition 3.5 and Proposition 3.6 we conclude the proof. $\qquad\square$

Note that combining Proposition 3.5 and Proposition 3.6 one can obtain an analogous result for the range $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$. Our last result concerns the case $K = \overline{B}_q^N$, $2 \leqslant q \leqslant \infty$; we can obtain a sharp asymptotic estimate for the expected volume of $T_{\mathbf{x}}(K)$.

Theorem 3.11. *Let μ be an isotropic probability measure on $\mathbb{R}^n$. For any $N \geqslant n$ and any $2 \leqslant q \leqslant \infty$ we have*

$$c_1 \sqrt{N/n}\, \mathrm{vrad}(\overline{B}_q^N) \leqslant \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_{\mathbf{x}}(\overline{B}_q^N))^{\frac{1}{n}} \right) \leqslant c_2 L_\mu \sqrt{N/n}\, \mathrm{vrad}(\overline{B}_q^N),$$

where $c_1, c_2 > 0$ are absolute constants

Proof. In the proof of Theorem 3.8 we observed the general inequality

$$(3.12) \qquad \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K)) \right) \leqslant L_\mu^n N^{n/2} \mathbb{E}_{\mu^N} \left(\mathrm{vol}_n(P_{E_{\mathbf{x}}}(K))^2 \right)^{\frac{1}{2}}$$

where $E_{\mathbf{x}} = \ker(T_{\mathbf{x}})^\perp = \mathrm{Range}(T_{\mathbf{x}}^*)$, which holds for any centrally symmetric convex body K in $\mathbb{R}^N$.

Note that if $2 \leqslant q \leqslant \infty$ then $R(B_q^N) \approx N^{\frac{1}{2} - \frac{1}{q}}$ and $\mathrm{vol}_n(B_q^N)^{1/N} \approx N^{-\frac{1}{q}}$. Therefore, $\overline{B}_q^N \subseteq c\sqrt{N} B_2^N$. It follows that

$$\mathrm{vol}_n(P_{E_{\mathbf{x}}}(\overline{B}_q^N))^{1/n} \leqslant c_1 \mathrm{vol}_n(P_{E_{\mathbf{x}}}(\sqrt{N} B_2^N))^{1/n} \leqslant c_2 \sqrt{N/n}$$

for all $\mathbf{x} = (x_1, \ldots, x_N)$, where $c_2 > 0$ is an absolute constant. Taking into account (3.12) we see that

$$\mathbb{E}_{\mu^n} \left(\mathrm{vol}_n(T_{\mathbf{x}}(\overline{B}_q^N))^{\frac{1}{n}} \right) \leqslant c_3 L_\mu \sqrt{N}\, \sqrt{N/n} \leqslant c_4 L_\mu \sqrt{N/n}\, \mathrm{vrad}(\overline{B}_q^N).$$

because $\mathrm{vrad}(\overline{B}_q^N) \approx \sqrt{N}$. For the lower bound we may apply Theorem 3.10, since $\overline{B}_q^N$ is 1-unconditional and isotropic. $\qquad\square$

Remark 3.12. Note that the property of B_q^N that was really used in the previous argument is that B_q^N is contained in a ball αB_2^N such that $\left(\mathrm{vol}_N(\alpha B_2^N)/\mathrm{vol}_N(B_q^N) \right)^{1/N} \leqslant C$, for a constant $C > 0$ that does not depend on N or q. In other words, we can also state the next result: Let μ be an isotropic probability measure on $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^N$. If $K \subseteq \alpha B_2^N$ and

$$\left(\mathrm{vol}_N(\alpha B_2^N)/\mathrm{vol}_n(K) \right)^{1/N} \leqslant \beta$$

then for any $N \geqslant n$ we have

$$\mathbb{E}_{\mu^n} \left(\mathrm{vol}_n(T_{\mathbf{x}}(K))^{\frac{1}{n}} \right) \leqslant c_1 \beta L_\mu \sqrt{N/n}\, \mathrm{vrad}(K),$$

where $c_1 > 0$ is an absolute constant.

4 Random ball polyhedra

In this section we prove Theorem 1.5. Our argument works in the following more general setting. We consider two centrally symmetric convex bodies K and C in $\mathbb{R}^n$; for any $N \geqslant 1$, $r_1, \ldots, r_N > 0$ and $x_1, \ldots, x_N \in K$ we consider the convex body

$$\bigcap_{i=1}^{N} (x_i + r_i C).$$

The next result provides upper and lower bounds for the expectation of $\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + r_i C) \right)$ with respect to the uniform measure $\mu_K(A) = \frac{\mathrm{vol}_n(K \cap A)}{\mathrm{vol}_n(K)}$ on K.

Theorem 4.1. *Let K, C be centrally symmetric convex bodies in $\mathbb{R}^n$ and $x_1, \ldots, x_N$ be independent random points uniformly distributed in K. Then, for any $r_1, \ldots, r_N > 0$,*

$$\binom{nN + n}{n}^{-1} \frac{\mathrm{vol}_n(K + rC) \prod_{i=1}^{N} \mathrm{vol}_n(K \cap r_i C)}{\mathrm{vol}_n(K)^N}$$

$$\leqslant \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + r_i C) \right) \right) \leqslant \frac{\mathrm{vol}_n(K + rC) \prod_{i=1}^{N} \mathrm{vol}_n(K \cap r_i C)}{\mathrm{vol}_n(K)^N},$$

where $r = \min\{r_1, \ldots, r_N\}$.

The proof is based on the next simple formula for the expectation.

Lemma 4.2. *Let K, C be centrally symmetric convex bodies in $\mathbb{R}^n$. For any $r_1, \ldots, r_N > 0$,*

$$\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + r_i C) \right) \right) = \frac{1}{\mathrm{vol}_n(K)^N} \int_{K + rC} \prod_{i=1}^{N} \mathrm{vol}_n((K - y) \cap r_i C) \, dy,$$

where $r = \min\{r_1, \ldots, r_N\}$.

Proof. Let $r_1, \ldots, r_N > 0$. We write

$$\mathrm{vol}_n(K)^N \cdot \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + r_i C) \right) \right)$$

$$= \int_K \cdots \int_K \int_{\mathbb{R}^n} \mathbf{1}_{\bigcap_{i=1}^{N} (x_i + r_i C)}(y) \, dy \, dx_N \cdots dx_1 = \int_K \cdots \int_K \int_{\mathbb{R}^n} \prod_{i=1}^{N} \mathbf{1}_{x_i + r_i C}(y) \, dy \, dx_N \cdots dx_1$$

$$= \int_{\mathbb{R}^n} \int_K \cdots \int_K \prod_{i=1}^{N} \mathbf{1}_{y + r_i C}(x_i) \, dx_N \cdots dx_1 \, dy = \int_{\mathbb{R}^n} \prod_{i=1}^{N} \left(\int_K \mathbf{1}_{y + r_i C}(x_i) \, dx_i \right) dy$$

$$= \int_{\mathbb{R}^n} \prod_{i=1}^{N} \mathrm{vol}_n(K \cap (y + r_i C)) \, dy.$$

The lemma follows from the fact that $\mathrm{vol}_n(K \cap (y + r_i C)) = \mathrm{vol}_n((K - y) \cap r_i C)$ and that $(K - y) \cap r_i C = \emptyset$ for some $1 \leqslant i \leqslant N$ if and only if $y \notin K + rC$. $\square$

Proof of Theorem 4.1. For each $i = 1, \ldots, N$ consider the function $u_i : K + rC \to [0, \infty)$ with $u_i(y) = \mathrm{vol}_n((K - y) \cap r_i C)^{1/n}$. Using the Brunn-Minkowski inequality and the convexity of K and C we easily check that u_i is an even concave function. Note that

$$\max(u_i) = u_i(0) = \mathrm{vol}_n(K \cap r_i C)^{1/n}$$

for every $i = 1, \ldots, N$, which gives immediately the upper bound: we have

$$\frac{1}{\mathrm{vol}_n(K)^N} \int_{K+rC} \prod_{i=1}^{N} \mathrm{vol}_n((K-y) \cap r_i C) \, dy = \frac{1}{\mathrm{vol}_n(K)^N} \int_{K+rC} \prod_{i=1}^{N} u_i(y)^n \, dy$$

$$\leqslant \frac{\mathrm{vol}_n(K+rC)}{\mathrm{vol}_n(K)^N} \prod_{i=1}^{N} u_i^n(0) = \frac{\mathrm{vol}_n(K+rC) \prod_{i=1}^{N} \mathrm{vol}_n(K \cap r_i C)}{\mathrm{vol}_n(K)^N}.$$

For the lower bound, let ϱ denote the radial function of $K + rC$ on S^{n-1}. Then,

$$\mathrm{vol}_n(K)^N \cdot \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \Big(\bigcap_{i=1}^{N} (x_i + rC) \Big) \right) = n\omega_n \int_{S^{n-1}} \int_0^{\varrho(\xi)} t^{n-1} \prod_{i=1}^{N} u_i^n(t\xi) \, dt \, d\sigma(\xi).$$

Since each u_i is concave, we have

$$u_i(t\xi) \geqslant (1 - t/\varrho(\xi)) u_i(0) + (t/\varrho(\xi)) u_i(\varrho(\xi)\xi) \geqslant (1 - t/\varrho(\xi)) u_i(0),$$

therefore

$$\mathrm{vol}_n(K)^N \cdot \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \Big(\bigcap_{i=1}^{N} (x_i + rC) \Big) \right)$$

$$\geqslant n\omega_n \prod_{i=1}^{N} u_i^n(0) \int_{S^{n-1}} \int_0^{\varrho(\xi)} t^{n-1} \Big(1 - \frac{t}{\varrho(\xi)} \Big)^{nN} \, dt \, d\sigma(\xi)$$

$$= n\omega_n \prod_{i=1}^{N} \mathrm{vol}_n(K \cap r_i C) \int_{S^{n-1}} \int_0^1 \varrho^n(\xi) s^{n-1} (1-s)^{nN} \, ds \, d\sigma(\xi)$$

$$= n \prod_{i=1}^{N} \mathrm{vol}_n(K \cap r_i C) \cdot \omega_n \int_{S^{n-1}} \varrho^n(\xi) \, d\sigma(\xi) \cdot \int_0^1 s^{n-1} (1-s)^{nN} \, ds$$

$$= n \, B(n, nN+1) \mathrm{vol}_n(K+rC) \prod_{i=1}^{N} \mathrm{vol}_n(K \cap r_i C)$$

$$= \binom{nN+n}{n}^{-1} \mathrm{vol}_n(K+rC) \prod_{i=1}^{N} \mathrm{vol}_n(K \cap r_i C),$$

and the result follows. $\qquad \square$

Remark 4.3. Note that in the case $N = 1$ we have $\mathrm{vol}_n(x + rC) = \mathrm{vol}_n(rC)$ for every $x \in \mathbb{R}^n$, and hence Theorem 4.1 takes the following form: If K, C are centrally symmetric convex bodies in $\mathbb{R}^n$ then, for any $r > 0$,

$$\binom{2n}{n}^{-1} \mathrm{vol}_n(K+rC)\mathrm{vol}_n(K \cap rC) \leqslant \mathrm{vol}_n(rC)\mathrm{vol}_n(K) \leqslant \mathrm{vol}_n(K+rC)\mathrm{vol}_n(K \cap rC),$$

which is a well-known inequality of Rogers and Shephard (see [1, Chapter 4]). The constant $\binom{2n}{n}$ is optimal.

Remark 4.4. An interesting question is to determine the best constants in the inequality of Theorem 4.1. The behavior of $\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\cap_{i=1}^{N} (x_i + rC) \right) \right)$ is of course different for small and large values of r. In the case $C = B_2^n$, Gorbovickis has proved in [10] that for any $n \geqslant 2$ and any $x_1, \ldots, x_N \in \mathbb{R}^n$ one has

$$\mathrm{vol}_n \Big(\bigcap_{i=1}^{N} B(x_i, r) \Big) = \mathrm{vol}_n(rB_2^n) - n\omega_n w(\mathrm{conv}(x_1, \ldots, x_N)) r^{n-1} + o(r^{n-1})$$

as $r \to \infty$. The next natural analogue of this result is not hard to check:

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Proposition 4.5. *Let K, C be centrally symmetric convex bodies in $\mathbb{R}^n$. Then,*

$$\lim_{r \to \infty} \frac{1}{\mathrm{vol}_n(rC)} \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + rC) \right) \right) = 1.$$

Proof. A classical theorem of Minkowski states that the function $\mathrm{vol}_n(K + rC)$ is a polynomial in $r \in [0, \infty)$; one has

$$\mathrm{vol}_n(K + rC) = \sum_{j=0}^{n} \binom{n}{j} V_j(K, C) \, r^j,$$

where $V_j(K, C) = V(K; n - j, C; j)$ is the j-th mixed volume of K and C (we use the notation $C; j$ for $C, \ldots, C$ j-times). One has $V_n(K, C) = \mathrm{vol}_n(C)$. From Lemma 4.2 we see that

$$\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + rC) \right) \right) = \frac{1}{\mathrm{vol}_n(K)^N} \int_{K+rC} \left(\mathrm{vol}_n(K \cap (y + rC)) \right)^N dy \leqslant \mathrm{vol}_n(K + rC).$$

It follows that

$$\limsup_{r \to \infty} \frac{1}{\mathrm{vol}_n(rC)} \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + rC) \right) \right) \leqslant \lim_{r \to \infty} \frac{1}{r^n \mathrm{vol}_n(C)} \sum_{j=0}^{n} \binom{n}{j} V_j(K, C) \, r^j = 1.$$

On the other hand, let $r_0 = \min\{t > 0 : K \subseteq tC\}$. Then, if $r > r_0$ and $y \in (r - r_0)C$ we easily check that $K \subseteq r_0 C \subseteq y + rC$. It follows that

$$\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + rC) \right) \right) = \frac{1}{\mathrm{vol}_n(K)^N} \int_{K+rC} \left(\mathrm{vol}_n(K \cap (y + rC)) \right)^N dy \geqslant \mathrm{vol}_n((r - r_0)C)$$

for all $r > r_0$, and hence

$$\liminf_{r \to \infty} \frac{1}{\mathrm{vol}_n(rC)} \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + rC) \right) \right) \geqslant \lim_{r \to \infty} \frac{(r - r_0)^n \mathrm{vol}_n(C)}{r^n \mathrm{vol}_n(C)} = 1.$$

This completes the proof. $\qquad\square$

It is also not hard to check that the dependence on r is different as $r \to 0$:

Proposition 4.6. *Let K, C be centrally symmetric convex bodies in $\mathbb{R}^n$. Then,*

$$\lim_{r \to 0^+} \frac{\mathrm{vol}_n(K)^{N-1}}{\mathrm{vol}_n(rC)^N} \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + rC) \right) \right) = 1.$$

Proof. From Lemma 4.2 we see that

$$\mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + rC) \right) \right) = \frac{1}{\mathrm{vol}_n(K)^N} \int_{K+rC} \left(\mathrm{vol}_n((K - y) \cap rC) \right)^N dy \leqslant \frac{\mathrm{vol}_n(K + rC) \mathrm{vol}_n(rC)^N}{\mathrm{vol}_n(K)^N}.$$

It follows that

$$\limsup_{r \to 0^+} \frac{\mathrm{vol}_n(K)^{N-1}}{\mathrm{vol}_n(rC)^N} \mathbb{E}_{\mu_K^N} \left(\mathrm{vol}_n \left(\bigcap_{i=1}^{N} (x_i + rC) \right) \right) \leqslant \lim_{r \to 0^+} \frac{\mathrm{vol}_n(K + rC)}{\mathrm{vol}_n(K)} = 1.$$

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On the other hand, let $t_0 = \max\{t > 0 : C \subseteq \frac{1}{t}K\}$. Then, if $0 < r < t_0$ and $y \in \left(1 - \frac{r}{t_0}\right)K$ we easily check that $y + rC \subseteq \left(1 - \frac{r}{t_0}\right)K + \frac{r}{t_0}K = K$. It follows that

$$\frac{\mathrm{vol}_n(K)^{N-1}}{\mathrm{vol}_n(rC)^N}\mathbb{E}_{\mu_K^N}\left(\mathrm{vol}_n\Big(\bigcap_{i=1}^N(x_i + rC)\Big)\right) = \frac{1}{\mathrm{vol}_n(rC)^N\mathrm{vol}_n(K)}\int_{K+rC}\left(\mathrm{vol}_n(K \cap (y + rC))\right)^N dy$$

$$\geqslant \frac{\mathrm{vol}_n\left(\left(1 - \frac{r}{t_0}K\right)\right)}{\mathrm{vol}_n(K)} = \left(1 - \frac{r}{t_0}\right)^n,$$

for all $0 < r < t_0$, and hence

$$\liminf_{r \to 0^+}\frac{\mathrm{vol}_n(K)^{N-1}}{\mathrm{vol}_n(rC)^N}\mathbb{E}_{\mu_K^N}\left(\mathrm{vol}_n\Big(\bigcap_{i=1}^N(x_i + rC)\Big)\right) \geqslant \lim_{r \to 0^+}\left(1 - \frac{r}{t_0}\right)^n = 1.$$

This completes the proof. $\qquad\square$

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