# Small ball probability estimates for log-concave measures

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**Abstract**

We establish a small ball probability inequality for isotropic log-concave probability measures: there exist absolute constants $c_1, c_2 > 0$ such that if $X$ is an isotropic log-concave random vector in $\mathbb{R}^n$ with $\psi_2$ constant bounded by $b$ and if $A$ is a non-zero $n \times n$ matrix, then for every $\varepsilon \in (0, c_1)$ and $y \in \mathbb{R}^n$,

$$\mathbb{P}\left(\|Ax - y\|_2 \leqslant \varepsilon \|A\|_{\mathrm{HS}}\right) \leqslant \varepsilon^{\left(\frac{c_2}{b}\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^2},$$

where $c_1, c_2 > 0$ are absolute constants.

## 1 Introduction

Recently, there is an increasing interest in extending results for independent random variables, which are known from probability theory, to the setting of log-concave probability measures. A Central Limit Theorem for isotropic log-concave measures was established by B. Klartag in [12] for these measures (see also [2] for an alternative proof and [13], [7] for related developments). A "large deviation inequality" for isotropic log-concave measures was proved in [27]. In all these questions the main effort is put in trying to replace the notion of independence by the "geometry" of convex bodies, since a log-concave measure should be considered as the measure-theoretic equivalent of a convex body. Most of these recent results make heavy use of tools from the asymptotic theory of finite-dimensional normed spaces.

The purpose of this paper is to add a "small ball probability" estimate in this setting. The motivation for us was a question of N. Tomczak-Jaegermann initiated by results in [16]. In this paper the authors, motivated by questions on random polytopes, proved the following "small ball probability" estimate.

**Theorem 1.1 ([16]).** *Let $A$ be a non-zero $n \times n$ matrix and let $X = (\xi_1, \dots, \xi_n)$ be a random vector, where $\xi_i$ are independent subgaussian random variables with* $\mathrm{Var}(\xi_i) \geqslant 1$ *and subgaussian constants bounded by $\beta$. Then, for any $y \in \mathbb{R}^n$, one has*

$$\mathbb{P}\left(\|AX - y\|_2 \leqslant \frac{\|A\|_{\mathrm{HS}}}{2}\right) \leqslant 2\exp\left(-\frac{c_0}{\beta^4}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^2\right),$$

*where $c_0 > 0$ is a universal constant.*

It is pointed out in [16] that, in the special case where the $\xi_i$'s are independent standard Gaussian random variables, one can obtain the following stronger result.

**Theorem 1.2 ([16]).** *Let $A$ be a non-zero $n \times n$ matrix and let $X = (\xi_1, \dots, \xi_n)$ be a random vector, where the $\xi_i$'s are independent standard Gaussian random variables. Then, for any $\varepsilon \in (0, c_1)$ and any $y \in \mathbb{R}^n$, one has*

$$\mathbb{P}\left(\|AX - y\|_2 \leqslant \varepsilon \|A\|_{\mathrm{HS}}\right) \leqslant \varepsilon^{\left(c_2 \frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^2},$$

*where $c_1, c_2 > 0$ are universal constants.*

The proof of Theorem 1.2 makes use of the affirmative answer to the $B$-conjecture by Cordero-Erausquin, Fradelizi and Maurey (see [6]). The "B-Theorem" has been already applied for small ball probability estimates in [15] and [14].

The main result of this paper extends the previously mentioned results to the setting of log-concave probability measures, answering a question posed to us by N. Tomczak-Jaegermann.

**Theorem 1.3.** *Let $X$ be an isotropic* log-*concave random vector in $\mathbb{R}^n$, which has subgaussian constant $b$. Let $A$ be a non-zero $n \times n$ matrix. For any $y \in \mathbb{R}^n$ and $\varepsilon \in (0, c_1)$, one has*

$$\mathbb{P}\left(\|Ax - y\|_2 \leqslant \varepsilon \|A\|_{\mathrm{HS}}\right) \leqslant \varepsilon^{\left(\frac{c_2}{b} \frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^2},$$

*where $c_1, c_2 > 0$ are universal constants.*

Theorem 1.3 depends on a reverse Hölder inequality for the negative moments of the Euclidean norm with respect to a log-concave probability measure $\mu$ with density $f$. Let $-n < p < \infty$, $p \neq 0$ and $I_p(f) := \left(\int_{\mathbb{R}^n} \|x\|_2^p f(x) dx\right)^{\frac{1}{p}}$. A result of O. Guédon (see [9]) implies that for $p \in (-1, 0)$ one has

$$I_p(f) \geqslant c_p I_2(f),$$

where the constant $c_p$ depends only on $p$.

Actually, Guédon's result is more general and holds even if we replace the Euclidean norm by any other norm. Moreover, the result is sharp and can be achieved for a 1-dimensional density.

In order to reveal the role of the dimension we introduce the quantity $q_*(\mu)$:

$$q_*(f) := \max\{k \geqslant 1 : k_*(Z_k(f)) \geqslant k\},$$

where $k_*(Z_k(f))$ is the Dvoretzky number of the $L_k$-centroid body of $f$ (see Section 2 for precise definitions). Then, one can show the following.

**Theorem 1.4.** *Let $f$ a* log-*concave density in $\mathbb{R}^n$ with center of mass at the origin. Then for every $k \leqslant c_1 q_*(f)$ one has*

$$I_{-k}(f) \geqslant c_2 I_2(f) \ ,$$

*where $c_1, c_2 > 0$ are absolute constants.*

The paper is organized as follows. In §2 we gather some background material needed in the rest of the paper. In the next section we study a family of convex bodies associated to a log-concave measure. This family was introduced by K. Ball in [1]. In §4 we establish a volumetric relation between any marginal of a log-concave measure and the corresponding projection of its associated generalized centroid body. Precisely, we prove an $L_q$-version of the Rogers-Shephard inequality. This is one of the main steps towards the proof of Theorem 1.4. In §5 we give an exact formula (Proposition 5.4 in the main text) relating the negative moments of the norm of the polar $L_q$ centroid body on the sphere with the negative moments of the Euclidean norm with respect to the measure. This can be seen as a transfer principle permitting the use of known concentration results on the sphere. We stress the fact that all the results up to §5 are valid for an arbitrary log-concave measure and not just merely for an isotropic one. This special class of measures is treated in §6. The proof of Theorem 1.3 is completed in §7, where we discuss the sharpness of the estimate in Theorem 1.3 and its connections with the well-known "Hyperplane Conjecture" in Convex Geometry.

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# 2 Background material

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write $B_2^n$ for the Euclidean unit ball, and $S^{n-1}$ for the unit sphere. Volume is denoted by $| \cdot |$. We write $\omega_n$ for the volume of $B_2^n$ and $\sigma$ for the rotationally invariant probability measure on $S^{n-1}$. The Grassmann manifold $G_{n,k}$ of $k$-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. We write $P_F$ for the orthogonal projection onto the subspace $F$. We also write $\widetilde{A}$ for the homothetic image of volume 1 of a compact set $A \subseteq \mathbb{R}^n$, i.e. $\widetilde{A} := \frac{A}{|A|^{1/n}}$.

A convex body is a compact convex subset $C$ of $\mathbb{R}^n$ with non-empty interior. We say that $C$ is symmetric if $-x \in C$ whenever $x \in C$. We say that $C$ has center of mass at the origin if $\int_C \langle x, \theta \rangle dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of $C$ is defined by $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$. The mean width of $C$ is defined by

$$w(C) = \int_{S^{n-1}} h_C(\theta) \sigma(d\theta).$$

The radius of $C$ is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$, and the polar body $C^\circ$ of $C$ is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\}.$$

Whenever we write $a \simeq b$, we mean that there exist universal constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. The letters $c, c', c_1, c_2 > 0$ etc. denote universal positive constants which may change from line to line. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist universal constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

Let $A = (a_{i,j})_{1\leqslant i,j\leqslant n}$ be a $n \times n$ matrix. We write $\|A\|_{\mathrm{HS}}$ for the Hilbert-Schmidt norm of $A$:

$$\|A\|_{\mathrm{HS}}^2 := \sum_{i,j} a_{i,j}^2,$$

and $\|A\|_{\mathrm{op}}$ for the operator norm of $A$:

$$\|A\|_{\mathrm{op}} := \max_{\theta\in S^{n-1}} \|A\theta\|_2.$$

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function. We say that $f$ has center of mass at the origin if

$$\int_{\mathbb{R}^n} \langle x, y\rangle f(x)dx = 0 \quad \text{for all } y \in \mathbb{R}^n. \tag{1}$$

Given $f$ and $y \in \mathbb{R}^n$ we write $f_y$ for the function $f_y(x) := f(x + y)$.

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$. For every $0 < p \leqslant \infty$ and $\theta \in S^{n-1}$ we consider the quantities

$$h_{Z_p(f)}(\theta) := \left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p f(x)dx\right)^{1/p} \tag{2}$$

If $h_{Z_p(f)}(\theta) < \infty$ for every $\theta \in S^{n-1}$, we define the $L_p$-centroid body $Z_p(f)$ of $f$ to be the centrally symmetric convex body that has support function $h_{Z_p(f)}$.

$L_p$-centroid bodies were introduced in [18] (see also [19]), where a generalization of Santaló's inequality was proved. In [18] and [19] a different normalization (and notation) was used. Here, we follow the normalization (and notation) that appeared in [25], since it fits better in a probabilistic setting. These bodies played a crucial role in [27] and [2].

Note that for $0 < p \leqslant q \leqslant \infty$ one has $Z_p(f) \subseteq Z_q(f)$. If $f := \mathbf{1}_A$ for some compact set $A \subseteq \mathbb{R}^n$, then $Z_\infty(f) = \mathrm{co}\{A, -A\}$. Note that if $T \in SL_n$ then for all $p > 0$ one has

$$Z_p(f \circ T^{-1}) = TZ_p(f). \tag{3}$$

We refer to [27] for additional information on $Z_p$-bodies.

A random variable $\xi$ is called subgaussian if there exists a constant $0 < \beta < \infty$ such that

$$\|\xi\|_{2k} \leqslant \beta\|\gamma\|_{2k} \;\; k = 1, 2, \ldots,$$

where $\gamma$ is a standard Gaussian random variable.

Let $\mu$ be a probability measure in $\mathbb{R}^n$ with density $f \geqslant 0$ and let $\alpha \geqslant 1$. We say that $\mu$ (or $f$) is $\psi_\alpha$ with constant $b_\alpha$ if for every $p \geqslant \alpha$ one has

$$Z_p(f) \subseteq b_\alpha p^{1/\alpha} Z_\alpha(f),$$

or, equivalently, if for every $\theta \in S^{n-1}$ and $t > 0$,

$$\mu\left(\left\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geqslant t\left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^\alpha f(x)dx\right)^{1/\alpha}\right\}\right) \leqslant 2\exp\left(-\frac{t}{b_a}\right)^\alpha.$$

For $-n < p \leqslant \infty$ we define the quantities $I_p(f)$ as

$$I_p(f) := \left(\int_{\mathbb{R}^n} \|x\|_2^p f(x)dx\right)^{1/p}. \tag{4}$$

We say that a function $f : \mathbb{R}^n \to [0, \infty]$ is log-concave if, for every $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \geqslant f(x)^\lambda f(y)^{1-\lambda}.$$

Note that if $f$ is log-concave and finite then, $I_p(f) < \infty$ for $-n < p < \infty$ and $\left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p f(x)dx\right)^{1/p} < \infty$ for $p > 0$.

It is well known that the level sets of a log-concave function are convex sets. Also, if $K \subseteq \mathbb{R}^n$ is a convex body, the Brunn-Minkowski inequality implies that the measure $\mu$ with $d\mu := \mathbf{1}_{\frac{K}{|K|^{1/n}}}(x)dx$ is a log-concave probability measure in $\mathbb{R}^n$.

We refer to the books [31], [23] and [28] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

# 3 Keith Ball's Bodies

K. Ball introduced a way to "pass" from a log-concave function to a convex body (see [1]). In this section we focus on the interaction between K. Ball's bodies $K_p(f)$ of some function $f$ and the $L_q$-centroid bodies $Z_q(f)$ of this function.

Let $f$ be an integrable function in $\mathbb{R}^n$ and let $p > 0$. We define a set $K_p(f)$ by

$$K_p(f) := \left\{x \in \mathbb{R}^n : p\int_0^\infty f(rx)r^{p-1}dr \geqslant f(0)\right\}. \tag{5}$$

$K_p(f)$ is a star shaped body and we can write

$$\|x\|_{K_p(f)} := \left(\frac{p}{f(0)}\int_0^\infty f(rx)r^{p-1}dr\right)^{-1/p}. \tag{6}$$

Indeed, for any $\lambda > 0$ and $x \in \mathbb{R}^n$ we have

$$\|\lambda x\|_{K_p(f)}^{-p} = \frac{p}{f(0)}\int_0^\infty f(r\lambda x)r^{p-1}dr = \frac{p}{\lambda^p f(0)}\int_0^\infty f(rx)r^{p-1}dr = \frac{1}{\lambda^p}\|x\|_{K_p(f)}^{-p}.$$

Note that if $f$ is even, then $K_p(f)$ is symmetric for all $p > 0$.

Integrating in polar coordinates we see that, for any $\theta \in S^{n-1}$,

$$\begin{aligned}
\int_{K_{n+1}(f)} \langle x, \theta \rangle dx &= n\omega_n \int_{S^{n-1}} \langle \phi, \theta \rangle \int_0^{1/\|\phi\|_{K_{n+1}(f)}} r^n dr d\sigma(\phi) \\
&= \frac{n\omega_n}{f(0)} \int_{S^{n-1}} \langle \phi, \theta \rangle \int_0^\infty r^n f(r\theta) dr d\sigma(\phi) \\
&= \frac{1}{f(0)} \int_{\mathbb{R}^n} \langle x, \theta \rangle f(x) dx.
\end{aligned}$$

So, if $f$ has center of mass at the origin then $K_{n+1}(f)$ has also center of mass at the origin.

The same argument shows that, for every $p > 0$ and $\theta \in S^{n-1}$,

$$\begin{aligned}
\int_{K_{n+p}(f)} |\langle x, \theta \rangle|^p dx &= n\omega_n \int_{S^{n-1}} |\langle \phi, \theta \rangle|^p \int_0^{1/\|\phi\|_{K_{n+p}(f)}} r^{n+p-1} dr d\sigma(\phi) \\
&= \frac{n\omega_n}{f(0)} \int_{S^{n-1}} |\langle \phi, \theta \rangle|^p \int_0^\infty r^{n+p-1} f(r\theta) dr d\sigma(\phi) \\
&= \frac{1}{f(0)} \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p f(x) dx.
\end{aligned}$$

We also have

$$\int_{K_{n+p}(f)} |\langle x, \theta \rangle|^p dx = |K_{n+p}|^{1+\frac{p}{n}} \int_{\widetilde{K_{n+p}(f)}} |\langle x, \theta \rangle|^p dx.$$

So, we conclude that

$$Z_p(\widetilde{K_{n+p}(f)}) |K_{n+p}(f)|^{\frac{1}{p}+\frac{1}{n}} f(0)^{1/p} = Z_p(f). \tag{7}$$

Let $V$ be a star-shaped body in $\mathbb{R}^n$ and let $\|x\|_V$ be the gauge function of $V$. Working in the same manner we see that for $-n < p \leqslant \infty$,

$$\begin{aligned}
\int_{K_{n+p}(f)} \|x\|_V^p dx &= n\omega_n \int_{S^{n-1}} \|\phi\|_V^p \int_0^{1/\|\phi\|_{K_{n+p}(f)}} r^{n+p-1} dr d\sigma(\phi) \\
&= \frac{n\omega_n}{f(0)} \int_{S^{n-1}} \|\phi\|_V^p \int_0^\infty r^{n+p-1} f(r\theta) dr d\sigma(\phi) \\
&= \frac{1}{f(0)} \int_{\mathbb{R}^n} \|x\|_V^p f(x) dx.
\end{aligned}$$

Setting $V = B_2^n$ we get

$$I_p(\widetilde{K_{n+p}(f)}) |K_{n+p}(f)|^{\frac{1}{p}+\frac{1}{n}} f(0)^{1/p} = I_p(f). \tag{8}$$

The family of bodies $K_p$ was introduced by K. Ball in [1], where the following theorem was proved:

**Theorem 3.1.** *If $f$ is a* log*-concave function then $K_p(f)$ is a convex set for all $p > 0$.*

We will use the following standard lemma:

**Lemma 3.2.** *Let $f : [0, \infty) \to [0, \infty)$ be a* log*-concave function. Then, for all $1 < p \leqslant q$ we have*

$$\left(\frac{p}{\|f\|_\infty} \int_0^\infty t^{p-1} f(t) dt\right)^{1/p} \leqslant \left(\frac{q}{\|f\|_\infty} \int_0^\infty t^{q-1} f(t) dt\right)^{1/q} \tag{9}$$

*and*

$$\left(\frac{q}{\Gamma(q+1)f(0)} \int_0^\infty t^{q-1} f(t) dt\right)^{1/q} \leqslant \left(\frac{p}{\Gamma(p+1)f(0)} \int_0^\infty t^{p-1} f(t) dt\right)^{1/p}. \tag{10}$$

**Comment**. The proof of both facts is well-known to specialists and can be found in [21]. The first claim can be derived from Lemma 2.1 in [21, page 76], whereas the second claim can be derived from Corollary 2.7 in [21, page 81]. Both facts are also corollaries of a result of Borell (see [5]).

If $f$ is log-concave and even, then $\|f\|_\infty = f(0)$. If $f$ is log-concave and has center of mass at the origin then the quantities $\|f\|_\infty$ and $f(0)$ are comparable. More precisely, we have the following theorem of M. Fradelizi (see [8]).

**Theorem 3.3.** *Let $f : \mathbb{R}^n \to [0, \infty]$ be a* log*-concave function with center of mass at the origin. Then,*

$$\|f\|_\infty \leqslant e^n f(0). \tag{11}$$

Using (9), (10) and (5) we see that if $f$ is log-concave then

$$\|x\|_{K_p(f)} \leqslant \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} \|x\|_{K_q(f)} \tag{12}$$

and

$$\|x\|_{K_q(f)} \leqslant \left(\frac{\|f\|_\infty}{f(0)}\right)^{\frac{1}{p}-\frac{1}{q}} \|x\|_{K_p(f)}. \tag{13}$$

Moreover, if $f$ has center of mass at the origin, then (13) becomes

$$\|x\|_{K_q(f)} \leqslant e^{\frac{n}{p}-\frac{n}{q}} \|x\|_{K_p(f)}. \tag{14}$$

So, if $f$ is log-concave and has center of mass at the origin, we get the following volumetric estimates for $1 < p \leqslant q$:

$$e^{\frac{n^2}{q}-\frac{n^2}{p}} |K_p(f)| \leqslant |K_q(f)| \leqslant \left(\frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}}\right)^n |K_p(f)|. \tag{15}$$

Once again, integrating in polar coordinates we get

$$|K_n(f)| = \frac{1}{f(0)} \int_{\mathbb{R}^n} f(x) dx. \tag{16}$$

So, if $f$ is a log-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$ then, combining (16) and (15) we get that, for $p > 0$,

$$\frac{e^{-\frac{np}{n+p}}}{f(0)} \leqslant |K_{n+p}(f)| \leqslant ((n+p)!)^{\frac{n}{n+p}} \frac{1}{n!f(0)},$$

and hence,

$$\frac{1}{e} \leqslant f(0)^{\frac{1}{n}+\frac{1}{p}} |K_{n+p}(f)|^{\frac{1}{n}+\frac{1}{p}} \leqslant \frac{((n+p)!)^{1/p}}{(n!)^{\frac{n+p}{np}}}.$$

Using the bounds

$$\frac{((n+p)!)^{1/p}}{(n!)^{\frac{n+p}{np}}} \leqslant (n+p)\frac{(n!)^{1/p}}{(n!)^{\frac{n+p}{np}}} = \frac{n+p}{(n!)^{1/n}} \leqslant e\frac{n+p}{n},$$

we conclude that

$$\frac{1}{e} \leqslant f(0)^{\frac{1}{n}+\frac{1}{p}} |K_{n+p}(f)|^{\frac{1}{n}+\frac{1}{p}} \leqslant e\frac{n+p}{n}. \tag{17}$$

Working in the same way for $-(n-1) < p \leqslant 0$, we get

$$e^{-\frac{np}{n-p}} |K_{n-p}(f)| \leqslant \frac{1}{f(0)} \leqslant \frac{n!}{((n-p)!)^{\frac{n}{n-p}}} |K_{n-p}(f)|,$$

and hence,

$$\frac{1}{e} \leqslant f(0)^{\frac{1}{n}-\frac{1}{p}} |K_{n-p}(f)|^{\frac{1}{n}-\frac{1}{p}} \leqslant \frac{((n!))^{\frac{n-p}{np}}}{((n-p)!)^{1/p}}.$$

Using the bounds

$$\frac{((n!))^{\frac{n-p}{np}}}{((n-p)!)^{1/p}} \leqslant n^{\frac{n-p}{n}} \frac{((n-p)!)^{\frac{n-p}{np}}}{((n-p)!)^{1/p}} \leqslant e\frac{n}{n-p},$$

we conclude that

$$\frac{1}{e} \leqslant f(0)^{\frac{1}{n}-\frac{1}{p}} |K_{n-p}(f)|^{\frac{1}{n}-\frac{1}{p}} \leqslant e\frac{n}{n-p}. \tag{18}$$

So, combining (17), (18) and (7), (8) we get the following:

**Proposition 3.4.** *Let $f : \mathbb{R}^n \to [0, \infty]$ be a log-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for $p \geqslant 0$ one has*

$$\frac{1}{e} Z_p(\widetilde{K_{n+p}(f)}) \subseteq f(0)^{1/n} Z_p(f) \subseteq e\frac{n+p}{n} Z_p(\widetilde{K_{n+p}(f)}). \tag{19}$$

*Moreover, for $-(n-1) < p \leqslant \infty$,*

$$\frac{1}{e} I_p(\widetilde{K_{n+p}(f)}) \leqslant f(0)^{1/n} I_p(f) \leqslant e\frac{n+p}{n} I_p(\widetilde{K_{n+p}(f)}). \tag{20}$$

*Note that if $f$ is even then the constant on the left hand side in the previous two inclusions can be chosen to be 1 instead of $\frac{1}{e}$.*

Working in the same spirit we can also compare the symmetric convex bodies $Z_q(\widetilde{K_{n+r_1}(f)})$ and $Z_q(\widetilde{K_{n+r_2}(f)})$ for $-(n-1) < r_1 \leqslant r_2 \leqslant \infty$ and $q \geqslant 1$. Indeed, for every $\theta \in S^{n-1}$ we have

$$\begin{aligned}\frac{h^q_{Z_q(\widetilde{K_{n+r_1}(f)})}(\theta)}{h^q_{Z_q(\widetilde{K_{n+r_2}(f)})}(\theta)} &= \left(\frac{|K_{n+r_2}|}{|K_{n+r_1}|}\right)^{1+\frac{q}{n}} \frac{\int_{k_{n+r_1}} |\langle x,\theta\rangle|^q dx}{\int_{k_{n+r_2}} |\langle x,\theta\rangle|^q dx} \\ &= \left(\frac{|K_{n+r_2}|}{|K_{n+r_1}|}\right)^{1+\frac{q}{n}} \frac{\frac{n\omega_n}{n+q}\int_{S^{n-1}} |\langle \phi,\theta\rangle|^q \|\phi\|^{-(n+q)}_{K_{n+r_1}} d\sigma(\phi)}{\frac{n\omega_n}{n+q}\int_{S^{n-1}} |\langle \phi,\theta\rangle|^q \|\phi\|^{-(n+q)}_{K_{n+r_2}} d\sigma(\phi)}.\end{aligned}$$

Using (13) and (14) we get

$$\frac{(\Gamma(n+r_1))^{\frac{n+q}{n+r_1}}}{(\Gamma(n+r_2))^{\frac{n+q}{n+r_2}}} \leqslant \frac{\|\phi\|^{-(n+q)}_{K_{n+r_1}(f)}}{\|\phi\|^{-(n+q)}_{K_{n+r_2}(f)}} \leqslant e^{n\frac{(r_2-r_1)(n+q)}{(n+r_1)(n+r_2)}}.$$

Also, (15) implies that

$$e^{-n^2\frac{r_2-r_1}{(n+r_1)(n+r_2)}} \leqslant \frac{|K_{n+r_2}|}{|K_{n+r_1}|} \leqslant \frac{(\Gamma(n+r_2))^{\frac{n}{n+r_2}}}{(\Gamma(n+r_1))^{\frac{n}{n+r_1}}}.$$

So,

$$\begin{aligned}e^{-\frac{n(r_2-r_1)(n+q)}{q(n+r_1)(n+r_2)}} \frac{(\Gamma(n+r_1))^{\frac{n+q}{q(n+r_1)}}}{(\Gamma(n+r_2))^{\frac{n+q}{q(n+r_2)}}} &\leqslant \frac{h_{Z_q(\widetilde{K_{n+r_1}(f)})}(\theta)}{h_{Z_q(\widetilde{K_{n+r_2}(f)})}(\theta)} \\ &\leqslant e^{\frac{n(r_2-r_1)(n+q)}{q(n+r_1)(n+r_2)}} \frac{(\Gamma(n+r_2))^{\frac{n+q}{q(n+r_2)}}}{(\Gamma(n+r_1))^{\frac{n+q}{q(n+r_1)}}}.\end{aligned}$$

For $n \in \mathbb{N}$, $q > 0$ and $-n < r_1 \leqslant r_2 \leqslant \infty$ we define

$$A_{n,q,r_1,r_2} := e^{\frac{n(r_2-r_1)(n+q)}{q(n+r_1)(n+r_2)}} \frac{(\Gamma(n+r_2))^{\frac{n+q}{q(n+r_2)}}}{(\Gamma(n+r_1))^{\frac{n+q}{q(n+r_1)}}}. \tag{21}$$

So, we have shown that if $f$ is a log-concave function in $\mathbb{R}^n$ with center of mass at the origin, then for every $q \geqslant 1$, for every $-(n-1) < r_1 \leqslant r_2 \leqslant \infty$ and for all $\theta \in S^{n-1}$, one has

$$A^{-1}_{q,r_1,r_2,n} \leqslant \frac{h_{Z_q(\widetilde{K_{n+r_1}(f)})}(\theta)}{h_{Z_q(\widetilde{K_{n+r_2}(f)})}(\theta)} \leqslant A_{q,r_1,r_2,n}, \tag{22}$$

or equivalently

$$A^{-1}_{q,r_1,r_2,n} Z_q(\widetilde{K_{n+r_2}(f)}) \subseteq Z_q(\widetilde{K_{n+r_1}(f)}) \subseteq A_{q,r_1,r_2,n} Z_q(\widetilde{K_{n+r_2}(f)}). \tag{23}$$

We are interested in the case where $r_2 = q$ and $r_1 = 1$ or $r_2 = 2$. We have that

$$\begin{aligned} A_{n,q,1,q} &= e^{\frac{n(q-1)}{q(n+1)}} \frac{(\Gamma(n+q))^{\frac{1}{q}}}{(\Gamma(n+1))^{\frac{n+q}{q(n+1)}}} \\ &= \left( \frac{e^{\frac{n(q-1)}{n+1}}(n+1)\ldots(n+q-1)}{(n!)^{\frac{q-1}{n+1}}} \right)^{\frac{1}{q}} \\ &\leqslant \left( e^{2\frac{n(q-1)}{n+1}} \frac{(n+q-1)^{q-1}}{n^{q-1}} \right)^{\frac{1}{q}} \\ &\leqslant e^2 \frac{n+q}{n}. \end{aligned}$$

A similar computation shows that $A_{n,q,2,q} \leqslant e^2 \frac{n+q}{n}$. So, we get that that for $r = 1$ or $r = 2$,

$$\frac{n}{e^2(n+q)} Z_q(\widetilde{K_{n+q}(f)}) \subseteq Z_q(\widetilde{K_{n+r}(f)}) \subseteq e^2 \frac{n+q}{n} Z_q(\widetilde{K_{n+q}(f)}). \tag{24}$$

Then, for $q \leqslant n$, using (19), (23) we get the following.

**Proposition 3.5.** *Let $f : \mathbb{R}^n \to [0, \infty)$ be a* log*-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for every $1 \leqslant q \leqslant n$, one has*

$$c_1 f(0)^{1/n} Z_q(f) \subseteq Z_q(\widetilde{K_{n+1}(f)}) \subseteq c_2 f(0)^{1/n} Z_q(f) \tag{25}$$

*and*

$$c_3 f(0)^{1/n} Z_q(f) \subseteq Z_q(\widetilde{K_{n+2}(f)}) \subseteq c_4 f(0)^{1/n} Z_q(f), \tag{26}$$

*where $c_1, c_2, c_3, c_4 > 0$ are universal constants.*

We will also use the following:

**Lemma 3.6.** *Let $K$ be a convex body in $\mathbb{R}^n$ with volume one and center of mass at the origin. Then, for every $p \geqslant n$,*

$$Z_p(K) \supseteq c_1 \operatorname{co}\{K, -K\} \tag{27}$$

*and*

$$c_1 \leqslant |Z_p(K)|^{1/n} \leqslant c_2 \tag{28}$$

*where $c_1, c_2 > 0$ are universal constants.*

*Proof.* Under our assumptions, one can prove that for every $\theta \in S^{n-1}$,

$$h_{Z_p(K)}(\theta) \geqslant \left( \frac{\Gamma(p+1)\Gamma(n)}{2e\Gamma(n+p+1)} \right)^{1/p} \max\{h_K(\theta), h_K(-\theta)\}.$$

For a proof of this well-known fact see [25]. It follows that if $p \geqslant n$ then $h_{Z_p(K)} \geqslant c_1 \max\{h_K(\theta), h_K(-\theta)\}$, which proves (3.6).

This in turn means that $|Z_p(K)|^{1/n} \geqslant c_1 |\mathrm{co}\{K, -K\}|^{1/n} \geqslant c_1 |K| \geqslant c_1$. Taking into account the fact that $Z_p(K) \subseteq \mathrm{co}\{K, -K\}$ and using an inequality due to Rogers and Shephard (see [30]) we readily see that $|\mathrm{co}\{K, -K\}| \leqslant 2^n |K|$. This proves (28). $\square$

Recall that if $f$ has center of mass at the origin then $K_{n+1}(f)$ has also its center of mass at the origin. So, combining the previous Lemma with (25) we get the following:

**Proposition 3.7.** *Let $f : \mathbb{R}^n \to [0, \infty)$ be a* log*-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then,*

$$\frac{c_1}{f(0)^{1/n}} \leqslant |Z_n(f)|^{1/n} \leqslant \frac{c_2}{f(0)^{1/n}}, \tag{29}$$

*where $c_1, c_2 > 0$ are universal constants.*

# 4 Marginals and Projections

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function. Let $1 \leqslant k < n$ be an integer and let $F \in G_{n,k}$. We define the marginal $\pi_F(f) : F \to \mathbb{R}_+$ of $f$ with respect to $F$ by

$$\pi_F(f)(x) := \int_{x+F^\perp} f(y)dy. \tag{30}$$

Note that, by Fubini's theorem,

$$\int_F \pi_F(f)(x)dx = \int_{\mathbb{R}^n} f(x)dx \tag{31}$$

and, for every $\theta \in S_F$,

$$\int_F \langle x, \theta \rangle \pi_F(f)(x)dx = \int_{\mathbb{R}^n} \langle x, \theta \rangle f(x)dx. \tag{32}$$

In particular, if $f$ has center of mass at the origin then for every $F \in G_{n,k}$, $\pi_F(f)$ has the same property.

The same argument gives that, for every $p > 0$ and $\theta \in S_F$,

$$\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p f(x)dx = \int_F |\langle x, \theta \rangle|^p \pi_F(f)(x)dx$$

and, for every $-k < p \leqslant \infty$,

$$\int_{\mathbb{R}^n} \|P_F x\|_2^p f(x)dx = \int_F \|x\|_2^p \pi_F(f)(x)dx.$$

We will use the notation

$$I_p(f, F) := \left( \int_{\mathbb{R}^n} \|P_F x\|_2^p f(x)dx \right)^{1/p}.$$

So, we have the following:

**Proposition 4.1.** *Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for $1 \leqslant k \leqslant n$, $F \in G_{n,k}$ and $p > 0$, one has*

$$P_F(Z_p(f)) = Z_p(\pi_F(f)). \tag{33}$$

*Also, for any $-k < p \leqslant \infty$,*

$$I_p(f, F) = I_p(\pi_F(f)). \tag{34}$$

Let $f$ be a log-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for every $F \in G_{n,k}$, the same holds true for $\pi_F(f)$. So, we may apply Proposition 2.7 to get

$$\frac{c_1}{\pi_F(f)(0)^{1/k}} \leqslant |Z_k(\pi_F(f))|^{1/k} \leqslant \frac{c_2}{\pi_F(f)(0)^{1/k}}.$$

This last fact, combined with (33), proves the following.

**Proposition 4.2.** *Let $f$ be a* log*-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for any $1 \leqslant k < n$ and $F \in G_{n,k}$, one has*

$$c_1 \leqslant \pi_F(f)(0)^{1/k} |P_F Z_p(f)|^{1/k} \leqslant c_2, \tag{35}$$

*where $c_1, c_2 > 0$ are universal constants.*

Consider the special case where $K$ is a convex body of volume 1 and has center of mass at the origin and $f := \mathbf{1}_K$. Observe that $\pi_F(f)(0) = |K \cap F^{\perp}|$. Then, the previous proposition can be viewed as an "$L_q$-version" of the following inequality due to Rogers-Shephard [29] (see [32] or [22] for the lower bound).

**Theorem 4.3.** *Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$. Let $1 \leqslant k \leqslant n$ and let $F \in G_{n,k}$. Then,*

$$|P_F(K)||K \cap F^{\perp}| \leqslant \binom{n}{k}.$$

*If $K$ has center of mass at the origin, then*

$$1 \leqslant |P_F(K)||K \cap F^{\perp}|.$$

The term "Rogers-Shephard inequality" is usually used for the upper bound. A more general inequality can be easily obtained by the following formula for mixed volumes, which is due to Fedotov (see [4] or [31]): Let $F \in G_{n,k}$, let $K_1, \dots K_k$ be convex bodies in $\mathbb{R}^n$ and let $L_i, \dots, L_{n-k}$ be compact convex subsets of $F^{\perp}$. Then,

$$\binom{n}{k} V(K_1, \dots K_k, L_1, \dots L_{n-k}) = V(P_F K_1, \dots, P_F K_k) V(L_1, \dots, L_{n-k}). \tag{36}$$

In the special case where $L_1 = L_{n-k} = K \cap F^{\perp}$, (36) implies that

$$\binom{n}{k} V(K_1, \dots K_k, K \cap F^{\perp} \dots K \cap F^{\perp}) = V(P_F K_1, \dots, P_F K_k) |K \cap F^{\perp}|.$$

The Rogers-Shephard inequality follows if we take $K_1 = K_k = K$ and use the monotonicity property of mixed volumes.

Note that one can rewrite the inequality in the following form:

$$1 \leqslant \left(|P_F(K)||K \cap F^{\perp}|\right)^{1/k} \leqslant e\frac{n}{k}.$$

In the special case where $K$ is an ellipsoid of volume 1 one actually has

$$c_1\sqrt{\frac{n}{k}} \leqslant \left(|P_F(K)||K \cap F^{\perp}|\right)^{1/k} \leqslant c_2\sqrt{\frac{n}{k}},$$

where $c_1, c_2 > 0$ are universal constants.

The following direct consequence of Proposition 4.2 can be viewed as an "$L_q$ version of the Rogers-Shephard inequality":

**Theorem 4.4.** *Let $K$ be a convex body in $\mathbb{R}^n$ with center of mass at the origin and volume 1. Then, for every $F \in G_{n,k}$ one has*

$$c_1 \leqslant |K \cap F^{\perp}|^{1/k}|P_F(Z_k(K)|^{1/k} \leqslant c_2, \tag{37}$$

*where $c_1, c_2 > 0$ are universal constants.*

The inequality of Theorem 4.4 is sharp up to a universal constant. A disadvantage is that the constants are not optimal (in contrast, the equality cases in the classical Rogers-Shephard inequality are known).

The $L_q$-version of the Rogers-Shephard inequality played an important role in [27]. In that paper, our approach was based on the bodies $B_p(K,F)$ which had appeared already in the classical paper of Milman and Pajor [21]. Our approach in the present paper is a little more general. We will recall the definition in order to provide a unified setting for our results.

Let us first recall the definition of isotropicity for convex bodies: Let $K$ be a convex body in $\mathbb{R}^n$ with center of mass at the origin and volume 1. We define the isotropic constant of $K$ as follows:

$$L_K := \left(\frac{|Z_2(K)|}{|D_n|}\right)^{1/n}.$$

We will say that $K$ is isotropic if $Z_2(K) = L_K D_n$.

Next, let $K$ be a convex body of volume 1 in $\mathbb{R}^n$, let $1 \leqslant k < n$, $F \in G_{n,k}$ and $p > 0$. We define a convex body $B_p(K,F)$ in $F$ by

$$B_p(K,F) := K_{p+1}(\pi_F\left((\mathbf{1}_K)\right).$$

Then, we have the following:

**Theorem 4.5.** *Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leqslant k < n$, $F \in G_{n,k}$ and $p > 0$. Then,*

(i) *If $K$ has center of mass at the origin, then $B_k(K,F)$ has also center of mass at the origin.*

(ii) *If $K$ is symmetric, then $B_p(K,F)$ is also symmetric. Moreover, if $K$ is symmetric and isotropic, then $\widetilde{B_{k+1}}(K,F)$ is also isotropic.*

(iii) *If $K$ has center of mass at the origin then, for any $q \leqslant k$ we have*

$$Z_q(\widetilde{B_{k+1}}(K,F)) \simeq |K \cap F^{\perp}|^{\frac{1}{k}} P_F(Z_q(K)) \simeq Z_q(\widetilde{B_k}(K,F)).$$

(iv) *If $K$ is isotropic, then*

$$|K \cap F^{\perp}| \simeq \frac{L_{\widetilde{B_{k+1}}(K,F)}}{L_K}.$$

*Moreover, if $K$ has center of mass at the origin, then $L_{\widetilde{B_{k+1}}(K,F)} \simeq L_{\widetilde{B_k}(K,F)}$.*

*Proof.* (i) Recall that if $\mathbf{1}_K$ has center of mass at the origin, then $\pi_F(\mathbf{1}_K)$ has center of mass at the origin. This implies that $K_{k+1}(\pi_F(\mathbf{1}_K))$ has center of mass at the origin.

(ii) Since $\mathbf{1}_K$ is an even function, the same is true for $\pi_F(\mathbf{1}_K)$. This implies that $K_p(\pi_F(\mathbf{1}_K))$ is symmetric. We also have that if $K$ is isotropic then, for every $F \in G_{n,k}$, $L_K B_F = P_F Z_2(\mathbf{1}_K) = Z_2(\pi_F(\mathbf{1}_K))$, where we have also used (33). Moreover, (7) implies that if $Z_2(\pi_F(\mathbf{1}_K))$ is homothetic to $B_2^n$ then the same holds true for $Z_2(K_{k+2}(\pi_F(\mathbf{1}_K))$. So, if $K$ is also symmetric, then $\widetilde{B_{k+1}}(K,F)$ is isotropic.

(iii) Note that $|K \cap F^{\perp}| = \pi_F(\mathbf{1}_K)(0)$. Using (25) and (33) we get

$$Z_q\left(\widetilde{(K_{k+1}}(\pi_F\left((\mathbf{1}_K)\right)))\right) \simeq \pi_F(\mathbf{1}_K)(0)^{1/k} Z_q(\pi_F(\mathbf{1}_K)) \simeq |K \cap F^{\perp}|^{1/k} P_F(Z_q(\mathbf{1}_K)).$$

We work similarly for $\widetilde{K_{k+2}}(\pi_F\left(\mathbf{1}_K\right))$, this time using (26) instead of (25).

(iv) It follows immediately from (ii) and (iii). $\square$

Statement (iv) in the previous theorem can be found explicitly in [21]. Note that the body $B_p(K,F)$ that we have defined here is homothetic to the one defined in [21] or [27]. On the other hand, the assertions of the previous theorem are independent of homothety.

Let $f : \mathbb{R}^n \to R_+$ and $y \in \mathbb{R}^n$. Then, for every $F \in G_{n,k}$,

$$\begin{aligned} \pi_F(f_y)(x) &= \int_{x+F^{\perp}} f_y(z)dz = \int_{x+F^{\perp}} f(z + P_F y + P_{F^{\perp}} y)dz \\ &= \int_{x+F^{\perp}} f(z + P_F y)dz = \int_{x+P_F y+F^{\perp}} f(z)dz = \pi_F(f)(x + P_F y). \end{aligned}$$

In particular, if $f : \mathbb{R}^n \to R_+$ is a log-concave function with center of mass at the origin and if $y \in \mathbb{R}^n$, using Theorem 2.3 we get

$$\pi_F(f_y)(0) \leqslant \|\pi_F(f)\|_{\infty} \leqslant e^k \pi_F(f)(0). \tag{38}$$

Also, for any integrable $f$ and $1 \leqslant m < n$ we have that

$$\begin{aligned}
\int_{G_{n,n-m}} \pi_F(f)(0) d\mu_f &= \int_{G_{n,m}} \pi_{F^\perp}(f)(0) d\mu(F) = \int_{G_{n,m}} \int_F f(z) dz d\mu(F) \\
&= \int_{G_{n,m}} m\omega_m \int_{S_F} \int_0^\infty r^{k-1} f(r\theta) dr d\sigma_F(\theta) d\mu(F) \\
&= \frac{m\omega_m}{n\omega_n} n\omega_n \int_{S^{n-1}} \int_0^\infty r^{k-1} f(r\theta) dr d\sigma(\theta) \\
&= \frac{m\omega_m}{n\omega_n} \int_{\mathbb{R}^n} \frac{f(x)}{\|x\|_2^{n-m}} dx = \frac{m\omega_m}{n\omega_n} I_{-(n-m)}^{-(n-m)}(f).
\end{aligned}$$

Equivalently, we may write that, for every integer $k < n$,

$$I_{-k}(f) = c_{n,k} \left( \int_{G_{n,k}} \pi_F(f)(0) d\mu(F) \right)^{-1/k}, \tag{39}$$

where $c_{n,k} = \left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \simeq \sqrt{n}$.

Let $f : \mathbb{R}^n \to R_+$ be a log-concave function with center of mass at the origin and let $y \in \mathbb{R}^n$. Using (38) we get that, for every integer $k < n$,

$$\begin{aligned}
I_{-k}(f_y) &= c_{n,k} \left( \int_{G_{n,k}} \pi_F(f_y)(0) d\mu(F) \right)^{-1/k} \geqslant \frac{c_{n,k}}{e} \left( \int_{G_{n,k}} \pi_F(f)(0) d\mu(F) \right)^{-1/k} \\
&= \frac{1}{e} I_{-k}(f).
\end{aligned}$$

In summary:

**Proposition 4.6.** *Let $f$ be an integrable function on $\mathbb{R}^n$ and let $k < n$ be a positive integer. Then,*

$$I_{-k}(f) = c_{n,k} \left( \int_{G_{n,k}} \pi_F(f)(0) d\mu(F) \right)^{-1/k}, \tag{40}$$

*where $c_{n,k} = \left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \simeq \sqrt{n}$. Moreover, if $f$ is also* log-concave *and has center of mass at the origin then, for every $y \in \mathbb{R}^n$,*

$$I_{-k}(f_y) \geqslant \frac{1}{e} I_{-k}(f). \tag{41}$$

The following argument is a variation of an argument of Milman and Pajor (see [21]). Let $K$ be a subset of $\mathbb{R}^n$ of volume 1. Let $V$ be a star shaped body and write

$\|x\|_V$ for the gauge function of $V$. Then, for every $-n \leqslant p \leqslant \infty$, $p \neq 0$, one has

$$\begin{aligned}
\left(\int_K \|x\|_V^p dx\right)^{1/p} &= \left(\int_{K\cap\tilde{V}} \|x\|_V^p dx + \int_{K\setminus\tilde{V}} \|x\|_V^p dx\right)^{1/p} \\
&\geqslant \left(\int_{K\cap\tilde{V}} \|x\|_V^p dx + \int_{\tilde{V}\setminus K} \|x\|_V^p dx\right)^{1/p} \\
&= \left(\int_{\tilde{V}} \|x\|_V^p dx\right)^{1/p} \\
&= \left(\frac{n}{n+p}\right)^{\frac{1}{p}} |V|^{-\frac{1}{n}}.
\end{aligned}$$

If we choose $V = B_2^n$ we get:

**Proposition 4.7.** *Let $K$ be a compact set of volume 1 in $\mathbb{R}^n$. Then, if $-(n-1) \leqslant p \leqslant \infty$, $p \neq 0$,*

$$I_p(K) \geqslant I_p(\widetilde{B_2^n}) \simeq \sqrt{n}. \tag{42}$$

Note that for $-(n-1) \leqslant p \leqslant \infty$, $p \neq 0$, we have $I_p(\widetilde{B_2^n}) \simeq \sqrt{n}$.

Let $f : \mathbb{R}^n \to R_+$ be a log-concave function with $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, using (20) and (42) we see that

$$f(0)^{1/n} I_p(f) \geqslant \frac{1}{e} I_p(\widetilde{K_{n+p}(f)}) \geqslant \frac{1}{e} I_p(\widetilde{B_2^n}) \geqslant c\sqrt{n},$$

where $c > 0$ is a universal constant. So, we have proved the following proposition

**Proposition 4.8.** *Let $f : \mathbb{R}^n \to R_+$ be a* log*-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. For every $y \in \mathbb{R}^n$,*

$$I_{-k}(f_y) \geqslant \frac{c\sqrt{n}}{f(0)^{1/n}}, \tag{43}$$

*where $c > 0$ is a universal constant.*

# 5 Constant behavior of moments

Let $C$ be a symmetric convex body in $\mathbb{R}^n$ and let $-\infty \leqslant p \leqslant \infty$, $p \neq 0$. We define

$$W_p(C) := \left(\int_{S^{n-1}} h_C(\theta) d\sigma(\theta)\right)^{1/p}. \tag{44}$$

Also, we denote by $k_*(C)$ the "Dvoretzky number" of $C$: roughly speaking, this is the maximum dimension such that a random projection of $C$ is 4-Euclidean, i.e.

$$\frac{1}{2} W(C) B_F \subseteq P_F C \subseteq 2W(C) B_F.$$

A remarkable formula due to V. D. Milman (see [20]) states that the Dvoretzky number of $C$ is determined from "global" parameters of $C$ (see also [24]):

$$k_*(C) \simeq n \left(\frac{W(C)}{R(C)}\right)^2. \tag{45}$$

The following theorem was proved in [17]:

**Theorem 5.1.** *Let $C$ be a symmetric convex body in $\mathbb{R}^n$. Then,*
(i) *If $1 \leqslant q \leqslant k_*(C)$ then $W(C) \leqslant W_q(C) \leqslant c_1 W(C)$.*
(ii) *If $k_*(C) \leqslant q \leqslant n$ then $c_2\sqrt{q/n}\, R(C) \leqslant W_q(C) \leqslant c_3\sqrt{q/n}\, R(C)$.*
(iii) *If $k_*(C) \geqslant n$ then $c_2\, R(C) \leqslant W_q(C) \leqslant R(C)$.*
*In the statements above, $c_1, c_2 > 0$ are universal constants.*

In particular, we see that we have almost constant behavior of the moments $w_q(C)$ until $q$ becomes of the order of $k_*(C)$. The same phenomenon occurs also for negative moments: we have the following theorem (see [15] and [14]):

**Theorem 5.2.** *Let $C$ be a symmetric convex body. Then, for $p \leqslant c_1 k_*(C)$,*

$$W_{-p}(C) \geqslant c_2 W(C),$$

*where $c_1, c_2 > 0$ are universal constants.*

Combining Theorems 5.1 and 5.2, and adjusting the constants, we get:

**Proposition 5.3.** *Let $C$ be a symmetric convex body in $\mathbb{R}^n$. Then, $W_p(C) \simeq W_{-p}(C)$ if and only if $p \leqslant k \simeq k_*(C)$.*

**Remark**. To be more precise, Theorem 5.1 implies that if for some $\delta \geqslant 1$ one has that $W_{-p}(C) \geqslant \frac{1}{\delta} W_p(C)$ then $p \leqslant c\delta^2 k_*(C)$, where $c > 0$ is an universal constant.

The Santaló inequality asserts that, for every symmetric convex body $K$ in $\mathbb{R}^n$,

$$|K||K^\circ| \leqslant \omega_n^2.$$

The reverse Santaló inequality proved by Bourgain and Milman (see [3]) asserts that

$$|K||K^\circ| \geqslant c^n \omega_n^2,$$

where $c > 0$ is a universal constant. Combining the two results we may write

$$c \leqslant \left(\frac{|K||K^\circ|}{|B_2^n|^2}\right)^{1/n} \leqslant 1, \tag{46}$$

where $c > 0$ is a universal constant.

Using (46) we can express negative moments of the support function of a convex body as an average of volumes of projections. Indeed, for $1 \leqslant k \leqslant n$ and any symmetric convex body $C$ in $\mathbb{R}^n$,

$$\begin{aligned} W_{-k}^{-1}(C) &= \left( \int_{S^{n-1}} \frac{1}{h_C^k(\theta)} d\sigma(\theta) \right)^{1/k} \\ &= \left( \frac{1}{\omega_k} \int_{G_{n,k}} \omega_k \int_{S_F} \frac{1}{\|\theta\|_{(P_F C)^\circ}^k} d\sigma(\theta) d\mu(F) \right)^{1/k} \\ &= \left( \int_{G_{n,k}} \frac{|(P_F(C))^\circ|}{|B_2^k|} d\mu(F) \right)^{1/k} \\ &\simeq \left( \int_{G_{n,k}} \frac{|B_2^k|}{|P_F(C)|} d\mu(F) \right)^{1/k}, \end{aligned}$$

and hence,

$$W_{-k}(C) \simeq \sqrt{k} \left( \int_{G_{n,k}} |P_F C|^{-1} d\mu(F) \right)^{-\frac{1}{k}}. \tag{47}$$

Now, let $f : \mathbb{R}^n \to R_+$ be a log-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Consider an integer $k < n$ and let $F \in G_{n,k}$. Recall that (from 35)

$$\frac{1}{|P_F Z_k(f)|^{1/k}} \simeq \pi_F(f)(0)^{1/k}.$$

Then, (47) and (40) imply the following:

**Proposition 5.4.** *Let $f : \mathbb{R}^n \to R_+$ be a* log-*concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. For every integer $k < n$,*

$$W_{-k}(Z_k(f)) \simeq \sqrt{k} \left( \int_{G_{n,k}} \pi_F(f)(0) d\mu(F) \right)^{-\frac{1}{k}} \tag{48}$$

*and*

$$I_{-k}(f) \simeq \sqrt{\frac{n}{k}} W_{-k}(Z_k(f)). \tag{49}$$

We will use the following simple fact (see e.g. [25]): For any $x \in \mathbb{R}^n$ and any $p \geqslant 1$ one has

$$\left( \int_{S^{n-1}} |\langle x, \theta \rangle|^p d\sigma(\theta) \right)^{1/p} \simeq \sqrt{\frac{p}{n+p}} \|x\|_2. \tag{50}$$

So, if $f$ is an integrable function in $\mathbb{R}^n$, by Fubini's theorem we have that for every $p \geqslant 1$,

$$
\begin{aligned}
W_p(Z_p(f)) &= \left( \int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p f(x) dx d\sigma(\theta) \right)^{1/p} \\
&= \left( \int_{\mathbb{R}^n} \int_{S^{n-1}} |\langle x, \theta \rangle|^p d\sigma(\theta) f(x) dx \right)^{1/p} \\
&\simeq \sqrt{\frac{p}{n+p}} \left( \int_{\mathbb{R}^n} \|x\|_2^p f(x) dx \right)^{1/p} \\
&\simeq \sqrt{\frac{p}{n+p}} I_p(f).
\end{aligned}
$$

This proves the following.

**Proposition 5.5.** *Let $f$ be an integrable function on $\mathbb{R}^n$ and let $p \geqslant 1$. Then,*

$$W_p(Z_p(f)) \simeq \sqrt{\frac{p}{n+p}} I_p(f). \tag{51}$$

The formulae (51) and (49) lead us to the following definition (in the case of convex bodies it first appeared in [26]): Let $f$ be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$ and $\delta > 0$. We define

$$q_*(f) := \max\{k \leqslant n : k_*(Z_k(f)) \geqslant k\}, \tag{52}$$

and

$$q_*(f, \delta) := \max\{k \leqslant n : k_*(Z_k(f)) \geqslant \frac{k}{\delta^2}\}.$$

Combining (49) and (51) with Proposition 5.3 we get:

**Theorem 5.6.** *Let $f : \mathbb{R}^n \to R_+$ be a* log-*concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. For any integer $k < n$ we have $I_{-k}(f) \simeq I_k(f)$ if and only if $k \leqslant q \simeq q_*(f)$.*

In particular, from the previous theorem we see that for all $k \leqslant q_*(f)$ one has $I_k(f) \leqslant CI_2(f)$, where $C > 0$ is a universal constant. This was the main result of [27]. Moreover note that Theorem 5.6 implies Theorem 1.3.

**Remark**. To be more precise, if for some $\delta \geqslant 1$ and some integer $k$ one has that $I_{-k}(f) \geqslant \frac{1}{\delta} I_k(f)$, then $k \leqslant q_*(f, c\delta)$, where $c > 0$ is an universal constant.
The following bound for the quantity $q_*(f)$ was proved in [27]:

**Proposition 5.7.** *Let $f$ be an integrable function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} f(x)dx = 1$. Assume that $f$ is $\psi_\alpha$ with constant $b_\alpha$ for some $\alpha \geqslant 1$. Then,*

$$q_*(f) \geqslant \frac{c}{b_\alpha^\alpha} \left(k_*(Z_2(f))\right)^{\frac{\alpha}{2}}, \tag{53}$$

*where $c > 0$ is a universal constant.*

It is well known that there exists a universal constant $C > 0$ such that every log-concave function $f$ is $\psi_1$ with constant $C$. Note that (33) implies that if $f$ is a $\psi_\alpha$ function with constant $b_\alpha$ for some $\alpha \geqslant 1$, then the same is true for $\pi_F(f)$, for every $F \in G_{n,k}$.

We conclude this section with the following fact.

**Proposition 5.8.** *Let $f$ be an integrable function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} f(x)dx = 1$. Assume that $f$ is $\psi_\alpha$ with constant $b_\alpha$ for some $\alpha \geqslant 1$. Then, for every $F \in G_{n,k}$,*

$$q_*(\pi_F(f)) \geqslant \frac{c}{b_\alpha^\alpha} \left(k_*(Z_2(\pi_F(f))\right)^{\frac{\alpha}{2}}, \tag{54}$$

*where $c > 0$ is a universal constant.*

# 6 Isotropicity

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$. We say that $f$ is isotropic if $f$ has center of mass at the origin and $Z_2(f) = B_2^n$. Equivalently if, for every $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^2 f(x)dx = 1. \tag{55}$$

Note that if $f$ is isotropic then $I_2(f) = \sqrt{n}$.

It is known that given any $f$ one can find $T \in SL_n$ such that $f \circ T^{-1}$ is isotropic. Also, the isotropic condition (55) is known to be equivalent with the following:

$$\int_{\mathbb{R}^n} \langle x, Ax\rangle f(x)dx = \mathrm{tr}(A) \tag{56}$$

for every $n \times n$ matrix $A$. In particular, one has that, if $f$ is isotropic then

$$\int_{\mathbb{R}^n} \|Ax\|_2^2 f(x)dx = \|A\|_{\mathrm{HS}}^2. \tag{57}$$

Let $f$ be isotropic and let $T \in SL_n$. Then,

$$Z_2(f \circ T^{-1}) = T(Z_2(f)) = T(B_2^n).$$

Note that $W(T(B_2^n)) = \frac{\|T\|_{\mathrm{HS}}}{\sqrt{n}}$ and $R(T(B_2^n)) = \|T\|_{\mathrm{op}}$. So, using (45), we have that

$$k_*(Z_2(f \circ T^{-1}) \simeq \left(\frac{\|T\|_{\mathrm{HS}}}{\|T\|_{\mathrm{op}}}\right)^2. \tag{58}$$

Also, if $F \in G_{n,k}$ then $Z_2(\pi_F(f \circ T^{-1})) = P_F(Z_2(f \circ T^{-1})) = P_F(T(B_2^n))$. Therefore,

$$k_\star(Z_2(\pi_F(f \circ T^{-1})) \simeq \left(\frac{\|P_F T\|_{\mathrm{HS}}}{\|P_F T\|_{\mathrm{op}}}\right)^2. \tag{59}$$

A major open question in Convex Geometry is the Hyperplane Conjecture: Let $K$ be a convex body of volume 1, with center of mass at the origin. Then, there exists $\theta \in S^{n-1}$ such that

$$|K \cap \theta^{\perp}| \geqslant c,$$

where $c > 0$ is an universal constant.

An equivalent formulation of the problem is the following: There exists a universal constant $C > 0$ such that $L_K \leqslant C$ for every convex body $K$ with center of mass at the origin.

It is well known (it also follows from Proposition 3.5) that the previous statement is equivalent to the following:

**Hyperplane Conjecture:** *There exists a universal constant $C > 0$ such that, for every isotropic* log-*concave function $f$ on $\mathbb{R}^n$,*

$$f(0)^{1/n} \leqslant C. \tag{60}$$

The best known bound is due to B. Klartag: $f(0)^{1/n} \leqslant Cn^{1/4}$ (see [11]). For more informations on isotropicity and the Hyperplane Conjecture we refer to [21] or [10].

# 7 Small ball probability

Let $\alpha \geqslant 1$ and let $f$ be an isotropic log-concave function on $\mathbb{R}^n$, which is $\psi_\alpha$ with constant $b_\alpha$.

Let $T \in SL_n$ and $y \in \mathbb{R}^n$. We set

$$m := \frac{c}{b_\alpha^\alpha}\left(\frac{\|T\|_{\mathrm{HS}}}{\|T\|_{\mathrm{op}}}\right)^{\alpha}.$$

We have chosen $c > 0$ such that $m \in \mathbb{N}$ and (see (58) and (54) ),

$$m \leqslant \frac{C}{b_\alpha^\alpha}\left(\frac{\|T\|_{\mathrm{HS}}}{\|T\|_{\mathrm{op}}}\right)^{\alpha} \leqslant \frac{C_1}{b_\alpha^\alpha}\left(k_*(Z_2(f \circ T^{-1}))\right)^{\frac{\alpha}{2}} \leqslant q_*(f \circ T^{-1}).$$

Note that $m < n$. Then, Theorem 5.6 implies that

$$I_{-m}(f \circ T^{-1}) \geqslant cI_2(f \circ T^{-1}).$$

Using (41), (57) we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \|Tx - y\|_2^{-m} f(x)dx &\leqslant e^m \int_{\mathbb{R}^n} \|x\|_2^{-m} f(T^{-1}x)dx \\
&= I_{-m}^{-m}(f \circ T^{-1}) \\
&\leqslant c_1^m I_2^{-m}(f \circ T^{-1}) \\
&= c_1^m \left(\int_{\mathbb{R}^n} \|Tx\|_2^2 f(x)dx\right)^{-\frac{m}{2}} \\
&= (c_1 \|T\|_{\mathrm{HS}})^{-m}.
\end{aligned}$$

Then, from Markov's inequality we get that for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\|Tx-y\|_2 \leqslant \varepsilon c_1 \|T\|_{\mathrm{HS}}\right) \leqslant \varepsilon^m = \varepsilon^{\frac{c}{b_\alpha^\alpha}\left(\frac{\|T\|_{\mathrm{HS}}}{\|T\|_{\mathrm{op}}}\right)^\alpha}. \tag{61}$$

Given $S \in GL_n$, let $T := |\det S|^{-1/n} S$; then, $T \in SL_n$. Observe that (61) holds for every $y \in \mathbb{R}^n$ and is homogeneous in $T$. So we have the following:

**Proposition 7.1.** *Let $\alpha \geqslant 1$ and let $f$ be an isotropic* log-*concave function on $\mathbb{R}^n$, which is $\psi_\alpha$ with constant $b_\alpha$. Let $S \in GL_n$ and $y \in \mathbb{R}^n$. Then,*

$$\mathbb{P}\left(\|Sx-y\|_2 \leqslant \varepsilon c_1 \|S\|_{\mathrm{HS}}\right) \leqslant \varepsilon^{\frac{c}{b_\alpha^\alpha}\left(\frac{\|S\|_{\mathrm{HS}}}{\|S\|_{\mathrm{op}}}\right)^\alpha}. \tag{62}$$

Now, let $A$ be a non-zero $n \times n$ matrix. Let $1 \leqslant k := rmrank(A) < n$. There exist $F \in G_{n,k}$ ($F := \mathrm{Im}(A)$) and $B_1 \in GL_n$ such that $A = (\det B_1) P_F(B)$, where $B = (\det B_1)^{-1} B_1 \in SL_n$.

Let $m := \frac{c}{b_\alpha^\alpha}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^\alpha$. We have chosen $c>0$ so that $m \in \mathbb{N}$ and (see (59) and (54) ),

$$m \leqslant \frac{C}{b_\alpha^\alpha}\left(\frac{\|P_F B\|_{\mathrm{HS}}}{\|P_F B\|_{\mathrm{op}}}\right)^\alpha \leqslant \frac{C_1}{b_\alpha^\alpha}\left(k_*(Z_2(\pi_F(f \circ B^{-1})))\right)^{\frac{\alpha}{2}} \leqslant q_*(\pi_F(f \circ B^{-1})).$$

Note that $m < k$. Then, Theorem 5.6 implies that

$$I_{-m}(\pi_F(f \circ B^{-1})) \geqslant c I_2(\pi_F(f \circ B^{-1})).$$

Then, for every $y \in \mathbb{R}^n$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \|Ax-y\|_2^{-m} f(x)dx &\leqslant e^m \int_{\mathbb{R}^n} \|Ax\|_2^{-m} f(x)dx \\
&= \left(\frac{e}{\det B_1}\right)^m \int_{\mathbb{R}^n} \|P_F x\|_2^{-m} f(B^{-1}x)dx \\
&= \left(\frac{e}{\det B_1}\right)^m I_{-m}^{-m}(f \circ B^{-1}, F) \\
&= \left(\frac{e}{\det B_1}\right)^m I_{-m}^{-m}(\pi_F(f \circ B^{-1})) \\
&\leqslant \left(\frac{c_1}{\det B_1}\right)^m \left(I_2((\pi_F(f \circ B^{-1}))\right)^{-m} \\
&= \left(\frac{c_1}{\det B_1}\right)^m \left(I_2(F \circ B^{-1}, F)\right)^{-m} \\
&= \left(\frac{c_1}{\det B_1}\right)^m \left(\int_{\mathbb{R}^n} \|P_F Bx\|_2^2 f(x)dx\right)^{-\frac{m}{2}} \\
&= c_1^m \left(\int_{\mathbb{R}^n} \|Ax\|_2^2 f(x)dx\right)^{-\frac{m}{2}} \\
&= c_1^m \|A\|_{\mathrm{HS}}^{-m},
\end{aligned}$$

where we have also used (34) and (57).

So, from Markov's inequality again, we get that for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\|Ax-y\|_2 \leqslant \varepsilon c_1 \|A\|_{\mathrm{HS}}\right) \leqslant \varepsilon^m = \varepsilon^{\frac{c}{b_\alpha^\alpha}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^\alpha}. \tag{63}$$

In summary:

**Theorem 7.2.** *Let $X$ be an isotropic* log*-concave random vector in $\mathbb{R}^n$ which is $\psi_\alpha$ with constant $b_\alpha$ for some $\alpha \geqslant 1$. Let $A$ be a non-zero $n \times n$ matrix, let $y \in \mathbb{R}^n$ and $\varepsilon \in (0, c_1)$. Then, one has*

$$\mathbb{P}\left(\|Ax-y\|_2 \leqslant \varepsilon \|A\|_{\mathrm{HS}}\right) \leqslant \varepsilon^{\frac{c_2}{b_\alpha^\alpha}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^\alpha}, \tag{64}$$

*where $c_1, c_2 > 0$ are absolute constants.*

In the special case where $a = 2$, the previous theorem implies Theorem 1.3:

**Remark.** Note that the dependence in Theorem 1.3 is better that the one in Theorem 1.1, although it is not clear if it is the right one.

Let $A$ be a projection matrix and let $F := \mathrm{Im}(A)$ and $k = \mathrm{rank}(A) = \dim(F)$. Note that $\|A\|_{\mathrm{HS}} = \sqrt{k}$ and $\|A\|_{\mathrm{op}} = 1$. Assume that the Hyperplane Conjecture is true. Then, by Proposition 4.7 we have that, for every $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \|Ax-y\|_2^{-(k-1)} f(x)dx \leqslant \left(c\frac{\pi_F(f)(0)^{1/k}}{\sqrt{k}}\right)^{k-1} \leqslant \left(\frac{c_1}{\sqrt{k}}\right)^{k-1}.$$

So, from Markov's inequality, we get that for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\|Ax-y\|_2 \leqslant \varepsilon C \|A\|_{\mathrm{HS}}\right) \leqslant \varepsilon^{k-1}.$$

This means that in this case we have no dependence on the $\psi_\alpha$ constant! In fact, the Hyperplane Conjecture is closely related to the question of the dependence in the $\psi_\alpha$ constant in Theorem 7.2. To fully reveal this connection we need different tools; we will present this connection elsewhere.

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