Berry–Esseen Bounds for random tensors

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Joint work with P. Dodos

Central Limit Theorem

Let $(X_n)_n$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X^2] = 1$. Then

$$d_K\Big(\frac{X_1 + \ldots + X_n}{\sqrt{n}}, \mathcal{N}(0,1)\Big) \to 0,$$

where for every pair of X, Y of random variables we denote by

$$d_K(X,Y) = \sup_{x\in\mathbb{R}} \Big|\mathbb{P}([X \leqslant x]) - \mathbb{P}([Y \leqslant x])\Big|.$$

Berry–Esseen Theorem

Let n be a positive integer and let $(X_1, \ldots, X_n)$ be a random vector with i.i.d. entries satisfying $\mathbb{E}[X_1] = 0$, $\mathbb{E}[X^2] = 1$ and $\rho = \mathbb{E}[|X_1|^3] < \infty$.

- Then we have that

$$d_K\Big(\frac{X_1 + \ldots + X_n}{\sqrt{n}}, \mathcal{N}(0,1)\Big) \leqslant \frac{C\rho}{\sqrt{n}}.$$

- Moreover, for every $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n \theta_i^2 = 1$ we have that

$$d_k\Big(\sum_{i=1}^n \theta_i X_i, \mathcal{N}(0,1)\Big) \leqslant C\rho \sum_{i=1}^n |\theta_i|^3.$$

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Notation

Let d, n be positive integers.

- We denote by $[n]$ the set $\{1, ..., n\}$ and by $[n]^d$ the set of all maps form $[d]$ into $[n]$.
- By $[n]^d_{\text{Inj}}$, we denote the set of all injective maps from $[d]$ into $[n]$.
- By $\mathbb{S}_n$, we denote the symmetric group of $[n]$, that is, the set of all permutations of $[n]$.

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Framework

Let d, n be positive integers with $2d \leqslant n$. Also let $\boldsymbol{X} = \langle X_i : i \in [n]^d \rangle$ be a random tensor satisfying the following.

($\mathcal{A}1$) We have $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] \leqslant 1$ and $\mathbb{E}\big[|X_i|^3\big] < \infty$ for every $i \in [n]^d$.

($\mathcal{A}2$) The random tensor $\boldsymbol{X}$ is symmetric, exchangeable and its diagonal terms vanish ($X_i = 0$ for all $i \in [n]^d \setminus [n]^d_{\mathrm{Inj}}$).

- *symmetric*: $X_{(i_1,\dots,i_d)} = X_{(i_{\tau(1)},\dots,i_{\tau(d)})}$ for every $(i_1, \dots, i_d) \in [n]^d$ and every $\tau \in \mathbb{S}_d$
- *exchangeable*: for every $\pi \in \mathbb{S}_n$ the random tensors $\boldsymbol{X}$ and $\boldsymbol{X}_\pi = \langle X_{\pi\circ i} : i \in [n]^d \rangle$ have the same distribution.

Examples

- Let $(\xi_k)_k$ be a sequence of i.i.d. random variables which take values in a measurable space $\mathcal{E}$ and let and $h\colon \mathcal{E}^d \to \mathbb{R}$ be a measurable symmetric function. For every $i \in [n]^d_{\mathrm{Inj}}$ set $X_i = h(\xi_{i(1)}, \dots, \xi_{i(d)})$ and consider the random tensor $\boldsymbol{X} = \langle X_i : i \in [n]^d \rangle$.

- Let k, ℓ be positive integers with $k \leqslant \ell$. Let $(\zeta_1, \dots, \zeta_\ell)$ be a boolean random vector uniformly distributed on the "slice" $\{f \in \{0,1\}^\ell : \sum_{i=1}^{\ell} f(i) = k\}$. Also let n be an integer with $n \leqslant \ell$. Consider the random vector $X = (\zeta_1 - \frac{k}{\ell}, \dots, \zeta_n - \frac{k}{\ell})$.

- Let $(\zeta_1, \dots, \zeta_n)$ be an exchangeable random vector which takes values in $[0,1]^n$. For every $i \in [n]^d_{\mathrm{Inj}}$, set $X_i = \prod_{\ell=1}^{d} \zeta_{i(\ell)} - \mathbb{E}\big[\prod_{\ell=1}^{d} \zeta_{i(\ell)}\big]$. Consider the random tensor $\boldsymbol{X} = \langle X_i : i \in [n]^d \rangle$.

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Our goal is to estimate the quantity

$$d_K\big(\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle, \mathcal{N}(0, \sigma^2)\big),$$

where σ^2 denotes the variance of $\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle = \sum_{i \in [n]^d} \theta_i X_i$.

Connection to permutation statistics

Set $Y = \sum_{i\in[n]^d} \theta_i X_{\pi\circ i}$ where π is a random permutation, independent of $\boldsymbol{X}$, which is uniformly distributed on $\mathbb{S}_n$.
Since $\boldsymbol{X}$ is exchangeable, we have that $\langle \boldsymbol{\theta}, \boldsymbol{X}\rangle$ and Y have the same distribution and therefore

$$d_K\big(\langle \boldsymbol{\theta}, \boldsymbol{X}\rangle, \mathcal{N}(0,\sigma^2)\big) = d_K\big(Y, \mathcal{N}(0,\sigma^2)\big).$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the underlying probability space on which the random tensor $\boldsymbol{X}$ is defined. For every $\omega \in \Omega$ denote by Z_ω the random variable

$$Z_\omega = \sum_{i\in[n]^d} \theta_i \cdot X_{\pi\circ i}(\omega)$$

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Noticing that Y is the mixture with respect to $(\Omega, \mathcal{F}, \mathbb{P})$ of Z_ω, we have that

$$d_K\big(\langle \boldsymbol{\theta}, \boldsymbol{X}\rangle, \mathcal{N}(0,\sigma^2)\big) = d_K\big(Y, \mathcal{N}(0,\sigma^2)\big) \leqslant \mathbb{E}_\omega\big[d_K\big(Z_\omega, \mathcal{N}(0,\sigma^2)\big)\big]$$

With a deterministic tensor $\zeta : [n]^d \times [n]^d \to \mathbb{R}$ we associate the Z-statistic

$$Z = \sum_{i \in [n]^d} \zeta(i, \pi \circ i),$$

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W-statistics

Let s be a positive integer (with $s \leqslant d$), and let $\xi : [n]^s \times [n]^s \to \mathbb{R}$. We say that ξ is a *Hoeffding tensor* if for every $r \in [s]$, every $j_0, q_0 \in [n]^{[d]\setminus\{r\}}$ and every $i_0, p_0 \in [n]^d$ we have

$$\sum_{j_0 \sqsubseteq i \in [n]^d} \xi(i, p_0) = 0 \quad \text{and} \quad \sum_{q_0 \sqsubseteq p \in [n]^d} \xi(i_0, p) = 0$$

where $[n]^{[d]\setminus\{r\}}$ denotes the set of all maps from $[d] \setminus \{r\}$ to $[n]$.

A W-statistic is a statistic of the form

$$W = \sum_{s=1}^{d} \sum_{i \in [n]^s_{\mathrm{Inj}}} \xi_s(i, \pi \circ i)$$

where π is uniformly distributed in $\mathbb{S}_n$ and $\xi_s : [n]^s \times [n]^s \to \mathbb{R}$ is a Hoeffding tensor for every $s \in [d]$.

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Combinatorial CLT

- Classical results for matrix permutation statistics were obtained by Wald/Wolfowitz (1944) and Hoeffding (1951) who established asymptotic normality under general conditions.
- The optimal result in establishing quantitative normality of W-statistics of order one was obtained by Bolthausen (1984) who showed that

$$d_K\Big(\sum_{i=1}^{n}\xi\big(i,\pi(i)\big),\mathcal{N}(0,1)\Big)\leqslant\frac{C_1}{n}\sum_{i,j=1}^{n}|\xi(i,j)|^3$$

for every Hoeffding tensor $\xi\colon [n]\times[n]\to\mathbb{R}$ which satisfies $\sum_{i,j=1}^{n}\xi(i,j)^2=n-1$.
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- W-statistics of order two are also studied by Barbour and Eagleson (1986), as well as, Zhao, Bai, Chao and Liang (1997).
- The strongest quantitative normal approximation was obtained by Barbour and Chen (2005) who showed that if $\xi_1 : [n] \times [n] \to \mathbb{R}$ and $\xi_2 : [n]^2 \times [n]^2 \to \mathbb{R}$ are Hoeffding tensors with $\sum_{i,j=1}^n \xi_1(i,j)^2 = n-1$, and W is the W-statistic associated with ξ_1 and ξ_2, then

$$d_K\big(W, \mathcal{N}(0,1)\big) \leqslant \frac{aC_1}{n} \sum_{i,j=1}^{n} |\xi_1(i,j)|^3 + C_2 \sqrt{\frac{1}{n^2} \sum_{i,p \in [n]^2} \xi_2(i,p)^2}$$

- High-dimensional Z-statistics and W-statistics have been studied, for instance, by Bolthausen and Götze (1993), by Bloznelis and Götze (2002) and by Loh (1996).

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Combinatorial CLT

Theorem (P. Dodos, K.T.)

Let n, d be positive integers such that $n \geqslant 4d^2$. For every $s \in [d]$ let $\boldsymbol{\xi}_s : [n]^s \times [n]^s \to \mathbb{R}$ be a Hoeffding tensor, and set

$$\beta_s = \sum_{i,p \in [n]^s} \boldsymbol{\xi}_s(i,p)^2.$$

Assume that $\beta_1 = n - 1$, and let W be the W-statistic associated with $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_d$. Then we have

$$d_K\big(W, \mathcal{N}(0,1)\big) \leqslant \frac{2^{18} C_1}{n} \sum_{i,j=1}^{n} |\boldsymbol{\xi}_1(i,j)|^3 + C_d \sum_{s=2}^{d} \sqrt{\frac{\beta_s}{n^s}}$$

In fact, we can take $C_d = 5d^2 e^d (2d)!$.

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($\mathcal{A}1$) We have $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] \leqslant 1$ and $\mathbb{E}\big[|X_i|^3\big] < \infty$ for every $i \in [n]^d$.

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Our goal is to estimate the quantity

$$d_K\big(\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle, \mathcal{N}(0, \sigma^2)\big),$$

where σ^2 denotes the variance of $\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle$.

Relevant parameters

For every $s \in \{0, 1, \ldots, d\}$ we set

- $\|\!|\boldsymbol{\theta}|\!\|_s = \Big(\sum_{j \in [n]^s} \Big(\sum_{j \sqsubseteq i \in [n]^d} \theta_i \Big)^2 \Big)^{1/2}$
- $\delta_s = \delta_s(\boldsymbol{X}) = \mathbb{E}[X_{(1,\ldots,d)} X_{(1,\ldots,s,d+1,\ldots,2d-s)}]$
- $\Sigma_s = \Sigma_s(\boldsymbol{X}) = \sum_{t=0}^{s} \binom{s}{t} (-1)^{s-t} \delta_t.$

Then we have

$$\mathrm{Var}\big(\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle\big) = \sum_{s=0}^{d} \binom{d}{s}^2 s!\, \Sigma_s \, \|\!|\boldsymbol{\theta}|\!\|_s^2.$$

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- $|||\boldsymbol{\theta}|||_s = \Big(\sum_{j \in [n]^s} \big(\sum_{j \sqsubseteq i \in [n]^d} \theta_i \big)^2 \Big)^{1/2}$
- $\delta_s = \delta_s(\boldsymbol{X}) = \mathbb{E}\big[X_{(1,\ldots,d)} X_{(1,\ldots,s,d+1,\ldots,2d-s)}\big]$
- $\Sigma_s = \Sigma_s(\boldsymbol{X}) = \sum_{t=0}^{s} \binom{s}{t} (-1)^{s-t} \, \delta_t.$

Then we have

$$\mathrm{Var}\big(\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle\big) = \sum_{s=0}^{d} \binom{d}{s}^2 s! \, \Sigma_s \, |||\boldsymbol{\theta}|||_s^2 .$$

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Relevant parameters

Finally, we define the *oscillation* of $\boldsymbol{X}$ by

$$\operatorname{osc}(\boldsymbol{X}) = \Big\| \frac{1}{n} \sum_{j=1}^{n} \Big(\frac{1}{n^{d-1}} \sum_{\substack{i \in [n]^d \\ i(1)=j}} X_i \Big)^2 - \delta_1 \Big\|_{L_1}.$$

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Main result

Let $\boldsymbol{X}, \boldsymbol{\theta}$ satisfying ($\mathcal{A}$*1*), ($\mathcal{A}$*2*) and ($\mathcal{A}$*3*), and such that $|||\boldsymbol{\theta}|||_1 = 1$. Set $\kappa = \kappa(d) = 20d^3 18^d (2d)!$ and $B = \big\| \frac{1}{n^d} \sum_{i \in [n]^d} X_i \big\|_{L_2}^2$. Let $\alpha \in (0,1)$. Assume that the following non-degenericity condition holds true

$$\delta_1 \geqslant \max\left\{ \mathrm{osc}(\boldsymbol{X})^\alpha, B^\alpha, \left(\frac{\kappa}{n}\right)^\alpha \right\}.$$

Then, setting $\sigma^2 = \mathrm{Var}\big(\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle\big)$, we have

$$d_K\big(\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle, \mathcal{N}(0,\sigma^2)\big) \leqslant E_1 + E_2 + E_3$$

Main result

where

$$E_1 = 5\operatorname{osc}(\boldsymbol{X})^{1-\alpha} + 5|\delta_0|^{1-\alpha} + \left|\frac{\delta_0}{d^2\delta_1}\left(|||\boldsymbol{\theta}|||_0^2 - 1\right)\right| + \frac{6\kappa}{n^{1-\alpha}} + 4\,\frac{|||\boldsymbol{\theta}|||_0^2}{n}$$

$$E_2 = 2^{36}\,\frac{\mathbb{E}\left[|X_{(1,\dots,d)}|^3\right]}{\delta_1^{3/2}}\left(\sum_{j=1}^{n}\Big|\sum_{\substack{i\in[n]^d\\ i(1)=j}}\theta_i\Big|^3\right)$$

$$E_3 = 3\kappa\,\frac{1}{d\sqrt{\delta_1}}\sum_{s=2}^{d}\binom{d}{s}\sqrt{s!}\sqrt{\Sigma_s + \frac{16d^2 2^d}{n}}\,|||\boldsymbol{\theta}|||_s.$$

On the bounds

$$E_1 = 5\mathrm{osc}(\boldsymbol{X})^{1-\alpha} + 5|\delta_0|^{1-\alpha} + \left| \frac{\delta_0}{d^2\delta_1} \left(|||\boldsymbol{\theta}|||_0^2 - 1 \right) \right| + \frac{6\kappa}{n^{1-\alpha}} + 4\,\frac{|||\boldsymbol{\theta}|||_0^2}{n}$$

- The first term of E_1 is, essentially, the oscillation of $\boldsymbol{X}$.
- The second and third terms are quantitative measures of the correlation of the entries of $\boldsymbol{X}$.
- The fourth term is related to the non-degenericity assumption.
- The last term in E_1 is more subtle, and it is related to the extendability of $\boldsymbol{X}$.

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$$E_3 = 3\kappa \frac{1}{d\sqrt{\delta_1}} \sum_{s=2}^{d} \binom{d}{s} \sqrt{s!} \sqrt{\Sigma_s + \frac{16 d^2 2^d}{n}} \, \|\boldsymbol{\theta}\|_s .$$

Bounding the oscillation

Let $d = 1$ and assume that the entries of $\boldsymbol{X}$ are of unit variance. Then

$$\mathrm{osc}(\boldsymbol{X}) \leqslant \sqrt{\left|\mathbb{E}[X_1^2 X_2^2] - 1\right|} + \frac{4\mathbb{E}\left[|X_1|^3\right]}{\sqrt[4]{n}}.$$

Assuming that the entries of $\boldsymbol{X}$ are of finite forth moment, we have that

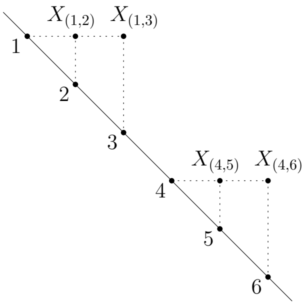
$$\mathrm{osc}(\boldsymbol{X}) \leqslant \sqrt{\left|\mathbb{E}[X_1^2 X_2^2] - 1\right|} + \frac{\mathbb{E}[X_1^4]^{1/2}}{\sqrt{n}}.$$

Bounding the oscillation

For general d define the *parallelepipedal correlation* of $\boldsymbol{X}$ by

$$\mathrm{pc}(\boldsymbol{X}) = \left|\mathbb{E}[X_{(1,\dots,d)}X_{(1,d+1,\dots,2d-1)}X_{(2d,\dots,3d-1)}X_{(2d,3d,\dots,4d-2)}] - \delta_1^2\right|.$$

For example, if $d = 2$, then $\mathrm{pc}(\boldsymbol{X}) = \left|\mathbb{E}[X_{(1,2)}X_{(1,3)}X_{(4,5)}X_{(4,6)}] - \delta_1^2\right|$.



Notice that for $d = 1$ we have that $\mathrm{pc}(\boldsymbol{X}) = \left|\mathbb{E}[X_1^2X_2^2] - 1\right|$.

Bounding the oscillation

Under the additional assumption that the entries of $\boldsymbol{X}$ are of finite forth moment, we have that

$$\mathrm{osc}(\boldsymbol{X}) \leqslant \sqrt{\mathrm{pc}(\boldsymbol{X})} + \frac{5d}{\sqrt{n}}\big(1 + \mathbb{E}[X^4_{(1,\dots,d)}]\big)^{1/2}.$$

Bounding the oscillation

If $\boldsymbol{X}$ is dissociated, then we have that $\mathrm{pc}(\boldsymbol{X}) = 0$ and

$$\mathrm{osc}(\boldsymbol{X}) \leqslant \frac{16d\left(\mathbb{E}\left[|X_{(1,\dots,d)}|^3\right] + 1\right)}{\sqrt[4]{n}}.$$

If $\boldsymbol{X}$ is a mixture of exchangeable, symmetric and dissociated random tensors and the entries of $\boldsymbol{X}$ have finite third moment, then we have

$$\mathrm{osc}(\boldsymbol{X}) \leqslant \sqrt{\mathrm{pc}(\boldsymbol{X})} + \frac{16d\left(\mathbb{E}\left[|X_{(1,\dots,d)}|^3\right] + 1\right)}{\sqrt[4]{n}}.$$

Thank you !!!