

# On a quantitative reversal of Alexandrov's inequality

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## Abstract

Alexandrov's inequalities imply that for any convex body  $A$ , the sequence of intrinsic volumes  $V_1(A), \dots, V_n(A)$  is non-increasing (when suitably normalized). Milman's random version of Dvoretzky's theorem shows that a large initial segment of this sequence is essentially constant, up to a critical parameter called the Dvoretzky number. We show that this near-constant behavior actually extends further, up to a different parameter associated with  $A$ . This yields a new quantitative reverse inequality that sits between the approximate reverse Urysohn inequality, due to Figiel–Tomczak-Jaegermann and Pisier, and the sharp reverse Urysohn inequality for zonoids, due to Hug–Schneider. In fact, we study concentration properties of the volume radius and mean width of random projections of  $A$  and show how these lead naturally to such reversals.

## 1 Introduction

For a convex body  $A \subseteq \mathbb{R}^n$ , the intrinsic volumes  $V_1(A), \dots, V_n(A)$  are fundamental quantities in convex geometry. Of special significance are  $V_1$ ,  $V_{n-1}$  and  $V_n$ , which are suitable multiples of the mean width, surface area and volume, respectively (precise definitions are recalled in §2). Alexandrov's inequalities imply that

$$\left(\frac{V_n(A)}{V_n(B)}\right)^{\frac{1}{n}} \leq \left(\frac{V_{n-1}(A)}{V_{n-1}(B)}\right)^{\frac{1}{n-1}} \leq \dots \leq \frac{V_1(A)}{V_1(B)}, \quad (1.1)$$

where  $B$  is the Euclidean unit ball in  $\mathbb{R}^n$ . The leftmost inequality is the isoperimetric inequality, while Urysohn's inequality is the comparison between the

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two endpoints. Thus (1.1) occupies a special role in convex geometry. For background and the more general Alexandrov-Fenchel inequality, we refer to Schneider's monograph [Sch14].

There are various reverse inequalities that complement (1.1) or some of its special cases. K. Ball's reverse isoperimetric inequality shows that any convex body  $A$  has an affine image  $\tilde{A}$  such that

$$\frac{V_{n-1}(\tilde{A})}{V_{n-1}(B)} \leq c_n \left( \frac{V_n(\tilde{A})}{V_n(B)} \right)^{\frac{n-1}{n}}, \quad (1.2)$$

where  $c_n$  is a constant which is attained when  $A$  is a simplex (and when  $A$  is a cube if one considers only origin-symmetric convex bodies) [Bal91]. A reverse form of Urysohn's inequality can be obtained by a result of Figiel and Tomczak-Jaegermann [FTJ79]: any symmetric convex body  $A$  has a linear image  $\tilde{A}$  satisfying

$$\frac{V_1(\tilde{A})}{V_1(B)} \leq CK(A) \left( \frac{V_n(\tilde{A})}{V_n(B)} \right)^{\frac{1}{n}}, \quad (1.3)$$

where  $C$  is an absolute constant and  $K(A)$  denotes the  $K$ -convexity constant of  $\mathbb{R}^n$  equipped with the norm  $\|\cdot\|_A$  associated to  $A$  (see also [AAGM15, Ch. 6]). A fundamental theorem of Pisier [Pis81] gives  $K(A) \leq C \log d(A, B)$ , where  $d$  denotes Banach-Mazur distance. By John's theorem [Joh48], one always has  $d(A, B) \leq \sqrt{n}$  and thus  $K(A) \leq C \log n$ . In a related direction, by a result of Milman [Mil86], any symmetric convex body  $A$  admits a linear image  $\tilde{A}$  such that

$$\left( \frac{V_{n/2}(\tilde{A})}{V_{n/2}(B)} \right)^{\frac{2}{n}} \leq c_1 \left( \frac{V_n(\tilde{A})}{V_n(B)} \right)^{\frac{1}{n}}, \quad (1.4)$$

where  $c_1$  is an absolute constant. The latter is based on the existence of Milman's ellipsoid, which in turn is intimately connected to the reverse Blaschke-Santaló inequality [BM87] and the reverse Brunn-Minkowski inequality [Mil86] (see also [Pis89, Ch. 7]). Each of (1.2), (1.3) and (1.4) share the common feature that to get reverse inequalities, one needs to consider affine (or linear) images of the convex body. Note that (1.2) is a sharp inequality while (1.3) and (1.4) are *isomorphic* reversals in that they hold up to constants without establishing extremizers. Moreover, (1.3) is a quantitative statement in the sense that a parameter associated with  $A$  quantifies the tightness of the reversal.

All of the reversed inequalities mentioned so far involve  $V_n(A)$ . Concerning the generalized Urysohn inequality, which compares  $V_1(A)$  with  $V_k(A)$  for  $1 \leq k \leq n$ , Hug and Schneider [HS11] have proved that for any zonoid  $A$ , there is a linear image  $\tilde{A}$  such that

$$\frac{V_1(\tilde{A})}{V_1(B)} \leq c(n, k) \left( \frac{V_k(\tilde{A})}{V_k(B)} \right)^{\frac{1}{k}}, \quad (1.5)$$

where  $c(n, k)$  is a constant that is obtained when  $A$  is a parallelepiped. As with Ball's inequality, (1.5) is sharp. The case  $k = n$  was proved earlier by Giannopoulos, Milman and Rudelson [GMR00].

Our first result is a quantitative reversal involving  $V_1(A)$  and  $V_k(A)$ . We show that when  $A$  is symmetric, (1.1) may be reversed up to a new parameter associated with  $A$ , studied recently in [PVb] and [PVc]. Specifically, let  $h_A$  denote the support function of  $A$  and let  $g$  be a standard Gaussian random vector in  $\mathbb{R}^n$ . We define a normalized variance of the random variable  $h_A(g)$  as follows:

$$\beta_*(A) = \frac{\text{Var}(h_A(g))}{(\mathbb{E}h_A(g))^2},$$

where  $\mathbb{E}$  denotes expectation and  $\text{Var}$  is the variance. With this notation, we have the following theorem.

**Theorem 1.1.** *There exists a constant  $c > 0$  such that if  $A$  is a symmetric convex body in  $\mathbb{R}^n$  and  $1 \leq k \leq c/\beta_*(A)$ , then*

$$\frac{V_1(A)}{V_1(B)} \leq \left(1 + c\sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right) \left(\frac{V_k(A)}{V_k(B)}\right)^{1/k}. \quad (1.6)$$

For comparison purposes, it will be convenient to write

$$W_{[k]}(A) = \left(\frac{V_k(A)}{V_k(B)}\right)^{1/k},$$

which is simply the radius of a Euclidean ball having the same  $k$ -th intrinsic volume as  $A$ . Then (1.1) says that  $k \mapsto W_{[k]}(A)$  is non-increasing, while the Hug-Schneider result (1.5) for zonoids implies that

$$W_{[1]}(A) \leq \left(1 + O\left(\frac{k}{n}\right)\right) W_{[k]}(A). \quad (1.7)$$

For our normalization, quantitative reversals comparing  $W_{[n]}(A)$  with  $W_{[k]}(A)$  (as opposed to  $W_{[k]}(A)$  with  $W_{[1]}(A)$ ) are somewhat easier tasks to achieve. For example, just using set inclusions and monotonicity of mixed volumes one has

$$W_{[n-k]}(A) \leq d^{\frac{k}{n-k}} W_{[n]}(A) \leq \left(1 + O\left(\frac{k \log d}{n-k}\right)\right) W_{[n]}(A),$$

as long as  $k \leq \frac{n}{1+\log d}$ , where  $d = d_G(A)$  is the geometric distance between  $A$  and  $B$  (i.e., the ratio of the circumradius of  $A$  over the inradius of  $A$ ). Thus we focus on comparisons between  $W_{[k]}(A)$  and  $W_{[1]}(A)$  in this paper.

Theorem 1.1 combines several features of the aforementioned inequalities: one has a quantitative reversal of (1.1) depending on the parameter  $\beta_*(A)$ . Unlike the reverse Urysohn inequality (1.3), (1.6) holds on an *almost isometric scale* as opposed to an isomorphic one.

To explain some of the ideas behind Theorem 1.1, recall that  $W_{[k]}(A)$  can be expressed through Kubota's integral recursion (e.g. [Sch14, Ch. 5]) via

$$W_{[k]}(A) = \left( \frac{1}{\omega_k} \int_{G_{n,k}} |P_E A| d\nu_{n,k}(E) \right)^{\frac{1}{k}}, \quad (1.8)$$

where  $\omega_k$  is the volume of the Euclidean unit ball in  $\mathbb{R}^k$ ,  $G_{n,k}$  is the Grassmanian of  $k$ -dimensional subspaces of  $\mathbb{R}^n$ , equipped with the Haar probability measure  $\nu_{n,k}$ ,  $P_E$  denotes the orthogonal projection onto  $E$  and  $|\cdot|$  denotes volume (on the subspace  $E$ ). Thus our interest is in tight lower bounds for the volume of random projections of  $A$ . By Milman's random version of Dvoretzky's theorem [Mil71], one has the following almost isometric inclusions

$$(1 - \varepsilon)W_{[1]}(A)P_E B \subseteq P_E A \subseteq (1 + \varepsilon)W_{[1]}(A)P_E B, \quad (1.9)$$

for a random subspace  $E \in G_{n,k}$  provided  $k \leq c(\varepsilon)k_*(A)$ , where  $k_*(A)$  denotes the Dvoretzky dimension (the definition is recalled in §3). The inclusions in (1.9) explain the almost constant behavior of  $k \mapsto W_{[k]}(A)$  for  $k$  up to  $k_*(A)$ . Theorem 1.1 goes further in that this near-constant behavior actually extends for dimensions  $k$  up to  $c/\beta_*(A)$ . In general,  $k_*(A) \leq c/\beta_*(A)$ , while for some convex bodies,  $c/\beta_*(A)$  is significantly larger than  $k_*(A)$ . An earlier indication of this phenomenon is suggested by work of Klartag and Vershynin in [KV07]. They proved that the lower inclusion in (1.9) on an isomorphic scale, i.e.,

$$c_1 W_{[1]}(A)P_E B \subseteq P_E A, \quad (1.10)$$

can hold for subspaces  $E$  of significantly larger dimensions, governed by a different parameter  $d_*(A)$  which satisfies  $d_*(A) \geq c_2 k_*(A)$ , where  $c_1, c_2$  are absolute constants (the precise definition of  $d_*(A)$  is in §3.2). In particular, they noted the following striking example: for  $A = B_1^n$ , the unit ball in  $\ell_1^n$ , one has  $d_*(A) \simeq n^{0.99}$  while  $k_*(A) \simeq \log n$ . The behavior of  $\beta_*(A)$  has been studied in [PVb] and [PVc] in connection with almost isometric Euclidean structure and concentration for convex functions. Theorem 1.1 shows that  $\beta_*(A)$  also plays a significant role in reversing (1.1).

More generally, we also show that  $\beta_*(A)$  arises in multi-dimensional concentration inequalities. In view of Kubota's formula (1.8), Theorem 1.1 concerns the expectation of the random variable

$$\text{vrad}(P_E A) := (|P_E A|/\omega_k)^{1/k},$$

where  $E$  is a random subspace distributed according to  $\nu_{n,k}$ . For families of convex bodies  $A = A_n \subseteq \mathbb{R}^n$  with  $n$  increasing (and  $k$  fixed), it is natural to study distributional properties of  $\text{vrad}(P_E A)$ . For example, when  $A_n$  is the cube  $[-1, 1]^n$ ,  $\text{vrad}(P_E A)$  is studied in [PPZ14] and a central limit theorem is proved. Here we treat concentration inequalities for arbitrary symmetric convex bodies. In this way, the next theorem can be seen as a more quantitative study of the intrinsic volumes.

**Theorem 1.2.** *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$  and let  $1 \leq k \leq c/\beta_*(A)$ . Then for all  $\varepsilon > c'_1 \sqrt{k\beta_*(A) \log(\frac{e}{k\beta_*(A)})}$ ,*

$$\nu_{n,k}(E \in G_{n,k} : \text{vrad}(P_E A) \geq (1 + \varepsilon)W_{[k]}(A)) \leq C_1 \exp(-c_1 \varepsilon^2 k k_*(A)); \quad (1.11)$$

*moreover, if  $c'_2 \sqrt{k\beta_*(A) \log(\frac{e}{k\beta_*(A)})} < \varepsilon < 1$ ,*

$$\nu_{n,k}(E \in G_{n,k} : \text{vrad}(P_E A) \leq (1 - \varepsilon)W_{[k]}(A)) \leq C_2 \exp(-c_2 \varepsilon^2 / \beta_*(A)), \quad (1.12)$$

*where  $c_i, C_i, c'_i > 0$ ,  $i = 1, 2$ , are absolute constants.*

If we take  $k = 1$  in Theorem 1.2, then  $E = \text{span}(\theta)$  for some  $\theta$  on the unit sphere  $S^{n-1}$  and  $\text{vrad}(P_E A) = h_A(\theta)$ , while  $W_{[k]}(A) = \int_{S^{n-1}} h_A(\theta) d\sigma(\theta)$ . Thus (1.11) recovers the standard concentration estimate on the sphere in terms of the Lipschitz constant of the support function  $h_A$  of  $A$  (up to constants), e.g., [MS86, Ch. 2]. Similarly, (1.12) recovers the new concentration inequality in terms of variance of the support function from [PVB] (stated below in Theorem 3.6). Thus Theorem 1.2 is a multi-dimensional extension of the latter results. Both Theorems 1.1 and 1.2 are based on new tight reverse Hölder inequalities for the random variables  $\text{vrad}(P_E A)$  and  $w(P_E A)$ . These improve the standard estimates following from the concentration of measure phenomenon in the current literature (this is discussed in §3.2).

We conclude the introduction with some examples where Theorem 1.1 gives the largest possible range of dimensions for the almost-constant behavior in (1.1). Recall that a Borel measure  $\mu$  on  $S^{n-1}$  is said to be isotropic if the covariance matrix of  $\mu$  is the identity matrix. For any such measure we associate the family of the  $L_q$ -zonoids  $\{Z_q(\mu)\}_{q \geq 1}$  which are defined through their support function:

$$h_{Z_q(\mu)}(x) = \left( \int_{S^{n-1}} |\langle x, \theta \rangle|^q d\mu(\theta) \right)^{1/q}, \quad x \in \mathbb{R}^n.$$

**Corollary 1.3.** *Let  $1 \leq q < \infty$ . Then there is a constant  $c_q > 0$  such that if  $k \leq c_q n$  and  $\mu$  is an isotropic Borel measure on  $S^{n-1}$ , then*

$$\left( 1 - \sqrt{\frac{c_q k}{n} \log \frac{n}{c_q k}} \right) W_{[1]}(Z_q(\mu)) \leq W_{[k]}(Z_q(\mu)) \leq W_{[1]}(Z_q(\mu)). \quad (1.13)$$

Lastly, the restriction to symmetric convex bodies in this paper seems to be inherent in the tools used in the proofs. However, we do not believe that symmetry is essential for such reverse inequalities.

The rest of the paper is organized as follows: In Section 2, we fix the notation and we provide necessary background information. In Section 3, we recall some auxiliary results from asymptotic convex geometry and from the concentration of measure for norms on Euclidean space. Some basic probabilistic facts are also considered. Finally, in Section 4 we present the proofs of our main results.

## 2 Notation and background material

We work in  $\mathbb{R}^n$  equipped with the usual inner-product  $\langle \cdot, \cdot \rangle$  and Euclidean norm  $\|x\|_2 := \sqrt{\langle x, x \rangle}$  for  $x \in \mathbb{R}^n$ ;  $B_2^n$  is the Euclidean ball of radius 1;  $S^{n-1}$  is the unit sphere, equipped with the Haar probability measure  $\sigma$ . For Borel sets  $A \subseteq \mathbb{R}^n$ , we use  $V_n(A)$  (or  $|A|$ ) for the Lebesgue measure of  $A$ ;  $\omega_n$  for the Lebesgue measure of  $B_2^n$ . The Grassmannian manifold of all  $n$ -dimensional subspaces of  $\mathbb{R}^n$  is denoted by  $G_{n,k}$ , equipped with the Haar probability measure  $\nu_{n,k}$ . For a subspace  $E \in G_{n,k}$ , we write  $P_E$  for the orthogonal projection onto  $E$ .

Throughout the paper we reserve the symbols  $c, c_1, c_2, \dots$  for absolute constants (not necessarily the same in each occurrence). We use the convention  $S \simeq T$  to signify that  $c_1 T \leq S \leq c_2 T$  for some positive absolute constants  $c_1$  and  $c_2$ . We also assume that  $n$  is larger than a fixed absolute constant. By adjusting the constants involved one can always ensure that the results to hold for all  $n$ .

A convex body  $K \subseteq \mathbb{R}^n$  is a compact, convex set with non-empty interior. The support function of a convex body  $K$  is given by

$$h_K(y) = \sup\{\langle x, y \rangle : x \in K\}, \quad y \in \mathbb{R}^n.$$

We say that  $K$  is (origin) symmetric if  $K = -K$ . For a symmetric convex body  $K$  the polar body  $K^\circ$  is defined by

$$K^\circ := \{x \in \mathbb{R}^n : |\langle x, y \rangle| \leq 1, y \in K\}.$$

For  $p \neq 0$ , we define the  $p$ -generalized mean width of  $A$  by

$$w_p(K) := \left( \int_{S^{n-1}} h_K^p(\theta) d\sigma(\theta) \right)^{1/p}. \quad (2.1)$$

The circumradius of  $K$  is defined by  $R(K) = \max_{\theta \in S^{n-1}} h_K(\theta) = \max_{x \in K} \|x\|_2$ . Note that  $R(K) = w_\infty(K) := \lim_{p \rightarrow \infty} w_p(K)$ . In addition, we denote by  $r(K)$  the inradius of  $K$ , i.e.  $r(K) = \min_{\theta \in S^{n-1}} h_K(\theta)$ . Again, we have:  $r(K) = w_{-\infty}(K) := \lim_{p \rightarrow \infty} w_{-p}(K)$ . Note that  $r(K^\circ) = 1/R(K)$ . Similarly, if  $\|\cdot\|_K$  is the norm induced by  $K$  we define, for  $p \neq 0$ ,

$$M_p(K) := \left( \int_{S^{n-1}} \|\theta\|_K^p d\sigma(\theta) \right)^{\frac{1}{p}}.$$

Note that  $M_p(K^\circ) = w_p(K)$ , by definition. We simply write  $w(K) := w_1(K)$  and  $M(K) := M_1(K)$ .

The intrinsic volumes of a convex body  $K \subseteq \mathbb{R}^n$  can be defined via the Steiner formula for the outer parallel volume of  $K$ :

$$|K + tB_2^n| = \sum_{k=0}^n \omega_k V_{n-k}(K) t^k, \quad t > 0.$$

Here  $V_k$ ,  $k = 1, \dots, n$ , is the  $n$ -th intrinsic volume of  $K$  (we set  $V_0 \equiv 1$ ).  $V_n$  is volume,  $2V_{n-1}$  is surface area and  $\frac{\omega_{n-1}}{n\omega_n} V_1 = w = w_1$  is the mean width (as we have

defined in (2.1)). Intrinsic volumes are also referred to as quermassintegrals (under an alternate labeling and normalization). For further background, see [Sch14, Ch. 4]. Here we prefer to work with a different normalization, similar to that used in [DP12], [PP13]. As in the introduction, for a convex body  $K \subseteq \mathbb{R}^n$  and  $1 \leq k \leq n-1$ , we write

$$W_{[k]}(K) := \left( \frac{1}{\omega_k} \int_{G_{n,k}} |P_E K| d\nu_{n,k}(E) \right)^{1/k}.$$

We will need the following generalization of this definition: for  $p \neq 0$  we write

$$W_{[k,p]}(K) := \left( \frac{1}{\omega_k^p} \int_{G_{n,k}} |P_E K|^p d\nu_{n,k}(E) \right)^{\frac{1}{pk}}.$$

Note that by Kubota's integral formula,

$$V_k(K) = \binom{n}{k} \frac{\omega_n}{\omega_{n-k}} W_{[k]}^k(K). \quad (2.2)$$

We also set  $W_{[n]}(K) = \text{vrad}(K) := \left( \frac{V_n(K)}{V_n(B_2^n)} \right)^{1/n}$ . For ease of reference, we will also explicitly recall Urysohn's inequality which is the endpoint inequality from (1.1):

$$w(K) = w_1(K) = W_{[1]}(K) \geq W_{[n]}(K) = \text{vrad}(K) = \left( \frac{|K|}{|B_2^n|} \right)^{1/n}. \quad (2.3)$$

### 3 Probabilistic and geometric tools

We start with a few elementary lemmas about moments of random variables. Since we need some refinements of standard inequalities, we include somewhat detailed proofs. We then combine these with Gaussian concentration inequalities to prove new sharp reverse-Hölder inequalities for norms of random vectors.

#### 3.1 Centered and noncentered moments of random variables

We begin with the following standard fact.

**Proposition 3.1.** *Let  $\xi$  be a random variable on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  with  $\xi \in L_2(\Omega)$ . If  $m = \text{med}(\xi)$  is a median of  $\xi$ , then*

$$\mathbb{E}|\xi - m| \leq \sqrt{\text{Var}(\xi)}.$$

*Proof.* Recall that  $\inf_{\lambda \in \mathbb{R}} \mathbb{E}|\xi - \lambda| = \mathbb{E}|\xi - m|$ . Thus, by the Cauchy-Schwarz inequality,

$$\sqrt{\text{Var}(\xi)} \geq \mathbb{E}|\xi - \mathbb{E}\xi| \geq \mathbb{E}|\xi - m|.$$

□

**Lemma 3.2.** Let  $\xi$  be a random variable on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  with  $\xi \in L_p(\Omega)$ ,  $p \geq 2$  and let  $k \in \mathbb{N}$  with  $2 \leq k \leq p$ . Then, for any  $a \neq 0$ ,

$$\frac{\mathbb{E}\xi^k}{a^k} = 1 + \sum_{s=1}^k \binom{k}{s} \frac{\mathbb{E}(\xi - a)^s}{a^s}.$$

In particular, if  $k \geq 2$  and we take  $a = \mu := \mathbb{E}\xi \neq 0$ , then

$$\frac{\mathbb{E}\xi^k}{\mu^k} = 1 + \sum_{s=2}^k \binom{k}{s} \frac{\mathbb{E}(\xi - \mu)^s}{\mu^s}.$$

*Proof.* Using the binomial expansion, we have

$$\mathbb{E}\xi^r = \mathbb{E}[(\xi - a) + a]^r = \sum_{s=0}^r \binom{r}{s} \mathbb{E}(\xi - a)^s a^{r-s} = a^r \left[ 1 + \sum_{s=1}^r \binom{r}{s} \frac{\mathbb{E}(\xi - a)^s}{a^s} \right],$$

for all positive integers  $r \geq 1$ . The result follows.  $\square$

**Proposition 3.3.** Let  $\xi$  be a non-negative random variable with  $\mathbb{E}\xi = \mu > 0$  and let  $A \geq 1$ ,  $k \geq 1$  and  $a > 0$  be constants with  $P(|\xi - \mu| > t\mu) \leq Ae^{-at^2k}$ , for all  $t > 0$ . Then for any  $s \geq 2$ ,

$$\frac{\mathbb{E}|\xi - \mu|^s}{\mu^s} \leq \left( \frac{As}{ak} \right)^{s/2}. \quad (3.1)$$

Moreover, for all  $r \geq 1$ ,

$$\|\xi\|_r = (\mathbb{E}\xi^r)^{1/r} \leq \sqrt{1 + \frac{CAr}{ak}} \mu, \quad (3.2)$$

where  $C > 0$  is an absolute constant.

*Proof.* Observe that

$$\begin{aligned} \mathbb{E}|\xi - \mu|^s &= s\mu^s \int_0^\infty z^{s-1} P(|\xi - \mu| > z\mu) dz \leq As\mu^s \int_0^\infty z^{s-1} e^{-az^2k} dz \\ &= \frac{As}{2} (ak)^{-s/2} \mu^s \int_0^\infty z^{\frac{s}{2}-1} e^{-z} dz = \frac{A\mu^s}{(ak)^{s/2}} \Gamma\left(\frac{s}{2} + 1\right) \\ &\leq \frac{A\mu^s}{(ak)^{s/2}} s^{s/2} \leq \mu^s \left( \frac{As}{ak} \right)^{s/2}, \end{aligned}$$

where we have used the rough estimate  $\Gamma(x+1) < (2x)^x$  for  $x \geq 1$ . This completes the proof of the first assertion.



Set  $c := A/a$ . Next, using formula (3.1) we have

$$\begin{aligned} \frac{\mathbb{E}\xi^r}{\mu^r} &\leq 1 + \sum_{s=2}^r \binom{r}{s} \frac{\mathbb{E}|\xi - \mu|^s}{\mu^s} \leq 1 + \sum_{s=2}^r \binom{r}{s} \left(\frac{cs}{k}\right)^{s/2} \\ &\leq 1 + \sum_{s=2}^r \left(\frac{ec^{1/2}r}{k^{1/2}}\right)^s \frac{1}{s^{s/2}} \leq 1 + \sum_{s=2}^{\infty} \frac{\theta^s}{s^{s/2}}, \end{aligned} \quad (3.3)$$

where we have used the estimate  $\binom{r}{s} \leq (en/k)^k$  and  $\theta := ec^{1/2}r/k^{1/2}$ . If  $r > k$ , then (3.2), follows from (3.1) and the triangle inequality (and possibly adjusting the constant). Thus we consider only  $r \leq k$  and distinguish two cases:

Case i:  $\theta < 1/2$ . In this case, we have

$$\frac{\mathbb{E}\xi^r}{\mu^r} \leq 1 + \sum_{s=2}^{\infty} \theta^s \leq 1 + 2\theta^2.$$

Case ii:  $\theta \geq 1/2$ . We write

$$\begin{aligned} \frac{\mathbb{E}\xi^r}{\mu^r} &\leq 1 + \sum_{s=1}^{\infty} \frac{\theta^{2s}}{(2s)^s} + \sum_{s=1}^{\infty} \frac{\theta^{2s+1}}{(2s+1)^{s+1/2}} \leq 1 + \sum_{s=1}^{\infty} \frac{(\theta^2/2)^s}{s!} + \theta \sum_{s=1}^{\infty} \frac{(\theta^2/2)^s}{s!} \\ &\leq e^{\theta^2/2} + \theta e^{\theta^2/2} \leq \exp(\theta + \theta^2/2) \leq \exp\left(\frac{5\theta^2}{2}\right). \end{aligned}$$

In either case, we have  $(\mathbb{E}\xi^r)^{1/r} \leq \mu e^{3\theta^2/r}$  and since  $r \leq k$ , the result follows.  $\square$

*Remark 3.4.* 1. If  $\xi$  satisfies  $\text{Var}(\xi) \geq c_1\mu^2/k$  (the maximal possible lower bound in light of (3.1)), a similar argument shows the reverse inequality

$$(\mathbb{E}\xi^r)^{1/r} \geq \left(1 + \frac{c_2 r}{k}\right)\mu,$$

for all  $2 \leq r \leq c_3\sqrt{k}$ , where  $c_1, c_2, c_3 > 0$  are constants depending only on  $A, a > 0$ . Thus (3.2) is essentially tight for  $2 \leq r \leq c_3\sqrt{k}$ .

2. If  $\xi$  has sub-exponential tails, i.e.  $P(|\xi - \mu| > t\mu) \leq A \exp(-atk)$  for all  $t > 0$ , then for all  $s \geq 1$ ,

$$(\mathbb{E}|\xi - \mu|^s)^{1/s} \leq \mu \frac{c_1 s}{k}. \quad (3.4)$$

Moreover, for  $1 \leq r \leq ck$ , we have

$$(\mathbb{E}\xi^r)^{1/r} \leq \left(1 + \frac{c_3 r}{k^2}\right)\mu.$$

As above, if  $\text{Var}(\xi) \geq c'_1\mu^2/k^2$ , then the reverse estimate also holds, i.e.,

$$(\mathbb{E}\xi^r)^{1/r} \geq \left(1 + \frac{c'_2 r}{k^2}\right)\mu,$$

for  $2 \leq r \leq c'k$ , where  $c'_1, c'_2, c, c' > 0$  are constants depending only on  $A, a > 0$ .

### 3.2 Sharp reverse-Hölder inequalities for norms

We now turn to norms on  $\mathbb{R}^n$ . If  $A$  is a symmetric convex body in  $\mathbb{R}^n$  with norm  $\|\cdot\|_A$ , we write  $M(A) := \int_{S^{n-1}} \|\theta\|_A d\sigma(\theta)$  and  $b(A) := \sup_{\theta \in S^{n-1}} \|\theta\|_A$ . Set  $v(A) := \text{Var}_{\gamma_n} \|x\|_A$  and write  $m(A)$  for the median of the function  $\|\cdot\|_A$  with respect to the Gaussian measure  $\gamma_n$ , i.e.,

$$\gamma_n(\{x : \|x\|_A \leq m(A)\}) \geq \frac{1}{2} \text{ and } \gamma_n(\{x : \|x\|_A \geq m(A)\}) \geq \frac{1}{2}. \quad (3.5)$$

For  $-n < p \neq 0$ , we also define

$$I_p(\gamma_n, A) := \left( \int_{\mathbb{R}^n} \|x\|_A^p d\gamma_n(x) \right)^{\frac{1}{p}}.$$

Using polar coordinates,

$$I_p(\gamma_n, A) = a_{n,p} M_p(A), \quad a_{n,p} := I_p(\gamma_n, B_2^n). \quad (3.6)$$

We will use the standard concentration inequality for  $\|\cdot\|_A$  on  $\mathbb{R}^n$  equipped with  $\gamma_n$ , as well as a recent refinement.

**Theorem 3.5.** *Let  $\|\cdot\|_A$  be a norm associated with a symmetric convex body  $A$  in  $\mathbb{R}^n$ . Then for any  $t \geq 0$ ,*

$$\max\left\{\gamma_n(\{x : \|x\|_A < m(A) - t\}), \gamma_n(\{x : \|x\|_A \geq m(A) + t\})\right\} \leq \frac{1}{2} e^{-ct^2/b(A)^2}, \quad (3.7)$$

where  $c > 0$  is an absolute constant.

For further background on the latter theorem, see e.g. [Pis86], [MS86]. Recently, it has been observed that in the lower small deviation regime  $\{x : \|x\|_A < m - t\}$ , the following refinement holds [PVb] (recall that  $v(A) \leq b(A)^2$ ).

**Theorem 3.6.** *Let  $\|\cdot\|_A$  be a norm associated with a symmetric convex body  $A$  in  $\mathbb{R}^n$ . Then for any  $t \geq 0$ ,*

$$\gamma_n(\{x : \|x\|_A \leq m(A) - t\}) \leq \frac{1}{2} e^{-ct^2/v(A)}, \quad (3.8)$$

where  $c > 0$  is an absolute constant.

For a convex body  $A \subseteq \mathbb{R}^n$ , the Dvoretzky number  $k(A)$  is the maximum  $k \leq n$  such that a  $\nu_{n,k}$ -random subspace  $E$  has the property that  $A \cap E$  is 4-isomorphic to the Euclidean ball of radius  $\frac{1}{M(A)}$  with probability at least 1/2.

Milman's formula (see [Mil71], [MS86]) states that  $k(A) \simeq n \frac{M(A)^2}{b(A)^2}$ . Moreover if  $A$  is in John's position then  $k(A) \geq c \log n$  (see [MS86]). We write  $k_*(A) = k(A^\circ)$ . In this case Milman's formula becomes

$$k_*(A) \simeq n \frac{w(A)^2}{R(A)^2}. \quad (3.9)$$

For completeness, we also recall a definition of Klartag and Vershynin from [KV07]. For a symmetric convex body  $A \subseteq \mathbb{R}^n$ , let

$$d(A) = \min(-\log \sigma\{\theta \in S^{n-1} : 2\|\theta\|_A \leq M(A)\}, n).$$

One can check that  $d(A) \geq ck(A)$  (see, e.g., [KV07]). We also set  $d_*(A) = d(A^\circ)$ .

We also define  $\beta(A)$  as the *normalized variance*, i.e.

$$\beta(A) = \frac{\text{Var}_{\gamma_n} \|g\|_A}{(\mathbb{E}_{\gamma_n} \|g\|_A)^2},$$

where  $g$  is an  $n$ -dimensional standard Gaussian random vector (see [PVb] and [PVc] for related background). We write  $\beta_*(A) = \beta(A^\circ)$  and note that  $\beta(A) \leq c/k(A)$  (see e.g., [PVc]). As an application of inequality (3.7) and Proposition 3.3 we get

$$M_q(A) \leq M(A) \sqrt{1 + \frac{c_1 q}{k(A)}}, \quad (3.10)$$

for every  $q \geq 1$  (see also [PVZ] for an alternative proof which uses the log-Sobolev inequality). For comparison, we note that similar reverse Hölder inequalities have often been stated in the form

$$M_q(A) \leq M(A) \left(1 + \sqrt{\frac{c_1 q}{k(A)}}\right). \quad (3.11)$$

See for example, [LMS98, Statement 3.1] or [Led01, Proposition 1.10, (1.19)]. Thus in the range  $1 \leq q \leq k(A)$ , (3.10) improves upon (3.11).

In [PVb], using (3.8) and a small ball probability estimate in terms of  $\beta(A)$ , the following reverse Hölder inequalities for the Gaussian moments of  $x \mapsto \|x\|_A$  are obtained:

$$I_{-q}(\gamma_n, A) \geq m(A) \exp\left(-c_1 \max\{\sqrt{\beta}, q\beta\}\right), \quad (3.12)$$

for all  $0 < q < c_2/\beta$ , where  $\beta \equiv \beta(A)$ .

We will also use the following application of Proposition 3.1.

**Lemma 3.7.** *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$ . Then*

$$1 \leq \frac{I_1(\gamma_n, A)}{m(A)} \leq 1 + c\sqrt{\beta(A)}. \quad (3.13)$$

*Proof.* The left-hand side follows from the fact that  $x \mapsto \|x\|_A$  is convex combined with the main result of [Kwa94]. The right-hand side follows by Proposition 3.1 and the definition of  $\beta$  (and the standard fact that  $m(A) \simeq \mathbb{E}_{\gamma_n} \|g\|_A$ ).  $\square$

**Proposition 3.8.** *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$ . Then for all  $q \geq 1$ ,*

$$w_q(A) \leq w(A) \sqrt{1 + \frac{cq}{k_*(A)}}, \quad (3.14)$$

where  $c > 0$  is an absolute constant. Moreover, for all  $0 < q < c_2/\beta_*(A)$ ,

$$w_{-q}(A) \geq \left(1 - c_1 \min \left\{ \frac{q}{k_*(A)}, \max \left\{ \sqrt{\beta_*(A)}, q\beta_*(A) \right\} \right\} \right) w(A), \quad (3.15)$$

where  $c_1, c_2 > 0$  are absolute constants.

*Proof.* Inequality (3.14) is simply a reformulation of (3.10). For proving (3.15) first we combine (3.12) with Proposition 3.1 to get

$$\frac{I_{-q}(\gamma_n, A^\circ)}{I_1(\gamma_n, A^\circ)} \geq 1 - c_1 \max \left\{ \sqrt{\beta_*(A)}, q\beta_*(A) \right\},$$

for  $0 < q < c_2/\beta_*(A)$ . Furthermore, it is known that

$$\frac{I_{-q}(\gamma_n, A^\circ)}{I_1(\gamma_n, A^\circ)} \geq 1 - \frac{c_3 q}{k_*(A)},$$

for all  $0 < q < c_4 k_*(C)$ . (A proof of this fact can be found e.g. in [PVZ]). Using (3.6) we find

$$\frac{I_{-q}(\gamma_n, A^\circ)}{I_1(\gamma_n, A^\circ)} = \frac{I_{-q}(\gamma_n, B_2^n) w_{-q}(A)}{I_1(\gamma_n, B_2^n) w(A)} \leq \frac{w_{-q}(A)}{w(A)}.$$

Combining all these estimates we arrive at (3.15).  $\square$

**Theorem 3.9** (Concentration for mean width). *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$  and let  $1 \leq k \leq n-1$ . Then for all  $t > 0$ ,*

$$\nu_{n,k}(\{E \in G_{n,k} : |w(P_E A) - w(A)| > tw(A)\}) \leq c_1 \exp(-c_2 t^2 k k_*(A)). \quad (3.16)$$

Moreover, for all  $r > 0$ ,

$$\left( \int_{G_{n,k}} w(P_E A)^r d\nu_{n,k}(E) \right)^{1/r} \leq w(A) \sqrt{1 + \frac{c_1 r}{k k_*(A)}}.$$

*Proof.* For a proof of the first part, we refer the reader to [PVb, Prop. 3.9] (which is stated in the Gaussian setting); see also [PVc, §6]. The second part follows from the concentration estimate (3.16) and Proposition 3.3.  $\square$

The next Lemma has its origins in [Kla04], [KV07]. However, our formulation takes into account the order of magnitude of the constants involved; see [PVb, pg. 14] for a proof in the Gaussian setting and [PVc] for an alternative proof.

**Lemma 3.10** (Dimension lift). *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$  and let  $1 \leq k \leq n-1$ . Then for any  $q \geq k$ , we have*

$$\left( \int_{G_{n,k}} [r(P_E A)]^{-q} d\nu_{n,k}(E) \right)^{1/q} \leq \left( 1 + \frac{ck}{q} \log \left( \frac{eq}{k} \right) \right) \frac{w(A)}{[w_{-3q}(A)]^2},$$

where  $c > 0$  is an absolute constant.

## 4 Multidimensional concentration for the volume of projections

Now we turn to proving the main results of the paper. First we study the almost constant behavior of the mapping  $E \mapsto |P_E A|$ ,  $E \in G_{n,k}$  by establishing reverse-Hölder inequalities for positive and negative moments. Second, we prove the deviation inequalities announced in Theorem 1.2.

### 4.1 Reverse-Hölder inequalities for generalized intrinsic volumes

We start with an inequality for positive moments, which follows from Uryshon's inequality (2.3) and Theorem 3.9.

**Proposition 4.1.** *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$ ,  $1 \leq k \leq n-1$ . Then, for all  $p > 0$ ,*

$$W_{[k,p]}(A) \leq w(A) \sqrt{1 + \frac{c_1 p}{k_* (A)}}. \quad (4.1)$$

*Proof.* Using Uryshon's inequality (2.3) we have

$$W_{[k,p]}(A) = \left( \int_{G_{n,k}} \text{vrad}(P_E A)^{pk} d\nu_{n,k}(E) \right)^{\frac{1}{pk}} \leq \left( \int_{G_{n,k}} w(P_E A)^{pk} d\nu_{n,k}(E) \right)^{\frac{1}{pk}}.$$

Now we apply Theorem 3.9 to get

$$\left( \int_{G_{n,k}} w(P_E A)^{pk} d\nu_{n,k}(E) \right)^{\frac{1}{pk}} \leq w(A) \sqrt{1 + \frac{c_1 pk}{k k_* (A)}}.$$

□

**Proposition 4.2.** *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$ . Let  $2 \leq k \leq \frac{c_1}{\beta_* (A)}$ . Then,*

$$W_{[k,-p]}(A) \geq \left( 1 - c_2 \max \left\{ \sqrt{k \beta_* \log \left( \frac{e}{k \beta_*} \right)}, pk \beta_* \right\} \right) w(A) \quad (4.2)$$

for all  $0 < p \leq \frac{c_3}{k \beta_*}$ .

*Proof.* We may assume that  $p \geq 1$  and  $pk \leq c'_1 / \beta_*$ . Then for any  $pk \leq q \leq c'_1 / \beta_*$  (which will be suitably chosen later) we have

$$W_{[k,-p]}(A) \geq W_{[k,-q/k]}(A) \geq \left( \int_{G_{n,k}} [r(P_E A)]^{-q} d\nu_{n,k}(E) \right)^{-1/q}. \quad (4.3)$$

Using Lemma 3.10 and (3.15), (4.3) becomes:

$$W_{[k,-p]}(A) \geq w(A) \exp\left(-\frac{ck}{q} \log\left(\frac{eq}{k}\right) - \tau(q)\right),$$

for all  $pk \leq q \leq c'_1/\beta_*$ , where  $\tau(q) = \min\{q/k_*, \max\{\sqrt{\beta_*}, q\beta_*\}\}$ . The choice  $q = \sqrt{\frac{k}{\beta_*} \log\left(\frac{e}{k\beta_*}\right)}$  yields the estimate:

$$\frac{ck}{q} \log\left(\frac{eq}{k}\right) + \tau(q) \simeq \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)},$$

for all  $0 < p \leq \frac{q}{k} = \sqrt{\frac{1}{k\beta_*} \log\left(\frac{e}{k\beta_*}\right)}$ . On the other hand when  $p \geq \sqrt{\frac{1}{k\beta_*} \log\left(\frac{e}{k\beta_*}\right)}$ , we choose  $q = pk$  to obtain the estimate:

$$\frac{ck}{q} \log\left(\frac{eq}{k}\right) + \tau(q) \simeq \max\left\{\frac{\log p}{p}, pk\beta_*\right\} \simeq pk\beta_*.$$

Combining the above we get the result.  $\square$

Theorem 1.1 is an immediate consequence of the following corollary (with possibly adjusting the constant  $c$ ).

**Corollary 4.3.** *There exists  $c > 0$  such that if  $A$  is a symmetric convex body in  $\mathbb{R}^n$  and  $1 \leq k \leq c/\beta_*(A)$ , then*

$$\left(1 - c \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right) w(A) \leq W_{[k]}(A) \leq w(A). \quad (4.4)$$

*Proof.* The right-hand side inequality follows from Urysohn's inequality (2.3) applied for the body  $P_E A$ . The left-hand side inequality follows from the fact that  $W_{[k]}(A) \geq W_{[k,-1]}(A)$  and (4.2).  $\square$

**Theorem 4.4.** *There exist  $c, c_1, c_2, c_3, c_4$  such that the following holds: Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$ . Let  $2 \leq k \leq c/\beta_*(A)$  and  $0 < p < c_1 k_*(A)$ . Then,*

$$\frac{W_{[k,p]}(A)}{W_{[k]}(A)} \leq 1 + c_2 \max\left\{\frac{p}{k_*}, \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right\}. \quad (4.5)$$

Moreover, if  $2 \leq k \leq c_3/\beta_*(A)$  and  $0 < p < \frac{c}{k\beta_*(A)}$ , we have

$$\frac{W_{[k,-p]}(A)}{W_{[k]}(A)} \geq 1 - c_4 \max\left\{pk\beta_*, \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right\}. \quad (4.6)$$

*Proof.* Let  $0 < p < ck_*(A)$ . Then using (4.1) and (1.6), we get

$$\frac{W_{[k,p]}(A)}{W_{[k]}(A)} \leq \frac{\left(1 + \frac{cp}{k_*(A)}\right)w(A)}{\left(1 - c' \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right)w(A)} \leq 1 + c \max\left\{\frac{p}{k_*}, \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right\}.$$

Moreover, using (4.2) and (1.6) we obtain

$$\frac{W_{[k,-p]}(A)}{W_{[k]}(A)} \geq \frac{\left(1 - c \max\left\{pk\beta_*, \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right\}\right)w(A)}{w(A)}.$$

□

## 4.2 Deviation inequalities

In this section we prove Theorem 1.2. We consider the upper and lower inequalities separately.

**Theorem 4.5.** *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$  and let  $1 \leq k \leq c/\beta_*(A)$ . Then for all  $\varepsilon \geq c \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}$ ,*

$$\nu_{n,k}\left(E \in G_{n,k} : \text{vrad}(P_E A) \geq (1 + \varepsilon)W_{[k]}(A)\right) \leq C_1 \exp\left(-c_1 \varepsilon^2 k k_*(A)\right).$$

*Proof.* For  $\varepsilon \geq c \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}$ , we apply Corollary 4.3 to get

$$\left\{E \in G_{n,k} : \text{vrad}(P_E A) \geq (1 + \varepsilon)W_{[k]}(A)\right\} \subseteq \left\{E : w(P_E A) \geq \left(1 + \frac{\varepsilon}{2}\right)w(A)\right\}.$$

The result follows if we use the estimate from Theorem 3.9. □

**Theorem 4.6.** *Let  $A$  be a symmetric convex body in  $\mathbb{R}^n$  and let  $1 \leq k \leq c/\beta_*(A)$ . Then for all  $c_1 \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)} < \varepsilon < 1$ ,*

$$\nu_{n,k}\left(E \in G_{n,k} : \text{vrad}(P_E A) \leq (1 - \varepsilon)W_{[k]}(A)\right) \leq \exp\left(-c\varepsilon^2/\beta_*\right).$$

*Proof.* Let  $\varepsilon \in (0, 1)$ . For any  $p \geq \sqrt{\frac{1}{k\beta_*} \log\left(\frac{e}{k\beta_*}\right)}$  we apply Markov's inequality and Theorem 4.4 to get

$$\begin{aligned} \nu_{n,k}\left(\left\{E : \text{vrad}(P_E A) \leq (1 - \varepsilon)W_{[k]}(A)\right\}\right) &\leq e^{-\varepsilon pk} \left(\frac{W_{[k]}(A)}{W_{[k,-p]}(A)}\right)^{pk} \\ &\leq \exp\left(-\varepsilon pk + cp^2 k^2 \beta_*\right). \end{aligned}$$

Choosing  $p \simeq \frac{\varepsilon}{k\beta_*}$  we get the assertion provided that  $\varepsilon \geq c_1 \sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}$ . □

### 4.3 Application to $L_q$ -zonoids

Here we explain how one obtains Corollary 1.3. Let us first recall that a Borel measure  $\mu$  on  $S^{n-1}$  is isotropic if the covariance matrix of  $\mu$  is the identity, or equivalently, for each  $x \in \mathbb{R}^n$ ,

$$\|x\|_2^2 = \int_{S^{n-1}} \langle x, \theta \rangle^2 d\mu(\theta).$$

For such measures  $\mu$ , we recall upper bounds for  $\beta_*(Z_q(\mu))$  proved in [PVa].

**Lemma 4.7.** *There exists an absolute constant  $c > 0$  such that for any  $n$ , for any  $1 \leq q < \infty$  and any isotropic Borel measure  $\mu$  on  $S^{n-1}$ ,*

$$\beta_*(Z_q(\mu)) \leq \frac{e^{cq}}{n}.$$

In view of Lemma 4.7 and Theorem 1.1 we readily get Corollary 1.3. Note that for  $q = 1$  this almost recovers the asymptotic version of Hug-Schneider's reverse inequality for zonoids.

Also of interest is the case when  $A = B_p^n$  and  $p = p(n)$ . Note that this is nothing more than  $B_p^n = Z_q(\nu)$  with  $\nu = \sum_{j=1}^n \delta_{e_j}$ , where  $(e_j)_{j \leq n}$  is the standard basis of  $\mathbb{R}^n$  and  $1/p + 1/q = 1$ . If  $q = c_0 \log n$  for some suitably chosen absolute constant  $c_0 \in (0, 1)$ , we have  $k_*(B_p^n) \simeq \log n$  and  $\beta_*(B_p^n) \simeq n^{-\alpha}$ , while the behavior of  $d_*(B_p^n)$  is unclear. Precise asymptotic estimates for  $\beta(B_q^n)$  were proved in [PVZ], which we now recall as they show the latter lemma is sharp.

**Lemma 4.8.** *There exist absolute constants  $0 < c_0 < 1 < C$  such that for any  $n \geq 2$  and for all  $1 \leq q \leq c_0 \log n$ ,*

$$\frac{2^q}{Cq^2n} \leq \beta(B_q^n) \leq \frac{C2^q}{q^2n}.$$

By invoking Lemma 4.8 we get a special case of Corollary 1.3 for  $B_p^n$ 's with  $\frac{c_0 \log n}{c_0 \log n - 1} \leq p \leq \infty$  and for all  $n$  large enough.

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