Threshold for the measure of random polytopes

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Abstract

Starting with the work of Dyer, Füredi and McDiarmid who established a sharp threshold for the expected volume of random polytopes with independent vertices uniformly distributed in the discrete cube $E_2^n = \{-1,1\}^n$, in this survey article we focus on a very general variant of the problem. Let μ be a log-concave probability measure on $\mathbb{R}^n$ and for any $N > n$ consider the random polytope $K_N = \mathrm{conv}\{X_1,\dots,X_N\}$, where $X_1, X_2, \dots$ are independent random points in $\mathbb{R}^n$ distributed according to μ. We discuss an approach to the question if there exists a threshold for the expected measure $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of K_N, based on joint works with S. Brazitikos and M. Pafis, via the Cramér transform Λ_μ^* of μ. We show that, under some conditions, one has a sharp threshold for the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the measure of K_N: it is close to 0 if $\ln N \ll \mathbb{E}_\mu(\Lambda_\mu^*)$ and close to 1 if $\ln N \gg \mathbb{E}_\mu(\Lambda_\mu^*)$. The main condition is that the parameter $\beta(\mu) = \mathrm{Var}_\mu(\Lambda_\mu^*)/(\mathbb{E}_\mu(\Lambda_\mu^*))^2$ should be small.

1 The case of the discrete cube

Let X be a random vector in $\mathbb{R}^n$ with independent coordinates that take each of the values ± 1 with probability $\frac{1}{2}$. Given $N > n$, we consider N independent copies $X_1,\dots,X_N$ of the random vector X and define the random "0/1 polytope"

$$K_N = \mathrm{conv}\{X_1,\dots,X_N\} \subseteq Q_n := [-1,1]^n. \tag{1.1}$$

Dyer, Füredi and McDiarmid established in [16] a sharp threshold for the volume of these random 0/1 polytopes.

Theorem 1.1 (Dyer-Füredi-McDiarmid). *Let $\kappa = \ln 2 - \frac{1}{2}$ and for any $N > n$ consider the random polytope K_N defined by* (1.1)*. For every $\varepsilon \in (0,\kappa)$ we have that*

$$\lim_{n\to\infty} \sup\left\{2^{-n}\mathbb{E}\,|K_N| \colon N \leqslant \exp((\kappa-\varepsilon)n)\right\} = 0 \tag{1.2}$$

and

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}\,|K_N| \colon N \geqslant \exp((\kappa+\varepsilon)n)\right\} = 1. \tag{1.3}$$

In the statement of the theorem we denote by $|K_N|$ the n-dimensional volume of K_N; since $|Q_n| = 2^n$, the ratio $2^{-n}\mathbb{E}\,|K_N|$ is the "expected percentage" of Q_n occupied by K_N. In this section we present the main points of the proof of Theorem 1.1. Several of the lemmas that are used will be proved in a more general and stronger form later on.

The function $\varphi : (-1,1)^n \to \mathbb{R}$ defined by

$$\varphi(x) := \inf\left\{\mathrm{Prob}\big(X \in H\big) \colon x \in H,\ H \text{ is a closed half-space}\right\} \tag{1.4}$$

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plays a key role in the argument. Note that the infimum in (1.4) is determined by those half-spaces H for which x lies on the boundary $\partial(H)$ of H. Next, for any origin symmetric convex body $A \subset (-1,1)^n$ we define

$$\varphi_+(A) = \sup_{x \notin A} \varphi(x) \quad \text{and} \quad \varphi_-(A) = \inf_{x \in A} \varphi(x). \tag{1.5}$$

Lemma 1.2. *Let $N > n$ and let A be an origin symmetric convex body contained in $(-1,1)^n$. Then,*

$$\mathbb{E}(|K_N|) \leqslant |A| + 2^n N \varphi_+(A).$$

Proof. We write

$$\mathbb{E}\left(|K_N|\right) = \mathbb{E}\left(|K_N \cap A|\right) + \mathbb{E}\left(|K_N \setminus A|\right) \leqslant |A| + \mathbb{E}\left(|K_N \setminus A|\right). \tag{1.6}$$

Note that if H is a closed half-space containing x, and if $x \in K_N$, then we may find $i \leqslant N$ such that $X_i \in H$ (otherwise, we would have $x \in K_N \subseteq H'$, where H' is the complementary half-space of H). It follows that $\text{Prob}\big(x \in K_N\big) \leqslant N\,\varphi(x)$. Using Fubini's theorem we see that

$$\mathbb{E}\left(|K_N \setminus A|\right) = \int_{Q_n \setminus A} \text{Prob}(x \in K_N)\, dx \leqslant \int_{Q_n \setminus A} N\varphi(x)\, dx \leqslant N\,\varphi_+(A)\,|Q_n \setminus A|\,.$$

where in the last inequality we use the fact that $\varphi(x) \leqslant \varphi_+(A)$ for every $x \notin A$. Going back to (1.6) we get the lemma. $\square$

Lemma 1.2 will be useful for the proof of (1.1). It remains to choose, if possible, suitable A (depending on N and n) such that for all $N \leqslant \exp((\kappa - \varepsilon)n)$ we will have simultaneously $|A|/2^n \to 0$ and $N\varphi_+(A) \to 0$ as $n \to \infty$.

A second basic observation is given by the following lemma.

Lemma 1.3. *Let A be an origin symmetric convex body contained in $(-1,1)^n$. Then,*

$$1 - \text{Prob}\big(K_N \supseteq A\big) \leqslant \binom{N}{n} 2^{-(N-n)} + 2\binom{N}{n}\big(1 - \varphi_-(A)\big)^{N-n}.$$

We skip the proof since we shall discuss a more general version of Lemma 1.3 in Section 3 (see Lemma 3.8). What is important is that Lemma 1.3 allows us to use the function φ in order to prove (1.2). Roughly speaking, it remains to choose, if possible, suitable A (depending on N and n) so that for all $N \geqslant \exp((\kappa{+}\varepsilon)n)$ we will have simultaneously $|A|/2^n \to 1$ and $1 - \text{Prob}\big(K_N \supseteq A\big) \to 0$ as $n \to \infty$.

Given a bounded random variable X, consider the moment generating function $M(s) := \mathbb{E}\left(e^{sX}\right)$ and the logarithmic moment generating function $\Lambda(s) := \ln M(s)$ of X. Since X is bounded, we see that $M(s) < \infty$ for every $s \in \mathbb{R}$. By the symmetry of X it also follows that M and Λ are even. Using Hölder's inequality we easily check that Λ, and hence also M, is convex. Finally, M is C^∞ on $\mathbb{R}$; the n-th derivative of M is the function $M^{(n)}(s) = \mathbb{E}\left(X^n e^{sX}\right)$.

Returning to our case, where X takes the values ± 1 with probability $\frac{1}{2}$, direct computation shows that

$$M(s) = \cosh(s) \quad \text{and} \quad \Lambda(s) = \ln\cosh(s).$$

Consider the Legendre transform of Λ: this is the function

$$f(x) := \sup\left\{sx - \Lambda(s) \colon s \in \mathbb{R}\right\}, \qquad x \in (-1,1).$$

Direct computation shows that

$$f(x) = \tfrac{1}{2}(1+x)\ln(1+x) + \tfrac{1}{2}(1-x)\ln(1-x)$$

for every $x \in (-1,1)$. Note that f is an even convex function and $\lim\limits_{x \to \pm 1} f(x) = \ln 2$.

From the definition of f and Markov's inequality we get the next upper bound for $\varphi(x)$ in terms of $\sum_{i=1}^n f(x_i)$, for every $x \in (-1,1)^n$.

Lemma 1.4. *For every $x \in (-1,1)^n$ we have that $\varphi(x) \leqslant \exp\left(-\sum_{i=1}^n f(x_i)\right)$.*

Proof. Let H be a closed half-space such that $x \in \partial(H)$. Then, there exists $s = (s_1, \dots, s_n) \in \mathbb{R}^n$ such that $H = H(s) = \{y : \langle s, y - x\rangle \geqslant 0\}$. From Markov's inequality,

$$\begin{aligned}\mathrm{Prob}\big(X \in H(s)\big) &= \mathrm{Prob}\left(\sum_{i=1}^n s_i(X_i - x_i) \geqslant 0\right) \leqslant \mathbb{E}\left(\exp\Big(\sum_{i=1}^n s_i(X_i - x_i)\Big)\right) \\ &= \prod_{i=1}^n \mathbb{E}\left(\exp(s_i(X_i - x_i)\right) = \prod_{i=1}^n \exp(\Lambda(s_i) - s_i x_i).\end{aligned}$$

By the definition of $\varphi(x)$ we have

$$\varphi(x) \leqslant \inf_{s\in\mathbb{R}^n} \prod_{i=1}^n \exp(\Lambda(s_i) - s_i x_i) = \prod_{i=1}^n \exp(-\sup\{s x_i - \Lambda(s) :\, s \in \mathbb{R}\}) = \prod_{i=1}^n \exp(-f(x_i)).$$

This proves the lemma. $\square$

We extend f continuously on $[-1,1]$ setting $f(\pm 1) = \ln 2$ and for every $x = (x_1, \dots, x_n) \in Q_n$ we set

$$F(x) = \frac{1}{n}\sum_{i=1}^n f(x_i).$$

For every $0 < t < \ln 2$, we define

$$F_t = \{x \in (-1,1)^n \colon F(x) \leqslant t\}.$$

Since f is even and convex on $(-1,1)$, the set F_t is an origin symmetric convex body contained in $(-1,1)^n$. From the definition of F_t we see that $\sum_{i=1}^n f(x_i) = nF(x) = tn$ for all $x \in \partial(F_t)$. Therefore, Lemma 1.4 and the definition of φ_- give us the next fact.

Lemma 1.5. *Let $0 < t < \ln 2$. For every $x \in (-1,1)^n$ we have $\varphi(x) \leqslant \exp(-nF(x))$. In particular,*

$$\varphi_+(F_t) \leqslant \exp(-tn).$$

Let $U_1, \dots, U_n$ be independent random variables, uniformly distributed in $(-1,1)$. Then, for every $0 < t < \ln 2$,

$$2^{-n}\,|F_t| = \mathrm{Prob}((U_1, \dots, U_n) \in F_t) = \mathrm{Prob}\left(\frac{1}{n}\sum_{i=1}^n f(U_i) \leqslant t\right).$$

Note that

$$\kappa = \mathbb{E}\left(f(U_i)\right) = \tfrac{1}{2}\int_{-1}^1 f(x)dx = \ln 2 - \tfrac{1}{2}.$$

By the law of large numbers we conclude the following.

Lemma 1.6. *For every $t \in (0, \kappa)$ we have $\lim_{n\to\infty} 2^{-n}\,|F_t| = 0$ and, similarly, for every $t \in (\kappa, \ln 2)$ we have $\lim_{n\to\infty} 2^{-n}\,|F_t| = 1$.*

Now, we can prove the first part of the theorem of Dyer, Füredi and McDiarmid.

Proposition 1.7. *For every $\varepsilon \in (0, \kappa)$,*

$$\lim_{n\to\infty} \sup\left\{2^{-n} E(|K_N|) \colon N \leqslant \exp((\kappa - \varepsilon)n)\right\} = 0.$$

Proof. We choose $t = \kappa - \varepsilon/2$. From Lemma 1.6 we have that $\lim_{n\to\infty} 2^{-n}|F_t| = 0$. On the other hand, if $N \leqslant \exp((\kappa - \varepsilon)n)$, then Lemma 1.5 gives

$$N\varphi_+(F_t) \leqslant \exp(-\varepsilon n/2).$$

Applying Lemma 1.2 with $A = F_t$ we get

$$2^{-n}\mathbb{E}\,(|K_N|) \leqslant 2^{-n}\,|F_t| + \exp(-\varepsilon n/2),$$

and Lemma 1.6 shows that the right hand side tends to 0 as $n \to \infty$. $\square$

For the proof of (1.2) we need to estimate $\varphi(x)$ from below in order to use Lemma 1.3. The basic technical step is the next proposition, which will be discussed, in a more general context, in Section 3.

Proposition 1.8. *For every $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$, depending only on ε, such that for every $0 < t < \ln 2$ and every $n \geqslant n(\varepsilon)$ we have*

$$\varphi_-(F_t) \geqslant \exp(-t(1+\varepsilon)n).$$

Then, the proof of (1.2) is simple.

Proposition 1.9. *For every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}\,(|K_N|) \colon N \geqslant \exp((\kappa+\varepsilon)n)\right\} = 1.$$

Proof. Fix $\varepsilon > 0$ and choose $t = \kappa + \varepsilon/3$. Combining Lemma 1.3 with Proposition 1.8 we see that if $n \geqslant n(\varepsilon)$ and $N \geqslant \exp((\kappa+\varepsilon)n) \geqslant \exp((t + 2\varepsilon/3)n)$, then

$$\mathbb{E}\,(|K_N|) \geqslant |F_t|\ \mathrm{Prob}\big(K_N \supseteq F_t\big) \geqslant |F_t|\,(1 - 2^{-n+1}).$$

Since $t > \kappa$, Lemma 1.6 shows that $\lim_{n\to\infty} 2^{-n}\,|F_t| = 1$, and the result follows. $\square$

Special cases of the threshold problem have been studied in various works. In addition to the case of the discrete cube, Dyer, Füredi and McDiarmid established in [16] a sharp threshold for the expected volume of random polytopes with vertices uniformly distributed in the solid cube $Q_n = [-1,1]^n$. If $\kappa = \ln(2\pi) - \gamma - \frac{1}{2}$, where γ is the Euler constant, then for every $\varepsilon \in (0,\kappa)$ one has

$$\lim_{n\to\infty} \sup\left\{2^{-n}\mathbb{E}|K_N| \colon N \leqslant \exp((\kappa-\varepsilon)n)\right\} = 0$$

and

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}|K_N| \colon N \geqslant \exp((\kappa+\varepsilon)n)\right\} = 1.$$

These results were generalized in [20] to the setting where the vertices of K_N have independent coordinates whose distribution is a fixed even measure with compact support in $\mathbb{R}$ that satisfies some mild condition (see Section 8).

The articles [35] and [4], [5] address the same question for a number of cases where X_i have a rotationally invariant density supported on the Euclidean unit ball B_2^n. More precisely, Pivovarov proved in [35] that if the vertices of K_N are uniformly distributed on the unit ball B_2^n then, for any $\varepsilon \in (0,1)$, if $N \leqslant \exp\big((1-\varepsilon)\frac{n}{2}\ln n\big)$ then $\mathbb{E}\,|K_N|/|B_2^n|$ tends to 0 and if $N \geqslant \exp\big((1+\varepsilon)\frac{n}{2}\ln n\big)$ then $\mathbb{E}\,|K_N|/|B_2^n|$ tends to 1 as $n \to \infty$. In the same work, he studied the case where the vertices of K_N are distributed according to the uniform measure on the unit sphere S^{n-1} or according to the standard Gaussian measure γ_n on $\mathbb{R}^n$; in the latter case, for a large range of $r = r_n > 0$ he established a sharp threshold for the ratio $\mathbb{E}\,|K \cap rB_2^n|/|rB_2^n|$ as $n \to \infty$. The work [4] of Bonnet, Chasapis, Grote, Temesvari and Turchi deals with the case where the vertices of K_N are distributed according to the measure with density $(1-|x|^2)^\beta$ or $(1-|x|^2/\sigma^2)^{-\beta}$ on B_2^n, where $\beta > -1$ in the first case and $\beta > \frac{n}{2}$ and $\sigma > 0$ in the second case. Sharper estimates for these models, describing the phase transition as well as its shape, were obtained in [5].

Exponential in the dimension upper and lower thresholds are obtained in [19] for the case where X_i are uniformly distributed in a simplex (then, the result can be extended to simplicial polytopes in $\mathbb{R}^n$). Let $\Omega_n = \{x = (x_1, \ldots, x_n) : x_1 + \cdots + x_n = 1,\ x_i \geqslant 0\}$ be the standard embedding of the $(n-1)$-dimensional simplex in n-dimensional space. If $N > C_0^n$, where $C_0 > 0$ is an absolute constant, then

$$\mathbb{E}\,|\mathrm{conv}\{x_1, \ldots, x_N\}| \geqslant (1 - e^{-c_0\sqrt{n}})|\Omega_n|.$$

A second main result of the same paper provides an upper threshold. For every $\varepsilon > 0$, if $N < e^{(\gamma-\varepsilon)n}$, where γ is the Euler constant, then the convex hull of N random points $x_1, \ldots, x_N$ uniformly distributed in Ω_n satisfies $\mathbb{E}\,|\mathrm{conv}\{x_1, \ldots, x_N\}|/|\Omega_n| \to 0$ as $n \to \infty$. To this end, the authors compute the Legendre transform of the log-moment generating function of a random vector X uniformly distributed in the simplex.

2 Log-concave probability measures

We would like to formulate and study the question how to obtain a threshold for the expected measure of a random polytope defined as the convex hull of independent random points with a log-concave distribution. Consider a log-concave probability measure μ on $\mathbb{R}^n$ and let $X_1, X_2, \ldots$ be a sequence of independent random points in $\mathbb{R}^n$ distributed according to μ. Then, for any $N > n$ we may define the random polytope

$$K_N = \mathrm{conv}\{X_1, \ldots, X_N\}.$$

We are interested in the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the μ-measure of K_N with respect to the product measure $\mu^N = \mu \otimes \cdots \otimes \mu$ (N times). This is an affinely invariant quantity, and hence we may assume that μ is centered, i.e. the barycenter of μ is at the origin.

Given $\delta \in (0,1)$ we say that μ satisfies a "δ-upper threshold" with constant ϱ_1 if

$$\sup\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \leqslant \exp(\varrho_1 n)\} \leqslant \delta \tag{2.1}$$

and that μ satisfies a "δ-lower threshold" with constant ϱ_2 if

$$\inf\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \geqslant \exp(\varrho_2 n)\} \geqslant 1 - \delta. \tag{2.2}$$

Then, we define $\varrho_1(\mu, \delta) = \sup\{\varrho_1 : \text{(2.1) holds true}\}$ and $\varrho_2(\mu, \delta) = \inf\{\varrho_2 : \text{(2.2) holds true}\}$. Our main goal is to obtain upper bounds for the difference

$$\varrho(\mu, \delta) := \varrho_2(\mu, \delta) - \varrho_1(\mu, \delta)$$

for any fixed $\delta \in \left(0, \frac{1}{2}\right)$.

One may also consider a sequence $\{\mu_n\}_{n=1}^{\infty}$ of log-concave probability measures, where μ_n is on $\mathbb{R}^n$, and say that $\{\mu_n\}_{n=1}^{\infty}$ exhibits a "sharp threshold" if there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$ of positive reals such that $\delta_n \to 0$ and $\varrho(\mu_n, \delta_n) \to 0$ as $n \to \infty$.

We shall describe a general approach to the problem, that was proposed in [11], working with an arbitrary log-concave probability measure μ on $\mathbb{R}^n$. We present the main ideas, the progress that has been achieved (especially in the case of the uniform measure on a convex body) and several remaining open questions. In the remaining part of this section we provide the necessary background information.

Throughout this article we write $\langle \cdot, \cdot \rangle$ for the standard inner product in $\mathbb{R}^n$ and denote the Euclidean norm by $|\cdot|$, the Euclidean unit ball by B_2^n and the unit sphere by S^{n-1}. Lebesgue measure in $\mathbb{R}^n$ is denoted by $|\cdot|$ and σ is the rotationally invariant probability measure on S^{n-1}. We use the letters c, c', c_j, c_j' etc. to denote absolute positive constants whose value may change from line to line.

A convex body in $\mathbb{R}^n$ is a compact convex set $K \subset \mathbb{R}^n$ with non-empty interior. We often consider bounded convex sets K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$; since the closure of such a set is a convex body, we shall call these sets convex bodies too. We say that K is centrally symmetric if $-K = K$ and that K is centered if the barycenter $\mathrm{bar}(K) = \frac{1}{|K|}\int_K x\,dx$ of K is at the origin. The radial function ϱ_K of a convex body K with $0 \in \mathrm{int}(K)$ is defined for all $x \neq 0$ by $\varrho_K(x) = \sup\{\lambda > 0 : \lambda x \in K\}$ and the support function of K is given by $h_K(x) = \sup\{\langle x, y\rangle : y \in K\}$ for all $x \in \mathbb{R}^n$. The polar body K° of a convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$ is the convex body

$$K^\circ := \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\big\}.$$

A Borel measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(H) < 1$ for every hyperplane H in $\mathbb{R}^n$ and $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any compact subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$. A theorem of Borell [6] shows that under these assumptions, μ has a log-concave density f_μ. A function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set in $\mathbb{R}^n$ and the restriction of $\ln f$ to it is concave. If f has finite positive integral then there exist constants $A, B > 0$ such that $f(x) \leqslant Ae^{-B|x|}$ for all $x \in \mathbb{R}^n$ (see [12, Lemma 2.2.1]). In particular, f has finite moments of all orders. We say that μ is even if $\mu(-B) = \mu(B)$ for every Borel subset B of $\mathbb{R}^n$ and that μ is centered if

$$\mathrm{bar}(\mu) := \int_{\mathbb{R}^n} \langle x, \xi\rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \xi\rangle f_\mu(x) dx = 0$$

for all $\xi \in S^{n-1}$. We shall use the fact that if μ is a centered log-concave probability measure on $\mathbb{R}^k$ then

$$\|f_\mu\|_\infty \leqslant e^k f_\mu(0). \tag{2.3}$$

This is a result of Fradelizi from [17]. Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that the indicator function $\mathbb{1}_K$ of K is the density of a log-concave measure, the Lebesgue measure on K.

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x\in\mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}.$$

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered and $\mathrm{Cov}(\mu) = I_n$, where I_n is the identity $n \times n$ matrix. In this case, $L_\mu = \|f_\mu\|_\infty^{1/n}$. For every μ there exists an invertible affine transformation T such that the push forward $T_*\mu$ of μ defined by $T_*\mu(B) = \mu(T^{-1}(B))$ for every Borel subset B of $\mathbb{R}^n$ is isotropic. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_\mu : \mu \text{ is an isotropic log-concave probability measure on } \mathbb{R}^n\} \leqslant C$$

for all $n \geqslant 1$. Bourgain [8] established the upper bound $L_n \leqslant c\sqrt[4]{n}\ln n$; later, Klartag, in [26], improved this estimate to $L_n \leqslant c\sqrt[4]{n}$. In a breakthrough work, Chen [14] proved that for any $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $L_n \leqslant n^\varepsilon$ for every $n \geqslant n_0(\varepsilon)$. Subsequently, Klartag and Lehec [29] showed that $L_n \leqslant c(\ln n)^4$, and very recently Klartag [28] achieved the best known bound $L_n \leqslant c\sqrt{\ln n}$.

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. The logarithmic Laplace transform of μ is the function

$$\Lambda_\mu(\xi) = \ln\left(\int_{\mathbb{R}^n} e^{\langle \xi, z\rangle} f_\mu(z)dz\right).$$

It is easy to check that Λ_μ is convex and $\Lambda_\mu(0) = 0$. Since $\mathrm{bar}(\mu) = 0$, Jensen's inequality shows that $\Lambda_\mu(\xi) \geqslant 0$ for all ξ. One can also check that the set $A(\mu) = \{\Lambda_\mu < \infty\}$ is open and Λ_μ is C^∞ and strictly convex on $A(\mu)$. The Legendre transform of Λ_μ defined by

$$\Lambda_\mu^*(x) = \sup_{\xi\in\mathbb{R}^n} \{\langle x, \xi\rangle - \Lambda_\mu(\xi)\}$$

is called the Cramér transform of μ and plays a crucial role in the theory of large deviations (see [15]). For every $t>0$ we also define the convex set

$$B_t(\mu) := \{x\in\mathbb{R}^n : \Lambda_\mu^*(x)\leqslant t\}.$$

A second important family of convex bodies associated to any log-concave probability measure μ on $\mathbb{R}^n$ is the family of L_t-centroid bodies $Z_t(\mu)$. For every $t\geqslant 1$ the body $Z_t(\mu)$ is the convex body with support function

$$h_{Z_t(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x,y\rangle|^t \, d\mu(x)\right)^{1/t}.$$

Note that $Z_t(\mu)$ is always centrally symmetric, and $Z_t(T_*\mu)=T(Z_t(\mu))$ for every $T\in GL(n)$ and $t\geqslant 1$. A centered log-concave probability measure μ is isotropic if and only if $Z_2(\mu)=B_2^n$. Paouris (see [12, Theorem 5.1.17]) established upper bounds for the volume of the L_t-centroid bodies of isotropic log-concave probability measures.

Theorem 2.1. *If μ is a centered log-concave probability measure on $\mathbb{R}^n$, then for every $2\leqslant t\leqslant n$ we have that*

$$|Z_t(\mu)|^{1/n}\leqslant c\sqrt{t/n}[\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $c>0$ is an absolute constant. In particular, if μ is isotropic then $|Z_t(\mu)|^{1/n}\leqslant c\sqrt{t/n}$ for all $2\leqslant t\leqslant n$.

A variant of the L_t-centroid bodies of μ is defined as follows. For every $t\geqslant 1$ we consider the convex body $Z_t^+(\mu)$ with support function

$$h_{Z_t^+(\mu)}(y) = \left(2\int_{\mathbb{R}^n}\langle x,y\rangle_+^t f_\mu(x)dx\right)^{1/t},$$

where $a_+=\max\{a,0\}$. When f_μ is even, it is clear that $Z_t^+(\mu)=Z_t(\mu)$. In any case, we easily verify that $Z_t^+(\mu)\subseteq 2^{1/t}Z_t(\mu)$. Moreover, if μ is isotropic then $Z_2^+(\mu)\supseteq cB_2^n$ for an absolute constant $c>0$. One can also check that if $1\leqslant t<s$ then

$$\left(\frac{2}{e}\right)^{\frac{1}{t}-\frac{1}{s}} Z_t^+(\mu)\subseteq Z_s^+(\mu)\subseteq c_1\left(\frac{2e-2}{e}\right)^{\frac{1}{t}-\frac{1}{s}}\frac{s}{t}Z_t^+(\mu).$$

The right-hand side inequality gives

$$\mathbb{E}_\mu(2\langle z,\xi\rangle_+^{2t}) = [h_{Z_{2t}^+(\mu)}(\xi)]^{2t}\leqslant C^{2t}[h_{Z_t^+(\mu)}(\xi)]^{2t} = C^{2t}[\mathbb{E}_\mu(2\langle z,\xi\rangle_+^t)]^2, \tag{2.4}$$

for all $\xi\in S^{n-1}$, where $C>1$ is an absolute constant. For a proof of all these claims see [22].

The next proposition compares $B_t(\mu)$ with $Z_t(\mu)$ (for a proof see [12, Proposition 15.3.7]).

Proposition 2.2. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Then, for any $t\geqslant 2$ we have*

$$B_t(\mu)\subseteq cZ_t(\mu)$$

where $c>0$ is an absolute constant.

Finally, if μ is a log-concave probability measure on $\mathbb{R}^n$ then, for every $t>0$, we define

$$K_t(\mu) := K_t(f_\mu) = \left\{x\in\mathbb{R}^n : \int_0^\infty r^{t-1}f_\mu(rx)\,dr\geqslant \frac{f_\mu(0)}{t}\right\}.$$

From the definition it follows that the radial function of $K_t(\mu)$ is given by

$$\varrho_{K_t(\mu)}(x) = \left(\frac{1}{f_\mu(0)}\int_0^\infty tr^{t-1}f_\mu(rx)\,dr\right)^{1/t} \tag{2.5}$$

for $x \neq 0$. The bodies $K_t(\mu)$ were introduced by K. Ball [2] who also established their convexity. If μ is additionally assumed centered then, for every $0 < t \leqslant s$,

$$\frac{\Gamma(t+1)^{\frac{1}{t}}}{\Gamma(s+1)^{\frac{1}{s}}} K_s(\mu) \subseteq K_t(\mu) \subseteq e^{\frac{n}{t}-\frac{n}{s}} K_s(\mu). \tag{2.6}$$

A proof is given in [12, Proposition 2.5.7]. It is easily checked that

$$|K_n(f)|\, f_\mu(0) = \int_{\mathbb{R}^n} f_\mu(x)dx = 1 \tag{2.7}$$

(see e.g. [12, Lemma 2.5.6]) and then we can use the inclusions (2.6) in order to estimate the volume of $K_t(\mu)$. For every $t > 0$ we have

$$e^{-1} \leqslant f_\mu(0)^{\frac{1}{n}+\frac{1}{t}} |K_{n+t}(\mu)|^{\frac{1}{n}+\frac{1}{t}} \leqslant e\frac{n+t}{n}. \tag{2.8}$$

We are mainly interested in the convex body $K_{n+1}(\mu)$. We shall use the fact that $K_{n+1}(\mu)$ is centered (see [12, Proposition 2.5.3 (v)]) and that

$$f_\mu(0)|K_{n+1}(\mu)| \approx 1. \tag{2.9}$$

The last estimate follows immediately from (2.7) and (2.8).

We conclude this section with some basic facts about κ-concave measures. Given $\kappa \in [-\infty, 1/n]$ we say that a measure μ on $\mathbb{R}^n$ is κ-concave if

$$\mu((1-\lambda)A + \lambda B) \geqslant ((1-\lambda)\mu^\kappa(A) + \lambda\mu^\kappa(B))^{1/\kappa} \tag{2.10}$$

for all compact subsets A, B of $\mathbb{R}^n$ with $\mu(A)\mu(B) > 0$ and all $\lambda \in (0,1)$. The limiting cases are defined appropriately. For $\kappa = 0$ the right hand side in (2.10) becomes $\mu(A)^{1-\lambda}\mu(B)^\lambda$ (therefore, 0-concave measures are the log-concave measures). In the case $\kappa = -\infty$ the right hand side in (2.10) becomes $\min\{\mu(A), \mu(B)\}$. Note that if μ is κ-concave and $\kappa_1 \leqslant \kappa$ then μ is κ_1-concave.

Next, let $\gamma \in [-\infty, \infty]$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called γ-concave if

$$f((1-\lambda)x + \lambda y) \geqslant ((1-\lambda)f^\gamma(x) + \lambda f^\gamma(y))^{1/\gamma}$$

for all $x, y \in \mathbb{R}^n$ with $f(x)f(y) > 0$ and all $\lambda \in (0,1)$. Again, we define the cases $\gamma = 0, +\infty$ appropriately. Borell [7] studied the relation between κ-concave probability measures and γ-concave functions and showed that if μ is a measure on $\mathbb{R}^n$ and the affine subspace F spanned by the support $\mathrm{supp}(\mu)$ of μ has dimension $\dim(F) = n$ then for every $-\infty \leqslant \kappa < 1/n$ we have that μ is κ-concave if and only if it has a non-negative density $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$ and ψ is γ-concave, where $\gamma = \frac{\kappa}{1-\kappa n} \in [-1/n, +\infty)$.

We refer to Schneider's book [38] for basic facts from the Brunn-Minkowski theory and to the book [1] for basic facts from asymptotic convex geometry. We also refer to [12] for more information on isotropic convex bodies and log-concave probability measures.

3 Tukey's half-space depth

Let μ be a probability measure on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x. The function

$$\varphi_\mu(x) = \inf\{\mu(H) : H \in \mathcal{H}(x)\}$$

is called Tukey's half-space depth. Tukey introduced the half-space depth for data sets in [40] as a measure of centrality for multivariate data that enables efficient visualization of random samples (some form of this notion had appeared in [23]). The term "depth" also comes from Tukey's article. The survey article of Nagy, Schütt and Werner [31] provides an overview of this topic, with an emphasis on its connections with convex geometry, and many references.

Tukey's half-space depth plays a key role in the study of the problem that we address in these notes. In this section we prove the basic results that we need, starting with estimates for the expectation

$$\mathbb{E}_\mu(\varphi_\mu) := \int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x)$$

of φ_μ with respect to μ. The question to provide an upper bound for this quantity was asked in [30] in connection with stochastic separability and applications to machine learning and error-correction mechanisms in artificial intelligence systems; motivation is given in [21] and in the references therein. More precisely, it was asked if there exists an absolute constant $c \in (0,1)$ such that $\mathbb{E}_\mu(\varphi_\mu) \leqslant c^n$ for all $n \geqslant 1$ and all log-concave probability measures μ on $\mathbb{R}^n$. The next theorem from [10] provides an affirmative answer (up to a $\ln n$-term).

Theorem 3.1. *Let μ be a log-concave probability measure on $\mathbb{R}^n$, $n \geqslant n_0$. Then, $\mathbb{E}_\mu(\varphi_\mu) \leqslant \exp\left(-cn/L_\mu^2\right)$ where L_μ is the isotropic constant of μ and $c > 0$, $n_0 \in \mathbb{N}$ are absolute constants.*

We shall use the next basic (and simple) lemma that generalizes Lemma 1.5.

Lemma 3.2. *Let μ be a Borel probability measure on $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ we have $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$. In particular, for any $t > 0$ and for all $x \notin B_t(\mu)$ we have that $\varphi_\mu(x) \leqslant \exp(-t)$.*

Proof. Let $x \in \mathbb{R}^n$. For any $\xi \in \mathbb{R}^n$ the half-space $\{z : \langle z - x, \xi\rangle \geqslant 0\}$ is in $\mathcal{H}(x)$, therefore

$$\varphi_\mu(x) \leqslant \mu(\{z : \langle z, \xi\rangle \geqslant \langle x, \xi\rangle\}) \leqslant e^{-\langle x,\xi\rangle}\mathbb{E}_\mu\big(e^{\langle z,\xi\rangle}\big) = \exp\big(-[\langle x, \xi\rangle - \Lambda_\mu(\xi)]\big),$$

and taking the infimum over all $\xi \in \mathbb{R}^n$ we see that $\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x))$, as claimed. □

Proof of Theorem 3.1. The quantity $\mathbb{E}_\mu(\varphi_\mu)$ is affinely invariant and hence for the proof of Theorem 3.1 we may assume that μ is isotropic. Using Lemma 3.2 we write

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) &\leqslant \int_{\mathbb{R}^n} e^{-\Lambda_\mu^*(x)} f_\mu(x)\, dx = \int_{\mathbb{R}^n} \left(\int_{\Lambda_\mu^*(x)}^\infty e^{-t} dt\right) f_\mu(x) dx \\ &= \int_0^\infty e^{-t} \int_{\mathbb{R}^n} \mathbb{1}_{B_t(\mu)}(x) f_\mu(x) dx\, dt = \int_0^\infty e^{-t}\mu(B_t(\mu))\, dt.\end{aligned}$$

Fix $b \in (2/n, 1/2]$ that will be specified in the end. Since $\mu(B_t(\mu)) \leqslant 1$ and also $\mu(B_t(\mu)) \leqslant \|f_\mu\|_\infty |B_t(\mu)|$ for all $t > 0$, we may write

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) &\leqslant \int_{bn}^\infty e^{-t}\mu(B_t(\mu)) dt + \|f_\mu\|_\infty \int_0^{bn} e^{-t}|B_t(\mu)|\, dt \\ &\leqslant \int_{bn}^\infty e^{-t}\, dt + L_\mu^n \int_0^2 e^{-t}|B_t(\mu)|\, dt + L_\mu^n \int_2^{bn} e^{-t}|B_t(\mu)|\, dt \\ &\leqslant e^{-bn} + L_\mu^n |B_2(\mu)| + L_\mu^n \int_2^{bn} e^{-t}|B_t(\mu)|\, dt.\end{aligned}$$

Applying Proposition 2.2 and Theorem 2.1 we get

$$|B_t(\mu)|^{1/n} \leqslant c_1 |Z_t(\mu)|^{1/n} \leqslant c_2\sqrt{t/n}$$

for all $2 \leqslant t \leqslant n$, where $c_1, c_2 > 0$ are absolute constants. It is also known that $L_\mu \geqslant c_3$ where $c_3 > 0$ is an absolute constant (see [12, Proposition 2.3.12] for a proof). So, we may assume that $c_2 L_\mu \geqslant \sqrt{2}$. Choosing $b_0 := 1/(c_2 L_\mu)^2 \leqslant 1/2$ we write

$$L_\mu^n \int_2^{b_0 n} e^{-t}|B_t(\mu)|\, dt \leqslant c_2^n L_\mu^n \int_2^{b_0 n} (t/n)^{n/2} e^{-t} dt = (c_2 L_\mu)^n \int_2^{b_0 n} (t/n)^{n/2} e^{-t} dt,$$

and since $b_0 n \leqslant n/2$ and the function $t \mapsto t^{n/2}e^{-t}$ is increasing on $[0, n/2]$, we get

$$(c_2 L_\mu)^n \int_2^{b_0 n} e^{-t}|B_t(\mu)|\,dt \leqslant (b_0 n - 2)\,(c_2 L_\mu)^n b_0^{n/2} e^{-b_0 n} = (b_0 n - 2)e^{-b_0 n}.$$

Moreover, $|B_2(\mu)|^{1/n} \leqslant c_2\sqrt{2/n}$, therefore

$$L_\mu^n |B_2(\mu)| \leqslant (c_4 L_\mu^2/n)^{n/2} \leqslant e^{-b_0 n},$$

because $c_4 L_\mu^2/n \leqslant e^{-2}$ if $n \geqslant n_0$. Combining the above we see that

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\,d\mu(x) \leqslant e^{-b_0 n} + e^{-b_0 n} + (b_0 n - 2)e^{-b_0 n} = b_0 n e^{-b_0 n},$$

and hence

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\,d\mu(x) \leqslant n \exp\left(-n/(c_2 L_\mu)^2\right)$$

which implies the result. □

The next theorem shows that, modulo the isotropic constant L_μ, the exponential estimate of Theorem 3.1 is sharp.

Theorem 3.3. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then,*

$$\int_{\mathbb{R}^n} \varphi_\mu(x) d\mu(x) \geqslant e^{-cn},$$

where $c > 0$ is an absolute constant.

The proof is based on a number of observations. First, by the affine invariance of $\mathbb{E}_\mu(\varphi_\mu)$, we may assume that μ is centered. As an application of the Paley-Zygmund inequality we obtain the next lemma.

Lemma 3.4. *Let $t \geqslant 1$ and $\delta \in (0,1)$. For every $x \in \delta Z_t^+(\mu)$ we have that*

$$\varphi_\mu(x) \geqslant (1-\delta^t)^2/C_1^t,$$

where $C_1 > 1$ is an absolute constant.

Proof. Let $x \in \delta Z_t^+(\mu)$. It is enough to show that

$$\inf \mu(\{z \in \mathbb{R}^n : \langle z, \xi\rangle \geqslant \langle x, \xi\rangle\}) \geqslant (1-\delta^t)^2/C_1^t \tag{3.1}$$

where the infimum is over all $\xi \in S^{n-1}$ with $\langle x, \xi\rangle \geqslant 0$, because if $\langle x, \xi\rangle < 0$ then Grünbaum's lemma (see [12, Lemma 2.2.6]) implies that $\mu_\xi(\{z : \langle z - x, \xi\rangle \geqslant 0\}) \geqslant 1/e$.

Since $x \in \delta Z_t^+(\mu)$, we have $\langle x, \xi\rangle \leqslant \delta h_{Z_t^+(\mu)}(\xi)$ for any such $\xi \in S^{n-1}$, so it is enough to show that

$$\mu(\{z \in \mathbb{R}^n : \langle z, \xi\rangle \geqslant \delta h_{Z_t^+(\mu)}(\xi)\}) \geqslant (1-\delta^t)^2/C_1^t. \tag{3.2}$$

We apply the Paley-Zygmund inequality

$$\mu(\{z : g(z) \geqslant \delta^t \mathbb{E}_\mu(g)\}) \geqslant (1-\delta^t)^2 \frac{[\mathbb{E}_\mu(g)]^2}{\mathbb{E}_\mu(g^2)}$$

for the function $g(z) = \langle z, \xi\rangle_+^t$. From (2.4) we see that

$$\mathbb{E}_\mu(g^2) \leqslant C_1^t\, [\mathbb{E}_\mu(g)]^2$$

for some absolute constant $C_1 > 0$, and the lemma follows. □

For every $t \geqslant 1$ we consider the convex set

$$R_t(\mu) = \{x \in \mathbb{R}^n : f_\mu(x) \geqslant e^{-t} f_\mu(0)\}.$$

Since f_μ is log-concave, we easily check that $R_t(\mu)$ is convex. Note also that $R_t(\mu)$ is bounded and $0 \in \mathrm{int}(R_t(\mu))$.

Lemma 3.5. *For every $t \geqslant 5n$ we have $R_t(\mu) \supseteq c_0 K_{n+1}(\mu)$, where $c_0 > 0$ is an absolute constant.*

Proof. Let $t \geqslant 5n$. For any $\xi \in S^{n-1}$ consider the log-concave function $h : [0, \infty) \to [0, \infty)$ with $h(t) = f_\mu(t\xi)$. Klartag has proved in [27, Lemma 5.2] that

$$\int_0^{\varrho_{R_t(\mu)}(\xi)} r^{n-1} h(r) dr \geqslant (1 - e^{-t/8}) \int_0^\infty r^{n-1} h(r) dr.$$

The definition of $K_n(\mu)$ gives

$$\int_0^\infty r^{n-1} h(r) dr = \frac{f_\mu(0)}{n} [\varrho_{K_n(\mu)}(\xi)]^n$$

and

$$\int_0^{\varrho_{R_t(\mu)}(\xi)} r^{n-1} h(r) dr \leqslant \|f\|_\infty \int_0^{\varrho_{R_t(\mu)}(\xi)} r^{n-1} dr = \frac{\|f\|_\infty}{n} [\varrho_{R_t(\mu)}(\xi)]^n.$$

Combining the above with the inequality $\|f\|_\infty \leqslant e^n f_\mu(0)$ from (2.3) we get

$$e^n [\varrho_{R_t(\mu)}(\xi)]^n \geqslant (1 - e^{-t/8}) [\varrho_{K_n(\mu)}(\xi)]^n.$$

This shows that $R_t(\mu) \supseteq c_0 K_n(\mu)$, where $c_0 > 0$ is an absolute constant. From (2.6) we know that $K_n(\mu) \approx K_{n+1}(\mu)$, and the lemma follows. $\square$

We can also compare $Z_t^+(\mu)$ with $K_{n+1}(\mu)$ when $t \geqslant 5n$.

Lemma 3.6. *For every $t \geqslant 5n$ we have that $Z_t^+(\mu) \supseteq c_0' K_{n+1}(\mu)$, where $c_0' > 0$ is an absolute constant.*

Proof. From Lemma 3.5 we know that $c_0 K_{n+1}(\mu) \subseteq R_t(\mu)$ for all $t \geqslant 5n$, where $c_0 > 0$ is an absolute constant. Let $\xi \in S^{n-1}$ and set $m_\xi := h_{c_0 K_{n+1}(\mu)}(\xi) = c_0 h_{K_{n+1}(\mu)}(\xi)$. Define

$$A_\xi = c_0 K_{n+1}(\mu) \cap \{x : \langle x, \xi \rangle \geqslant m_\xi / 2\}.$$

Since $K_{n+1}(\mu)$ is centered, one can check (see e.g. [25, Lemma 2.2] or [10, Proposition 4.1]) that

$$|A_\xi| \geqslant |c_0 K_{n+1}(\mu)| / C^n$$

for some absolute constant $C > c_0$. Moreover, if $x \in A_\xi$ then $x \in R_t(\mu)$ and hence $f_\mu(x) \geqslant e^{-t} f_\mu(0)$. We write

$$\begin{aligned} \int_{\mathbb{R}^n} \langle x, \xi \rangle_+^t d\mu(x) &\geqslant \int_{A_\xi} \langle x, \xi \rangle_+^t d\mu(x) \\ &\geqslant \left(\frac{m_\xi}{2}\right)^t e^{-t} f_\mu(0) |A_\xi| \geqslant \left(\frac{m_\xi}{2e}\right)^t \left(\frac{c_0}{C}\right)^n f_\mu(0) |K_{n+1}(\mu)|. \end{aligned}$$

Using also the fact that $(c_0/C)^n \geqslant (c_0/C)^t$ because $t \geqslant 5n$, we get

$$\int_{\mathbb{R}^n} \langle x, \xi \rangle_+^t d\mu(x) \geqslant (c_1 m_\xi)^t f_\mu(0) |K_{n+1}(\mu)|,$$

where $c_1 > 0$ is an absolute constant. Finally, $f_\mu(0)|K_{n+1}(\mu)| \approx 1$ by (2.9), which implies that

$$h_{Z_t^+(\mu)}(\xi) \geqslant c_2 m_\xi = c_0' h_{K_{n+1}(\mu)}(\xi),$$

where $c_0' = c_2 c_0$. $\square$

Proof of Theorem 3.3. Combining Lemma 3.5 and Lemma 3.6 we see that

$$R_{5n}(\mu) \cap Z^+_{5n}(\mu) \supseteq c_1 K_{n+1}(\mu)$$

for some absolute constant $c_1 > 0$. We apply Lemma 3.4 with $t = 5n$ and $\delta = \frac{1}{2}$. For every $x \in \frac{1}{2} Z^+_{5n}(\mu)$ we have

$$\varphi_\mu(x) \geqslant C_1^{-n}$$

for some absolute constant $C_1 > 1$. It follows that

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) \geqslant C_1^{-n} \mu\left(\tfrac{1}{2} Z^+_{5n}(\mu)\right).$$

Then, by Lemma 3.6 we have $\frac{1}{2} Z^+_{5n}(\mu) \supseteq \frac{c_1}{2} K_{n+1}(\mu)$. Since $\frac{c_1}{2} K_{n+1}(\mu) \subseteq R_{5n}(\mu)$, we know that $f_\mu(x) \geqslant e^{-5n} f_\mu(0)$ for all $x \in \frac{c_1}{2} K_{n+1}(\mu)$. Using also (2.9), we get

$$\begin{aligned}\mu\left(\tfrac{1}{2} Z^+_{5n}(\mu)\right) \geqslant \mu\left(\frac{c_1}{2} K_{n+1}(\mu)\right) &= \int_{\frac{c_1}{2} K_{n+1}(\mu)} f_\mu(x)\, dx \geqslant e^{-5n} f_\mu(0) \left|\frac{c_1}{2} K_{n+1}(\mu)\right| \\ &= e^{-5n} (c_1/2)^n f_\mu(0) |K_{n+1}(\mu)| \geqslant e^{-5n} c_2^n.\end{aligned}$$

Combining the above we conclude that

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) \geqslant C_1^{-n} e^{-5n} c_2^n \geqslant e^{-cn},$$

for some absolute constant $c > 0$. □

The half-space depth plays a key role in the study of the threshold problem. Let μ be a log-concave probability measure on $\mathbb{R}^n$. Let $X_1, X_2, \ldots$ be independent random points in $\mathbb{R}^n$ distributed according to μ and for any $N > n$ consider the random polytope $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$. We shall generalize Lemma 1.2 and Lemma 1.3 in this setting. To this end, for every convex body A in $\mathbb{R}^n$ with $0 \in \mathrm{int}(A)$ we define

$$\varphi_+(A) = \sup_{x \notin A} \varphi_\mu(x) \quad \text{and} \quad \varphi_-(A) = \inf_{x \in A} \varphi_\mu(x).$$

Recall that $B_t(\mu) = \{v \in \mathbb{R}^n : \Lambda^*_\mu(v) \leqslant t\}$, where Λ^*_μ is the Cramér transform of μ.

Lemma 3.7. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. For every convex body A in $\mathbb{R}^n$ and every $N > n$ we have that*

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \mu(A) + N\varphi_+(A).$$

In particular, for every $t > 0$,

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \mu(B_t(\mu)) + N \exp(-t).$$

Proof. We write

$$\mathbb{E}_{\mu^N}(\mu(K_N)) = \mathbb{E}_{\mu^N}(\mu(K_N \cap A)) + \mathbb{E}_{\mu^N}(\mu(K_N \setminus A)) \leqslant \mu(A) + \mathbb{E}_{\mu^N}(\mu(K_N \setminus A)).$$

Arguing as in the proof of Lemma 1.2 we see that $\mu^N\big(x \in K_N\big) \leqslant N\varphi_\mu(x)$ for any $x \in \mathbb{R}^n$. Then, Fubini's theorem shows that

$$\mathbb{E}_{\mu^N}(\mu(K_N \setminus A)) = \int_{\mathbb{R}^n \setminus A} \mu^N(x \in K_N)\, d\mu(x) \leqslant \int_{\mathbb{R}^n \setminus A} N\varphi_\mu(x)\, d\mu(x) \leqslant N\varphi_+(A).$$

The last claim follows if we set $A = B_t(\mu)$ because, by Lemma 3.2, $\varphi_\mu(x) \leqslant \exp(-\Lambda^*_\mu(x)) \leqslant e^{-t}$ for all $x \notin B_t(\mu)$. □

For the lower threshold we shall use the next lemma which is in the spirit of Lemma 1.3.

Lemma 3.8. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. For every convex body A in $\mathbb{R}^n$ and every $N > n$ we have that*

$$1-\mu^N(K_N \supseteq A) \leqslant 2\binom{N}{n}(1-\varphi_-(A))^{N-n}.$$

Therefore,

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \geqslant \mu(A)\left(1-2\binom{N}{n}(1-\varphi_-(A))^{N-n}\right).$$

Proof. Note that, with probability equal to 1 the random polytope K_N has non-empty interior. For every subset $J=\{j_1,\dots,j_n\}$ of $\{1,\dots,N\}$, of cardinality n, note that $X_{j_1},\dots,X_{j_n}$ are affinely independent with probability 1, and define the event L_J as follows: for one of the two closed half-spaces H_1,H_2 they determine, say H_i, we have simultaneously $K_N\subset H_i$ and $\mu\big(\mathbb{R}^n\setminus H_i\big)\geqslant \varphi_-(A)$.

If $A\nsubseteq K_N$, then there exists $x\in\partial(A)\setminus K_N$. Since $x\notin K_N$, there exists a facet F of K_N with the following property: one of the two closed half-spaces H_1 and H_2 determined by F contains K_N but does not contain x. Thus, if H_i is this half-space, we have simultaneously $K_N\subset H_i$ and $\mu\big(\mathbb{R}^n\setminus H_i\big)\geqslant\varphi_\mu(x)\geqslant\varphi_-(A)$. Since the hyperplane bounding H_i is determined by some affinely independent vertices $X_{j_1},\dots,X_{j_n}$ of K_N which lie in F, this shows that

$$\{A\nsubseteq K_N\}\subseteq\bigcup_J L_J.$$

It follows that

$$\mathrm{Prob}\big(A\nsubseteq K_N\big)\leqslant\sum_J\mathrm{Prob}(L_J)=\binom{N}{n}\mathrm{Prob}(L'),$$

where $L':=L_{\{1,\dots,n\}}$. It is not hard to see that

$$\mathrm{Prob}(L')\leqslant 2(1-\varphi_-(A))^{N-n}.$$

Indeed, $X_1,\dots,X_n$ determine two closed half-spaces $H_i=H_i(X_1,\dots,X_n)$, $i=1,2$. Let L^i be the event that $\mu\big(\mathbb{R}^n\setminus H_i\big)\geqslant\varphi_-(A)$. Then, with Exp denoting expectation with respect to the measure Prob,

$$\begin{aligned}\mathrm{Prob}(L')&\leqslant\sum_{i=1}^2\mathrm{Prob}\big(\{X_{n+1},\dots,X_N\in H_i\}\cap L^i\big)\\&=\sum_{i=1}^2\mathrm{Exp}\big(\mathrm{Prob}\big(\{X_{n+1},\dots,X_N\in H_i\}\mid X_1,\dots,X_n\big)\mathbb{1}_{L^i}\big)\\&\leqslant(1-\varphi_-(A))^{N-n}\sum_{i=1}^2\mathrm{Prob}(L^i).\end{aligned}$$

The second claim of the lemma follows from Markov's inequality. □

4 Rough upper and lower thresholds

Rough upper and lower thresholds were obtained by Chakraborti, Tkocz and Vritsiou in [13] for some general families of distributions. If μ is an even log-concave probability measure supported on a convex body K in $\mathbb{R}^n$ and if $X_1,X_2,\dots$ are independent random points distributed according to μ, then for any $n<N\leqslant\exp(c_1n/L_\mu^2)$ we have that

$$\frac{\mathbb{E}_{\mu^N}(|K_N|)}{|K|}\leqslant\exp\left(-c_2n/L_\mu^2\right),$$

where $c_1, c_2 > 0$ are absolute constants.

We shall describe a variant of this result for log-concave probability measures. We consider independent random points $X_1, X_2, \ldots$ in $\mathbb{R}^n$ distributed according to a log-concave probability measure μ and the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the μ-measure of K_N. Recall that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible affine transformation and $T_*\mu$ is the push-forward of μ then

$$\mathbb{E}_{(T_*\mu)^N}[(T_*\mu)(K_N)] = \mathbb{E}_{\mu^N}[\mu(K_N)].$$

So, we may assume that μ is isotropic.

Theorem 4.1. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$, $n \geqslant n_0$. For any $N \leqslant \exp(c_1 n/L_\mu^2)$ we have that*

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant 2\exp\left(-c_2 n/L_\mu^2\right),$$

where $c_1, c_2 > 0$ and $n_0 \in \mathbb{N}$ are absolute constants.

Proof. Using the estimate $\mu(B_t(\mu)) \leqslant \|f_\mu\|_\infty |B_t(\mu)|$, Proposition 2.2 and Theorem 2.1, from Lemma 3.7 we get

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \left(c_1\|f_\mu\|_\infty^{1/n}\sqrt{t/n}\right)^n + N\exp(-t)$$

for every $N > n$ and $2 \leqslant t \leqslant n$. Recall that μ is isotropic, therefore $\|f_\mu\|_\infty^{2/n} = L_\mu^2 = O(\ln n)$. Then, if $n \geqslant n_0$ where $n_0 \in \mathbb{N}$ is an absolute constant, we see that $t := (c_1 e)^{-2} n/\|f_\mu\|_\infty^{2/n}$ satisfies $2 \leqslant t \leqslant n$ and

$$\left(c_1\|f_\mu\|_\infty^{1/n}\sqrt{t/n}\right)^n \leqslant e^{-n}.$$

It follows that

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant e^{-n} + N\exp(-c_2 n/\|f_\mu\|_\infty^{2/n}),$$

where $c_2 = (c_1 e)^{-2}$. Then, if $N \leqslant \exp(c_3 n/\|f_\mu\|_\infty^{2/n})$ where $c_3 = c_2/2$, we see that

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant e^{-n} + \exp(-c_3 n/\|f_\mu\|_\infty^{2/n})$$

and the result follows from the fact that $\|f_\mu\|_\infty^{2/n} = L_\mu^2 \geqslant c$. $\square$

We pass now to the lower threshold. It was proved in [13] that if μ is an even κ-concave measure on $\mathbb{R}^n$ with $0 < \kappa < 1/n$, supported on a convex body K in $\mathbb{R}^n$, if $X_1, X_2, \ldots$ are independent random points in $\mathbb{R}^n$ distributed according to μ and $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$ as before, then for any $M \geqslant C$ and any $N \geqslant \exp\left(\frac{1}{\kappa}(\ln n + 2\ln M)\right)$ we have that

$$\frac{\mathbb{E}_{\mu^N}(|K_N|)}{|K|} \geqslant 1 - \frac{1}{M}, \tag{4.1}$$

where $C > 0$ is an absolute constant.

Since the family of log-concave probability measures corresponds to the case $\kappa = 0$, it is natural to ask for analogues of this result for 0-concave, i.e. log-concave, probability measures. In order to have a feeling, we should note that in the case where $X_1, X_2, \ldots$ are uniformly distributed in the Euclidean unit ball the sharp threshold for the problem (see [35] and [4]) is

$$\exp\left((1 \pm \varepsilon)\tfrac{1}{2}n\ln n\right), \qquad \varepsilon > 0.$$

We shall establish a weak lower threshold of this order.

Theorem 4.2. *Let $\delta \in (0,1)$. Then,*

$$\inf_{\mu}\Big(\inf\Big\{\mathbb{E}_{\mu^N}\big[\mu((1+\delta)K_N)\big] : N \geqslant \exp\big(C\delta^{-1}\ln(2/\delta)\, n\ln n\big)\Big\}\Big) \longrightarrow 1$$

as $n \to \infty$, where the first infimum is over all centered log-concave probability measures μ on $\mathbb{R}^n$ and $C > 0$ is an absolute constant.

This is a weak threshold in the sense that we consider the expected measure of $(1+\delta)K_N$ instead of K_N, where $\delta > 0$ is arbitrarily small. The reason for this is the dependence on δ in the next technical proposition (we omit the proof; see [10, Proposition 5.6] for the details).

Proposition 4.3. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $\delta \in (0,1)$ and any $t \geqslant C_\delta n\ln n$ we have that*

$$\mu((1+\delta)Z_t^+(\mu)) \geqslant 1 - e^{-c_\delta t}$$

where $C_\delta = C\delta^{-1}\ln(2/\delta)$ and $c_\delta = c\delta$ are positive constants depending only on δ.

Proof of Theorem 4.2. Let $0 < \delta < 1$ and set $\varepsilon = \delta/3$. Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Since the expectation $\mathbb{E}_{\mu^N}\big[\mu((1+\delta)K_N)\big]$ is a linearly invariant quantity, we may assume that μ is isotropic. From Lemma 3.4 we know that for every $x \in (1-\varepsilon)Z_t^+(\mu)$ we have

$$\varphi_\mu(x) \geqslant \frac{(1-(1-\varepsilon)^t)^2}{C_1^t},$$

where $C_1 > 1$ is an absolute constant. Then, taking into account the fact that $1-\varepsilon > 2/3$, we get

$$\mu^N\Big(K_N \supseteq (1-\varepsilon)Z_t^+(\mu)\Big) \geqslant 1 - 2\binom{N}{n}\left[1 - \frac{(1-(1-\varepsilon)^t)^2}{C_1^t}\right]^{N-n}.$$

By the mean value theorem we have $1-(1-\varepsilon)^t = t\varepsilon z^{t-1}$ for some $z \in (1-\varepsilon, 1)$, and hence $1-(1-\varepsilon)^t \geqslant t\varepsilon(1-\varepsilon)^{t-1}$. Taking also into account the fact that $1-\varepsilon > 2/3$, we get

$$\begin{aligned}\mu^N\Big(K_N \supseteq (1-\varepsilon)Z_t^+(\mu)\Big) &\geqslant 1 - 2\binom{N}{n}\left[1 - \frac{(t\varepsilon(1-\varepsilon)^{t-1})^2}{C_1^t}\right]^{N-n}\\ &\geqslant 1 - \left(\frac{2eN}{n}\right)^n \exp\left(-(N-n)\frac{(t\varepsilon)^2}{(3C_1)^t}\right).\end{aligned}$$

This last quantity tends to 1 as $n \to \infty$ if

$$(3C_1)^t n\ln(4eN/n) < (N-n)(t\varepsilon)^2, \tag{4.2}$$

and assuming that $\delta \in (1/n^2, 1)$ and $t \geqslant C_\varepsilon n\ln n$ where C_ε is the constant from Proposition 4.3, we check that (4.2) holds true if $N \geqslant \exp(C_2 t)$ for a large enough absolute constant $C_2 > 0$.

Note that $\varepsilon = \delta/3$ implies that $1+\delta > \frac{1+\varepsilon}{1-\varepsilon}$. Then, if $N \geqslant \exp(C_2 C_\varepsilon n\ln n)$ we see that

$$\begin{aligned}\mathbb{E}_{\mu^N}\left[\mu\left((1+\delta)K_N\right)\right] &\geqslant \mathbb{E}_{\mu^N}\left[\mu\left(\frac{1+\varepsilon}{1-\varepsilon}K_N\right)\right] \geqslant \mu((1+\varepsilon)Z_t^+(\mu)) \times \mu^N\Big(K_N \supseteq (1-\varepsilon)Z_t^+(\mu)\Big)\\ &\geqslant \left(1 - e^{-c\varepsilon t}\right)\left[1 - \left(\frac{2eN}{n}\right)^n \exp\left(-(N-n)\frac{(t\varepsilon)^2}{(3C_1)^t}\right)\right] \longrightarrow 1\end{aligned}$$

as $n \to \infty$. $\square$

The next theorem provides an estimate where "δ is removed", however the dependence on n becomes worse.

Theorem 4.4. *There exists an absolute constant $C>0$ such that*

$$\inf_{\mu}\Big(\inf\Big\{\mathbb{E}_{\mu^N}\big[\mu(K_N)\big]:N\geqslant\exp(C(n\ln n)^2u(n))\Big\}\Big)\longrightarrow 1$$

as $n\to\infty$, where the first infimum is over all log-concave probability measures μ on $\mathbb{R}^n$ and $u(n)$ is any function with $u(n)\to\infty$ as $n\to\infty$.

Proof. Let μ be a log-concave probability measure on $\mathbb{R}^n$. Since the expectation $\mathbb{E}_{\mu^N}\big[\mu(K_N)\big]$ is an affinely invariant quantity, we may assume that μ is centered. Note that if $A\subset\mathbb{R}^n$ is a Borel set, then

$$\mu((1+\delta)A)=\int_{(1+\delta)A}f_\mu(x)\,dx=(1+\delta)^n\int_A f_\mu((1+\delta)x)\,dx.$$

Since f_μ is log-concave, we see that

$$f_\mu((1+\delta)x)\leqslant f_\mu(x)\left(\frac{f_\mu(x)}{f_\mu(0)}\right)^{\delta}\leqslant e^{n\delta}f_\mu(x)$$

for every $x\in\mathbb{R}^n$, because $f_\mu(x)\leqslant e^nf_\mu(0)$ by (2.3). It follows that

$$\mu((1+\delta)A)\leqslant(1+\delta)^ne^{n\delta}\mu(A)\leqslant e^{2n\delta}\mu(A). \tag{4.3}$$

Given a function $u(n)$ with $u(n)\to\infty$ as $n\to\infty$, choose $\delta_n=(nu(n))^{-1}$. From (4.3) we see that

$$\mathbb{E}_{\mu^N}\big[\mu(K_N)\big]\geqslant e^{-2n\delta_n}\mathbb{E}_{\mu^N}\big[\mu((1+\delta_n)K_N)\big].$$

Therefore, we see that

$$\begin{aligned}&\inf_{\mu}\Big(\inf\Big\{\mathbb{E}_{\mu^N}\big[\mu(K_N)\big]:N\geqslant\exp\big(C\delta_n^{-1}\ln(2/\delta_n)\,n\ln n\big)\Big\}\Big)\\ &\qquad\geqslant e^{-2n\delta_n}\inf_{\mu}\Big(\inf\Big\{\mathbb{E}_{\mu^N}\big[\mu((1+\delta_n)K_N)\big]:N\geqslant\exp\big(C\delta_n^{-1}\ln(2/\delta_n)\,n\ln n\big)\Big\}\Big)\longrightarrow 1\end{aligned}$$

as $n\to\infty$, using Theorem 4.2 and the fact that $e^{-2n\delta_n}=e^{-2/u(n)}\to 1$. We may clearly assume that $u(n)=O(n)$. Then,

$$\delta_n^{-1}\ln(2/\delta_n)\,n\ln n=n^2\ln n\ln(2nu(n))u(n)\approx(n\ln n)^2u(n),$$

and the result follows. $\square$

5 Comparing half-space depth with the Cramér transform

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$ with density $f:=f_\mu$. Recall that for every $t>0$ we consider the convex set $B_t(\mu):=\{x\in\mathbb{R}^n:\Lambda_\mu^*(x)\leqslant t\}$, and for any $x\in\mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x and consider Tukey's half-space depth $\varphi_\mu(x)=\inf\{\mu(H):H\in\mathcal{H}(x)\}$. In Lemma 3.2 we showed that for every $x\in\mathbb{R}^n$ we have $\varphi_\mu(x)\leqslant\exp(-\Lambda_\mu^*(x))$, which implies that

$$\varphi_+(B_t(\mu))\leqslant e^{-t}$$

for any $t>0$. Our aim in this section is to obtain a lower bound for $\varphi_-(B_t(\mu))$, or equivalently for $\varphi_\mu(x)$ when $x\in B_t(\mu)$. First, we consider the case where $\mu=\mu_K$ is the uniform measure on a centered convex body K of volume 1 in $\mathbb{R}^n$, and prove the following.

Theorem 5.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $t>0$ we have that*

$$\inf\{\varphi_{\mu_K}(x):x\in B_t(\mu_K)\}\geqslant\frac{1}{10}\exp(-t-2\sqrt{n}).$$

The first part of the argument works for any centered log-concave probability measure μ with density f on $\mathbb{R}^n$. For every $\xi \in \mathbb{R}^n$ we define the probability measure μ_ξ with density

$$f_\xi(z) = e^{-\Lambda_\mu(\xi) + \langle \xi, z\rangle} f(z).$$

One can compute that the barycenter of μ_ξ is $x = \nabla\Lambda_\mu(\xi)$ and $\mathrm{Cov}(\mu_\xi) = \mathrm{Hess}\,(\Lambda_\mu)(\xi)$ (a proof is given in [12, Proposition 7.2.1]). Next, we set

$$\sigma_\xi^2 = \int_{\mathbb{R}^n} \langle z - x, \xi\rangle^2 d\mu_\xi(z) = \mathrm{Var}_{\mu_\xi}(\langle \xi, z\rangle).$$

Let $t > 0$. Since $B_t(\mu)$ is convex, in order to give a lower bound for $\inf\{\varphi_\mu(x) : x \in B_t(\mu)\}$ it suffices to give a lower bound for $\mu(H)$, where H is any closed half-space whose bounding hyperplane supports $B_t(\mu)$. In that case,

$$\mu(H) = \mu(\{z : \langle z - x, \xi\rangle \geqslant 0\}) \tag{5.1}$$

for some $x \in \partial(B_t(\mu))$, with $\xi = \nabla\Lambda_\mu^*(x)$, or equivalently $x = \nabla\Lambda_\mu(\xi)$ (see e.g. Theorem 23.5 and Corollary 23.5.1 in [36]). Note that

$$\begin{aligned}
\mu(\{z : \langle z - x, \xi\rangle \geqslant 0\}) &= \int_{\mathbb{R}^n} \mathbb{1}_{[0,\infty)}(\langle z - x, \xi\rangle) f(z)\, dz \\
&= e^{\Lambda_\mu(\xi)} \int_{\mathbb{R}^n} \mathbb{1}_{[0,\infty)}(\langle z - x, \xi\rangle) e^{-\langle z, \xi\rangle}\, d\mu_\xi(z) \\
&= e^{\Lambda_\mu(\xi)} e^{-\langle x, \xi\rangle} \int_{\mathbb{R}^n} \mathbb{1}_{[0,\infty)}(\langle z - x, \xi\rangle) e^{-\langle z - x, \xi\rangle}\, d\mu_\xi(z) \\
&\geqslant e^{-\Lambda_\mu^*(x)} \int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(\{z : 0 \leqslant \langle z - x, \xi\rangle \leqslant \sigma_\xi t\})\, dt.
\end{aligned} \tag{5.2}$$

Using Markov's inequality we check that $\mu_\xi(\{z : \langle z - x, \xi\rangle \geqslant 2\sigma_\xi\}) \leqslant \frac{1}{4}$, and since x is the barycenter of μ_ξ, from Grünbaum's lemma (see [12, Lemma 2.2.6]) we get that $\mu_\xi(\{z : \langle z - x, \xi\rangle \geqslant 0\}) \geqslant \frac{1}{e}$. Therefore,

$$\int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(\{z : 0 \leqslant \langle z - x, \xi\rangle \leqslant \sigma_\xi t\})\, dt \geqslant \int_2^\infty \sigma_\xi e^{-\sigma_\xi t} \left(\frac{1}{e} - \frac{1}{4}\right) dt \geqslant \frac{4 - e}{4e} e^{-2\sigma_\xi}. \tag{5.3}$$

We would like to have an upper bound for $\sup_\xi \sigma_\xi$. This is the point where we need to restrict ourselves to the case where $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 on $\mathbb{R}^n$: then, we can exploit a theorem of Nguyen [33] which was proved independently by Wang [42] (the sketch of its proof below follows [18]).

Theorem 5.2. *Let ν be a log-concave probability measure on $\mathbb{R}^n$ with density $g = \exp(-p)$, where $p : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function. Then,*

$$\mathrm{Var}_\nu(p) \leqslant n.$$

Sketch of the proof. Note that

$$\mathrm{Var}_\nu(p) = V(g) := \int_{\mathbb{R}^n} g(\ln g)^2 - \left(\int_{\mathbb{R}^n} g \ln g\right)^2.$$

Define $F : (0, \infty) \to \mathbb{R}$ with $F(s) = \ln\left(\int_{\mathbb{R}^n} g^s(x)\, dx\right)$. A careful computation of $F''(s)$ and $V(g_s)$ shows that $F''(s) = V(g_s)/s^2$, where g_s is the log-concave density

$$g_s = \frac{g^s}{\int_{\mathbb{R}^n} g^s}.$$

Next, observe that the function $w : \mathbb{R}^n \times (0,\infty) \to (-\infty,\infty]$ with $w(z,s) = s\psi(z/s)$ is convex. It follows that the function $G : (0,\infty) \to \mathbb{R}$ defined by

$$G(s) = s^n \int_{\mathbb{R}^n} g^s(x)\, dx$$

is log-concave. In order to check this, one can make the change of variables $x = z/s$ and use the convexity of w as well as the fact that marginals of a log-concave measure are log-concave, therefore they have a log-concave density. This implies that $V(g_s) \leqslant n$ for every $s > 0$. To see this, note that $\ln G(s) = n \ln s + F(s)$ and differentiate twice. In particular, for $s = 1$, we get $V(g) \leqslant n$.

In the case where $g = e^{-p}$ as above, and p is positively homogeneous of degree 1. one can check that $G(s) = 1$ for all $s > 0$, and hence $V(g_s) = n$. In particular, we have that $V(g) = n$, which shows that the inequality of the theorem is sharp. □

Proof of Theorem 5.1. Set $\mu := \mu_K$. Since $f(z) = \mathbb{1}_K(z)$, the density f_ξ of μ_ξ is proportional to $e^{\langle \xi, z\rangle}\mathbb{1}_K(z)$. From Theorem 5.2 we get

$$\sigma_\xi^2 = \mathbb{E}_{\mu_\xi}(\langle z - x, \xi\rangle)^2 = \mathrm{Var}_{\mu_\xi}(\langle \xi, z\rangle) = \mathrm{Var}_{\mu_\xi}(-\ln f_\xi) \leqslant n.$$

Then, combining (5.1), (5.2) and (5.3), for any bounding hyperplane H of $B_t(\mu)$ we have

$$\begin{aligned}\mu(H) &\geqslant e^{-\Lambda_\mu^*(x)} \int_0^\infty \sigma_\xi e^{-\sigma_\xi t} \mu_\xi(0 \leqslant \langle z - x, \xi\rangle \leqslant \sigma_\xi t)\, dt \\ &\geqslant \frac{4-e}{4e} e^{-\Lambda_\mu^*(x) - 2\sigma_\xi} \geqslant \frac{1}{10}\exp(-t - 2\sqrt{n}),\end{aligned}$$

as claimed. □

Theorem 5.1 shows that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then

$$10\varphi_{\mu_K}(x) \geqslant \exp(-\Lambda_{\mu_K}^*(x) - 2\sqrt{n})$$

for all $x \in \mathbb{R}^n$. Setting

$$\omega_{\mu_K}(x) = \ln\left(\frac{1}{\varphi_{\mu_K}(x)}\right) \tag{5.4}$$

and taking into account Lemma 3.2 we have the next two-sided estimate.

Corollary 5.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $x \in \mathrm{int}(K)$ we have that*

$$\omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda_{\mu_K}^*(x) \leqslant \omega_{\mu_K}(x). \tag{5.5}$$

A basic question that arises is whether an analogue of (5.5) holds true for any centered log-concave probability measure μ on $\mathbb{R}^n$. This would allow us to apply the next steps of our procedure to all log-concave probability measures. Brazitikos and Chasapis [9] have recently obtained such a variant of Theorem 5.1 which is valid in this more general setting.

Theorem 5.4. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. For every $x \in \mathrm{supp}(\mu)$ and any $\varepsilon \in (0,1)$ we have that*

$$\Lambda_\mu^*(x) \geqslant (1-\varepsilon)\ln\left(\frac{1}{\varphi_\mu(x)}\right) + \ln\left(\frac{\varepsilon}{2^{1-\varepsilon}}\right).$$

Sketch of the proof. We start with the case $n = 1$. Note that $\varphi_\mu(y) = \min\{\mu((-\infty, y]), \mu([y, \infty))\}$ is a log-concave function. Then, the function $g(y) = -(1-\varepsilon)\ln \varphi_\mu(y)$ is convex, and hence we may find $t, b \in \mathbb{R}$ such that the $\ell(y) = ty + b$ satisfies $\ell(x) = g(x)$ and $\ell(y) \leqslant g(y)$ for all y. Then, $\mathbb{E}_\mu(e^{\ell(y)}) \leqslant \mathbb{E}_\mu(e^{g(y)})$, or equivalently $-\ln\left(\mathbb{E}_\mu(e^{g(y)})\right) \leqslant -\ln\left(\mathbb{E}_\mu(e^{\ell(y)})\right) = -b - \Lambda_\mu(t)$. Since $\ell(x) = b(x)$, we see that

$$g(x) - \ln\left(\mathbb{E}_\mu(e^{g(y)})\right) \leqslant \ell(x) - b - \Lambda_\mu(t) = xt - \Lambda_\mu(t) \leqslant \Lambda_\mu^*(x).$$

Finally, one can check that $\mathbb{E}(\varphi_\mu(y)^{\varepsilon-1}) \leqslant (\varepsilon 2^{\varepsilon-1})^{-1}$, which implies that

$$-\ln\left(\mathbb{E}_\mu(e^{g(y)})\right) = -\ln\left(\mathbb{E}_\mu(\phi_\mu(y)^{\varepsilon-1})\right) \geqslant \ln\left(\frac{\varepsilon}{2^{1-\varepsilon}}\right)$$

and the claim follows.

Next, assume that $n > 1$. If X is a random vector in $\mathbb{R}^n$ which is distributed according to μ, for any $\xi \in S^{n-1}$ consider the random variable $\xi_X = \langle X, \xi\rangle$. Then, the distribution μ_ξ of ξ_X is log-concave and from the one-dimensional result we see that

$$\Lambda_{\mu_\xi}^*(\langle x, \xi\rangle) \geqslant (1-\varepsilon)\ln\left(\frac{1}{\varphi_{\mu_\xi}(\langle x, \xi\rangle)}\right) + \ln\left(\frac{\varepsilon}{2^{1-\varepsilon}}\right)$$

for all $x \in \mathbb{R}^n$. We observe that, for any $x \in \mathbb{R}^n$,

$$\Lambda_\mu^*(x) = \sup_{y \in \mathbb{R}^n}\left(\langle x, y\rangle - \Lambda_\mu(y)\right) = \sup_{(\xi, t) \in S^{n-1}\times\mathbb{R}}\left(t\langle x, \xi\rangle - \Lambda_{\mu_\xi}(t)\right) = \sup_{\xi \in S^{n-1}} \Lambda_{\mu_\xi}^*(\langle x, \xi\rangle).$$

Since

$$\sup_{\xi \in S^{n-1}} \ln\left(\frac{1}{\varphi_{\mu_\xi}(\langle x, \xi\rangle)}\right) = \ln\left(\frac{1}{\inf_{\xi \in S^{n-1}}\mu(\{z : \langle z, \xi\rangle \geqslant \langle x, \xi\rangle\})}\right) = \ln\left(\frac{1}{\varphi_\mu(x)}\right),$$

combining the above we obtain the result. □

Note. The inequality of Theorem 5.4 can be rewritten as $\varphi_\mu(x)^{1-\varepsilon} \geqslant (\varepsilon/2^{1-\varepsilon})e^{-\Lambda_\mu^*(x)}$, which gives

$$\left[\varphi_-(B_t(\mu))\right]^{1-\varepsilon} \geqslant \frac{\varepsilon}{2^{1-\varepsilon}}e^{-t} \tag{5.6}$$

for all $\varepsilon \in (0, 1)$. This last inequality, which is valid for all log-concave probability measures, may be viewed as a substitute of the estimate in Theorem 5.1.

6 Moments of the Cramér transform

Our approach to the threshold problem for a given centered log-concave probability measure μ on $\mathbb{R}^n$ requires to know that the Cramér transform Λ_μ^* has finite variance. We can give an affirmative answer to this question in the case where $\mu = \mu_K$ is the uniform measure on a centered convex body K of volume 1 in $\mathbb{R}^n$. Actually, one can show that, for a more general class of measures, it is still true that Λ_μ^* has finite moments of all orders.

Theorem 6.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with* $\mathrm{supp}(\mu) = K$. *Then,*

$$\int_{\mathbb{R}^n} \exp(\kappa\Lambda_\mu^*(x)/2)\, d\mu(x) < \infty.$$

In particular, for all $p \geqslant 1$ we have that $\mathbb{E}_\mu\left((\Lambda_\mu^*(x))^p\right) < \infty$.

For the proof we need a lemma, which is proved in [13, Lemma 7] in the symmetric case.

Lemma 6.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0,1/n]$ and let μ be a centered κ-concave probability measure with* $\mathrm{supp}(\mu)=K$. *Then,*

$$\varphi_\mu(x) \geqslant e^{-2}\kappa(1-\|x\|_K)^{1/\kappa} \tag{6.1}$$

for every $x\in K$, where $\|x\|_K$ is the Minkowski functional of K.

Sketch of the proof. Let X be a random vector distributed according to μ. Given $\theta\in S^{n-1}$ let $b=h_K(\theta)$ and $a=h_K(-\theta)$. If g_θ is the density of $\langle X,\theta\rangle$ then $g_\theta^{\frac{\kappa}{1-\kappa}}$ is concave on $[-a,b]$, therefore

$$g_\theta(t)\geqslant g_\theta(0)\left(1-\frac{t}{b}\right)^{\frac{1-\kappa}{\kappa}}$$

for all $t\in[0,b]$. It follows that, for every $0<s<b$,

$$\mathbb{P}(\langle X,\theta\rangle\geqslant s)=\int_s^b g_\theta(t)\,dt\geqslant g_\theta(0)\int_s^b\left(1-\frac{t}{b}\right)^{\frac{1-\kappa}{\kappa}}\,dt=\kappa g_\theta(0)b\left(1-\frac{s}{b}\right)^{\frac{1}{\kappa}}.$$

Note that g_θ is a centered log-concave density. Therefore, $g_\theta(0)\geqslant e^{-1}\|g_\theta\|_\infty$ by (2.3) and $\|g_\theta\|_\infty b\geqslant \mathbb{P}(\langle X,\theta\rangle\geqslant 0)\geqslant e^{-1}$ by Grünbaum's lemma [12, Lemma 2.2.6], which implies that $g_\theta(0)b\geqslant e^{-2}$. It follows that

$$\mathbb{P}(\langle X,\theta\rangle\geqslant s)=\int_s^b g_\theta(t)\,dt\geqslant e^{-2}\kappa\left(1-\frac{s}{b}\right)^{\frac{1}{\kappa}}.$$

Now, let $x\in K$. Then $\langle x,\theta\rangle\leqslant\|x\|_K h_K(\theta)=\|x\|_K b$, therefore

$$\mathbb{P}(\langle X,\theta\rangle\geqslant\langle x,\theta\rangle)\geqslant\mathbb{P}(\langle X,\theta\rangle\geqslant\|x\|_K b)\geqslant e^{-2}\kappa\,(1-\|x\|_K)^{\frac{1}{\kappa}}$$

as claimed. $\square$

Proof of Theorem 6.1. From Lemma 3.2 we know that $\varphi_\mu(x)\leqslant\exp(-\Lambda_\mu^*(x))$, or equivalently,

$$\exp(\kappa\Lambda_\mu^*(x)/2)\leqslant\varphi_\mu(x)^{-\kappa/2}$$

for all $x\in K$. Lemma 6.2 shows that

$$\varphi_\mu(x)\geqslant e^{-2}\kappa(1-\|x\|_K)^{1/\kappa}$$

for every $x\in K$, and hence

$$\int_K\exp(\kappa\Lambda_\mu^*(x)/2)\,d\mu(x)\leqslant(e^2/\kappa)^{\kappa/2}\int_K\frac{1}{(1-\|x\|_K)^{1/2}}d\mu(x).$$

Recall that the cone probability measure ν_K on the boundary $\partial(K)$ of a convex body K with $0\in\mathrm{int}(K)$ is defined by

$$\nu_K(B)=|\{rx:x\in B,0\leqslant r\leqslant 1\}|/|K|$$

for all Borel subsets B of $\partial(K)$. We shall use the identity

$$\int_{\mathbb{R}^n}g(x)\,dx=n|K|\int_0^\infty r^{n-1}\int_{\partial(K)}g(rx)\,d\nu_K(x)\,dr$$

which holds for every integrable function $g:\mathbb{R}^n\to\mathbb{R}$ (see [32, Proposition 1]). Let f denote the density of μ on K. We write

$$\begin{aligned}\int_K \frac{1}{(1-\|x\|_K)^{1/2}}d\mu(x) &= \int_{\mathbb{R}^n}\frac{f(x)}{(1-\|x\|_K)^{1/2}}\mathbb{1}_K(x)\,dx\\ &= n|K|\int_0^\infty r^{n-1}\int_{\partial(K)}\frac{f(ry)}{(1-\|ry\|_K)^{1/2}}\mathbb{1}_K(ry)\,d\nu_K(y)\,dr\\ &= n|K|\int_0^1\frac{r^{n-1}}{\sqrt{1-r}}\int_{\partial(K)}f(ry)\,d\nu_K(y)\,dr\\ &\leqslant n|K|\|f\|_\infty\int_0^1\frac{r^{n-1}}{\sqrt{1-r}}\,dr = n|K|B(n,1/2)\|f\|_\infty\leqslant c\sqrt{n}\|f\|_\infty<+\infty,\end{aligned}$$

and the proof is complete. $\square$

In the case of the uniform measure $\mu=\mu_K$ on a centered convex body K of volume 1 in $\mathbb{R}^n$ we see that

$$\int_K\big(\Lambda^*_{\mu_K}(x)/2n\big)^p dx\leqslant(c_1p)^p\int_K\exp(\Lambda^*_{\mu_K}(x)/2n)\,dx\leqslant(c_2p)^p\sqrt{n},$$

where $c_1,c_2>0$ are absolute constants. This shows that

$$\|\Lambda^*_{\mu_K}\|_{L^p(\mu_K)}\leqslant cpn^{1+\frac{1}{2p}}$$

for all $p\geqslant 1$. However, the argument that we used for Theorem 6.1 leads to sharp estimates in the cases $p=1,2$. We shall use the fact that

$$\int_0^1 r^{n-1}\ln(1-r)\,dr=-\frac{1}{n}H_n\quad\text{and}\quad\int_0^1 r^{n-1}\ln^2(1-r)\,dr=\frac{1}{n}H_n^2+\frac{1}{n}\sum_{k=1}^n\frac{1}{k^2}, \tag{6.2}$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$.

Theorem 6.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $n\geqslant 2$. Let $\kappa\in(0,1/n]$ and let μ be a centered κ-concave probability measure with* $\mathrm{supp}(\mu)=K$. *Then,*

$$\mathbb{E}_\mu(\Lambda^*_\mu)\leqslant\big(\mathbb{E}_\mu[(\Lambda^*_\mu)^2]\big)^{1/2}\leqslant\frac{c\ln n}{\kappa}\|f\|_\infty^{1/2},$$

where $c>0$ is an absolute constant and f is the density of μ.

Proof. As in the proof of Theorem 6.1 we write

$$\int_K\big(\Lambda^*_\mu(x)\big)^2\,d\mu(x)\leqslant\int_K\ln^2\left(\frac{e^2}{\kappa}\frac{1}{(1-\|x\|_K)^{1/\kappa}}\right)d\mu(x).$$

If f is the density of μ on K and ν_K is the cone measure of K, using the inequality $\ln^2(ab)\leqslant 2(\ln^2 a+\ln^2 b)$ where $a,b>0$, we may write

$$\begin{aligned}&\frac{1}{2}\int_K\ln^2\left(\frac{e^2}{\kappa}\frac{1}{(1-\|x\|_K)^{1/\kappa}}\right)d\mu(x)-\ln^2\left(\frac{e^2}{\kappa}\right)\\ &\quad\leqslant\int_{\mathbb{R}^n}f(x)\ln^2\left(\frac{1}{(1-\|x\|_K)^{1/\kappa}}\right)\mathbb{1}_K(x)\,dx\\ &\quad= n|K|\int_0^\infty r^{n-1}\int_{\partial(K)}f(ry)\ln^2\left(\frac{1}{(1-\|ry\|_K)^{1/\kappa}}\right)\mathbb{1}_K(ry)\,d\nu_K(y)\,dr\\ &\quad=\frac{n}{\kappa^2}\int_0^1 r^{n-1}\ln^2(1-r)\int_{\partial(K)}f(ry)\,d\nu_K(y)\,dr\\ &\quad\leqslant\frac{n}{\kappa^2}\|f\|_\infty\int_0^1 r^{n-1}\ln^2(1-r)\,dr.\end{aligned}$$

Since $1 \leqslant \int_K f(x)\,dx \leqslant \|f\|_\infty$, using also (6.2) we get

$$\int_K \left(\Lambda_\mu^*(x)\right)^2 d\mu(x) \leqslant \frac{2n}{\kappa^2}\left(\frac{1}{n}H_n^2 + \frac{1}{n}\sum_{k=1}^n \frac{1}{k^2}\right)\|f\|_\infty + 2\ln^2\left(\frac{e^2}{\kappa}\right)$$
$$\leqslant \left(\frac{4H_n^2}{\kappa^2} + 2\ln^2(e^2/\kappa)\right)\|f\|_\infty \leqslant \frac{c_1 \ln^2 n}{\kappa^2}\|f\|_\infty,$$

where $c_1 > 0$ is an absolute constant. □

In fact, for the uniform measure on a convex body, we can summarize in a sharp two sided estimate.

Theorem 6.4. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $n \geqslant 2$. Then,*

$$c_1 n/L_{\mu_K}^2 \leqslant \|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)} \leqslant \|\Lambda_{\mu_K}^*\|_{L^2(\mu_K)} \leqslant c_2 n \ln n,$$

where L_{μ_K} is the isotropic constant of the uniform measure μ_K on K and $c_1, c_2 > 0$ are absolute constants.

Proof. For the left-hand side inequality recall that in Theorem 3.1 we saw that

$$\int_{\mathbb{R}^n} \varphi_\mu(x)\,d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right),$$

where $c > 0$ is an absolute constant. In fact, the proof of this estimate starts with Lemma 3.2 and follows from the next stronger result: If $n \geqslant n_0$ then

$$\int_{\mathbb{R}^n} \exp(-\Lambda_\mu^*(x))\,d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right)$$

where L_μ is the isotropic constant of μ and $c > 0$, $n_0 \in \mathbb{N}$ are absolute constants. Then, Jensen's inequality implies that

$$e^{-\mathbb{E}_\mu(\Lambda_\mu^*)} \leqslant \int_{\mathbb{R}^n} \exp(-\Lambda_\mu^*(x))\,d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right)$$

and the result follows. □

Both the lower and the upper bound are of optimal order with respect to the dimension. This can be seen e.g. from the example of the uniform measure on the cube or the Euclidean ball, respectively.

An alternative approach to the question of moments of Λ_μ^* may be based on the notion of affine surface area. Let K be a convex body in $\mathbb{R}^n$. The affine surface area of K is the quantity

$$\mathrm{as}(K) = \int_{\partial(K)} \kappa(x)^{\frac{1}{n+1}}\,d\mu_{\partial(K)}(x),$$

where $\kappa(x)$ is the generalized Gauss-Kronecker curvature at x and $\mu_{\partial(K)}$ is the surface measure on $\partial(K)$ (see [31] and the references therein). The affine surface area of a Euclidean ball of radius 1 is equal to its surface area and the affine surface area of all polytopes is equal to 0. The affine isoperimetric inequality states that

$$\left(\frac{\mathrm{as}(K)}{\mathrm{as}(B_2^n)}\right)^{n+1} \leqslant \left(\frac{|K|}{|B_2^n|}\right)^{n-1}$$

with equality if and only if K is an ellipsoid (see [38, Section 10.5]). Since $\mathrm{as}(B_2^n) = n\omega_n$ where $\omega_n = |B_2^n|$, we see that if $|K| = |B_2^n|$ then $\mathrm{as}(K) \leqslant \mathrm{as}(B_2^n)$. Thus, if $|K| = 1$ then

$$\mathrm{as}(K) \leqslant c_1\sqrt{n}.$$

For every $\delta \in (0, 1/2)$ we define the floating body

$$K_\delta = \bigcap \{H^+ : H^+ \text{ is a closed half-space with } |K \cap H^-| = \delta\},$$

where H^- is the complementary half-space of H^+. We also define

$$T_\delta = \{x \in K : \varphi_{\mu_K}(x) \geqslant \delta\}$$

Note that T_δ is convex: if $x, y \in T_\delta$ then for any $z \in [x, y]$ and any $H \in \mathcal{H}(z)$ we have that either x or y belongs to H, and hence $|K \cap H| \geqslant \delta$, therefore $\varphi_{\mu_K}(z) \geqslant \delta$, In fact, it is not hard to show that

$$T_\delta = K_\delta.$$

Schütt and Werner proved in [37] that for every convex body K in $\mathbb{R}^n$ one has that

$$\lim_{\delta \to 0} \frac{|K| - |K_\delta|}{\delta^{\frac{2}{n+1}}} = \frac{1}{2} \left(\frac{n+1}{\omega_{n-1}} \right)^{\frac{2}{n+1}} \mathrm{as}(K).$$

In particular, if $|K| = 1$ then there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ then

$$1 - |K_\delta| \leqslant c_2 n^{3/2} \delta^{\frac{2}{n+1}}.$$

We can exploit these results as follows. Recall that $\varphi_{\mu_K}(x) \leqslant \exp(-\Lambda^*_{\mu_K}(x))$ and hence, for any $s > 0$ we have that

$$\exp(s\Lambda^*_{\mu_K}(x)) \leqslant (\varphi_{\mu_K}(x))^{-s}.$$

Then,

$$\begin{aligned}\int_K \exp(s\Lambda^*_{\mu_K}(x))\, dx &\leqslant \int_K \frac{1}{(\varphi_{\mu_K}(x))^s} dx = \int_0^\infty \mu_K(\{1/\varphi^s_{\mu_K} \geqslant t\})\, dt \\ &= \int_0^\infty \mu_K(\{\varphi_{\mu_K} \leqslant 1/t^{1/s}\})\, dt = \int_0^\infty (1 - \mu_K(T_{t^{-1/s}}))\, dt \\ &= \int_0^\infty (1 - \mu_K(T_{t^{-1/s}}))\, dt.\end{aligned}$$

Let $t_0 = \delta_0^{-s}$. Then,

$$\int_{t_0}^\infty (1 - \mu_K(T_{t^{-1/s}}))\, dt \leqslant c_2 n^{3/2} \int_{t_0}^\infty t^{-\frac{2}{s(n+1)}}\, dt < +\infty$$

provided that $s < \frac{2}{n+1}$. This proves the following.

Theorem 6.5. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then, for any $0 < s < \frac{2}{n+1}$,*

$$\int_{\mathbb{R}^n} \exp(s\Lambda^*_{\mu_K}(x))\, d\mu_K(x) < \infty.$$

In particular, for all $p \geqslant 1$ we have that $\mathbb{E}_{\mu_K}\big((\Lambda^*_{\mu_K}(x))^p\big) < \infty$.

One can also obtain a generalization of this fact, with a similar argument, using results of Besau, Ludwig and Werner on weighted floating bodies. More precisely, from [3, Theorem 1.1] it follows that the conclusion of Theorem 6.5 still holds if we replace μ_K by any probability measure with continuous and strictly positive density $\psi : K \to (0, +\infty)$. These ideas might prove useful in the study of the question for an arbitrary log-concave probability measure.

The next result concerns the one-dimensional case. Let μ be a centered probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure and consider a random variable X, on some probability space $(\Omega, \mathcal{F}, P)$, with distribution μ, i.e., $\mu(B) := P(X \in B),\ B \in \mathcal{B}(\mathbb{R})$. We define

$$\alpha_+ = \alpha_+(\mu) := \sup\{x \in \mathbb{R} \colon \mu([x,\infty)) > 0\}) \quad \text{and} \quad \alpha_- = \alpha_-(\mu) := \sup\{x \in \mathbb{R} \colon \mu((-\infty, -x])) > 0\}).$$

Thus, $-\alpha_-, \alpha_+$ are the endpoints of the support of μ. Note that we may have $\alpha_\pm = +\infty$. We define $I_\mu = (-\alpha_-, \alpha_+)$. Recall that

$$\Lambda_\mu^*(x) := \sup\{tx - \Lambda_\mu(t) \colon t \in \mathbb{R}\}, \qquad x \in \mathbb{R}.$$

In fact, since $tx - \Lambda_\mu(t) < 0$ for $t < 0$ when $x \in [0, \alpha_+)$, we have that $\Lambda_\mu^*(x) = \sup\{tx - \Lambda_\mu(t) \colon t \geqslant 0\}$ in this case, and similarly $\Lambda_\mu^*(x) := \sup\{tx - \Lambda_\mu(t) \colon t \leqslant 0\}$ when $x \in (-\alpha_-, 0]$. One can also check that $\Lambda_\mu^*(\alpha_\pm) = +\infty$. See [20, Lemma 2.8] for the case $\alpha_\pm < +\infty$. In the case $\alpha_\pm = \pm\infty$, the convexity and monotonicity properties of Λ_μ^* imply again that $\lim\limits_{t \to \pm\infty} \Lambda_\mu^*(t) = +\infty$.

Proposition 6.6. *Let μ be a centered probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure and let $I_\mu = \mathrm{supp}(\mu)$. Then,*

$$\int_{I_\mu} \exp(\Lambda_\mu^*(x)/2)\, d\mu(x) \leqslant 4.$$

Proof. Let $F(x) = \mu(-\infty, x]$. For any $x \in [0, \alpha_+)$ and $t \geqslant 0$ we have

$$\min\{F(x), 1 - F(x)\} = \varphi_\mu(x) \leqslant e^{-\Lambda_\mu^*(x)}.$$

It follows that

$$\begin{aligned} \int_{I_\mu} e^{\Lambda_\mu^*(x)/2} d\mu(x) &\leqslant \int_{I_\mu} \frac{1}{\sqrt{\min\{F(x), 1-F(x)\}}} f(x)\, dx \\ &\leqslant \int_{I_\mu} \frac{1}{\sqrt{F(x)}} f(x)\, dx + \int_{I_\mu} \frac{1}{\sqrt{1-F(x)}} f(x)\, dx. \end{aligned} \tag{6.3}$$

Write f for the density of μ with respect to Lebesgue measure. Then, $(1-F)'(x) = -f(x)$, which implies that

$$\int_0^{\alpha_+} \frac{1}{\sqrt{1-F(x)}} f(x)\, dx \leqslant -\int_0^{\alpha_+} \frac{1}{\sqrt{1-F(x)}} (1-F)'(x) dx = -2\sqrt{1-F(x)}\Big|_0^{\alpha_+} = 2\sqrt{1-F(0)}$$

since $F(\alpha_+) = 1$. In the same way we check that

$$\int_{-\alpha_-}^0 \frac{1}{\sqrt{1-F(x)}} f(x)\, dx \leqslant -\int_{-\alpha_-}^0 \frac{1}{\sqrt{1-F(x)}} (1-F)'(x) dx = -2\sqrt{1-F(x)}\Big|_{-\alpha_-}^0 = 2 - 2\sqrt{1-F(0)}.$$

This shows that

$$\int_{I_\mu} \frac{1}{\sqrt{1-F(x)}} f(x)\, dx \leqslant 2.$$

In a similar way we obtain the same upper bound for the second summand in (6.3) and the result follows. $\square$

Proposition 6.6 can be extended to products. Let μ_i, $1 \leqslant i \leqslant n$ be centered probability measures on $\mathbb{R}$, all of them absolutely continuous with respect to Lebesgue measure. If $\overline{\mu} = \mu_1 \otimes \cdots \otimes \mu_n$ then $I_{\overline{\mu}} = \prod_{i=1}^n I_{\mu_i}$ and we can easily check that

$$\Lambda_{\overline{\mu}}^*(x) = \sum_{i=1}^n \Lambda_{\mu_i}^*(x_i)$$

for all $x=(x_1,\dots,x_n)\in I_{\overline{\mu}}$, which implies that

$$\int_{I_{\overline{\mu}}} e^{\Lambda^*_{\overline{\mu}}(x)/2} d\overline{\mu}(x) = \prod_{i=1}^n \left(\int_{I_{\mu_i}} e^{\Lambda^*_{\mu_i}(x_i)/2} d\mu_i(x_i) \right) \leqslant 4^n.$$

In particular, for all $p\geqslant 1$ we have that

$$\int_{I_{\overline{\mu}}} (\Lambda^*_{\overline{\mu}}(x))^p \, d\overline{\mu}(x) < +\infty.$$

We close this section with one more case where we can establish that Λ^*_μ has finite moments of all orders. We consider an arbitrary centered log-concave probability measure on $\mathbb{R}^n$ but we have to assume that it satisfies some additional condition on the growth of its one-sided L_t-centroid bodies $Z_t^+(\mu)$; namely, that the family of the one-sided L_t-centroid bodies grows with some mild rate as $t\to\infty$ (note that the assumption in the next proposition can be satisfied only for log-concave probability measures μ with support $\mathrm{supp}(\mu)=\mathbb{R}^n$).

Proposition 6.7. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Assume that there exists an increasing function $g:[1,\infty)\to[1,\infty)$ with $\lim_{t\to\infty} g(t)/\ln(t+1)=+\infty$ such that $Z_t^+(\mu)\supseteq g(t)Z_2^+(\mu)$ for all $t\geqslant 2$. Then,*

$$\int_{\mathbb{R}^n} |\Lambda^*_\mu(x)|^p d\mu(x) < +\infty$$

for every $p\geqslant 1$.

Proof. In Lemma 3.4 we saw that if $t\geqslant 1$ then for every $x\in \frac{1}{2}Z_t^+(\mu)$ we have

$$\varphi_\mu(x)\geqslant e^{-c_1 t},$$

where $c_1>1$ is an absolute constant. Since $\Lambda^*_\mu(x)\leqslant \ln\frac{1}{\varphi_\mu(x)}$, this shows that $\Lambda^*_\mu(x)\leqslant c_1 t$ for all $x\in\frac{1}{2}Z_t^+(\mu)$. In other words,

$$\frac{1}{2}Z^+_{t/c_1}(\mu)\subseteq B_t(\mu), \qquad t\geqslant c_1. \tag{6.4}$$

Since $\lim_{t\to\infty} g(t)=+\infty$, there exists $t_0\geqslant c_1$ such that $\mu\left(\frac{g(t_0/c_1)}{2}Z_2^+(\mu)\right)\geqslant 2/3$. From Borell's lemma [12, Lemma 2.4.5] we know that, for all $t\geqslant t_0$,

$$1-\mu\left(\frac{g(t/c_1)}{2}Z_2^+(\mu)\right)\leqslant e^{-c_2 g(t/c_1)/g(t_0/c_1)},$$

where $c_2>0$ is an absolute constant. We write

$$\int_{\mathbb{R}^n} |\Lambda^*_\mu(x)|^p d\mu(x) = \int_0^\infty pt^{p-1}\mu(\{x:\Lambda^*_\mu(x)\geqslant t\})\,dt = p\int_0^\infty t^{p-1}(1-\mu(B_t(\mu)))\,dt.$$

From (6.4) it follows that

$$1-\mu(B_t(\mu))\leqslant 1-\mu\left(\frac{1}{2}Z^+_{t/c_1}(\mu)\right)\leqslant 1-\mu\left(\frac{g(t/c_1)}{2}Z_2^+(\mu)\right)\leqslant e^{-c_2 g(t/c_1)/g(t_0/c_1)}$$

for all $t\geqslant t_0$. Since $\lim_{t\to\infty} g(t)/\ln(t+1)=+\infty$, there exists $t_p\geqslant t_0$ such that

$$(p-1)\ln(t)\leqslant \frac{c_2}{2g(t_0/c_1)}g(t/c_1)$$

for all $t \geqslant t_p$. Assume that $p > 2$. Then, from the previous observations we get

$$p \int_{t_p}^{\infty} t^{p-1}(1-\mu(B_t(\mu)))\,dt \leqslant p \int_{t_p}^{\infty} t^{p-1}\left(1-\mu\left(\tfrac{g(t/c_1)}{2} Z_2^+(\mu)\right)\right)\,dt$$
$$\leqslant p \int_{t_p}^{\infty} t^{p-1}t^{-2(p-1)}\,dt = p\int_{t_p}^{\infty} t^{-(p-1)}\,dt < \infty.$$

This proves the result for $p > 2$ and then from Hölder's inequality it is clear that the assertion of the proposition is also true for all $p \geqslant 1$. □

Note. It is not hard to construct examples of log-concave probability measures, even on the real line, for which $\mathrm{supp}(\mu) = \mathbb{R}^n$ but the assumption of Proposition 6.7 is not satisfied. Consider for example a measure μ on $\mathbb{R}$ with density $f(x) = c \cdot \exp(-\psi)$ where ψ is an even convex function rapidly increasing to infinity, e.g. $\psi(t) = e^{t^2}$.

However, this does not exclude the possibility that for every centered log-concave probability measure μ on $\mathbb{R}^n$ the function Λ_μ^* has finite second or higher moments.

7 Threshold for the measure: the approach and examples

Let μ be a log-concave probability measure on $\mathbb{R}^n$ such that $\|\Lambda_\mu^*\|_{L^2(\mu)} < \infty$. We define the parameter

$$\beta(\mu) = \frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{(\mathbb{E}_\mu(\Lambda_\mu^*))^2} \tag{7.1}$$

We shall give a general estimate for the upper threshold $\varrho_1(\mu,\delta)$ in terms of $\beta(\mu)$.

Theorem 7.1. *Let $\beta,\delta > 0$ with $8\beta < \delta < 1$. If μ is a log-concave probability measure on $\mathbb{R}^n$ with $\beta(\mu) = \beta$ and $n/L_\mu^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta(\mu)}$, then*

$$\varrho_1(\mu,\delta) \geqslant \left(1 - \sqrt{8\beta(\mu)/\delta}\right)\frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}.$$

Proof. Recall that $B_t(\mu) = \{x \in \mathbb{R}^n : \Lambda_\mu^*(x) \leqslant t\}$. We use Lemma 3.7 in the following way. Let $m := \mathbb{E}_\mu(\Lambda_\mu^*)$. Then, for all $\varepsilon \in (0,1)$, from Chebyshev's inequality we have that

$$\mu(\{\Lambda_\mu^* \leqslant m - \varepsilon m\}) \leqslant \mu(\{|\Lambda_\mu^* - m| \geqslant \varepsilon m\}) \leqslant \frac{\mathbb{E}_\mu|\Lambda_\mu^* - m|^2}{\varepsilon^2 m^2} = \frac{\beta(\mu)}{\varepsilon^2}.$$

Equivalently,

$$\mu(B_{(1-\varepsilon)m}(\mu)) \leqslant \frac{\beta(\mu)}{\varepsilon^2}.$$

Let $\delta \in (\beta(\mu),1)$. Since $8\beta(\mu) < \delta < 1$, choosing $\varepsilon = \sqrt{2\beta(\mu)/\delta}$ we have that

$$\mu(B_{(1-\varepsilon)m}(\mu)) \leqslant \frac{\delta}{2}.$$

Then, from Lemma 3.7 we see that

$$\sup\{\mathbb{E}_{\mu^N}(\mu(K_N)) : N \leqslant e^{(1-2\varepsilon)m}\} \leqslant \mu(B_{(1-\varepsilon)m}(\mu)) + e^{(1-2\varepsilon)m}e^{-(1-\varepsilon)m}$$
$$\leqslant \frac{\delta}{2} + e^{-\varepsilon m} \leqslant \delta,$$

provided that $\varepsilon m \geqslant \ln(2/\delta)$. Since $m \geqslant c_1 n/L_\mu^2$, this condition is satisfied if $n/L_\mu^2 \geqslant c_2\ln(2/\delta)\sqrt{\delta/\beta(\mu)}$. By the choice of ε we conclude that $\varrho_1(\mu,\delta) \geqslant \left(1-\sqrt{8\beta(\mu)/\delta}\right)\frac{\mathbb{E}_\mu(\Lambda_\mu^*)}{n}$. □

For the proof of the lower threshold we work in a similar way, using Lemma 3.8. In the case of the uniform measure on a convex body we obtain the next theorem.

Theorem 7.2. *Let $\beta,\delta > 0$ with $2\beta < \delta < 1$. If K is a centered convex body of volume 1 in $\mathbb{R}^n$ with $\beta(\mu_K) = \beta$ and $n/L_{\mu_K}^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta}$ then*

$$\varrho_2(\mu_K,\delta) \leqslant \left(1+\sqrt{8\beta/\delta}\right)\frac{\mathbb{E}_{\mu_K}(\Lambda^*_{\mu_K})}{n}.$$

Proof. Note that if $m := \mathbb{E}_\mu(\Lambda^*_\mu)$ then as before, for all $\varepsilon \in (0,1)$, from Chebyshev's inequality we have that

$$\mu(\{\Lambda^*_\mu \geqslant m+\varepsilon m\}) \leqslant \mu(\{|\Lambda^*_\mu - m| \geqslant \varepsilon m\}) \leqslant \frac{\beta(\mu)}{\varepsilon^2}.$$

If $\beta(\mu) < 1/2$ and $2\beta(\mu) < \delta < 1$ then, choosing $\varepsilon = \sqrt{2\beta(\mu)/\delta}$ we have that

$$\mu(B_{(1+\varepsilon)m}(\mu)) \geqslant 1 - \frac{\delta}{2}.$$

Therefore, we will have that

$$\varrho_2(\mu,\delta) \leqslant (1+2\varepsilon)m/n$$

if our lower bound for $\inf_{x\in B_{(1+\varepsilon)m}(\mu)} \varphi_\mu(x)$ gives

$$2\binom{N}{n}\left(1-\inf_{x\in B_{(1+\varepsilon)m}(\mu)}\varphi_\mu(x)\right)^{N-n} \leqslant \frac{\delta}{2} \tag{7.2}$$

for all $N \geqslant N_0 := \exp((1+2\varepsilon)m)$. Recall that in the case of the uniform measure on a centered convex body of volume 1, Theorem 5.1 shows that

$$\inf_{x\in B_{(1+\varepsilon)m}(\mu_K)} \varphi_{\mu_K}(x) \geqslant \frac{1}{10}\exp(-(1+\varepsilon)m - 2\sqrt{n}).$$

We require that n and m are large enough so that $1/2^n < \delta/2$ and $2\sqrt{n} \leqslant \frac{\varepsilon m}{2}$. Using also the fact that $\binom{N}{n} \leqslant e^{-1}\left(\frac{eN}{n}\right)^n$ we see that (7.2) will be satisfied if we also have

$$\left(\frac{2eN}{n}\right)^n e^{-\frac{N-n}{10}e^{-(1+3\varepsilon/2)m}} < 1.$$

Setting $x := N/n$ we see that this last is equivalent to

$$e^{(1+3\varepsilon/2)m} < \frac{x-1}{10\ln(2ex)}.$$

One can now check that if $N \geqslant \exp((1+2\varepsilon)m)$ then all the restrictions are satisfied if we assume that $n/L_{\mu_K}^2 \geqslant c_2 \ln(2/\delta)\sqrt{\delta/\beta(\mu_K)}$. □

For an arbitrary log-concave probability measure we can use the recent estimate (5.6) of Brazitikos and Chasapis. Using this lower bound for $\varphi_-(B_t(\mu))$ instead of Theorem 5.1 we arrive at a similar conclusion.

Theorem 7.3. *Let $\beta,\delta > 0$ with $128\beta < \delta < 1$. If $n \geqslant n_0(\beta,\delta)$ then for any log-concave probability measure μ on $\mathbb{R}^n$ with $\beta(\mu) = \beta$ we have that*

$$\varrho_2(\mu,\delta) \leqslant \left(1+c\sqrt{\beta(\mu)/\delta}\right)\frac{\mathbb{E}_\mu(\Lambda^*_\mu)}{n},$$

where $c > 0$ is an absolute constant.

From the discussion in this section it is clear that our approach is able to provide good bounds for the threshold $\varrho(\mu,\delta)$ if the parameter $\beta(\mu)$ is small, especially if $\beta(\mu)=o_n(1)$ as the dimension increases. We illustrate this with a number of examples, starting with the uniform measure on the cube. Let μ_{C_n} be the uniform measure on $C_n=\left[-\frac{1}{2},\frac{1}{2}\right]^n$. Since $\mu_{C_n}=\mu_{C_1}\otimes\cdots\otimes\mu_{C_1}$ we have

$$\mathrm{Var}_{\mu_{C_n}}(\Lambda^*_{\mu_{C_n}})=n\mathrm{Var}_{\mu_{C_1}}(\Lambda^*_{\mu_{C_1}})\qquad\text{and}\qquad\mathbb{E}_{\mu_{C_n}}(\Lambda^*_{\mu_{C_n}})=n\mathbb{E}_{\mu_{C_1}}(\Lambda^*_{\mu_{C_1}}).$$

Therefore,

$$\beta(\mu_{C_n})=\frac{\mathrm{Var}_{\mu_{C_n}}(\Lambda^*_{\mu_{C_n}})}{(\mathbb{E}_{\mu_{C_n}}(\Lambda^*_{\mu_{C_n}}))^2}=\frac{\beta(\mu_{C_1})}{n}\longrightarrow 0.$$

as $n\to\infty$. Then, Theorem 7.1 and Theorem 7.2 show that for any $\delta\in(0,1)$ there exists $n_0(\delta)$ such that, for any $n\geqslant n_0$,

$$\varrho(\mu_{C_n},\delta)\leqslant\frac{c}{\sqrt{\delta n}},$$

where $c>0$ is an absolute constant. This estimate provides a sharp threshold for the measure of a random polytope K_N with independent vertices uniformly distributed in C_n. It provides a direct proof of the result of Dyer, Füredi and McDiarmid in [16] with a stronger estimate for the "width of the threshold".

Next, let is consider the example of the standard n-dimensional Gaussian measure γ_n with density $f_{\gamma_n}(x)=(2\pi)^{-n/2}e^{-|x|^2/2}$, $x\in\mathbb{R}^n$. Note that $\gamma_n=\gamma_1\otimes\cdots\otimes\gamma_1$, and hence we may argue as in the previous example. We may also use direct computation to see that $\Lambda_{\gamma_n}(\xi)=|\xi|^2/2$ for all $\xi\in\mathbb{R}^n$ and $\Lambda^*_{\gamma_n}(x)=|x|^2/2$ for all $x\in\mathbb{R}^n$. It follows that $B_t(\gamma_n)=\sqrt{2t}B_2^n$. We check that if $x\in\partial(B_t(\gamma_n))$ then

$$\varphi_{\gamma_n}(x)=\frac{1}{\sqrt{2\pi}}\int_{\sqrt{2t}}^{\infty}e^{-u^2/2}du\geqslant\frac{c}{\sqrt{t}}e^{-t}$$

for all $t\geqslant 1$ (see [24, p. 17] for a refined form of the lower bound that we use). By the standard concentration estimate for the Euclidean norm with respect to γ_n (see [41, Theorem 3.1.1]), for all $s>0$ we have that

$$\gamma_n(\{x\in\mathbb{R}^n:|\,|x|-\sqrt{n}\,|\geqslant s\sqrt{n}\})\leqslant 2\exp(-cs^2n),$$

where $c>0$ is an absolute constant. This shows that

$$\max\{\gamma_n((1-s)\sqrt{n}B_2^n),1-\gamma_n((1+s)\sqrt{n}B_2^n)\}\leqslant 2\exp(-cs^2n)$$

for every $s\in(0,1)$. From Lemma 3.7 we see that $\varrho_1(\gamma_n,\delta)\geqslant\frac{1}{2}-\frac{c_1\sqrt{\ln(4/\delta)}}{\sqrt{n}}$, and applying Lemma 3.8 we get $\varrho_2(\gamma_n,\delta)\leqslant\frac{1}{2}+\frac{c_2\sqrt{\ln(4/\delta)}}{\sqrt{n}}$. Combining the above we get

$$\varrho(\gamma_n,\delta)\leqslant\frac{C\sqrt{\ln(4/\delta)}}{\sqrt{n}},$$

where $C>0$ is an absolute constant.

Finally, we discuss the example of the uniform measure on the Euclidean ball. It was proved in [4] that if $\varepsilon\in(0,1)$ and $K_N=\mathrm{conv}\{x_1,\ldots,x_N\}$ where $x_1,\ldots,x_N$ are random points independently and uniformly chosen from B_2^n then

$$\lim_{n\to\infty}\sup\left\{\frac{\mathbb{E}|K_N|}{|B_2^n|}:N\leqslant\exp\left((1-\varepsilon)\left(\frac{n+1}{2}\right)\ln n\right)\right\}=0$$

and

$$\lim_{n\to\infty}\inf\left\{\frac{\mathbb{E}|K_N|}{|B_2^n|}:N\geqslant\exp\left((1+\varepsilon)\left(\frac{n+1}{2}\right)\ln n\right)\right\}=1.$$

We shall obtain a similar conclusion with the approach of this work.

For any centered convex body K of volume 1 in $\mathbb{R}^n$ recall the definition of $\omega_{\mu_K} = \ln(1/\varphi_{\mu_K})$ in (5.4) and consider the parameter

$$\tau(\mu_K) = \frac{\mathrm{Var}_{\mu_K}(\omega_{\mu_K})}{(\mathbb{E}_{\mu_K}(\omega_{\mu_K}))^2}. \tag{7.3}$$

From (5.5) we know that $\omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda^*_{\mu_K}(x) \leqslant \omega_{\mu_K}(x)$ for every $x \in \mathrm{int}(K)$. This allows us to compare $\beta(\mu_K)$ with $\tau(\mu_K)$; some simple calculations show that

$$\beta(\mu_K) = \big(\tau(\mu_K) + O(L^2_{\mu_K}/\sqrt{n})\big)\big(1 + O(L^2_{\mu_K}/\sqrt{n})\big) \tag{7.4}$$

Note also that if K is a centered convex body in $\mathbb{R}^n$ and $r > 0$ then $\Lambda^*_{\mu_{rK}}(x) = \Lambda^*_{\mu_K}(x/r)$ for all $x \in \mathbb{R}^n$, where μ_{rK} is the uniform measure on rK. It follows that

$$\frac{1}{|rK|}\int_{rK}[\Lambda^*_{\mu_{rK}}(x)]^p dx = \frac{1}{|K|}\int_K [\Lambda^*_{\mu_K}(x)]^p dx$$

for every $p > 0$ and $r > 0$. This shows that if D_n is the centered Euclidean ball of volume 1 in $\mathbb{R}^n$ then in order to compute $\beta(\mu_{D_n})$ it suffices to compute the ratio

$$\beta(\mu_{D_n}) + 1 = \frac{\frac{1}{|B_2^n|}\int_{B_2^n}\Lambda^*(x)^2 dx}{\left(\frac{1}{|B_2^n|}\int_{B_2^n}\Lambda^*(x)dx\right)^2}$$

where $\Lambda^* := \Lambda^*_{\mu_{B_2^n}}$ and, because of (7.4), it is enough to compute $\tau(\mu_{B_2^n})$. Set $\omega := \omega_{\mu_{B_2^n}}$. Then, $\omega(x) = \ln(1/\varphi(x))$ where $\varphi(x) = F(|x|)$,

$$F(r) = c_n \int_r^1 (1-t^2)^{\frac{n-1}{2}} dt, \qquad r \in [0,1]$$

and $c_n = \pi^{-1/2}\Gamma(\frac{n}{2}+1)/\Gamma(\frac{n+1}{2})$. From [4, Lemma 2.2] we know that

$$F(r) = (1-r^2)^{\frac{n+1}{2}} h(r,n),$$

where

$$\frac{1}{\sqrt{2\pi(n+2)}} \leqslant h(r,n) \leqslant \frac{1}{r\sqrt{2\pi n}} \tag{7.5}$$

for all $r \in (0,1]$. We assume that n is even (the case where n is odd can be treated in a similar way). Using polar coordinates we compute

$$\frac{1}{|B_2^n|}\int_{B_2^n}\omega(x)\,dx = \frac{n+1}{2}H_{\frac{n}{2}} + O(\ln n) \tag{7.6}$$

and

$$\frac{1}{|B_2^n|}\int_{B_2^n}(\omega(x))^2\,dx = \frac{(n+1)^2}{4}H^2_{\frac{n}{2}} + O(n^2). \tag{7.7}$$

From (7.6) and (7.7) we finally get

$$\tau(\mu_{B_2^n}) = O(1/(\ln n)^2).$$

Then, (7.4) and a simple computation show that

$$\beta(\mu_{D_n}) = \Big(\tau(\mu_{B_2^n}) + O(L^2_{\mu_{B_2^n}}/\sqrt{n})\Big)\Big(1 + O(L^2_{\mu_{B_2^n}}/\sqrt{n})\Big) = O(1/(\ln n)^2),$$

because $L_{\mu_{B_2^n}} \approx 1$. Finally, note that by the estimate (5.5) in Corollary 5.3 we have

$$\mathbb{E}_{\mu_n}(\Lambda^*_{\mu_n}) = \frac{1}{|B_2^n|}\int_{B_2^n} \omega(x)\,dx + O(\sqrt{n}) = \frac{(n+1)}{2}H_{\frac{n}{2}} + O(\sqrt{n})$$

as $n \to \infty$. Combining the above we see that $\beta(\mu_{D_n}) = o_n(1)$ but $\varrho(\mu_{D_n}, \delta) = O(1/\sqrt{\delta})$.

Note. The above discussion leaves open the following question: to estimate

$$\beta_n^* := \sup\{\beta(\mu_K) : K \text{ is a centered convex body of volume 1 in } \mathbb{R}^n\}$$

or, more generally,

$$\beta_n := \sup\{\beta(\mu) : \mu \text{ is a centered log-concave probability measure on } \mathbb{R}^n\}.$$

8 Product measures

In [20] a threshold for $\mathbb{E}_{\mu^N}|K_N|/(2\alpha)^n$ was established for the case where X_i have independent identically distributed coordinates supported on a bounded interval, under some mild additional assumptions (see below for a more precise description). This result was generalized by Pafis in [34] as follows. Let μ be an even Borel probability measure on the real line and let $X_1, \ldots, X_n$ be independent and identically distributed random variables, defined on some probability space $(\Omega, \mathcal{F}, P)$, each with distribution μ. Consider the random vector $\vec{X} = (X_1, \ldots, X_n)$ and, for a fixed N satisfying $N > n$, consider N independent copies $\vec{X}_1, \ldots, \vec{X}_N$ of $\vec{X}$. The distribution of $\vec{X}$ is $\mu_n := \mu \otimes \cdots \otimes \mu$ (n times) and the distribution of $(\vec{X}_1, \ldots, \vec{X}_N)$ is $\mu_n^N := \mu_n \otimes \cdots \otimes \mu_n$ (N times). The goal is to obtain a sharp threshold for the expected μ_n-measure of the random polytope

$$K_N := \mathrm{conv}\{\vec{X}_1, \ldots, \vec{X}_N\}.$$

Assume that μ is non-degenerate, i.e. $\mathrm{Var}(X) > 0$. Let

$$x^* = x^*(\mu) := \sup\{x \in \mathbb{R} \colon \mu([x,\infty)) > 0\}$$

be the right endpoint of the support of μ and set $I_\mu = (-x^*, x^*)$. Note that since μ is non-degenerate and even, we have that $x^* > 0$. As usual, let

$$M_\mu(t) := \mathbb{E}\big(e^{tX}\big) := \int_{\mathbb{R}} e^{tx}\,d\mu(x), \qquad t \in \mathbb{R}$$

denote the moment generating function of X, and let $\Lambda_\mu(t) := \ln M_\mu(t)$ be its logarithmic moment generating function. Finally, consider the Legendre transform $\Lambda^*_\mu : I_\mu \to \mathbb{R}$ of Λ_μ.

We say that μ is *admissible* if it is non-degenerate, i.e. $\mathrm{Var}_\mu(X) > 0$, and satisfies the following conditions:

(i) There exists $r > 0$ such that $\mathbb{E}\big(e^{tX}\big) < \infty$ for all $t \in (-r, r)$; in particular, X has finite moments of all orders.

(ii) One of the following holds: (1) $x^* < +\infty$ and $P(X = x^*) = 0$, or (2) $x^* = +\infty$ and $\{\Lambda_\mu < \infty\} = \mathbb{R}$, or (3) $x^* = +\infty$, $\{\Lambda_\mu < \infty\}$ is bounded and μ is log-concave.

Finally, we say that μ satisfies *the Λ^*-condition* if

$$\lim_{x \uparrow x^*} \frac{-\ln \mu([x,\infty))}{\Lambda^*_\mu(x)} = 1.$$

Theorem 8.1. *Let μ be an admissible even probability measure on $\mathbb{R}$ that satisfies the Λ^*-condition. Then, for any $\delta \in \big(0, \frac{1}{2}\big)$ and any $\varepsilon \in (0,1)$ there exists $n_0(\mu, \delta, \varepsilon)$ such that*

$$\varrho_1(\mu_n, \delta) \geqslant (1-\varepsilon)\mathbb{E}_\mu(\Lambda^*_\mu) \quad \textit{and} \quad \varrho_2(\mu_n, \delta) \leqslant (1+\varepsilon)\mathbb{E}_\mu(\Lambda^*_\mu)$$

for every $n \geqslant n_0(\mu, \delta, \varepsilon)$. In particular, $\{\mu_n\}_{n=1}^\infty$ exhibits a sharp threshold, i.e. $\lim_{n\to\infty} \varrho(\mu_n, \delta) = 0$, with "threshold constant" $\mathbb{E}_\mu(\Lambda^*_\mu)$.

An application of Theorem 8.1 is also given to the case of the product p-measure $\nu_p^n := \nu_p^{\otimes n}$. For any $p \geqslant 1$ we denote by ν_p the probability distribution on $\mathbb{R}$ with density $(2\gamma_p)^{-1}\exp(-|x|^p)$, where $\gamma_p = \Gamma(1+1/p)$. We show that ν_p satisfies the Λ^*-condition.

Theorem 8.2. *For any $p \geqslant 1$ we have that*

$$\lim_{x\to\infty} \frac{-\ln(\nu_p[x,\infty))}{\Lambda^*_{\nu_p}(x)} = 1.$$

Note that the measure ν_p is admissible for all $1 \leqslant p < \infty$; it satisfies condition (ii-3) if $p = 1$ and condition (ii-2) for all $1 < p < \infty$. Therefore, Theorem 8.2 implies that if K_N is the convex hull of $N > n$ independent random vectors $\vec{X}_1, \ldots, \vec{X}_N$ with distribution ν_p^n then the expected measure $\mathbb{E}_{(\nu_p^n)^N}(\nu_p^n(K_N))$ exhibits a sharp threshold at $N = \exp((1\pm\varepsilon)\mathbb{E}_{\nu_p}(\Lambda^*_{\nu_p})n)$.

The variant of this question that was studied in [20] dealt with the case where μ is an even, compactly supported, Borel probability measure on the real line, $\mu_n(K_N)$ is replaced by the volume of K_N, and

$$\kappa = \kappa(\mu) := \frac{1}{2x^*}\int_{-x^*}^{x^*} \Lambda^*_\mu(x)dx.$$

If $0 < \kappa(\mu) < \infty$ then one has that, for every $\varepsilon \in (0,\kappa)$,

$$\lim_{n\to\infty} \sup\left\{(2x^*)^{-n}\mathbb{E}(|K_N|) \colon N \leqslant \exp((\kappa-\varepsilon)n)\right\} = 0 \tag{8.1}$$

and if the distribution μ satisfies the Λ^*-condition then one also has

$$\lim_{n\to\infty} \inf\left\{(2x^*)^{-n}\mathbb{E}(|K_N|) \colon N \geqslant \exp((\kappa+\varepsilon)n)\right\} = 1. \tag{8.2}$$

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