On the average volume of sections of convex bodies

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Abstract

The average section functional $\mathrm{as}(K)$ of a centered convex body in $\mathbb{R}^n$ is the average volume of the central hyperplane sections of K:

$$\mathrm{as}(K) = \int_{S^{n-1}} |K \cap \xi^\perp| \, d\sigma(\xi).$$

We study the question if there exists an absolute constant $C > 0$ such that for every n, for every centered convex body K in $\mathbb{R}^n$ and for every $1 \leqslant k \leqslant n-1$,

$$\mathrm{as}(K) \leqslant C^k |K|^{\frac{k}{n}} \max_{E \in \mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E).$$

We observe that the case $k = 1$ is equivalent to the hyperplane conjecture. We show that this inequality holds true in full generality if one replaces C by CL_K or $Cd_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)$, where L_K is the isotropic constant of K and $d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)$ is the outer volume ratio distance from K to the class $\mathcal{BP}_k^n$ of generalized k-intersection bodies. We also compare $\mathrm{as}(K)$ to the average of $\mathrm{as}(K \cap E)$ over all k-codimensional sections of K. We examine separately the dependence of the constants on the dimension in the case where K is in some of the classical positions as well as the natural lower dimensional analogue of the average section functional.

1 Introduction

Let K be a convex body in $\mathbb{R}^n$, with barycenter at the origin (we call these bodies centered). We denote by $\mathrm{as}(K)$ the average volume of the central hyperplane sections of K:

$$\mathrm{as}(K) = \int_{S^{n-1}} |K \cap \xi^\perp| \, d\sigma(\xi), \tag{1.1}$$

where $|\cdot|$ denotes volume in the appropriate dimension, $\xi^\perp$ is the subspace perpendicular to ξ, and σ is the rotationally invariant probability measure on S^{n-1}. More generally, for any $1 \leqslant r \leqslant n-1$ we define

$$\mathrm{as}_r(K) = \int_{\mathrm{Gr}_{n-r}} |K \cap E| \, d\nu_{n-r}(E), \tag{1.2}$$

where ν_{n-r} is the Haar probability measure on the Grassmannian Gr_{n-r} of $(n-r)$-dimensional subspaces of $\mathbb{R}^n$. Thus, $\mathrm{as}_r(K)$ is the average volume of r-codimensional central sections of K; note that $\mathrm{as}(K) = \mathrm{as}_1(K)$.

The fourth named author proved in [13] that if K is an intersection body in $\mathbb{R}^n$ (see Section 2 for definitions and background information) then

$$\mathrm{as}(K) \leqslant b_{n,1} \, |K|^{\frac{1}{n}} \max_{\xi \in S^{n-1}} \mathrm{as}(K \cap \xi^\perp), \tag{1.3}$$

where

$$b_{n,1} := \frac{\omega_{n-1}}{\omega_{n-2}\omega_n^{\frac{1}{n}}} \simeq 1$$

(and ω_m denotes the volume of the Euclidean unit ball B_2^m in $\mathbb{R}^m$). Whenever we write $a \lesssim b$ we mean that there exists an absolute constant $c > 0$ such that $a \leqslant cb$, and whenever we write $a \simeq b$, we mean that $a \lesssim b$ and $b \lesssim a$. Note that (1.3) is sharp: it becomes equality if $K = B_2^n$.

The purpose of this article is to discuss similar inequalities for the average volume of hyperplane sections of an arbitrary centered convex body K in $\mathbb{R}^n$. More precisely, we study the following question.

Question 1.1. *Let $1 \leqslant k < n$ and define $\gamma_{n,k}$ as the smallest constant $\gamma > 0$ for which the following holds true: for every centered convex body K in $\mathbb{R}^n$ we have*

$$\mathrm{as}(K) \leqslant \gamma^k |K|^{\frac{k}{n}} \max_{E\in \mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E). \tag{1.4}$$

Is it true that $\sup_{n,k} \gamma_{n,k} < \infty$?

In Section 3 we generalize (1.3) using as a parameter the outer volume ratio distance $d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)$ from an origin-symmetric convex body K to the class $\mathcal{BP}_k^n$ of generalized k-intersection bodies. Our estimates are based on the next more general theorem which is valid for the larger class of origin-symmetric star bodies in $\mathbb{R}^n$ and for any even continuous density on $\mathbb{R}^n$.

Theorem 1.2. *Let $1 \leqslant k \leqslant n-1$, let K be an origin-symmetric star body in $\mathbb{R}^n$, and let f be a non-negative even continuous function on $\mathbb{R}^n$. Then*

$$\int_{S^{n-1}} \rho_K^{n-1}(\theta) f(\rho_K(\theta)\theta)\, d\theta \leqslant c_{n,k}^k\, d_{\mathrm{ovr}}^k(K, \mathcal{BP}_k^n)\, |K|^{\frac{k}{n}} \max_{E\in \mathrm{Gr}_{n-k}} \int_{S^{n-1}\cap E} \rho_K^{n-k-1}(\theta) f(\rho_K(\theta)\theta)\, d\theta. \tag{1.5}$$

In the statement above, ρ_K is the radial function of a star body K and we use the notation $d\theta$ for the non-normalized measure on the sphere with density 1. The constant $c_{n,k}$ is given by

$$c_{n,k}^k = \frac{n\omega_n^{\frac{n-k}{n}}}{(n-k)\omega_{n-k}},$$

and one can check that $c_{n,k} \simeq 1$.

Theorem 1.2 provides a first estimate on the constants $\gamma_{n,k}$ of Question 1.1. Choosing $f \equiv 1$ we see that (1.5) implies the following.

Theorem 1.3. *Let $1 \leqslant k \leqslant n-1$, and let K be an origin-symmetric star body in $\mathbb{R}^n$. Then,*

$$\mathrm{as}(K) \leqslant b_{n,k}^k d_{\mathrm{ovr}}^k(K, \mathcal{BP}_k^n)\, |K|^{\frac{k}{n}} \max_{E\in Gr_{n-k}} \mathrm{as}(K \cap E). \tag{1.6}$$

In other words, $\gamma_{n,k} \leqslant b_{n,k} d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)$.

The constant $b_{n,k}$ in Theorem 1.3 is given by

$$b_{n,k}^k = \frac{\omega_{n-1}}{\omega_{n-k-1}\omega_n^{\frac{k}{n}}},$$

and one can also check that $b_{n,k} \simeq 1$.

In the case where the body K is convex, the distance $d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)$ was estimated in [17]. In particular, the available bounds for $d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)$ show that $\gamma_{n,k}$ is bounded by a function of n/k, and hence it remains bounded as long as k is proportional to n. More generally, we have:

Theorem 1.4. *For every origin-symmetric convex body K, for every $1 \leqslant k \leqslant n-1$ and every even non-negative continuous function f on $\mathbb{R}^n$,*

$$\int_{S^{n-1}} \rho_K^{n-1}(\theta) f(\rho_K(\theta)\theta)\, d\theta \leqslant \big(c_1 h(n/k)\big)^k\, |K|^{\frac{k}{n}} \max_{E\in \mathrm{Gr}_{n-k}} \int_{S^{n-1}\cap E} \rho_K^{n-k-1}(\theta) f(\rho_K(\theta)\theta)\, d\theta, \tag{1.7}$$

where $c_1 > 0$ is an absolute constant and $h(t) = \sqrt{t}\cdot (\log(et))^{\frac{3}{2}}$, $t \geqslant 1$. In particular,

$$\gamma_{n,k} \leqslant c_1 \sqrt{n/k}\, [\log(en/k)]^{\frac{3}{2}}. \tag{1.8}$$

It is also known that for many classes of convex bodies the distance $d_{\rm ovr}(K,\mathcal{BP}_k^n)$ is bounded by an absolute constant. This includes unconditional bodies, unit balls of subspaces of L_p, and others. Therefore, the restriction of Question 1.1 to all these classes has an affirmative answer.

Theorem 1.2 also allows us to prove an analogue of Theorem 1.3 for the quantities $\mathrm{as}_r(K)$.

Theorem 1.5. *Let $1\leqslant k<n-2$ and $1\leqslant r<n-k$. For any origin-symmetric star body K in $\mathbb{R}^n$ we have that*

$$\mathrm{as}_r(K)\leqslant \phi_{n,k,r}^k d_{\rm ovr}^k(K,\mathcal{BP}_k^n)\,|K|^{\frac{k}{n}}\max_{E\in Gr_{n-k}}\mathrm{as}_r(K\cap E). \tag{1.9}$$

Here

$$\phi_{n,k,r}^k=\frac{\omega_{n-r}}{\omega_{n-k-r}\omega_n^{\frac{k}{n}}},$$

and one can check that $\phi_{n,k,r}\simeq\sqrt{\frac{n}{n-r}}$.

In Section 4 we show that an analogue of (1.3) holds true in full generality, up to the value of the isotropic constant of K. In order to state our main result we recall the definition of the isotropic position. A centered convex body K of volume 1 in $\mathbb{R}^n$ is called isotropic if there exists a constant $L_K>0$ such that

$$\int_K\langle x,\xi\rangle^2dx=L_K^2 \tag{1.10}$$

for every $\xi\in S^{n-1}$. Every centered convex body K has an *isotropic position* $T(K)$, $T\in GL(n)$, which is uniquely defined modulo orthogonal transformations, and hence the *isotropic constant* L_K is an invariant of the linear class of K. A well-known question in asymptotic convex geometry asks if there exists an absolute constant $C>0$ such that $L_K\leqslant C$ for every n and every centered convex body K in $\mathbb{R}^n$. The best known upper bound

$$L_n:=\sup\{L_K:K\text{ isotropic in }\mathbb{R}^n\}\leqslant c\sqrt[4]{n} \tag{1.11}$$

is due to Klartag [11] (see also [6] for the history of the problem and recent developments in this area). On the other hand, one always has $L_K\geqslant L_{B_2^n}\geqslant c$, where $c>0$ is an absolute constant. In other words, the question is if $L_K\simeq 1$ for all centered convex bodies.

Theorem 1.6. *Let K be a centered convex body in $\mathbb{R}^n$. Then, for every $1\leqslant k\leqslant n-1$,*

$$\mathrm{as}(K)\leqslant (c_2L_K)^k\,|K|^{\frac{k}{n}}\max_{E\in\mathrm{Gr}_{n-k}}\mathrm{as}(K\cap E), \tag{1.12}$$

where $c_2>0$ is an absolute constant and L_K is the isotropic constant of K.

It is known that for many classes of convex bodies the isotropic constant L_K is bounded by an absolute constant (see [6, Chapter 4]). Theorem 1.6 provides an affirmative answer to Question 1.1 for all these classes.

On the other hand, it is interesting to note that (1.12) is essentially the best bound we can hope for. We show in Proposition 4.3 that if K is an isotropic convex body in $\mathbb{R}^n$ then

$$\mathrm{as}(K)\simeq L_K\max_{\xi\in S^{n-1}}\mathrm{as}(K\cap\xi^\perp)\,|K|^{\frac{1}{n}}. \tag{1.13}$$

This shows that the estimate of Theorem 1.6 is asymptotically sharp: if $\gamma>0$ is a constant such that (1.4) holds for $k=1$ and all K then we must have $\gamma\geqslant cL_K$. Combining this fact with Theorem 1.6 we actually conclude that

$$\gamma_{n,k}\lesssim\gamma_{n,1}\simeq L_n \tag{1.14}$$

for all $1 \leqslant k \leqslant n-1$ (see Proposition 4.5).

One of the tools that are used in the proof of Theorem 1.6 is a variant of Meyer's dual Loomis-Whitney inequality [22] that was recently obtained in [5]; see (4.2). The second tool is a lower bound for the dual affine quermassintegrals

$$\tilde{\Phi}_k(K) := \frac{\omega_n}{\omega_{n-k}} \left(\int_{\mathrm{Gr}_{n-k}} |K \cap E|^n \, d\nu_{n-k}(E) \right)^{\frac{1}{n}} \tag{1.15}$$

of a convex body K in $\mathbb{R}^n$ in terms of the isotropic constant of K (see [7]). In fact, one can check that the problem to obtain asymptotically sharp lower bounds for $\tilde{\Phi}_k(K)$ is equivalent to the question whether $\gamma_{n,1} \simeq L_n \simeq 1$ (see Remark 4.7). When the codimension k is proportional to n the available lower bounds are independent from the isotropic constant of K (see [7] and [6, Section 6.4]). Thus, we get a variant of Theorem 1.4.

Theorem 1.7. *Let $1 \leqslant k \leqslant n-1$ and let K be a centered convex body in $\mathbb{R}^n$. Then,*

$$\mathrm{as}(K) \leqslant \big(c_3 h(n/k)\big)^k \, |K|^{\frac{k}{n}} \max_{E \in Gr_{n-k}} \mathrm{as}(K \cap E), \tag{1.16}$$

where $c_2 > 0$ is an absolute constant and $h(t) = \sqrt{t} \cdot (\log(et))^{\frac{3}{2}}$, $t \geqslant 1$.

The methods that are used for the proof of Theorem 1.6 and Theorem 1.3 are independent. Note that the first method allows us to work with (not necessarily symmetric) centered convex bodies while the second method allows us to work with origin-symmetric (not necessarily convex) star bodies and to consider even continuous densities in place of volume. Therefore, the two results complement each other. A link between the two bounds is given by the inequality

$$L_K \leqslant c L_k \cdot d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \tag{1.17}$$

which is due to E. Milman (see [23, Corollary 5.4]). However, since we only know that $L_k = O(\sqrt[4]{k})$, the estimates of Theorem 1.6 and Theorem 1.3 are incomparable for $k \gg 1$.

In Section 5 we discuss the mean value of the average section functional $\mathrm{as}(K \cap E)$ over all $E \in \mathrm{Gr}_{n-k}$, $1 \leqslant k \leqslant n-1$. We obtain the next general upper and lower bounds.

Theorem 1.8. *Let K be a centered convex body in $\mathbb{R}^n$ and define $p(K) := R(K)/|K|^{\frac{1}{n}}$, where $R(K)$ is the circumradius of K. Then, for every $1 \leqslant k \leqslant n-1$ we have that*

$$\left(\frac{c_4 \sqrt{n}}{p(K)} \right)^k \mathrm{as}(K) \leqslant |K|^{\frac{k}{n}} \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E) \, d\nu_{n-k}(E) \leqslant \left(\frac{c_5 p(K)}{\sqrt{n}} \right)^{\frac{k}{n-1}} \mathrm{as}(K), \tag{1.18}$$

where $c_4, c_5 > 0$ are absolute constants.

Since $R(K)$ is polynomial in n for all the classical positions of a convex body K (isotropic position, minimal surface position, minimal mean width position, John's and Löwner's position) the right hand side inequality of (1.18) implies the following.

Theorem 1.9. *Let K be a centered convex body in $\mathbb{R}^n$. If K is in one of the classical positions then*

$$|K|^{\frac{k}{n}} \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E) \, d\nu_{n-k}(E) \leqslant c_6^k \, \mathrm{as}(K) \tag{1.19}$$

for every $1 \leqslant k \leqslant n-1$, where $c_6 > 0$ is an absolute constant.

Closing this introductory section we would like to note that the results of this article are dual to the ones in [9]. In that work, the main question was to compare the surface area $S(K)$ of a convex body K in $\mathbb{R}^n$ to the minimal, average or maximal surface area of its hyperplane or lower dimensional projections. One of the main results in [9] states that there exists an absolute constant $c_1 > 0$ such that, for every convex body K in $\mathbb{R}^n$,

$$|K|^{\frac{1}{n}} \min_{\xi\in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant \frac{c_7\partial_K}{\sqrt{n}} S(K), \tag{1.20}$$

where $c_7 > 0$ is an absolute constant and

$$\partial_K := \min\left\{S(T(K))/|T(K)|^{\frac{n-1}{n}} : T \in GL(n)\right\} \tag{1.21}$$

is the *minimal surface area parameter* of K. Another result from [9] asserts that if K is in some of the classical positions mentioned above, then

$$|K|^{\frac{1}{n}} \int_{S^{n-1}} S(P_{\xi^\perp}(K))\, d\sigma(\xi) \geqslant c_8 S(K), \tag{1.22}$$

where $c_9 > 0$ is an absolute constant. The analogy with Theorem 1.6 and Theorem 1.9 is clear; the role of the average section functional $\mathrm{as}(K)$ is played by the surface area $S(K)$, and the role of the isotropic constant is played by the minimal surface area parameter.

2 Notation and background

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We use the notation $d\theta$ for the non-normalized measure on the sphere with density 1.

If $\xi \in S^{n-1}$, then $\xi^\perp = \{x \in \mathbb{R}^n : \langle x,\xi\rangle = 0\}$. The Grassmann manifold Gr_m of m-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure ν_m. For every $1 \leqslant m \leqslant n-1$ and $E \in \mathrm{Gr}_m$ we write P_E for the orthogonal projection from $\mathbb{R}^n$ onto E, and we set $B_E = B_2^n \cap F$ and $S_E = S^{n-1} \cap E$. The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line.

We refer to the books [8] and [25] for basic facts from the Brunn-Minkowski theory and to the book [1] for basic facts from asymptotic convex geometry.

2.1. Star bodies and convex bodies. A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$, and that K is centered if its barycenter $\frac{1}{|K|}\int_K x\,dx$ is at the origin. A compact set K in $\mathbb{R}^n$ will be called star-shaped at 0 if it contains the origin in its interior and every line through 0 meets K in a line segment. For such a set, the radial function ρ_K is defined on S^{n-1} by

$$\rho_K(\theta) = \max\{\lambda > 0 : \lambda\theta \in K\}, \qquad \theta \in S^{n-1}. \tag{2.1}$$

If ρ_K is continuous, then we say that K is a star body. Then, the volume of K in polar coordinates is given by

$$|K| = \omega_n \int_{S^{n-1}} \rho_K^n(\theta)\, d\sigma(\theta). \tag{2.2}$$

The radial sum $K\tilde{+}D$ of two star bodies K and D is defined by

$$\rho_{K\tilde{+}D} = \rho_K + \rho_D. \tag{2.3}$$

We equip the class $\mathcal{S}_n$ of star bodies with the radial metric

$$d_r(K,D) := \sup_{\xi\in S^{n-1}} |\rho_K(\xi) - \rho_D(\xi)|. \tag{2.4}$$

The support function of a convex body K is defined by $h_K(y) = \max\{\langle x,y\rangle : x \in K\}$, and the mean width of K is

$$w(K) = \int_{S^{n-1}} h_K(\theta)\, d\sigma(\theta). \tag{2.5}$$

The circumradius of K is the smallest $R > 0$ for which $K \subseteq RB_2^n$. If $0 \in \mathrm{int}(K)$ then we write $r(K)$ for the inradius of K (the largest $r > 0$ for which $rB_2^n \subseteq K$) and we define the polar body K° of K by

$$K^\circ := \{y \in \mathbb{R}^n : \langle x,y\rangle \leqslant 1 \text{ for all } x \in K\}. \tag{2.6}$$

The section of a star body K with $\xi^\perp$ is denoted by $K \cap \xi^\perp$, and we write $P_{\xi^\perp}(K)$ for the orthogonal projection of K onto $\xi^\perp$.

The volume radius of K is the quantity $\mathrm{vrad}(K) = (|K|/|B_2^n|)^{1/n}$. We also define $\|\theta\|_K = \min\{t > 0 : \theta \in tK\}$ and

$$M(K) = \int_{S^{n-1}} \rho_K^{-1}(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} \|\theta\|_K\, d\sigma(\theta). \tag{2.7}$$

2.2. Dual mixed volumes. Lutwak introduced dual mixed volumes in [18]; he first considered convex bodies, but then extended his definition to the class $\mathcal{S}_n$ of star bodies. Given $K_1,\dots,K_n \in \mathcal{S}_n$, their dual mixed volume is the integral

$$\tilde{V}(K_1,\dots,K_n) = \omega_n \int_{S^{n-1}} \rho_{K_1}(\theta)\cdots\rho_{K_n}(\theta) d\sigma(\theta). \tag{2.8}$$

The observation is that such integrals have properties analogous to those of mixed volumes if one replaces Minkowski addition by radial addition. The function $\tilde{V}$ is clearly non-negative, symmetric and monotone with respect to its arguments, positive linear with respect to $\tilde{+}$ in each of its arguments, and has volume as its diagonal. A simple calculation shows that if $K_1,\dots,K_m \in \mathcal{S}_n$ and $\lambda_1,\dots,\lambda_m > 0$, then

$$|\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_m K_m| = \sum_{i_1,\dots,i_n=1}^{m} \tilde{V}(K_{i_1},\dots,K_{i_n})\lambda_{i_1}\dots\lambda_{i_n}. \tag{2.9}$$

In particular, if $K, D \in \mathcal{S}_n$ and $t > 0$ then

$$|K \tilde{+} tD| = \sum_{j=0}^{n} \binom{n}{j} \tilde{V}_j(K,D)\, t^j, \tag{2.10}$$

where $\tilde{V}_j(K,D) := \omega_n \int_{S^{n-1}} \rho_K^{n-j}(\theta)\rho_D^j(\theta) d\sigma(\theta)$, is the j-th dual mixed volume of K and D.

An inequality which further illustrates the analogy with the mixed volumes is the dual Minkowski's inequality: for every $K, D \in \mathcal{S}_n$ an application of Hölder's inequality gives

$$\tilde{V}_1(K,D) \leqslant \left(\omega_n \int_{S^{n-1}} \rho_K^n(\theta) d\sigma(\theta)\right)^{\frac{n-1}{n}} \left(\omega_n \int_{S^{n-1}} \rho_D^n(\theta) d\sigma(\theta)\right)^{\frac{1}{n}} \leqslant |K|^{\frac{n-1}{n}} |D|^{\frac{1}{n}}. \tag{2.11}$$

2.3. Intersection bodies. The class of intersection bodies was introduced by Lutwak [20]. The intersection body of a star body K in $\mathbb{R}^n$ with $\rho_K \in C(S^{n-1})$ is the star body IK with radial function

$$\rho_{IK}(\xi) = |K \cap \xi^\perp| = \omega_{n-1} \int_{S(\xi^\perp)} \rho_K^{n-1}(\theta) d\sigma_\xi(\theta), \tag{2.12}$$

where $S(\xi^\perp) = S^{n-1} \cap \xi^\perp$ is the Euclidean unit sphere of $\xi^\perp$ and σ_ξ denotes the rotationally invariant probability measure on $S(\xi^\perp)$. If K is a centered convex body then IK is a symmetric convex body. It is known that

$$I(TK) = |\det T| \, (T^{-1})^*(IK) \tag{2.13}$$

for every $T \in GL(n)$. In particular, if $T \in SL(n)$ we see that $|I(TK)| = |IK|$. The class of intersection bodies $\mathcal{I}_n$ is defined as the closure in the radial metric of intersection bodies of star bodies.

Zhang introduced more general classes of bodies in [26]. For $1 \leqslant k \leqslant n-1$, the $(n-k)$-*dimensional spherical Radon transform* $R_{n-k} : C(S^{n-1}) \to C(\mathrm{Gr}_{n-k})$ is a linear operator defined by

$$R_{n-k}g(E) = \int_{S^{n-1}\cap E} g(\theta)\, d\theta, \qquad E \in \mathrm{Gr}_{n-k} \tag{2.14}$$

for every function $g \in C(S^{n-1})$. We say that an origin-symmetric star body D in $\mathbb{R}^n$ is a *generalized k-intersection body*, and write $D \in \mathcal{BP}_k^n$, if there exists a finite non-negative Borel measure μ_D on Gr_{n-k} so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \rho_D^k(\theta) g(\theta)\, d\theta = \int_{\mathrm{Gr}_{n-k}} R_{n-k}g(H)\, d\mu_D(H). \tag{2.15}$$

For a star body K in $\mathbb{R}^n$ and $1 \leqslant k \leqslant n-1$, we denote by

$$d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) = \inf \left\{ \left(\frac{|D|}{|K|} \right)^{1/n} : \ K \subset D, \ D \in \mathcal{BP}_k^n \right\}$$

the outer volume ratio distance from K to the class $\mathcal{BP}_k^n$. The reader will find more information on the Radon transform and intersection bodies in the book [12].

3 Bounds in terms of the outer volume ratio distance to the class of generalized k-intersection bodies

The main result of this section is Theorem 1.2 which is valid for the larger class of origin-symmetric star bodies in $\mathbb{R}^n$ and for any even continuous density on $\mathbb{R}^n$. By an appropriate choice of the density f we obtain Theorem 1.3 and its generalization in Theorem 1.5.

Proof of Theorem 1.2. Let $\varepsilon > 0$. For every $E \in \mathrm{Gr}_{n-k}$, we have

$$\int_{(K \tilde{+} \varepsilon B_2^n)\cap E} f(x)dx - \int_{K\cap E} f(x)dx \leqslant \max_{F\in \mathrm{Gr}_{n-k}} \left(\int_{(K \tilde{+} \varepsilon B_2^n)\cap F} f(x)dx - \int_{K\cap F} f(x)dx \right). \tag{3.1}$$

Note that $\rho_{K \tilde{+} \varepsilon B_2^n} = \rho_K + \varepsilon$. Expressing the integrals in polar coordinates we get

$$R_{n-k}\left(\int_{\rho_K(\cdot)}^{\rho_K(\cdot)+\varepsilon} r^{n-k-1} f(r\cdot)dr \right)(E) \leqslant \max_{F\in \mathrm{Gr}_{n-k}} \left(\int_{S^{n-1}\cap F} \int_{\rho_K(\theta)}^{\rho_K(\theta)+\varepsilon} r^{n-k-1} f(r\theta) dr d\theta \right). \tag{3.2}$$

Let $D \in \mathcal{BP}_k^n$ such that $K \subset D$. Integrating the latter inequality by E over Gr_{n-k} with the measure μ_D corresponding to D by (2.14), we get

$$\begin{aligned} &\int_{S^{n-1}} \rho_D^k(\theta) \int_{\rho_K(\theta)}^{\rho_K(\theta)+\varepsilon} r^{n-k-1} f(r\theta) dr d\theta \\ &\qquad \leqslant \mu_D(\mathrm{Gr}_{n-k}) \cdot \max_{F\in \mathrm{Gr}_{n-k}} \left(\int_{S^{n-1}\cap F} \int_{\rho_K(\theta)}^{\rho_K(\theta)+\varepsilon} r^{n-k-1} f(r\theta) dr d\theta \right). \end{aligned} \tag{3.3}$$

We divide both sides by ε and send ε to zero. Note that we can interchange the limit with the maximum, because the convergence is uniform with respect to F. Thus, we get

$$\int_{S^{n-1}} \rho_D^k(\theta)\rho_K^{n-k-1}(\theta) f(\rho_K(\theta)\theta) d\theta \leqslant \mu_D(\mathrm{Gr}_{n-k}) \cdot \max_{F\in \mathrm{Gr}_{n-k}} \left(\int_{S^{n-1}\cap F} \rho_K^{n-k-1}(\theta) f(\rho_K(\theta)\theta) d\theta \right). \tag{3.4}$$

The integral in the left hand side can be estimated from below by $\int_{S^{n-1}} \rho_K^{n-1}(\theta) f(\rho_K(\theta)\theta)\ d\theta$, because $K \subset D$.

To estimate $\mu_D(\mathrm{Gr}_{n-k})$ from above, we combine the fact that $1 = R_{n-k}\mathbf{1}(E)/|S^{n-k-1}|$ for every $E \in \mathrm{Gr}_{n-k}$ with Definition (2.14) and Hölder's inequality to write

$$\begin{aligned} \mu_D(\mathrm{Gr}_{n-k}) &= \frac{1}{|S^{n-k-1}|} \int_{\mathrm{Gr}_{n-k}} R_{n-k}\mathbf{1}(E) d\mu_D(E) \\ &= \frac{1}{|S^{n-k-1}|} \int_{S^{n-1}} \|\theta\|_D^{-k}\ d\theta \\ &\leqslant \frac{1}{|S^{n-k-1}|} \left|S^{n-1}\right|^{\frac{n-k}{n}} \left(\int_{S^{n-1}} \|\theta\|_D^{-n}\ d\theta \right)^{\frac{k}{n}} \\ &= \frac{1}{|S^{n-k-1}|} \left|S^{n-1}\right|^{\frac{n-k}{n}} n^{\frac{k}{n}} |D|^{\frac{k}{n}}. \end{aligned} \tag{3.5}$$

These estimates show that

$$\begin{aligned} &\int_{S^{n-1}} \rho_K^{n-1}(\theta) f(\rho_K(\theta)\theta) d\theta \\ &\qquad \leqslant \frac{1}{|S^{n-k-1}|} \left|S^{n-1}\right|^{\frac{n-k}{n}} n^{\frac{k}{n}} |D|^{\frac{k}{n}} \max_{F\in \mathrm{Gr}_{n-k}} \left(\int_{S^{n-1}\cap F} \rho_K^{n-k-1}(\theta) f(\rho_K(\theta)\theta) d\theta \right). \end{aligned} \tag{3.6}$$

Finally, we choose D so that $|D|^{1/n} \leqslant (1+\delta) d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)|K|^{1/n}$, and then send δ to zero. $\square$

All the other results of this section are consequences of Theorem 1.2.

Proof of Theorem 1.3. First, we express the average section functionals $\mathrm{as}(K)$ and $\mathrm{as}(K\cap E)$ in terms of the radial function of K. Using (2.2) we write

$$\begin{aligned} \mathrm{as}(K) &= \int_{S^{n-1}} |K \cap \xi^\perp|\, d\sigma(\xi) = \omega_{n-1} \int_{S^{n-1}} \int_{S(\xi^\perp)} \rho_K^{n-1}(\theta)\, d\sigma_\xi(\theta)\, d\sigma(\xi) \\ &= \omega_{n-1} \int_{S^{n-1}} \rho_K^{n-1}(\theta)\, d\sigma(\theta). \end{aligned} \tag{3.7}$$

Similarly, for every $1 \leqslant k \leqslant n-1$ and any $E \in \mathrm{Gr}_{n-k}$, we have

$$\mathrm{as}(K\cap E) = \omega_{n-k-1} \int_{S_E} \rho_K^{n-k-1}(\theta)\, d\sigma_E(\theta), \tag{3.8}$$

where σ_E is the rotationally invariant probability measure on $S_E = S^{m-1} \cap E$. Applying Theorem 1.2 for the density $f \equiv \mathbf{1}$ we get

$$\int_{S^{n-1}} \rho_K^{n-1}(\theta)\ d\theta \leqslant c_{n,k}^k\ d_{\mathrm{ovr}}^k(K, \mathcal{BP}_k^n)\ |K|^{\frac{k}{n}} \max_{E\in \mathrm{Gr}_{n-k}} \int_{S^{n-1}\cap E} \rho_K^{n-k-1}(\theta)\ d\theta, \tag{3.9}$$

and Theorem 1.3 follows from (3.7) and (3.8) and an adjustment of the constants. $\square$

Remark 3.1. For certain classes of origin-symmetric convex bodies the distance $d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)$ is bounded by an absolute constant. These classes include unconditional convex bodies and duals of bodies with bounded volume ratio (see [14]) and the unit balls of normed spaces that embed in L_p, $-n < p < \infty$ (see [15], [23] and [16]). If we restrict Question 1.1 to any of these classes then Theorem 1.3 provides an affirmative answer.

Proof of Theorem 1.4. We combine Theorem 1.2 with the following result from [17]: For every origin-symmetric convex body K in $\mathbb{R}^n$,

$$d_{\rm ovr}(K,\mathcal{BP}_k^n) \leqslant c\sqrt{n/k}\,[\log(en/k)]^{\frac{3}{2}}, \tag{3.10}$$

where $c>0$ is an absolute constant. □

Proof of Theorem 1.5. We choose $f(x)=\|x\|_2^{-r+1}$ in Theorem 1.2 to get

$$\int_{S^{n-1}} \rho_K^{n-r}(\theta)\ d\theta \leqslant c_{n,k}^k\ d_{\rm ovr}^k(K,\mathcal{BP}_k^n)\ |K|^{\frac{k}{n}} \max_{E\in {\rm Gr}_{n-k}} \int_{S^{n-1}\cap E} \rho_K^{n-k-r}(\theta)\ d\theta. \tag{3.11}$$

Then, we apply the formula

$$\text{as}_r(K) = \omega_{n-r}\int_{S^{n-1}} \rho_K^{n-r}(\theta) d\sigma(\theta) \tag{3.12}$$

which generalizes (3.7) and is easily verified in the same way. □

4 Bounds in terms of the isotropic constant

Let K be a centered convex body in $\mathbb{R}^n$. In this section we compare $\text{as}(K)$ with the corresponding average section functional $\text{as}(K\cap E)$ for any k-codimensional subspace E of $\mathbb{R}^n$. Our main tool will be a recent result from [5] which is a restricted version of Meyer's dual Loomis-Whitney inequality

$$|K|^{n-1} \geqslant \frac{n!}{n^n}\prod_{i=1}^n |K\cap e_i^{\perp}| \tag{4.1}$$

where $\{e_1,\dots,e_n\}$ is any orthonormal basis of $\mathbb{R}^n$ (see [22]) and in a sense dualizes the uniform cover inequality of Bollobás and Thomason (see [4]). In order to give the precise statement, we introduce some notation. For every non-empty $\tau\subset[n]:=\{1,\dots,n\}$ we set $F_\tau=\text{span}\{e_j : j\in\tau\}$ and $E_\tau=F_\tau^{\perp}$. Given $s\geqslant 1$ and $\sigma\subseteq[n]$, following the terminology of [4] we say that the (not necessarily distinct) sets $\sigma_1,\dots,\sigma_t\subseteq\sigma$ form an s-uniform cover of σ if every $j\in\sigma$ belongs to exactly s of the sets σ_i. Then, [5, Theorem 1.3] states that for any centered convex body K in $\mathbb{R}^n$, for any $t\geqslant 1$ and any s-uniform cover $(\sigma_1,\dots,\sigma_t)$ of a subset σ of $[n]$ we have

$$\prod_{i=1}^t |K\cap E_{\sigma_i}| \leqslant \left(\frac{c_0 t}{s}\right)^{ds} |K\cap E_\sigma|^s |K|^{t-s}, \tag{4.2}$$

where $d=|\sigma|$. We will need a special case of this inequality. We consider $1\leqslant k\leqslant n-1$ and a $(k+1)$-tuple of orthonormal vectors $e_1,\dots,e_k,e_{k+1}:=\xi$ in $\mathbb{R}^n$. Note that the sets $\sigma_1=[k]$ and $\sigma_2=\{k+1\}$ form a 1-uniform cover of the set $\sigma=[k+1]$. Applying (4.2) with $t=2$, $s=1$ and $d=k+1$ we obtain the next lemma.

Lemma 4.1. *Let K be a centered convex body in $\mathbb{R}^n$. For every $1\leqslant k\leqslant n-1$, for any $E\in{\rm Gr}_{n-k}$ and any $\xi\in S^{n-1}\cap E$ we have*

$$|K\cap E|\cdot|K\cap\xi^{\perp}| \leqslant c_0^{k+1}|K\cap E\cap\xi^{\perp}|\cdot|K|, \tag{4.3}$$

where $c_0>0$ is an absolute constant.

Using Lemma 4.1 we can compare $\text{as}(K)$ to $\text{as}(K\cap E)$ for every $E\in{\rm Gr}_{n-k}$. We need the next well-known properties of the parameter M which was defined by (2.7). If D is a symmetric convex body in $\mathbb{R}^m$ then for every $1\leqslant s\leqslant m-1$ and $F\in{\rm Gr}_s(\mathbb{R}^m)$ we have that

$$M(D\cap F) = \int_{S_F}\|\xi\|_D\, d\sigma_F(\xi) \leqslant c_1\sqrt{m/s}\int_{S^{m-1}}\|\xi\|_D\, d\sigma(\xi) = c_1\sqrt{m/s}\, M(D), \tag{4.4}$$

where $c_1 > 0$ is an absolute constant. It is also known that

$$\int_{S^{m-1}} \rho_D(\theta)\, d\sigma(\theta) = \int_{S^{m-1}} \|\theta\|_D^{-1}\, d\sigma(\theta) \simeq \frac{1}{M(D)}. \tag{4.5}$$

For a proof of (4.4) and (4.5) see [1, Section 5.2.1] and [1, Theorem 5.8.7] respectively.

Theorem 4.2. *Let K be a centered convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ and $E \in \mathrm{Gr}_{n-k}$ we have*

$$|K \cap E| \cdot \mathrm{as}(K) \leqslant c_2^k\, \mathrm{as}(K \cap E) \cdot |K|, \tag{4.6}$$

where $c_2 > 0$ is an absolute constant.

Proof. We consider an orhonormal basis $\{e_1, \ldots, e_k\}$ of $E^\perp$ and any unit vector $\xi \in E$. From Lemma 4.1 we have

$$|K \cap E| \cdot |K \cap \xi^\perp| \leqslant c_0^{k+1} |K \cap E \cap \xi^\perp| \cdot |K|. \tag{4.7}$$

Integrating (4.7) with respect to $\xi \in S_E$ we see that

$$\begin{aligned} |K \cap E| \cdot \int_{S_E} |K \cap \xi^\perp| d\sigma_E(\xi) &\leqslant c_0^{k+1} \int_{S_E} |(K \cap E) \cap \xi^\perp| d\sigma_E(\xi) \cdot |K| \\ &= c_0^{k+1}\, \mathrm{as}(K \cap E) \cdot |K|. \end{aligned} \tag{4.8}$$

Applying (4.5) for the symmetric convex body $IK \cap E$ we see that

$$\int_{S_E} |K \cap \xi^\perp| d\sigma_E(\xi) = \int_{S_E} \rho_{IK}(\xi)\, d\sigma_E(\xi) \simeq \frac{1}{M(IK \cap E)}, \tag{4.9}$$

and hence, using (4.4) with $m = n$ and $s = n - k$ and then applying (4.5) for the body IK this time, we obtain

$$\begin{aligned} \int_{S_E} |K \cap \xi^\perp| d\sigma_E(\xi) &\geqslant \frac{c\sqrt{n-k}}{\sqrt{n}} \frac{1}{M(IK)} \simeq \frac{\sqrt{n-k}}{\sqrt{n}} \int_{S^{n-1}} \rho_{IK}(\xi)\, d\sigma(\xi) \\ &= \frac{\sqrt{n-k}}{\sqrt{n}} \int_{S^{n-1}} |K \cap \xi^\perp|\, d\sigma(\xi) = \frac{\sqrt{n-k}}{\sqrt{n}} \mathrm{as}(K). \end{aligned} \tag{4.10}$$

Therefore,

$$|K \cap E|\, \mathrm{as}(K) \leqslant \frac{c_1\sqrt{n}}{\sqrt{n-k}} c_0^{k+1}\, \mathrm{as}(K \cap E) \cdot |K| \leqslant c_2^k\, \mathrm{as}(K \cap E) \cdot |K| \tag{4.11}$$

for every $E \in \mathrm{Gr}_{n-k}$. $\square$

For the proof of Theorem 1.6 we use Theorem 4.2 and estimates for the dual affine quermassintegrals of a centered convex body K: these are defined, for any $1 \leqslant k \leqslant n-1$, as follows:

$$\tilde{R}_k(K) := \frac{1}{|K|^{n-k}} \int_{\mathrm{Gr}_{n-k}} |K \cap E|^n\, d\nu_{n-k}(E). \tag{4.12}$$

The quantities $\tilde{R}_k(K)$ were introduced by Lutwak in [19] and [20]. More precisely, he considered the quantities $\tilde{\Phi}_k(K)$ that were introduced in (1.15), which clearly satisfy the identity

$$\tilde{\Phi}_k(K) = \frac{\omega_n}{\omega_{n-k}} |K|^{\frac{n-k}{n}} \left[\tilde{R}_k(K)\right]^{\frac{1}{n}}. \tag{4.13}$$

Grinberg proved in [10] that the quantity $\tilde{R}_k(K)$ is invariant under $T\in GL(n)$: one has

$$\tilde{R}_k(T(K)) = \tilde{R}_k(K) \tag{4.14}$$

for every $T\in GL(n)$. He also proved that

$$\tilde{R}_k(K) \leqslant \tilde{R}_k(B_2^n) := \frac{\omega_{n-k}^n}{\omega_n^{n-k}} \leqslant e^{\frac{kn}{2}}. \tag{4.15}$$

On the other hand, it was observed by Dafnis and Paouris in [7] that

$$\tilde{R}_k(K) \geqslant \left(\frac{c_4}{L_K}\right)^{kn}, \tag{4.16}$$

where $c_4>0$ is an absolute constant. We will use this lower bound, which is an immediate consequence of (4.14) and of the fact that if K is isotropic then $|K\cap E|^{\frac{1}{k}} \geqslant \frac{c_4}{L_K}$ for every $E\in \mathrm{Gr}_{n-k}$ (see [6, Proposition 5.1.15] for a proof).

Proof of Theorem 1.6. Let K be a centered convex body in $\mathbb{R}^n$ and fix $1\leqslant k\leqslant n-1$. From Theorem 4.2 we know that for every $E\in \mathrm{Gr}_{n-k}$ we have

$$|K\cap E|\cdot \mathrm{as}(K) \leqslant c_2^k\, \mathrm{as}(K\cap E)\cdot |K|, \tag{4.17}$$

where $c_2>0$ is an absolute constant. Therefore,

$$\max_{E\in \mathrm{Gr}_{n-k}} |K\cap E|\cdot \mathrm{as}(K) \leqslant c_2^k \max_{E\in \mathrm{Gr}_{n-k}} \mathrm{as}(K\cap E)\cdot |K|. \tag{4.18}$$

Next, from (4.14) we see that

$$\max_{E\in \mathrm{Gr}_{n-k}} |K\cap E| \geqslant \left(\int_{\mathrm{Gr}_{n-k}} |K\cap E|^n \, d\nu_{n-k}(E)\right)^{\frac{1}{n}} \geqslant \left(\frac{c_4}{L_K}\right)^k |K|^{\frac{n-k}{n}}. \tag{4.19}$$

Going back to (4.18) we see that

$$\left(\frac{c_4}{L_K}\right)^k |K|^{\frac{n-k}{n}} \mathrm{as}(K) \leqslant c_2^k |K| \max_{E\in \mathrm{Gr}_{n-k}} \mathrm{as}(K\cap E), \tag{4.20}$$

and this proves Theorem 1.6. $\square$

The next proposition shows that if K is isotropic and if we consider the hyperplane case (where $k=1$) then the estimate of Theorem 1.6 is sharp: we have an asymptotic formula.

Proposition 4.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, $\mathrm{as}(K)\simeq L_K^{-1}$ and $\mathrm{as}(K\cap \xi^\perp)\simeq L_K^{-2}$ for all $\xi\in S^{n-1}$. In particular,*

$$\mathrm{as}(K) \simeq L_K\, |K|^{\frac{1}{n}} \max_{\xi\in S^{n-1}} \mathrm{as}(K\cap \xi^\perp). \tag{4.21}$$

Proof. It is a general fact (following from [6, Proposition 5.1.15]) that if K is an isotropic convex body then, for every $E\in \mathrm{Gr}_{n-k}$ we have

$$\frac{c_1}{L_K} \leqslant |K\cap E|^{\frac{1}{k}} \leqslant \frac{cL_k}{L_K} \leqslant \frac{c_2(k)}{L_K}, \tag{4.22}$$

where $c_1>0$ is an absolute constant and $c_2(k)$ is a positive constant depending only on k (in fact, $c_2(k)\leqslant c\sqrt[4]{k}$ by Klartag's estimate on L_k). Applying (4.22) with $k=1$ we see that all hyperplane sections $K\cap\xi^\perp$ of K have volume equal (up to an absolute constant) to L_K^{-1}. In particular,

$$\mathrm{as}(K) = \int_{S^{n-1}} |K\cap \xi^\perp|\, d\sigma(\xi) \simeq L_K^{-1}. \tag{4.23}$$

Applying (4.22) with $k=2$ we see that all 2-codimensional sections $K\cap E$ of K have volume equal (up to an absolute constant) to L_K^{-2}. In particular, for every $\xi\in S^{n-1}$ we get

$$\text{as}(K\cap\xi^{\perp})=\int_{S(\xi^{\perp})}|K\cap E_{\xi,\theta}|\,d\sigma_{\xi}(\theta)\simeq L_K^{-2}, \tag{4.24}$$

where $E_{\xi,\theta}=[\text{span}\{\xi,\theta\}]^{\perp}$. This shows that

$$\text{as}(K)\simeq L_K\,\text{as}(K\cap\xi^{\perp})=L_K\,\text{as}(K\cap\xi^{\perp})\,|K|^{\frac{1}{n}} \tag{4.25}$$

In particular, (4.25) implies (4.21). $\square$

Remark 4.4. Proposition 4.3 and the definition of $\gamma_{n,1}$ show that

$$L_K^{-1}\simeq\text{as}(K)\leqslant\gamma_{n,1}\max_{\xi\in S^{n-1}}\text{as}(K\cap\xi^{\perp})\simeq\gamma_{n,1}L_K^{-2} \tag{4.26}$$

for every isotropic convex body K in $\mathbb{R}^n$. Therefore, $L_K\leqslant c\gamma_{n,1}$ for some absolute constant $c>0$, which implies that

$$L_n\leqslant c\gamma_{n,1}. \tag{4.27}$$

Note that by Theorem 1.6 we can then conclude that $\gamma_{n,k}\leqslant cL_n\leqslant c\gamma_{n,1}$. Finally, Theorem 1.6 shows that $\gamma_{n,1}\leqslant c'L_n$. We summarize in the next proposition.

Proposition 4.5. *For any* $1\leqslant k\leqslant n-1$ *we have*

$$\gamma_{n,k}\lesssim\gamma_{n,1}\simeq L_n, \tag{4.28}$$

where $c>0$ *is an absolute constant.*

Proposition 4.5 shows that a positive answer to Question 1.1 (actually, in the case $k=1$) is equivalent to the uniform boundedness of the isotropic constants of all convex bodies in all dimensions (this is exactly the hyperplane conjecture). It also shows that the question becomes "easier" when the codimension k increases, in the sense that $\gamma_{n,k}\lesssim\gamma_{n,1}$. In fact, we can show that if k is proportional to n then $\gamma_{n,k}$ is bounded (this is precisely the content of Theorem 1.7):

Theorem 4.6. *For any* $1\leqslant k\leqslant n-1$ *we have*

$$\gamma_{n,k}\leqslant c\sqrt{n/k}\,[\log(en/k)]^{\frac{3}{2}}, \tag{4.29}$$

where $c>0$ *is an absolute constant.*

Proof. We repeat the proof of Theorem 1.6 using the estimate (see [7, Theorem 1.3])

$$\tilde{R}_k(K)\geqslant\left(\frac{c_5}{\sqrt{n/k}\,[\log(en/k)]^{\frac{3}{2}}}\right)^{kn} \tag{4.30}$$

instead of (4.16). $\square$

Remark 4.7. Let $\alpha_{n,k}$ be the largest constant $\alpha>0$ with the property that $\tilde{R}_k(K)\geqslant\alpha^{kn}$ for every centered convex body K in $\mathbb{R}^n$. Repeating the proof of Theorem 1.6 or Theorem 4.6 we see that

$$\gamma_{n,k}\leqslant\frac{c_1}{\alpha_{n,k}}, \tag{4.31}$$

where $c_1 > 0$ is an absolute constant. In particular,

$$\gamma_{n,k} \lesssim \gamma_{n,1} \simeq L_n \lesssim \alpha_{n,1}^{-1}. \tag{4.32}$$

On the other hand, (4.16) shows that $\alpha_{n,k} \geqslant c/L_n$ for all $1 \leqslant k \leqslant n-1$, and hence $\alpha_{n,1}^{-1} \lesssim L_n$. Therefore,

$$\gamma_{n,1} \simeq L_n \simeq \alpha_{n,1}^{-1}. \tag{4.33}$$

In other words, the question whether

$$\tilde{R}_k(K) \geqslant c^{kn} \tag{4.34}$$

for all $1 \leqslant k \leqslant n-1$ which is studied in [7] (see also [8, Section 9.4]) is equivalent to the hyperplane conjecture and to Question 1.1.

5 Reverse inequalities in the classical positions

Next, we pass to estimates for the mean value of the average section functional of hyperplane sections of K. We start by expressing $\mathrm{as}(K)$ in terms of dual mixed volumes. Note that by (3.7) we have

$$\mathrm{as}(K) = \omega_{n-1} \int_{S^{n-1}} \rho_K^{n-1}(\theta)\, d\sigma(\theta) = \frac{\omega_{n-1}}{\omega_n} \tilde{V}(K, \ldots, K, B_2^n), \tag{5.1}$$

and using (3.8) we see that

$$\int_{G_{n,,n-k}} \mathrm{as}(K \cap E)\, d\nu_{n-k}(E) = \omega_{n-k-1} \int_{S^{n-1}} \rho_K^{n-k-1}(\theta)\, d\sigma(\theta) = \frac{\omega_{n-k-1}}{\omega_n} \tilde{V}(K\,[n-k-1], B_2^n\,[k+1]), \tag{5.2}$$

where $A\,[s]$ means $A, \ldots, A$ repeated s-times.

Theorem 5.1. *Let K be a centered convex body in $\mathbb{R}^n$. Then,*

$$\mathrm{as}(K)^{k+1} \leqslant c^k\, |K|^k \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E)\, d\nu_{n-k}(E) \tag{5.3}$$

and

$$\int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E)\, d\nu_{n-k}(E) \leqslant c^k\, \mathrm{as}(K)^{\frac{n-k-1}{n-1}}, \tag{5.4}$$

where $c > 0$ is an absolute constant.

Proof. From Hölder's inequality we see that

$$\left(\int_{S^{n-1}} \rho_K^{n-1}(\theta)\, d\sigma(\theta)\right)^{k+1} \leqslant \left(\int_{S^{n-1}} \rho_K^{n}(\theta)\, d\sigma(\theta)\right)^{k} \left(\int_{S^{n-1}} \rho_K^{n-k-1}(\theta)\, d\sigma(\theta)\right), \tag{5.5}$$

which can be equivalently written as

$$\tilde{V}(K, \ldots, K, B_2^n)^{k+1} \leqslant |K|^k\, \tilde{V}(K\,[n-k-1], B_2^n\,[k+1]). \tag{5.6}$$

Taking into account (5.1) and (5.2) we rewrite (5.6) in the form

$$\mathrm{as}(K)^{k+1} \leqslant |K|^k\, \frac{\omega_{n-1}^{k+1}}{\omega_n^k \omega_{n-k-1}} \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E)\, d\nu_{n-k}(E). \tag{5.7}$$

A simple computation shows that $\frac{\omega_{n-1}^{k+1}}{\omega_n^k\omega_{n-k-1}} < c^k$ for an absolute constant $c>0$, and (5.3) follows.

On the other hand, by Hölder's inequality,

$$\begin{aligned}(5.8)\qquad \frac{1}{\omega_{n-k-1}}\int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K\cap E)\,d\nu_{n-k}(E) &= \int_{S^{n-1}} \rho_K^{n-k-1}(\theta)\,d\sigma(\theta)\\ &\leqslant \left(\int_{S^{n-1}} \rho_K^{n-1}(\theta)\,d\sigma(\theta)\right)^{\frac{n-k-1}{n-1}} = \left(\frac{\mathrm{as}(K)}{\omega_{n-1}}\right)^{\frac{n-k-1}{n-1}},\end{aligned}$$

therefore

$$(5.9)\qquad \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K\cap E)\,d\nu_{n-k}(E) \leqslant \varrho_{n,k}\,\mathrm{as}(K)^{\frac{n-k-1}{n-1}},$$

where $\varrho_{n,k} = \omega_{n-k-1}\cdot\omega_{n-1}^{-\frac{n-k-1}{n-1}} \leqslant c^k$ for an absolute constant $c>0$, which gives (5.4). □

Let K be a convex body in $\mathbb{R}^n$ with $0\in\mathrm{int}(K)$. Recall that the radius $R(K)$ of K is the smallest $R>0$ for which $K\subseteq RB_2^n$. Using the monotonicity and homogeneity of dual mixed volumes and (5.1) we may write

$$\begin{aligned}(5.10)\qquad \mathrm{as}(K) &= \frac{\omega_{n-1}}{\omega_n}\tilde{V}(K,\dots,K,B_2^n) \geqslant \frac{\omega_{n-1}}{\omega_n R(K)}\tilde{V}(K,\dots,K,R(K)B_2^n)\\ &\geqslant \frac{\omega_{n-1}}{\omega_n R(K)}\tilde{V}(K,\dots,K,K) = \frac{\omega_{n-1}}{\omega_n R(K)}\,|K|.\end{aligned}$$

In this way we obtain the following general lower bound for $\mathrm{as}(K)$.

Lemma 5.2. *Let K be a centered convex body in $\mathbb{R}^n$. If we define $p(K)=R(K)/|K|^{\frac{1}{n}}$ then*

$$(5.11)\qquad \frac{c\sqrt{n}}{p(K)} \leqslant \frac{\mathrm{as}(K)}{|K|^{\frac{n-1}{n}}},$$

where $c>0$ is an absolute constant.

Proof. From (5.10) we see that

$$(5.12)\qquad \frac{\mathrm{as}(K)}{|K|^{\frac{n-1}{n}}} \geqslant \frac{\omega_{n-1}}{\omega_n R(K)}|K|^{1/n} \geqslant \frac{c\sqrt{n}}{R(K)}|K|^{1/n},$$

and the lemma follows by the definition of $p(K)$. □

Going back to Theorem 5.1 we immediately get the following.

Theorem 5.3. *Let K be a centered convex body in $\mathbb{R}^n$. Then, for every $1\leqslant k\leqslant n-1$ we have that*

$$(5.13)\qquad \left(\frac{c_1\sqrt{n}}{p(K)}\right)^k \mathrm{as}(K) \leqslant |K|^{\frac{k}{n}} \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K\cap E)\,d\nu_{n-k}(E) \leqslant \left(\frac{c_2 p(K)}{\sqrt{n}}\right)^{\frac{k}{n-1}} \mathrm{as}(K),$$

where $c_1,c_2>0$ are absolute constants.

Proof. The left hand side inequality follows from Lemma 5.2 and (5.3). We have

$$|K|^{\frac{k(n-1)}{n}} \left(\frac{c_1\sqrt{n}}{p(K)}\right)^k \mathrm{as}(K) \leqslant \mathrm{as}(K)^{k+1} \leqslant c^k|K|^k \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K\cap E)\,d\nu_{n-k}(E),$$

which implies that

$$\left(\frac{c_1\sqrt{n}}{cp(K)}\right)^k \mathrm{as}(K) \leqslant |K|^{\frac{k}{n}} \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K\cap E)\, d\nu_{n-k}(E).$$

Next, we observe that

$$|K|^{\frac{k}{n}} = \left(|K|^{\frac{n-1}{n}}\right)^{\frac{k}{n-1}} \leqslant \left(\frac{p(K)\mathrm{as}(K)}{c\sqrt{n}}\right)^{\frac{k}{n-1}}, \tag{5.14}$$

which implies that

$$|K|^{\frac{k}{n}}\, \mathrm{as}(K)^{\frac{n-k-1}{n-1}} \leqslant \left(\frac{c_2 p(K)}{\sqrt{n}}\right)^{\frac{k}{n-1}} \mathrm{as}(K). \tag{5.15}$$

Then, the right hand side inequality of (5.13) follows from (5.4) in Theorem 5.1. $\square$

Remark 5.4. We will discuss the estimates that one can get from Theorem 5.3 if the centered convex body K in $\mathbb{R}^n$ is assumed to be in some of the classical positions; we introduce these below. For a detailed presentation and references see [1].

(i) We say that K is in minimal mean width position if $w(K) \leqslant w(T(K))$ for every $T \in SL(n)$. It was proved by V. Milman and the second named author that K has minimal mean width if and only if

$$w(K) = n\int_{S^{n-1}} h_K(\theta)\langle \xi, \theta\rangle^2 d\sigma(\theta) \tag{5.16}$$

for every $\xi \in S^{n-1}$. From results of Figiel-Tomczak, Lewis and Pisier (see [1, Chapter 6]) we know that if a convex body K in $\mathbb{R}^n$ has minimal mean width then $w(K) \leqslant c|K|^{\frac{1}{n}}\sqrt{n}\log n$. From the general fact that $R(K) \leqslant c\sqrt{n}w(K)$ for every centered convex body, we conclude that $R(K) \leqslant c|K|^{\frac{1}{n}} n\log n$ in the minimal mean width position.

(ii) We say that K is in John's position if the ellipsoid of maximal volume inscribed in K is a multiple of the Euclidean unit ball B_2^n and that K is in Löwner's position if the ellipsoid of minimal volume containing K is a multiple of the Euclidean unit ball B_2^n. One can check that this holds true if and only if K° is in John's position. The volume ratio of a centered convex body K in $\mathbb{R}^n$ is the quantity

$$\mathrm{vr}(K) = \inf\left\{\left(\frac{|K|}{|\mathcal{E}|}\right)^{\frac{1}{n}} : \mathcal{E} \text{ is an ellipsoid and } \mathcal{E} \subseteq K\right\}. \tag{5.17}$$

The outer volume ratio of a centered convex body K in $\mathbb{R}^n$ is the quantity $\mathrm{ovr}(K) = \mathrm{vr}(K^\circ)$. K. Ball proved in [2] that $\mathrm{vr}(K) \leqslant \mathrm{vr}(C_n) \simeq \sqrt{n}$ in the symmetric case and $\mathrm{vr}(K) \leqslant \mathrm{vr}(\Delta_n) \simeq \sqrt{n}$ in the not necessarily symmetric case, where $C_n = [-1,1]^n$ and Δ_n is a regular simplex in $\mathbb{R}^n$. Assume that K is in John's position. Then, from a theorem of Barthe [3] we know that if Δ_n is the regular simplex whose maximal volume ellipsoid is B_2^n and rB_2^n is the maximal volume ellipsoid of K we have $w(r^{-1}K) \leqslant w(\Delta_n) \leqslant c\sqrt{\log n}$. Since $|K|^{1/n} \geqslant r|B_2^n|^{1/n} \geqslant cr/\sqrt{n}$, we get

$$R(K) \leqslant c\sqrt{n}w(K) = cr\sqrt{n}w(r^{-1}K) \leqslant cr\sqrt{n\log n} \leqslant c'|K|^{\frac{1}{n}} n\sqrt{\log n}. \tag{5.18}$$

Next, assume that K is in Löwner's position; we know that $R(K)B_2^n$ is the minimal volume ellipsoid of K, and hence

$$R(K)|B_2^n|^{1/n} = |K|^{\frac{1}{n}}\mathrm{ovr}(K) = |K|^{\frac{1}{n}}\mathrm{vr}(K^\circ) \leqslant c\sqrt{n}|K|^{\frac{1}{n}}, \tag{5.19}$$

which implies that $R(K) \leqslant cn|K|^{\frac{1}{n}}$.

(iii) We say that K has minimal surface area if $S(K) \leqslant S(T(K))$ for every $T \in SL(n)$. Recall that the area measure σ_K of K is the Borel measure on S^{n-1} defined by

$$\sigma_K(A) = \lambda(\{x \in \mathrm{bd}(K) : \text{ the outer normal to } K \text{ at } x \text{ belongs to } A\}), \tag{5.20}$$

where λ is the usual surface measure on K. Petty proved in [24] that K has minimal surface area if and only if σ_K satisfies the isotropic condition

$$S(K) = n \int_{S^{n-1}} \langle \xi, \theta \rangle^2 d\sigma_K(\theta) \tag{5.21}$$

for every $\xi \in S^{n-1}$. It is known that if K has minimal surface area then $w(K) \leqslant cn|K|^{\frac{1}{n}}$ (this was observed by Markessinis, Paouris and Saroglou in [21]). Therefore, $R(K) \leqslant cn^{\frac{3}{2}}|K|^{\frac{1}{n}}$.

(iv) Finally, if K is in the isotropic position then we know that $R(K) \leqslant |K|^{\frac{1}{n}}(n+1)L_K$. This estimate is due to Kannan, Lovász and Simonovits (the asymptotically sharp bound $R(K) \leqslant cnL_K\,|K|^{\frac{1}{n}}$ can be obtained with an elementary argument).

Since $R(K)$ is polynomial in n for all the classical positions of a convex body K, from the right hand side inequality (5.13) of Theorem 5.3 we obtain the next result.

Theorem 5.5. *Let K be a centered convex body in $\mathbb{R}^n$. If K is in any of the classical positions that we discussed in Remark* 5.4*, then*

$$|K|^{\frac{k}{n}} \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E)\, d\nu_{n-k}(E) \leqslant C^k\, \mathrm{as}(K) \tag{5.22}$$

for all $1 \leqslant k \leqslant n-1$, where $C > 0$ is an absolute constant.

Remark 5.6. Similarly, from the left hand side inequality (5.13) of Theorem 5.3 we see that if K is in some of the classical positions that we discussed in Remark 5.4 then

$$c_n^{-k}\, \mathrm{as}(K) \leqslant |K|^{\frac{k}{n}} \int_{\mathrm{Gr}_{n-k}} \mathrm{as}(K \cap E)\, d\nu_{n-k}(E) \tag{5.23}$$

for every $1 \leqslant k \leqslant n-1$, where $c_n \simeq \sqrt{n}$ if K is in Löwner's position, $c_n \simeq \sqrt{n \log n}$ if K is in John's position, $c_n \simeq \sqrt{n}(\log n)$ if K is in the minimal mean width position, $c_n \simeq n$ if K is in the minimal surface area position, and $c_n \simeq \sqrt{n}L_K$ if K is in the isotropic position.

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