A note on a problem of H. Busemann and C.M. Petty concerning sections of symmetric convex bodies

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Abstract

Let $A_n(a,b) = \{(x_i) \in \mathbb{R}^n : \sum_{i=1}^{n-1} |x_i|^2 \leq a^2,\ |x_n| \leq b\}$. It is proved that for suitable a and b, $n \geq 7$, one can have $V_n(A_n) = V_n(B_n)$ and $V_{n-1}(H \cap A_n) < V_{n-1}(H \cap B_n)$ for every $(n-1)$-dimensional subspace H of $\mathbb{R}^n$, where B_n is the unit ball of $\mathbb{R}^n$. This strengthens previous negative results on a problem of H. Busemann and C.M. Petty.

1 Introduction

In [1] Busemann and Petty asked the question: "Let A, B be two convex symmetric bodies in $\mathbb{R}^n$ with their common centre of symmetry at the origin. If, for each $(n-1)$-dimensional subspace H of $\mathbb{R}^n$, $V_{n-1}(H \cap A) < V_{n-1}(H \cap B)$, is it then true that $V_n(A) < V_n(B)$?" (V_k is the k-dimensional volume function). If $n = 2$, it is clear that this is true. However, in [2], Larman and Rogers showed that the above assertion is false for $n \geq 12$. In their counterexample, B is the ball of $\mathbb{R}^n$, while the existence of A is established by probabilistic arguments. Ball [3] showed that if $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ is the unit cube of $\mathbb{R}^n$, then $V_{n-1}(H \cap Q_n) \leq \sqrt{2}$ for every $(n-1)$-dimensional subspace H of $\mathbb{R}^n$, and in [4] he observed that his result implies a negative answer to the problem of Busemann and Petty for $n \geq 10$ (with $A = Q_n$, B the ball of volume 1).

In this note we give an elementary counterexample for $n \geq 7$, by considering the sections of cylinders of the form

$$A_n(a,b) = \{(x_i) \in \mathbb{R}^n : \sum_{i=1}^{n-1} |x_i|^2 \leq a^2,\ |x_n| \leq b\},$$

for suitable $a,\ b > 0$.

2 Sections of $A_n(a,b)$

We shall use the notation $v_k = \pi^{k/2}/\Gamma(1+\frac{1}{2}k)$ for the volume of the unit ball of $\mathbb{R}^k$, $\{e_i\}_{i\le k}$ for the usual basis of $\mathbb{R}^k$ and

$$I_k = \int_{-1}^{1}(1-t^2)^{\frac{k-1}{2}}\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^k\theta\,d\theta = \frac{v_k}{v_{k-1}}.$$

We first prove the following.

Proposition 2.1 *Let*

$$A_n(a,b) = \{(x_i)\in\mathbb{R}^n : \sum_{i=1}^{n-1}|x_i|^2 \le a^2,\ |x_n|\le b\},\qquad n\ge 3,$$

$$m = \left(\frac{a}{b}\right)^2 \quad \textit{and}\ \ f(x) = \sqrt{1+mx^2}\,\frac{1}{x}\int_0^x (1-t^2)^{\frac{n-2}{2}}dt,\ 0<x\le 1,\ \ f(0)=1.$$

Then,

$$\sup_H V_{n-1}(H\cap A_n) = 2v_{n-2}a^{n-2}b\sup f(x),$$

where H runs over all $(n-1)$-dimensional subspaces of $\mathbb{R}^n$.

Proof: Let H be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ and $u=(u_1,\dots,u_n)$ be the unit vector normal to H. We can clearly assume that $u_n>0$. Set $\varphi_0 = \operatorname{arccot}(a/b)$ and $\varphi_u = \arccos u_n$. As we shall see, $V_{n-1}(H\cap A_n)$ depends only on $|u_n|$.

We examine two cases separately.

(i) $0\le\varphi_u\le\varphi_0$, that is $\cot^2\varphi_u \ge a^2/b^2$ or $|u_n|\ge a/(a^2+b^2)^{1/2}$. Let $P_n:\mathbb{R}^n\to\mathbb{R}^n$ be the orthogonal projection onto the subspace $\{x:\langle x,e_n\rangle = 0\}$. It is easy to see that

$$P_n(H\cap A_n) = B_{n-1}(a) = \{(x_i)\in\mathbb{R}^n : \sum_{i=1}^{n-1}|x_i|^2\le a^2,\ x_n=0\}.$$

So,

$$(1)\qquad V_{n-1}(H\cap A_n) = \frac{1}{|\langle u,e_n\rangle|}V_{n-1}[P_n(H\cap A_n)] = \frac{1}{|u_n|}v_{n-1}a^{n-1}$$

$$\le v_{n-1}a^{n-2}(a^2+b^2)^{1/2} = 2v_{n-2}a^{n-2}bf(1).$$

(ii) $\varphi_0\le\varphi_u\le\frac{\pi}{2}$, that is $a^2\ge\cot^2\varphi_u b^2$. Let

$$v = \frac{1}{\sqrt{1-u_n^2}}(u_1,\dots,u_{n-1},0),\qquad |t|\le b,$$

and $P_v:\mathbb{R}^n\to\mathbb{R}^n$ be the orthogonal projection onto the subspace $\{x:\langle x,v\rangle=0\}$.

We set $[P_v(H\cap A_n)]_t = P_v(H\cap A_n)\cap\{x : x_n = t\}$. If $x\in H\cap A_n, x_n = t, \sum_{i\le n-1}|x_i|^2 = d^2 \le a^2$, then $P_v(x) = x - \langle x,v\rangle v = x + (t\cot\varphi_u)v$ and $\|P_v(x) - te_n\|^2 = d^2 - t^2\cot^2\varphi_u$. It is also clear that $P_v(x) - te_n \in [v,e_n]^\perp$.

Conversely, if $\langle y,u\rangle = 0, y_n = t, \|y - te_n\|^2 = d^2 - t^2\cot^2\varphi_u$, then $x = y - (t\cot\varphi_u)v \in H\cap A_n, x_n = t, \sum_{i\le n-1}|x_i|^2 = d^2$ and $P_v(x) = y$. So, $[P_v(H\cap A_n)]_t$ is an $(n-2)$-dimensional ball of radius $(a^2 - t^2\cot^2\varphi_u)^{1/2}$, centered on te_n and lying on $te_n + [v,e_n]^\perp$.

It follows that

$$V_{n-1}(P_v(H\cap A_n)) = v_{n-2}\int_{-b}^{b}(a^2 - \cot^2\varphi_u t^2)^{\frac{n-2}{2}}dt,$$

and

$$V_{n-1}(H\cap A_n) = \frac{1}{|\langle u,v\rangle|}v_{n-2}\int_{-b}^{b}(a^2 - \cot^2\varphi_u t^2)^{\frac{n-2}{2}}dt \tag{2}$$

$$= \frac{1}{\sin\varphi_u}v_{n-2}\int_{-b}^{b}(a^2 - \cot^2\varphi_u t^2)^{\frac{n-2}{2}}dt$$

$$= \sqrt{1+\cot^2\varphi_u}\,v_{n-2}\int_{-b}^{b}(a^2 - \cot^2\varphi_u t^2)^{\frac{n-2}{2}}dt.$$

Now, setting $x = (b/a)\cot\varphi_u$, (2) becomes

$$V_{n-1}(H\cap A_n) = \sqrt{1+mx^2}\; v_{n-2}a^{n-2}b\int_{-1}^{1}(1 - x^2t^2)^{\frac{n-2}{2}}dt \tag{3}$$

$$= \sqrt{1+mx^2}\; v_{n-2}a^{n-2}b\;\frac{1}{x}\int_{-1}^{1}(1-t^2)^{\frac{n-2}{2}}dt = 2v_{n-2}a^{n-2}b\; f(x).$$

On observing that $\varphi_0 \le \varphi_u \le \pi/2$ is equivalent to $0\le x\le 1$, the result follows from (1) and (3). □

3 Case of $n \ge 8$

Consider two sequences, $p_n = (I_n/I_{n-1})^{n-1}$ and $s_n = \sqrt{n}I_n$. As Ball shows in [4], p_n is decreasing. We set $t_n = s_{n+1}/s_n$. Then $t_n \to 1$ and since $I_{n+2} = ((n+1)/(n+2))\,I_n$ we have

$$\frac{t_{n+2}}{t_n} = \left[\frac{n(n+2)}{(n+1)(n+3)}\right]^{1/2}\frac{n+2}{n+1} < 1.$$

So $t_n > 1$, i.e, s_n is increasing. By Stirling's formula, $s_n \to \sqrt{2\pi}$.

Lemma 3.1 *Let f be as in Proposition 2.1. Then f is decreasing on $[0,1]$ if and only if $m \le \frac{n-2}{3}$ $(n\ge 4)$.*

Proof: Differentiating we see that f is decreasing if and only if

$$x(1+mx^2)(1-x^2)^{\frac{n-2}{2}} \leq \int_0^x (1-t^2)^{\frac{n-2}{2}} dt, \qquad 0 \leq x \leq 1. \tag{4}$$

We set

$$g(x) = x(1+mx^2)(1-x^2)^{\frac{n-2}{2}} - \int_0^x (1-t^2)^{\frac{n-2}{2}} dt,$$

and observe that $g'(x) \leq 0$ if and only if $3m(1-x^2) \leq (n-2)(1+mx^2)$. Since $g(0) = 0$, (4) is true if and only if $m \leq \frac{n-2}{3}$. □

Lemma 3.2 *Suppose that $n \geq 8$. We can choose a and b so that*
(i) $V_n(A_n(a,b)) = v_n$,
(ii) $2v_{n-2}a^{n-2}b\, f(0) < v_{n-1}$, *and*
(iii) $m \leq \frac{n-2}{3}$.

Proof: Since

$$V_n(A_n(a,b)) = 2v_{n-1}a^{n-1}b, \tag{5}$$

(i) is true if $a^{n-1}b = I_n/2$. In view of (5), (ii) is satisfied if $v_{n-2}I_n/a < v_{n-1}$, i.e, if

$$a > I_n/I_{n-1}. \tag{6}$$

Set $a = I_n/I_{n-1}$. From (5),

$$m = \left(\frac{a}{b}\right)^2 = \left(\frac{2a^n}{I_n}\right)^2 = \left[2\left(\frac{I_n}{I_{n-1}}\right)^{n-1} \frac{1}{I_{n-1}}\right]^2 = \left(\frac{2p_n}{I_{n-1}}\right)^2. \tag{7}$$

So, if (iii) is to be true, we must have

$$2p_n < \sqrt{\frac{n-2}{3}} I_{n-1}, \qquad \text{or} \qquad 2p_n < \frac{1}{\sqrt{3}}\sqrt{\frac{n-2}{n-1}} s_{n-1}. \tag{8}$$

Now, the sequence on the left is decreasing, while the one on the right is increasing. So, we only need to check the inequality for $n = 8$.

But,

$$2p_8 = 2\left(\frac{I_8}{I_7}\right)^7 = 2\left(\frac{1225\pi}{4096}\right)^7 \simeq 1.29274\ldots,$$

while

$$\sqrt{2} I_7 = \sqrt{2}\,\frac{32}{35} \simeq 1.29299\ldots .$$

It is now clear that choosing a slightly larger than I_n/I_{n-1} and b from (5), conditions (i)-(iii) are satisfied. □

Proposition 2.1 and the above two lemmas imply our counterexample for $n \geq 8$:

Theorem 3.1 *Let*

$$A_n(a,b) = \{(x_i) \in \mathbb{R}^n : \sum_{i \le n-1} |x_i|^2 \le a^2,\ |x_n| \le b\}, \quad n \ge 7.$$

We can choose $a, b > 0$ so that
(i) $V_n(A_n(a,b)) = v_n$,
(ii) *for each $(n-1)$-dimensional subspace H of $\mathbb{R}^n$, $V_{n-1}(H \cap A_n) < v_{n-1}$.* □

Note that in our example ($n \ge 8$), $V_{n-1}(H \cap A_n)$ is a decreasing function of $|u_n|$, where $u = (u_1, \ldots, u_n)$ is the unit vector normal to H.

4 Case of $n = 7$

If $3 \le n \le 7$, conditions (i)-(iii) of Lemma 3.2 are incompatible. However, in case $n = 7$, we can also find a and b for which the theorem is true.

Let $C = \sup_H V_6(H \cap A_7(a,b))$. From $V_7(A_7(a,b)) = v_7$, we get

$$a^6 b = \frac{1}{2} I_7, \tag{9}$$

and hence $2v_5 a^5 b = v_6 \frac{I_7}{I_6} \frac{1}{a}$, and $m = \left(\frac{2a^7}{I_7}\right)^2$, i.e,

$$a = \left(\frac{I_7}{2}\right)^{1/7} m^{1/14}. \tag{10}$$

So, by Proposition 2.1,

$$\frac{C}{v_6} = \frac{I_7}{I_6} \left(\frac{2}{I_7}\right)^{1/7} \frac{1}{m^{1/14}} \sup_{0 \le x \le 1} \sqrt{1 + mx} \int_0^1 (1 - xt^2)^{5/2} dt.$$

Now,

$$(1 - xt^2)^{5/2} = \sum_{n=0}^{\infty} \binom{5/2}{n} (-1)^n x^n t^{2n} \le 1 - \frac{5}{2} xt^2 + \frac{15}{8} x^2 t^4,$$

and hence

$$\int_0^1 (1 - xt^2)^{5/2} dt \le 1 - \frac{5}{6} x + \frac{3}{8} x^2.$$

Let $r_m(x) = \sqrt{1 + mx}\,\left[1 - \frac{5}{6}x + \frac{3}{8}x^2\right]$. One must choose m such that

$$\frac{I_7}{I_6} \left(\frac{2}{I_7}\right)^{1/7} \frac{1}{m^{1/14}} \sup_{0 \le x \le 1} r_m(x) < 1.$$

This can be done with m a little larger than $\frac{5}{3} = \frac{n-2}{3}$. For example, if we look for m such that $\sup_{0 \le x \le 1} r_m(x) = r_m(1/20)$, then, as we can easily see by differentiating

r_m, $m = m_0$ with $m_0 = 3056/1689$ (for this value of m, r_m has two local maxima: $r_m(1/20) > 1$ and $r_m(1) < 1$).

On observing that the number

$$\frac{I_7}{I_6}\left(\frac{2}{I_7}\right)^{1/7}\frac{1}{m_0^{1/14}}\,r_{m_0}\left(\frac{1}{20}\right) = \left(\frac{512}{175\pi}\right)\left(\frac{35}{16}\right)^{1/7}\left(\frac{1689}{3056}\right)^{1/14}\left(\frac{36836}{33780}\right)^{1/2}\left(\frac{9209}{9600}\right)$$

$\simeq 0.999998\ldots$ is smaller than 1, we conclude the proof for $n = 7$. □

ACKNOWLEDGEMENT: I thank Professor S. Papadopoulou for her encouragement and advice.

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