How small can the intersection of a few rotations of a symmetric convex body be?

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Abstract

Let $t \geq 2$ be an integer. We show that the minimal circumradius of an intersection of t rotations of a symmetric convex body K and the "upper radius" of a random $\lceil n/f(t) \rceil$–dimensional section of K are equivalent up to a constant depending only on t, where f is a function explicitely defined. We also give lower and upper bounds for the diameter of a random λn–dimensional section of K in terms of average parameters of the body.

Sur le plus petit diamètre de l'intersection des images d'un corps convexe symétrique par un certain nombre de rotations

Résumé – Soit $t \geq 2$ un entier. On montre que le plus petit rayon de l'intersection des images d'un corps convexe symétrique K par t rotations et le "rayon supérieur" d'une section $\lceil n/f(t) \rceil$–dimensionnelle de K sont équivalents à une constante multiplicative près (qui ne dépend que de t), où f est une fonction explicite. On donne aussi des bornes inférieures et supérieures du diamètre d'une section λn–dimensionnelle de K en fonction de paramètres de K.

Version française abrégée – Soit K un corps convexe symétrique dans $\mathbb{R}^n$. Supposons que $\frac{1}{b}D_n \subseteq K \subseteq aD_n$, où $a, b > 0$ et D_n est la boule unité euclidienne. On définit

$$M(K) = \int_{S^{n-1}} \|x\|_K \sigma(dx)$$

où $\|.\|_K$ est la norme induite sur $\mathbb{R}^n$ par K, S^{n-1} est la sphère euclidienne, et σ est la probabilité invariante par rotation sur S^{n-1}.

Pour tout entier $t \geq 2$ nous désignons par $r_t(K)$ le plus petit rayon de l'intersection des images de K par t rotations:

$$r_t(K) = \min\{\rho > 0 : \text{ il existe } u_1, \ldots, u_t \in SO(n) \text{ tels que } u_1(K) \cap \ldots \cap u_t(K) \subseteq \rho D_n\}.$$

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L'étude de la fonction $r_t(K)$ et de ses relations avec d'autres paramètres du corps K est motivée par [5], où $r_t(K)$ est calculé pour de grandes valeurs de t: on démontre que si $t \geq c(b/M(K))^2$, on a $r_t(K) \leq 2/M(K)$. D'autre part, le cas d'un entier t petit (à partir de $t = 2$) est totalement différent. Les paramètres du corps K en rapport avec $r_t(K)$ semblent d'une nature plus locale, et les techniques utilisées ne sont pas les mêmes.

Pour tout entier $k \geq 2$ on définit le "rayon supérieur" d'une section centrale aléatoire de K de dimension $\lceil n/k \rceil$ par

$$R_k(K) = \min\{R > 0 : \mathrm{Prob}(\xi \in G_{n,\lceil n/k \rceil} : K \cap \xi \subseteq RD_\xi) \geq 1 - \frac{1}{k+1}\}.$$

Il est bien connu (voir [8]) que

$$r_{2t}(K) \leq \sqrt{t} R_t(K).$$

Un des principaux résultats de cette note est la démonstration d'une inégalité inverse:

THÉORÈME 1. *Soit $t \geq 2$ un entier. Pour tout corps convexe symétrique K dans $\mathbb{R}^n$ (où $n \geq n_0(t)$ est suffisamment grand), une section aléatoire $K \cap \xi$ de K de dimension $\lfloor c^t n \rfloor$ satisfait*

$$\mathrm{diam}(K \cap \xi) \leq 20 C^t r_t(K),$$

où $0 < c < 1$ et $C > 1$ sont des constantes universelles.

Autrement dit, il existe une constante universelle $C_1 > 1$ telle que pour tout corps convexe K de dimension suffisamment grande on a

$$R_{f(t)}(K) \leq g(t) r_t(K)$$

où $g(t) = C_1^t$ et $f(t) = \lfloor g(t) \rfloor$. Il serait très intéressant d'obtenir la meilleure estimation possible de ce type.

La démonstration du Théorème repose sur une "M–borne inférieure conditionnelle" nouvelle (Proposition 2) établie dans [3] où l'on étudie la question d'une borne inférieure du diamètre d'une section aléatoire d'un corps K (de dimension proportionnelle λn, $0 < \lambda < 1$). On utilise encore le Théorème antipodal de Borsuk de façon essentielle.

Finalement, on annonce quelques résultats récents concernant la question du diamètre des sections aléatoire: on donne des bornes supérieures et inférieures du diamètre moyen ou aléatoire des sections $\lfloor \lambda n \rfloor$–dimensionnelles d'un corps K en utilisant la largeur moyenne $S_K^*(\lambda)$ des λn–sections de K. Cela nous permet de déduire l'inégalité

$$t^{-\frac{1}{2}} r_{2t}(K) \leq R_t(K) \leq c(\alpha) t^{\alpha+1} r_t(K)$$

dans le cas où le corps K est en M–position d'ordre $\alpha > \frac{1}{2}$.

English version – Let K be a symmetric convex body in $\mathbb{R}^n$ with $\frac{1}{b}D_n \subseteq K \subseteq aD_n$, where D_n is the Euclidean unit ball. Consider the average parameter

$$M(K) = \int_{S^{n-1}} \|\theta\|_K \sigma(d\theta)$$

where S^{n-1} is the unit sphere, σ is the rotationally invariant probability measure on S^{n-1}, and $\|.\|_K$ is the norm induced to $\mathbb{R}^n$ by K. Then, the mean width of K is the quantity $M^*(K) = M(K^o)$ where K^o is the polar body of K. Let $G_{n,m}$ be the Grassmannian of m–dimensional subspaces ξ of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,m}$.

Let $t, k \geq 2$ be two integers. We define the minimal circumradius of an intersection of t rotations of K by

$$r_t(K) = \min\{\rho > 0 : \ u_1(K) \cap \ldots \cap u_t(K) \subseteq \rho D_n \text{ for some } u_1, \ldots, u_t \in SO(n)\},$$

and the "upper radius" of a random $\lceil n/k \rceil$-dimensional central section of K (where $\lceil x \rceil$ denotes the least integer which is greater than or equal to x) by

$$R_k(K) = \min\{R > 0 : \nu_{n,\lceil n/k \rceil}(\xi \in G_{n,\lceil n/k \rceil} : K \cap \xi \subseteq RD_\xi) \geq 1 - \frac{1}{k+1}\}.$$

It is proved in [8] that if $R > R_k(K)$ then starting from a set of pairwise orthogonal $\lceil n/k \rceil$–dimensional subspaces $\xi_1, \ldots, \xi_k$ which satisfy $K \cap \xi_i \subseteq RD_{\xi_i}$ and $\oplus \sum \xi_i = \mathbb{R}^n$, one can build rotations $u_1, \ldots, u_t \in SO(n)$, $t \leq 2k$, such that

$$\frac{1}{t} \sum_{j=1}^{t} u_j(K^o) \supseteq \frac{1}{R\sqrt{k}} D_n. \tag{1}$$

Dualizing this fact, one has that $u_1^*(K) \cap \ldots \cap u_t^*(K) \subseteq R\sqrt{k} D_n$, i.e

$$r_{2k}(K) \leq \sqrt{k} R_k(K). \tag{2}$$

In this note we show that an inverse inequality holds true:

Qualitative Statement: *Let $A = \{s \in \mathbf{Z} : s \geq 2\}$. There exist two functions $f : A \to A$ and $g : A \to \mathbb{R}^+$ such that*

$$R_{f(t)}(K) \leq g(t) r_t(K)$$

for every $t \in A$ and every symmetric convex body K in $\mathbb{R}^n$ with n large enough (depending on t).

The case where t or k is of the order of $(b/M(K))^2$ has been studied in detail (see for example [5] where the idea of studying $r_t(K)$ comes from). It is a well–known fact related to Dvoretzky's theorem (see [6]) that when $k \geq c(b/M(K))^2$, then most $\lceil n/k \rceil$-dimensional sections of K satisfy

$$\frac{1}{2M(K)} D_\xi \subseteq K \cap \xi \subseteq \frac{2}{M(K)} D_\xi, \tag{3}$$

therefore, $R_k(K) \leq 2/M(K)$. On the other hand, it is proved in [1] that if $t \geq c(b/M(K))^2$, then there exist $u_1, \ldots, u_t \in SO(n)$ for which

$$\frac{M(K)}{2}|x| \leq \frac{1}{t}\sum_{i=1}^{t} \|u_i(x)\|_K \leq 2M(K)|x| \ , \quad x \in \mathbb{R}^n, \tag{4}$$

which implies that $r_t(K) \leq 2/M(K)$. Rather surprisingly, it is observed in [10] that if $k(K)$ and $t(K)$ are the smallest integers for which (3) and (4) respectively hold, then $k(K) \simeq t(K)$ up to an absolute constant. The relation between $k(K)$ and $t(K)$ is further clarified in [5], where it is also extended to quasi–convex bodies. However, the quantity $r_t(K)$ has not been studied for fixed small integer values of t. The case $t = 2$ is already interesting: It appears that $r_t(K)$ is related to different parameters of K of a more local nature, and at the same time different techniques need to be invented.

In this paper we are interested in a fixed integer value of t (starting with $t = 2$), and prove the following version of our qualitative statement:

Theorem. *Let $t \geq 2$ be an integer. For every symmetric convex body K in $\mathbb{R}^n$ (where n is large enough depending on t), a random $c^t n$–dimensional section $K \cap \xi$ of K satisfies*

$$\mathrm{diam}(K \cap \xi) \leq 20C^t r_t(K),$$

where $0 < c < 1$ and $C > 1$ are absolute constants.

Observe that the Theorem implies that our general statement holds true with $g(t) = C_1^t$ and $f(t) = \lfloor g(t) \rfloor$, where $C_1 > 1$ is an absolute constant. The dependence of f and g on t is most probably not the best possible, and it would be very interesting to obtain sharper estimates in this direction.

The proof of the Theorem is based on a general double sided estimate of the diameter of the random proportional sections of a symmetric convex body K which was established in [3]: We proved that the computable function

$$M_K^*(r) = \frac{1}{r} M^*(K \cap rD_n) \quad , \quad r > 0$$

can be used to provide a "confidence interval" for the diameter of a random λn–dimensional section ($\frac{1}{2} < \lambda < 1$) of K in the following way:

" There exist two explicit functions $h_1, h_2 : (0,1) \to (0,1)$ such that for every $\lambda \in (1/2, 1)$ and every symmetric convex body K, the solutions of the equations $M_K^*(r) = h_1(\lambda)$ and $M_K^*(r) = h_2(\lambda)$ give an upper estimate r_1 and a lower estimate r_2 respectively for the diameter of a random λn–section of K."

The function h_1 is defined by $h_1(\lambda) \simeq \sqrt{1-\lambda}$, and the upper estimate follows from the precise probabilistic form of the Low M^*–estimate (see [2], [8]):

Proposition 1. *Let $\varepsilon, \lambda \in (0,1)$. If K is a symmetric convex body in $\mathbb{R}^n$, and if $r > 0$ is the solution of the equation $M^*(K \cap rD_n) = (1-\varepsilon)\sqrt{1-\lambda}\, r$, then*

$$\mathrm{diam}(K \cap \xi) \leq 2r$$

for all ξ *in a subset* L_λ *of* $G_{n,m}$ *with measure* $\nu_{n,m}(L_\lambda) \geq 1 - \frac{7}{2}\exp(-\frac{1}{72}a_{n-m}^2\varepsilon^2)$, *where* $m = \lfloor \lambda n \rfloor$ *and* $a_s = \sqrt{2}\Gamma(\frac{s+1}{2})/\Gamma(\frac{s}{2}) \simeq \sqrt{s}$. □

The existence of the second function h_2 is established through a "conditional Low M–estimate" which is a main new ingredient of [3]:

Proposition 2. *There exist two absolute constants* $0 < c < 1$ *and* $C > 1$ *which satisfy the following: If* $\lambda \in (0,1)$ *and* K *is a symmetric convex body in* $\mathbb{R}^n$ *with* n *large enough, and if* $r > 0$ *satisfies*

$$M^*(K \cap rD_n) = (1 - c^{\frac{1}{1-\lambda}})r,$$

then there exists a subset L_λ *of* $G_{n,m}$ *with* $\nu_{n,m}(L_\lambda) \geq 1 - c^m$, *where* $m = \lfloor \lambda n \rfloor$, *such that*

$$K^o \cap \xi \subseteq \frac{10}{r}\, C^{\frac{\lambda}{1-\lambda}} D_\xi$$

for all $\xi \in L_\lambda$. □

Having these two facts available, we can pass to the

Proof of the Theorem: Assume that for some body K in $\mathbb{R}^n$ and for some $\rho > 0$ there exist rotations $u_1, \ldots, u_t \in SO(n)$ for which

$$u_1(K) \cap \ldots \cap u_t(K) \subseteq \rho D_n.$$

We apply Proposition 2 with $\lambda > \frac{t}{t+1}$ to each $u_j(K)$ and set $m = \lfloor \lambda n \rfloor$. Since M_K^* is onto $(0,1]$, there exists $r > 0$ (clearly the same for all j) satisfying $M^*(u_j(K) \cap rD_n) = (1 - c^{\frac{1}{1-\lambda}})r$. We can then find subsets $L_{\lambda,j}$ of $G_{n,m}$ with $\nu_{n,m}(L_{\lambda,j}) \geq 1 - c^m$, such that

$$[u_j(K)]^o \cap \xi \subseteq \frac{10}{r}\, C^{\frac{\lambda}{1-\lambda}} D_\xi \tag{5}$$

for all $\xi \in L_{\lambda,j}$. Assuming that n is large enough, we can find $0 < c_1 < 1$ and some $L_\lambda \subseteq G_{n,m}$ with $\nu_{n,m}(L_\lambda) \geq 1 - c_1^m$ so that (5) holds for all $j \leq t$ and $\xi \in L_\lambda$. Taking polars in $\xi \in L_\lambda$ we get

$$P_\xi(u_j(K)) \supseteq \frac{r}{10}\, C^{-\frac{\lambda}{1-\lambda}} D_\xi \quad , \quad j = 1, \ldots, t. \tag{6}$$

Without loss of generality we may assume that K is strictly convex. We then define a map $T : S(\xi) \to \mathbb{R}^{t(n-m)}$ as follows: Given $\theta \in S(\xi)$ we find $x_j = a_j\theta \in \mathrm{bd}(P_\xi(u_j(K)))$, $j = 1, \ldots, t$. Then, we have $x_j = P_\xi(y_j)$ for a unique $y_j \in \mathrm{bd}(u_j(K))$. We define

$$T(\theta) = (y_1 - x_1, \ldots, y_t - x_t),$$

where we identify $(\xi^\perp)^t$ with $\mathbb{R}^{t(n-k)}$. It is easy to check that T is an odd continuous function on $S(\xi)$. From the choice of λ, if n is large we have $t(n-m) < m$. We can then apply Borsuk's antipodal theorem to find $\theta \in S(\xi)$ with $T(\theta) = 0$.

Consider an index $j_0 \le t$ for which $a_{j_0} = |x_{j_0}|$ is minimal. Since $x_{j_0} = y_{j_0}$, we have $x_{j_0} \in u_{j_0}(K) \cap \xi$, and since $a_{j_0} = \min_{j \le t} a_j$ we see that $x_{j_0} \in u_1(K) \cap \ldots \cap u_t(K) \cap \xi$. On the other hand, x_{j_0} is also on the boundary of $P_\xi(u_{j_0}(K))$, which gives

$$\frac{r}{10}\, C^{-\frac{\lambda}{1-\lambda}} \le |x_{j_0}| \le \frac{1}{2}\mathrm{diam}(u_1(K) \cap \ldots \cap u_t(K) \cap \xi) \le \rho.$$

This provides an upper bound for r in terms of ρ and t:

$$r \le 10 C^t \rho. \tag{7}$$

We choose $\varepsilon = 1 - (1 - c^{t+1})^{1/2}$ in Proposition 1, and find $\mu \in (0,1)$ such that $M^*(K \cap r D_n) = (1-\varepsilon)\sqrt{1-\mu}\, r$. The choice of ε gives $\mu = c^{t+1}$, and Proposition 1 implies that there is a subset L'_μ of $G_{n,\lfloor \mu n \rfloor}$ with $\nu_{n,\lfloor \mu n \rfloor}(L'_\mu) \ge 1 - c_2 \exp[-c_3(\mu, t) n]$, such that

$$\mathrm{diam}(K \cap \xi) \le 2r \le 20 C^t \rho$$

for all $\xi \in L'_\mu$. This completes the proof. □

Let us describe a second method for obtaining upper and lower bounds on the diameter of random λn–dimensional sections of a body K. We introduce a function $S^*_K : (0,1) \to \mathbb{R}^+$ defined by

$$S^*_K(\lambda) = \int_{G_{n,m}} M^*(K \cap \xi)\, \nu_{n,m}(d\xi),$$

where $m = \lfloor \lambda n \rfloor$. The value $S^*_K(\lambda)$ gives the average mean width of the λn-sections of K. Being a refinement of the single number $M^*(K)$, this function is more likely to give tighter bounds. On the other hand, it remains an average parameter of the body (although it is clearly more complicated). Imposing mild restrictions which prevent the body K from being degenerated, we can give double sided inequalities for the average or even the random diameter of the sections of K in terms of S^*_K:

1. If $\frac{1}{b} D_n \subseteq K \subseteq a D_n$ and ab is polynomial in n, then the average diameter

$$D_K(\lambda) = \int_{G_{n,m}} \mathrm{diam}(K \cap \xi) \nu_{n,m}(d\xi)$$

of the λn–sections of K satisfies the bounds

$$2 S^*_K(\lambda) \le D_K(\lambda) \le 5 S^*_K(\lambda/\theta)/(1-\theta)^{1/2}$$

for every $\lambda < \theta < 1$, provided that n is large enough.

2. If $ab = o(\sqrt{n})$, then the diameter of a *random* λn–section of K satisfies

$$c_1 S^*_K(\lambda\theta) \le \mathrm{diam}(K \cap \xi) \le c_2 S^*_K(\lambda/\theta)/(1-\theta)^{1/2}$$

for every $\lambda < \theta < 1$, provided that n is large enough (depending on θ). In particular, with the cost of a small (e.g logarithmic in n) factor this information is available for every body K whose the ellipsoid of maximal (or minimal) volume is a ball.

3. If K is in M–position of order $\alpha > \frac{1}{2}$ (that is, if K is "α–regular" in the terminology of [11]– see p.121), then we determine the function S_K^* up to constants depending only on λ and α: For every $\lambda \in (0,1)$,

$$c_1\lambda^\alpha \left(\frac{|K|}{|D_n|}\right)^{1/n} \leq S_K^*(\lambda) \leq c_2(1-\lambda)^{-\alpha}\left(\frac{|K|}{|D_n|}\right)^{1/n}.$$

Actually, the same estimates pass with high probability to the diameter of λn–sections of K:

$$c_1\lambda^\alpha(|K|/|D_n|)^{\frac{1}{n}} \leq \mathrm{diam}(K\cap\xi) \leq c_2(1-\lambda)^{-\alpha-\frac{1}{2}}(|K|/|D_n|)^{\frac{1}{n}}. \tag{8}$$

It should be emphasized that this approach gives bounds for *every* $\lambda \in (0,1)$ (while the use of Borsuk's theorem in [3] restricts us to $(1/2,1)$). Also, if K is assumed to satisfy a polynomial growth of ab, then S_K^* has many intervals of regular behavior, in which we obtain tight bounds for the average or random "diameter function". The detailed proofs of the statements above will be given elsewhere (see [4]).

In the case of a body K in M–position, the relation between the minimal radius $r_t(K)$ and the random diameter of the proportional sections of K can be demonstrated in a more or less optimal way:

Proposition 3. *Let K be a symmetric convex body in M-position of order α. For every integer $t \geq 2$ and for every $\lambda \in (0,1)$, the diameter of a random λn–dimensional section $K\cap\xi$ of K satisfies*

$$\mathrm{diam}(K\cap\xi) \leq c(\alpha)t^\alpha(1-\lambda)^{-\alpha-\frac{1}{2}}r_t(K).$$

Proof: Assume that for some $\rho > 0$ we can find $u_1,\ldots,u_t \in SO(n)$ such that $u_1(K)\cap\ldots\cap u_t(K) \subseteq \rho D_n$. Taking polars we obtain

$$v_1(K^o)+\ldots+v_t(K^o) \supseteq \frac{1}{\rho}D_n \tag{9}$$

where $v_j = u_j^*$. The inverse Brunn–Minkowski inequality [7], [11] in the "α–order" formulation gives

$$|\frac{1}{t}\sum_{j=1}^{t} v_j(K^o)|^{\frac{1}{n}} \leq c_1(\alpha)t^\alpha|K^o|^{\frac{1}{n}}.$$

Then, (8), (9) and an application of the Blaschke–Santaló inequality show that for *every* $\lambda \in (0,1)$ and for most $\xi \in G_{n,\lambda n}$,

$$\mathrm{diam}(K\cap\xi) \leq c_2(\alpha)(1-\lambda)^{-\alpha-\frac{1}{2}}\left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}} \leq c(\alpha)t^{\alpha+1}(1-\lambda)^{-\alpha-\frac{1}{2}}\rho. \quad \Box$$

An immediate consequence of the Proposition is that

$$t^{-\frac{1}{2}}r_{2t}(K) \leq R_t(K) \leq c_1(\alpha)t^{\alpha+1}r_t(K) \tag{10}$$

for every $t \geq 2$ and every body K in M–position of order $\alpha > \frac{1}{2}$. Note that when t is a power of 2, $r_{2t}(K)$ can be replaced by $r_t(K)$ in (10) – see [8].

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