On the existence of subgaussian directions for log-concave measures

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Abstract. We show that if μ is a centered log-concave probability measure on $\mathbb{R}^n$ then,

$$\frac{c_1}{\sqrt{n}} \leq |\Psi_2(\mu)|^{1/n} \leq \frac{c_2\sqrt{\log n}}{\sqrt{n}},$$

where $\Psi_2(\mu)$ is the ψ_2-body of μ, and $c_1, c_2 > 0$ are absolute constants. It follows that μ has "almost subgaussian" directions: there exists $\theta \in S^{n-1}$ such that

$$\mu\left(\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geq ct\mathbb{E}|\langle \cdot, \theta\rangle|\}\right) \leq e^{-\frac{t^2}{\log(t+1)}}$$

for all $1 \leq t \leq \sqrt{n\log n}$, where $c > 0$ is an absolute constant.

1. Introduction

Let μ be a log-concave probability measure on $\mathbb{R}^n$, with centre of mass at the origin. We say that a direction $\theta \in S^{n-1}$ is subgaussian for μ with constant $r > 0$ if

$$\|\langle \cdot, \theta\rangle\|_{\psi_2} \leq rm_\theta, \tag{1.1}$$

where m_θ is the median of $|\langle \cdot, \theta\rangle|$ with respect to μ, and

$$\|f\|_{\psi_2} = \inf\left\{t > 0 : \int_{\mathbb{R}^n} \exp\left((|f(x)|/t)^2\right)\, d\mu(x) \leq 2\right\}. \tag{1.2}$$

It is known that

$$\|f\|_{\psi_2} \simeq \sup_{q\geq 2} \frac{\|f\|_q}{\sqrt{q}}. \tag{1.3}$$

This article provides a new general estimate on the following question: is it true that every log-concave measure μ has at least one "subgaussian" direction (with constant $r = O(1)$)?

This question was posed by V. Milman in the setting of convex bodies and an affirmative answer was first given for some special classes. Bobkov and Nazarov (see [**3**] and [**4**]) proved that if K is an isotropic 1–unconditional convex body, then

1991 *Mathematics Subject Classification.* Primary 52A20; Secondary 46B07.

Key words and phrases. Log-concave probability measures, random marginals, isotropic constant.

The second named author is partially supported by an NSF grant.

$\|\langle\cdot,\theta\rangle\|_{\psi_2}\leq c\sqrt{n}\|\theta\|_\infty$ for every $\theta\in S^{n-1}$, which implies that the diagonal direction is a subgaussian direction with constant $O(1)$. In [**16**] it is proved that every zonoid has a subgaussian direction with a uniformly bounded constant. Another partial result was obtained in [**17**]: if K is isotropic and $K\subseteq(\gamma\sqrt{n}L_K)B_2^n$ for some $\gamma>0$, then

$$\sigma\big(\theta\in S^{n-1}:\|\langle\cdot,\theta\rangle\|_{\psi_2}\geq c_1\gamma tL_K\big)\leq\exp(-c_2\sqrt{n}t^2/\gamma) \tag{1.4}$$

for every $t\geq 1$, where σ is the rotationally invariant probability measure on S^{n-1} and $c_1,c_2>0$ are absolute constants.

In the general case, B. Klartag ([**9**]) established the existence of a "subgaussian" direction up to a logarithmic in the dimension factor (see also [**8**]). More precisely, there exists $\theta\in S^{n-1}$ such that

$$\mu\left(\{x:|\langle x,\theta\rangle|\geq ctm_\theta\}\right)\leq e^{-\frac{t^2}{\log^{2\alpha}(t+1)}}, \tag{1.5}$$

for all $1\leq t\leq\sqrt{n}\log^\alpha n$, where $\alpha=3$ in [**9**] and $\alpha=1$ in [**8**]. In this article we obtain a slightly better estimate.

A natural way to study this problem is to define the symmetric convex set $\Psi_2(\mu)$ with support function

$$h_{\Psi_2(\mu)}(\theta):=\sup_{2\leq q\leq n}\frac{\left(\mathbb{E}\,|\langle\cdot,\theta\rangle|^q\right)^{1/q}}{\sqrt{q}}, \tag{1.6}$$

and to estimate its volume. Actually, this was the strategy in [**9**] and [**8**]. Note that $\Psi_2(\mu)$ contains the ellipsoid $\frac{1}{\sqrt{2}}Z_2(\mu)$, where

$$h_{Z_2(\mu)}(\theta):=\left(\mathbb{E}\,|\langle\cdot,\theta\rangle|^2\right)^{1/2}. \tag{1.7}$$

It seems plausible that, in the case of centered log-concave probability measures, $\Psi_2(\mu)$ is a "bounded volume ratio" body, i.e.

$$\left(\frac{|\Psi_2(\mu)|}{|Z_2(\mu)|}\right)^{1/n}\leq C, \tag{1.8}$$

where $C>0$ is an absolute constant.

The main result of the paper establishes this volume estimate up to a $\sqrt{\log n}$ term.

Theorem 1.1. *Let μ be a centered* log-*concave probability measure on $\mathbb{R}^n$. Then,*

$$\frac{c_1}{\sqrt{n}}\leq|\Psi_2(\mu)|^{1/n}\leq\frac{c_2\sqrt{\log n}}{\sqrt{n}}, \tag{1.9}$$

where $c_1,c_2>0$ are absolute constants.

A direct consequence of Theorem 1.1 is the existence of subgauusian directions for μ with constant $r=O(\sqrt{\log n})$. A variant of the proof leads to the following:

Theorem 1.2. (i) *If K is a centered convex body of volume 1 in $\mathbb{R}^n$, then there exists $\theta\in S^{n-1}$ such that*

$$|\{x\in K:|\langle x,\theta\rangle|\geq cth_{Z_2(K)}(\theta)\}|\leq e^{-\frac{t^2}{\log(t+1)}} \tag{1.10}$$

for all $t\geq 1$, where $c>0$ is an absolute constant.

(ii) *If μ is a centered* log-*concave probability measure on $\mathbb{R}^n$, then there exists $\theta \in S^{n-1}$ such that*

$$\mu\left(\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geq ct\mathbb{E}|\langle \cdot, \theta\rangle|\}\right) \leq e^{-\frac{t^2}{\log(t+1)}} \tag{1.11}$$

for all $1 \leq t \leq \sqrt{n \log n}$, where $c > 0$ is an absolute constant.

The starting point of the proof is the same as in [**8**]. Assume, for simplicity, that μ is the Lebesgue measure on an isotropic convex body K with bounded isotropic constant. We approximate $\Psi_2(K)$ by the convex hull of a logarithmic in the dimension number of bodies of the form $Z_q(K)/\sqrt{q}$, where $Z_q(K)$ is the L_q-centroid body of K. We obtain an improved estimate for the covering numbers $N(Z_q(K), t\sqrt{q}B_2^n)$ by replacing the argument in [**8**], which was using quermassintegrals, by a variant of an argument of M. Talagrand which provides a bound for the dual covering numbers $N(B_2^n, t\sqrt{q}Z_q^\circ(K))$ in terms of widths of $Z_q(K)$ of negative order (see Proposition 4.4 and Corollary 4.5). Here we exploit the negative moments approach which has been developed by the second named author. Then, we apply the theorem of Artstein, Milman and Szarek (see [**1**]) on the duality of entropy numbers.

An additional feature of the proof is a reduction to the case of convex bodies with uniformly bounded isotropic constant. This reduction is obtained by a "convolution argument" which is presented in Section 3 and is of independent interest. An analogous reduction is an essential ingredient in Klartag's work [**9**, Section 4]; the main difference is that in the present paper convolution with a gaussian is replaced by convolution with a Euclidean ball.

The paper is organized as follows: In Section 2 we introduce notation, terminology and some background material which is needed for the rest of the paper. In Section 3 we describe the convolution procedure. In Section 4 we introduce the p-medians and describe a method for covering numbers estimates in a more general setting. In Section 5, this method is applied to the L_q-centroid bodies. The proof of the theorems is given in Section 6.

2. Preliminaries

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We also write $\widetilde{A}$ for the homothetic image of volume 1 of a compact set $A \subseteq \mathbb{R}^n$ of positive volume, i.e. $\widetilde{A} := \frac{A}{|A|^{1/n}}$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

A star-shaped body C with respect to the origin is a compact set that satisfies $tC \subseteq C$ for all $t \in [0, 1]$. We denote by $\|\cdot\|_C$ the gauge function of C:

$$\|x\|_C = \inf\{\lambda > 0 : x \in \lambda C\}. \tag{2.1}$$

A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if it has centre of mass at the origin: $\int_C \langle x, \theta\rangle\, dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y\rangle : y \in C\}$. We define the mean width of C by

$$W(C) = \int_{S^{n-1}} h_C(\theta)\sigma(d\theta), \tag{2.2}$$

and, for each $-\infty < p < \infty$, $p \neq 0$, we define the p-mean width of C by

$$W_p(C) = \left(\int_{S^{n-1}} h_C^p(\theta)\sigma(d\theta)\right)^{1/p}. \tag{2.3}$$

The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C, the polar body C° of C is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in C\}. \tag{2.4}$$

The geometric distance of two centered convex bodies A and B is the quantity $d(A, B) = \inf\left\{ts \mid t, s > 0, \frac{1}{t}A \subseteq B \subseteq sA\right\}$.

If A and B are compact sets in $\mathbb{R}^n$, then the covering number $N(A, B)$ of A by B is the smallest number of translates of B whose union covers A. We will use the duality of entropy numbers theorem of Artstein, Milman and Szarek (see [**1**]):

Theorem 2.1. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$\log N(K, B_2^n) \leq c_1 \log N(B_2^n, c_2K^\circ), \tag{2.5}$$

where $c_1, c_2 > 0$ are absolute constants.

We will also use Sudakov's inequality [**23**]: If C is a symmetric convex body in $\mathbb{R}^n$, then

$$N(C, tB_2^n) \leq \exp(cn(W(C)/t)^2) \tag{2.6}$$

for every $t > 0$, where $c > 0$ is an absolute constant.

We refer to the books [**22**], [**15**] and [**20**] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

We write $\mathcal{P}_{[n]}$ for the class of all probability measures in $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. The density of $\mu \in \mathcal{P}_{[n]}$ is denoted by f_μ. A probability measure $\mu \in \mathcal{P}_{[n]}$ is called symmetric if f_μ is an even function on $\mathbb{R}^n$. We say that $\mu \in \mathcal{P}_{[n]}$ is centered if for all $\theta \in S^{n-1}$, $\int_{\mathbb{R}^n} \langle x, \theta\rangle d\mu(x) = 0$. A measure μ on $\mathbb{R}^n$ is called log-concave if for any Borel subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0, 1)$, $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$. A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if $\log f$ is concave on its support $\{f > 0\}$. It is known that if μ is log-concave and $\mu(H) < 1$ for every hyperplane H, then $\mu \in \mathcal{P}_{[n]}$ and its density f_μ is log-concave (see [**5**]). Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_{\widetilde{K}}$ is the density of a log-concave measure.

Let $\mu \in \mathcal{P}_{[n]}$. For every $q \geq 1$ and $\theta \in S^{n-1}$ we define

$$h_{Z_q(\mu)}(\theta) := \left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^q f_\mu(x)\, dx\right)^{1/q}. \tag{2.7}$$

If μ is log-concave then $h_{Z_q(\mu)}(\theta) < \infty$ for every $q \geq 1$ and every $\theta \in S^{n-1}$. We define the L_q-centroid body $Z_q(\mu)$ of μ to be the centrally symmetric convex set with support function $h_{Z_q(\mu)}$. L_q–centroid bodies were introduced in [**12**]. Here we follow the normalization (and notation) that appeared in [**18**]. The original definition concerned the class of measures $\mathbf{1}_K$ where K is a convex body of volume 1. In this case, we also write $Z_q(K)$ instead of $Z_q(\mathbf{1}_K)$.

If K is a compact set in $\mathbb{R}^n$ and $|K| = 1$, it is easy to check that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \le p \le q \le \infty$, where $Z_\infty(K) = \mathrm{conv}\{K, -K\}$. Note that if $T \in SL_n$ then $Z_p(T(K)) = T(Z_p(K))$. Moreover, if K is convex body, as a consequence of the Brunn–Minkowski inequality (see, for example, [**18**]), one can check that

$$Z_q(K) \subseteq c\frac{q}{p} Z_p(K) \tag{2.8}$$

for all $1 \le p < q$, where $c \ge 1$ is an absolute constant. If K has its center of mass at the origin, then

$$Z_q(K) \supseteq c\, K \tag{2.9}$$

for all $q \ge n$, where $c > 0$ is an absolute constant. Additional information on L_q–centroid bodies can be found in [**16**] and [**19**].

A centered measure in $\mu \in \mathcal{P}_{[n]}$ is called isotropic if $Z_2(\mu) = B_2^n$. We say that a centered convex body K is isotropic if $Z_2(K)$ is a multiple of the Euclidean ball and we define the isotropic constant of K by

$$L_K := \left(\frac{|Z_2(K)|}{|B_2^n|}\right)^{1/n}. \tag{2.10}$$

So, K is isotropic if and only if $Z_2(K) = L_K B_2^n$. Note that K is isotropic if and only if $L_K^n \mathbf{1}_{\frac{K}{L_K}}$ is isotropic. A centered convex body K is called *almost isotropic* if K has volume one and $K \simeq T(K)$ where $T(K)$ is an isotropic linear transformation of K. In general, we define the isotropic constant of an isotropic $\mu \in \mathcal{P}_{[n]}$ by $L_\mu := f_\mu(0)^{\frac{1}{n}}$. We refer to [**14**], [**7**] and [**19**] for additional information on isotropic convex bodies.

Let μ be a centered measure in $\mathcal{P}_{[n]}$. For every star shaped body C in $\mathbb{R}^n$ and any $-n < p \le \infty$, $p \neq 0$, we set

$$I_p(\mu, C) := \left(\int_{\mathbb{R}^n} \|x\|_C^p d\mu(x)\right)^{1/p}. \tag{2.11}$$

As before, if K is a compact set of volume 1, we write $I_p(K, C)$ instead of $I_p(\mathbf{1}_K, C)$. We also define

$$R(K, C) := I_\infty(K, C) := \max_{x \in K} \|x\|_C. \tag{2.12}$$

This is the radius of K with respect to C. If $C = B_2^n$ we simply write $I_p(K)$ instead of $I_p(K, B_2^n)$.

Let $\mu \in \mathcal{P}_{[n]}$ and assume that $0 \in \mathrm{supp}(\mu)$. For every $p > 0$ we define a set $K_p(\mu)$ as follows:

$$K_p(\mu) := \left\{x \in \mathbb{R}^n : p\int_0^\infty f_\mu(rx) r^{p-1} dr \ge f_\mu(0)\right\}. \tag{2.13}$$

The bodies $K_p(\mu)$ were introduced in [**2**] and allow us to study log-concave measures using convex bodies. K. Ball proved that if μ is log-concave then $K_p(\mu)$ is convex.

If μ is centered then $K_{n+1}(\mu)$ is also centered. Moreover, if μ is centered and log-concave, then, for all$1 \leq p \leq n$,

$$c_1 f_\mu(0)^{1/n} Z_q(\mu) \subseteq Z_q(\widetilde{K}_{n+1}(\mu)) \subseteq c_2 f_\mu(0)^{1/n} Z_q(\mu), \tag{2.14}$$

where $c_1, c_2 > 0$ are absolute constants (see [**19**] for a proof). By the definition of $\Psi_2(\mu)$, it follows that

$$c_1 f_\mu(0)^{1/n} \Psi_2(\mu) \subseteq \Psi_2(\widetilde{K}_{n+1}(\mu)) \subseteq c_2 f_\mu(0)^{1/n} \Psi_2(\mu), \tag{2.15}$$

Note that, if μ is also isotropic, (2.14) implies that

$$L_\mu = f_\mu(0)^{1/n} \simeq L_{K_{n+1}(\mu)}. \tag{2.16}$$

3. Convolutions

The purpose of this Section is to show that for every isotropic convex body K there exists a second isotropic convex body K_1 with bounded isotropic constant and the "same behavior" with respect to linear functionals.

Theorem 3.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. There exists an isotropic convex body K_1 in $\mathbb{R}^n$ with the following properties:*

(1) $L_{K_1} \leq c_1$.
(2) $c_2 Z_p(K_1) \subseteq \frac{Z_p(K)}{L_K} + \sqrt{p} B_2^n \subseteq c_3 Z_p(K_1)$ *for all* $1 \leq p \leq n$.
(3) $c_4 \Psi_2(K_1) \subseteq \frac{\Psi_2(K)}{L_K} \subseteq c_5 \Psi_2(K_1)$.

The constants c_i, $i = 1, \ldots, 5$ are absolute positive constants.

We shall define K_1 as the "convolution" of K with a multiple of B_2^n. Before giving the necessary definitions, we recall some simple properties of the convolution $f * g$ of two non-negative integrable functions f and g on $\mathbb{R}^n$; recall that $f * g$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y)\, dy, \quad x \in \mathbb{R}^n \tag{3.1}$$

and satisfies

$$\int_{\mathbb{R}^n} (f * g)(x)\, dx = \left(\int_{\mathbb{R}^n} f(x)\, dx \right) \left(\int_{\mathbb{R}^n} g(x)\, dx \right). \tag{3.2}$$

If $\mu_1, \mu_2 \in \mathcal{P}_{[n]}$ we define $\mu_1 * \mu_2$ to be the probability measure with density $f_{\mu_1} * f_{\mu_2}$.

Lemma 3.2. *Let $f, g : \mathbb{R}^n \to \mathbb{R}^+$ be integrable functions with $\int_{\mathbb{R}^n} f(x)\, dx = \int_{\mathbb{R}^n} g(x)\, dx = 1$.*

(1) *If f and g are even, then $f * g$ is even.*
(2) *If f and g have their center of mass at the origin, then $f * g$ has its center of mass at the origin.*
(3) *If f and g are log-concave, then $f * g$ is log-concave.*

Proof. The first assertion follows directly from the definition and the third one is a consequence of the Prékopa–Leindler inequality (see e.g. [**20**]). Assuming that both f and g have center of mass at the origin, for every $\theta \in S^{n-1}$ we write

$$\begin{aligned}\int_{\mathbb{R}^n} \langle x,\theta\rangle (f*g)(x)dx &= \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n}\langle x,\theta\rangle g(x-y)dxdy \\ &= \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n}\langle z+y,\theta\rangle g(z)dzdy \\ &= \int_{\mathbb{R}^n}\langle y,\theta\rangle f(y)dy + \int_{\mathbb{R}^n}\langle z,\theta\rangle g(z)dz = 0.\end{aligned}$$

This proves the second claim. □

LEMMA 3.3. *Let $\mu_1,\mu_2 \in \mathcal{P}_{[n]}$. Assume that at least one of the densities $f := f_{\mu_1}$ and $g = g_{\mu_2}$ is even. Then, for every $k \in \mathbb{N}$,*

$$\frac{Z_{2k}(\mu_1)+Z_{2k}(\mu_2)}{2} \subseteq Z_{2k}(\mu_1*\mu_2) \subseteq Z_{2k}(\mu_1)+Z_{2k}(\mu_2). \tag{3.3}$$

In the case $k=1$ we have

$$h^2_{Z_2(\mu_1*\mu_2)} = h^2_{Z_2(\mu_1)} + h^2_{Z_2(\mu_2)}. \tag{3.4}$$

Proof. For every $\theta \in S^{n-1}$ we have

$$\begin{aligned} h^{2k}_{Z_{2k}(f*g)}(\theta) &= \int_{\mathbb{R}^n}\langle x,\theta\rangle^{2k}(f*g)(x)\,dx \\ &= \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n}\langle x,\theta\rangle^{2k} g(x-y)dxdy \\ &= \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n}(\langle y,\theta\rangle+\langle z,\theta\rangle)^{2k} g(z)dzdy \\ &= \sum_{s=0}^{2k}\binom{2k}{s}\left(\int_{\mathbb{R}^n}\langle z,\theta\rangle^s g(z)dz\right)\left(\int_{\mathbb{R}^n}\langle y,\theta\rangle^{2k-s}f(y)dy\right).\end{aligned}$$

Since at least one of f and g is even, for all odd s we have

$$\left(\int_{\mathbb{R}^n}\langle z,\theta\rangle^s g(z)dz\right)\left(\int_{\mathbb{R}^n}\langle y,\theta\rangle^{2k-s}f(y)dy\right) = 0, \tag{3.5}$$

and hence, all the terms in the above sum are non-negative. It follows that, for every $\theta \in S^{n-1}$,

$$h^{2k}_{Z_{2k}(\mu_1*\mu_2)}(\theta) \geq \left(\int_{\mathbb{R}^n}\langle z,\theta\rangle^{2k}g(z)dz + \int_{\mathbb{R}^n}\langle y,\theta\rangle^{2k}f(y)dy\right), \tag{3.6}$$

which shows that

$$h_{Z_{2k}(\mu_1*\mu_2)} \geq \left(h^{2k}_{Z_{2k}(\mu_1)} + h^{2k}_{Z_{2k}(\mu_2)}\right)^{1/2k} \geq \frac{h_{Z_{2k}(\mu_1)}+h_{Z_{2k}(\mu_2)}}{2}. \tag{3.7}$$

On the other hand, for all $0 \leq s \leq 2k$ we have

$$\int_{\mathbb{R}^n}\langle x,\theta\rangle^s f(y)dy \leq \left(\int_{\mathbb{R}^n}|\langle y,\theta\rangle|^{2k}f(y)dy\right)^{s/2k} = h^s_{Z_{2k}(\mu_1)}(\theta) \tag{3.8}$$

and similarly,

$$\int_{\mathbb{R}^n}\langle z,\theta\rangle^{2k-s}g(z)dz \leq \left(\int_{\mathbb{R}^n}|\langle z,\theta\rangle|^{2k}g(z)dz\right)^{(2k-s)/2k} = h^{2k-s}_{Z_{2k}(\mu_2)}(\theta). \tag{3.9}$$

This implies that

$$h^{2k}_{Z_{2k}(\mu_1*\mu_2)} \leq \sum_{s=0}^{2k} \binom{2k}{s} h^{s}_{Z_{2k}(\mu_2)} h^{2k-s}_{Z_{2k}(\mu_1)} = \left(h_{Z_{2k}(\mu_1)} + h_{Z_{2k}(\mu_2)}\right)^{2k}. \tag{3.10}$$

So, $Z_{2k}(\mu_1 * \mu_2) \subseteq Z_{2k}(\mu_1) + Z_{2k}(\mu_2)$. □

DEFINITION 3.4. Let $f : \mathbb{R}^n \to \mathbb{R}^+$ be an integrable function with $\int_{\mathbb{R}^n} f(x)\,dx = 1$. For every $\lambda > 0$ we define $f_{(\lambda)} : \mathbb{R}^n \to \mathbb{R}^+$ by

$$f_{(\lambda)}(x) := \lambda^n f(\lambda x). \tag{3.11}$$

Moreover, if $\mu \in \mathcal{P}_{[n]}$, we write $\mu_{(\lambda)}$ for the measure with density $f_{\mu(\lambda)} := (f_\mu)_{(\lambda)}$. It is easily checked that

$$\int_{\mathbb{R}^n} f_{(\lambda)}(x)dx = \int_{\mathbb{R}^n} f(x)dx = 1 \tag{3.12}$$

and

$$Z_q(\mu_{(\lambda)}) = \frac{1}{\lambda} Z_q(\mu) \tag{3.13}$$

for every $q > 0$. In particular, if K is a convex body of volume 1, for all $q, a > 0$ we have

$$Z_q(a^n \mathbf{1}_{\frac{K}{a}}) = \frac{1}{a} Z_q(K). \tag{3.14}$$

and for all $-n < q < \infty$, $q \neq 0$ and $a > 0$ we have

$$I_q(a^n \mathbf{1}_{\frac{K}{a}}) = \frac{1}{a} I_q(K). \tag{3.15}$$

DEFINITION 3.5. Let C_1 and C_2 be two centered convex bodies of volume 1 in $\mathbb{R}^n$. Assume that at least one of K_1, K_2 is symmetric. For any $a, b > 0$ we define a log-concave function $h_{\frac{C_1}{a}, \frac{C_2}{b}}$ on $\mathbb{R}^n$ by

$$h_{\frac{C_1}{a}, \frac{C_2}{b}}(x) := a^n b^n \left(\mathbf{1}_{\frac{C_1}{a}} * \mathbf{1}_{\frac{C_2}{b}}\right)(x). \tag{3.16}$$

From Lemma 3.2 we see that $h_{\frac{C_1}{a}, \frac{C_2}{b}}$ has center of mass at the origin and integral equal to 1. Observe that

$$h_{\frac{C_1}{a}, \frac{C_2}{b}}(0) = a^n b^n \int_{\mathbb{R}^n} \mathbf{1}_{\frac{C_1}{a}}(y) \mathbf{1}_{\frac{C_2}{b}}(-y)dy = |bC_1 \cap (-aC_2)| = |bC_1 \cap (aC_2)|. \tag{3.17}$$

We define the centered log-concave measure $\mu_{\frac{C_1}{a}, \frac{C_2}{b}}$ as the measure with density $h_{\frac{C_1}{a}, \frac{C_2}{b}}$.

PROPOSITION 3.6. *Let C_1, C_2 be two centered convex bodies of volume 1 in $\mathbb{R}^n$ and let $\mu_{\frac{C_1}{a}, \frac{C_2}{b}}$ be defined as above. If $C = \widetilde{K}_{n+1}\left(\mu_{\frac{C_1}{a}, \frac{C_2}{b}}\right)$, then*

1. *C is centered and has volume 1.*
2. *For any $1 \leq q \leq n$, $abZ_q(C) \simeq |bC_1 \cap aC_2|^{1/n} \left(bZ_q(C_1) + aZ_q(C_2)\right)$.*
3. *$ab\Psi_2(C) \simeq |bC_1 \cap aC_2|^{1/n} \left(b\Psi_2(C_1) + a\Psi_2(C_2)\right)$.*

Proof. The first assertion is clear. Let $1 \leq q \leq n$. Then, using Lemma 3.3 and (2.14), (3.17) and (3.14), we see that

$$\begin{aligned} Z_q(C) &= Z_q(\widetilde{K}_{n+1}(h_{\frac{C_1}{a},\frac{C_2}{b}})) \\ &\simeq h^{1/n}_{\frac{C_1}{a},\frac{C_2}{b}}(0) Z_q(h_{\frac{C_1}{a},\frac{C_2}{b}}) \\ &\simeq |bC_1 \cap aC_2|^{1/n} Z_q\left((a^n \mathbf{1}_{\frac{C_1}{a}}) * (b^n \mathbf{1}_{\frac{C_2}{b}})\right) \\ &\simeq |bC_1 \cap aC_2|^{1/n} \left(Z_q(a^n \mathbf{1}_{\frac{C_1}{a}}) + Z_q(b^n \mathbf{1}_{\frac{C_2}{b}})\right) \\ &\simeq |bC_1 \cap aC_2|^{1/n} \left(\frac{Z_q(C_1)}{a} + \frac{Z_q(C_2)}{b}\right). \end{aligned}$$

This proves the second claim. Finally, observe that, for every $\theta \in S^{n-1}$,

$$\begin{aligned} \frac{1}{|bC_1 \cap aC_2|^{1/n}} h_{\Psi_2(C)}(\theta) &\simeq \sup_{q\geq 2} \frac{a^{-1}h_{Z_q(C_1)}(\theta) + b^{-1}h_{Z_q(C_2)}(\theta)}{\sqrt{q}} \\ &\geq \max\left\{\frac{1}{a}\sup_{q\geq 2}\frac{h_{Z_q(C_1)}(\theta)}{\sqrt{q}}, \frac{1}{b}\sup_{q\geq 2}\frac{h_{Z_q(C_2)}(\theta)}{\sqrt{q}}\right\} \\ &\geq c\left(\frac{h_{\Psi_2(C_1)}(\theta)}{a} + \frac{h_{\Psi_2(C_2)}(\theta)}{b}\right), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{|bC_1 \cap aC_2|^{1/n}} h_{\Psi_2(C)}(\theta) &\simeq \sup_{q\geq 2} \frac{a^{-1}h_{Z_q(C_1)}(\theta) + b^{-1}h_{Z_q(C_2)}(\theta)}{\sqrt{q}} \\ &\leq \sup_{q\geq 2}\frac{a^{-1}h_{Z_q(C_1)}(\theta)}{\sqrt{q}} + \sup_{q\geq 1}\frac{b^{-1}h_{Z_q(C_2)}(\theta)}{\sqrt{q}} \\ &= \frac{h_{\Psi_2(C_1)}(\theta)}{a} + \frac{h_{\Psi_2(C_2)}(\theta)}{b}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 3.1. Let K_1 be the convex body that we obtain if we apply Proposition 3.6 with $C_1 = K$, $C_2 = \widetilde{B}_2^n$, $a = L_K$ and $b = 1$. Since $|K \cap L_K \widetilde{B}_2^n|^{1/n} \simeq 1$, we immediately get

$$Z_q(K_1) \simeq \frac{Z_q(K)}{L_K} + \sqrt{q}B_2^n \supseteq \sqrt{q}B_2^n \tag{3.18}$$

for all $1 \leq q \leq n$. Since $\Psi_2(K) \supseteq cL_K B_2^n$, we also have

$$\Psi_2(K_1) \simeq \frac{\Psi_2(K)}{L_K} + B_2^n \simeq \frac{\Psi_2(K)}{L_K}. \tag{3.19}$$

This already proves (2) and (3).

For (1) we use the fact that $Z_2(K) = L_K B_2^n$ to write

$$\frac{L_{K_1}}{\sqrt{n}} \simeq |Z_2(K_1)|^{1/n} \simeq |B_2^n + \sqrt{2}B_2^n|^{1/n} \simeq \frac{1}{\sqrt{n}}. \tag{3.20}$$

It follows that $L_{K_1} \leq c_1$ for some absolute constant $c_1 > 0$. □

Note: We conclude this Section by pointing out the following consequence of Theorem 3.1: If K is a centered convex body of volume 1 in $\mathbb{R}^n$, then there exists a centered convex body K_1 of volume 1 in $\mathbb{R}^n$, such that

(1) $L_{K_1} \simeq 1$
(2) $d(\Psi_2(K), \Psi_2(K_1)) \simeq 1$, where $d(A, B)$ is the geometric distance of A, B.

This means that the geometry of the "Ψ_2–body" cannot be a reason for L_K to be unbounded.

4. p-medians

In this section we describe an argument that can be used in order to give upper bounds for the covering numbers $N(K, tB_2^n)$ of a convex body K by multiples of the Euclidean ball. In fact, we will work in a more general setting than the one which is needed for our purpose. We start with the following definition.

Definition 4.1. Let μ be a centered log-concave probability measure on $\mathbb{R}^n$ with density f_μ and let C be a symmetric convex body in $\mathbb{R}^n$. For every $p > 0$ we define the *p-median* of C with respect to μ as the unique number $m_p(\mu, C) > 0$ for which

$$\mu\left(m_p(\mu, C)C\right) = 2^{-p}. \tag{4.1}$$

Similarly, if K is a centered convex body of volume 1 in $\mathbb{R}^n$, we define $m_p(K, C)$ by the equation

$$|K \cap m_p(K, C)C| = 2^{-p}. \tag{4.2}$$

Note that $m_p(K, C)$ is a decreasing function of p. If we set $m_0(K, C) = R(K, C)$ and $m_\infty(K, C) = 0$ then $m_p(K, C)$ is a continuous function of p on $[0, \infty]$. Note also that m_1 is the usual median of $\|\cdot\|_C$ on K; it is known that $m_1(K, C) \simeq I_1(K, C)$. Moreover, we have the following:

Lemma 4.2. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, let C be a symmetric convex body in $\mathbb{R}^n$ and let $p \geq 1$. Then,*

$$m_p(K, C) \geq \frac{1}{2} I_{-p}(K, C). \tag{4.3}$$

Moreover, if $p < n$ and if $I_{-p}(K, C) \leq aI_{-2p}(K, C)$ for some $a \geq 1$, then we have that

$$m_{2p\log(2a)}(K, C) \leq 2I_{-p}(K, C). \tag{4.4}$$

Proof. We set $I_{-p} := I_{-p}(K, C)$ and $m_p := m_p(K, C)$. From Markov's inequality we have that

$$\left| K \cap \frac{1}{2} I_{-p}(K, C)C \right| \leq 2^{-p} = |K \cap m_p(K, C)C|, \tag{4.5}$$

and hence, $m_p(K, C) \geq \frac{1}{2} I_{-p}(K, C)$. On the other hand, by the Paley–Zygmund inequality (see [**21**]) we have that

$$|\{x \in K : \|x\|_C^{-p} \geq s^p I_{-p}^{-p}\}| \geq (1 - s^p)^2 \left(\frac{I_{-p}^{-p}}{I_{-2p}^{-p}}\right)^2 = (1 - s^p)^2 \left(\frac{I_{-2p}}{I_{-p}}\right)^{2p}. \tag{4.6}$$

Choosing $s = \frac{1}{2}$, we get

$$|K \cap 2I_{-p}C| \geq (1 - 2^{-p})^2 a^{-2p} \geq (2a)^{-2p} = 2^{-2p\log 2a}. \tag{4.7}$$

This proves (4.4). □

Let L be a star-shaped body in $\mathbb{R}^n$. For every $r > 0$ we define a probability density $g_{L,r}$ on $\mathbb{R}^n$ by

$$g_{L,r}(x) := \frac{1}{|L|\Gamma(\frac{n+r}{r})} e^{-\|x\|_L^r}. \tag{4.8}$$

We write $g_L := g_{L,1}$. Note that the density γ_n of the standard Gaussian measure can be expressed as $\gamma_n := g_{\sqrt{2}B_2^n,2}$. Note also that, if $\mu_{L,r}$ is the measure with density $g_{L,r}$, then for every compact set C and any $a > 0$, one has that

$$g_{aL,r}(C) = g_{L,r}(\frac{C}{a}). \tag{4.9}$$

LEMMA 4.3. *Under the above assumptions, the measure $\mu_{L,r}$ with density $g_{L,r}$ is a probability measure on $\mathbb{R}^n$. If L is convex and $r \geq 1$ then $\mu_{L,r}$ is a* log-concave *measure. If L is symmetric, then $g_{L,r}$ is even. If L has center of mass at the origin then $g_{L,r}$ has center of mass at the origin.*

Moreover, if $|L| = 1$, if V is a star-shaped body, and if $p > 0$ and $q > -n$, we have that

$$Z_p(g_{L,r}) = \left(\frac{\Gamma\left(\frac{n+p+r}{r}\right)}{\Gamma\left(\frac{n+r}{r}\right)} \right)^{1/p} Z_p(L) \tag{4.10}$$

and

$$I_q(g_{L,r}, V) = \left(\frac{\Gamma\left(\frac{n+q+r}{r}\right)}{\Gamma\left(\frac{n+r}{r}\right)} \right)^{1/q} I_q(L, V). \tag{4.11}$$

Proof. Let $h : \mathbb{R}^n \to \mathbb{R}^+$ be homogeneous of degree p. Then,

$$\int_{\mathbb{R}^n} h(x) d\mu_{L,r}(x) = \frac{\Gamma\left(\frac{n+p+r}{r}\right)}{\Gamma\left(\frac{n+r}{r}\right)} \frac{1}{|L|} \int_L h(x)dx. \tag{4.12}$$

Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^n} h(x)e^{-\|x\|_L^r} dx &= \int_{\mathbb{R}^n} h(x) \int_{\|x\|_L}^{\infty} (-e^{-t^r})' dt dx \\
&= \int_0^{\infty} rt^{r-1}e^{-t^r} \int_{\|x\|_L \leq t} h(x) dx dt \\
&= \int_{\|x\|_L \leq 1} h(x)dx \int_0^{\infty} rt^{r+p+n-1}e^{-t^r} dt \\
&= \Gamma\left(\frac{n+p+r}{r}\right) \int_L h(x)dx.
\end{aligned}$$

The assertions of the Lemma follow if we choose h to be 1, $\langle x, \theta \rangle$, $|\langle x, \theta \rangle|^p$ or $\|x\|_V^q$ respectively. □

PROPOSITION 4.4. *Let K and C be star-shaped sets in $\mathbb{R}^n$. Assume that C is symmetric, K has volume 1 and K is r-convex: for every $x, y \in \mathbb{R}^n$,*

$$\|x+y\|_K^r + \|x-y\|_K^r \leq 2\|x\|_K^r + 2\|y\|_K^r. \tag{4.13}$$

Then, for every $p > 0$ we have that

$$\log N\left(K, \frac{c_1}{p^{1/r}} m_p(g_{K,r}, C)C\right) \leq c_2 p, \tag{4.14}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Let $\{z_1, \ldots, z_N\}$ be a maximal set of points in K with respect to the condition $\|z_i - z_j\|_C \geq t$. Then, the sets $z_i + \frac{t}{2}C$ have mutually disjoint interiors.

Let $A := |K|\Gamma(\frac{n+r}{r})$. Using the symmetry of C and (4.13) we see that, for every $s > 0$,

$$\begin{aligned} g_{K,r}\left(\frac{2s}{t}z_i + sC\right) &= \frac{1}{A}\int_{sC} e^{-\|x+\frac{2s}{t}z_i\|_K^r}dx \\ &\geq \frac{1}{A}\int_{sC} e^{-\left(\|x\|_K^r+\|\frac{2s}{t}z_i\|_K^r\right)}dx \\ &= \frac{1}{A}e^{-\|\frac{2s}{t}z_i\|_K^r}\int_{sC} e^{-\|x\|_K^r}dx \\ &\geq e^{-\left(\frac{2s}{t}\right)^r}g_{K,r}(sC). \end{aligned}$$

Choose $s = m_p(g_{K,r}, C)$ and $t := \frac{2s}{p^{1/r}}$. Then,

$$1 \geq \sum_{i=1}^{N} \mu_{K,r}\left(\frac{2s}{t} + sC\right) \geq N2^{-p}e^{-p}. \tag{4.15}$$

This implies that $N(K, tC) \leq N \leq e^{c_2 p}$, where $c_2 > 0$ is an absolute constant. □

Corollary 4.5. *Let C be a symmetric convex body in $\mathbb{R}^n$ and let $1 \leq p \leq n/2$ be such that $W_{-2p}(C) \simeq W_{-p}(C)$. Then,*

$$\log N\left(C, c_1\sqrt{n/p}W_{-p}(C)B_2^n\right) \leq c_2 p, \tag{4.16}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. We apply Proposition 4.4 with $K = \widetilde{B_2^n}$ and $r = 2$. Lemma 4.3 shows that, for any $1 \leq q \leq n/2$,

$$I_{-q}(g_{\widetilde{B_2^n},2}, C^\circ) \simeq \sqrt{n}I_{-q}(\widetilde{B_2^n}, C^\circ) \simeq nW_{-q}(C), \tag{4.17}$$

where the last equality follows by integration in polar coordinates.

Since $W_{-2p}(C) \simeq W_{-p}(C)$, Lemma 4.2 shows that

$$m_p(g_{\widetilde{B_2^n},2}, C^\circ) \simeq nW_{-p}(C). \tag{4.18}$$

Now, Proposition 4.4 gives

$$\log N\left(\widetilde{B_2^n}, c\frac{n}{\sqrt{p}}W_{-p}(C)C^\circ\right) \leq cp. \tag{4.19}$$

Therefore,

$$\log N\left(B_2^n, c\frac{\sqrt{n}}{\sqrt{p}}W_{-p}(C)C^\circ\right) \leq c'p. \tag{4.20}$$

The result follows from Theorem 2.1. □

Remarks. (i) The argument that we used is a variation of an argument of M. Talagrand (see [**10**] and [**11**]). The new ingredient is the use of the p-median.

(ii) There exist convex bodies that do not satisfy the "regularity" assumption $W_{-2p}(C) \simeq W_{-p}(C)$ of Corollary 4.4. Given any $a \geq 1$ and $1 \leq p \leq \frac{n}{4}$ one can find an ellipsoid $\mathcal{E}$ such that $W_{-2p}(\mathcal{E}) \leq \frac{1}{a}W_{-p}(\mathcal{E})$.

5. Covering numbers of the L_q-centroid bodies

Our goal in this section is to show the following:

Proposition 5.1. *Let K be an isotropic convex body in $\mathbb{R}^n$, let $1 \leq q \leq n$ and $t \geq 1$. Then,*

$$\log N\left(Z_q(K), c_1 t\sqrt{q} L_K B_2^n\right) \leq c_2 \frac{n}{t^2} + c_3 \frac{\sqrt{n}\sqrt{q}}{t}, \tag{5.1}$$

where $c_1, c_2, c_3 > 0$ are absolute constants.

We will use the following fact proved in [**19**]: if K is a centered convex body of volume 1, and if $1 \leq p \leq n/2$, then

$$I_{-p}(K) \simeq \sqrt{\frac{n}{p}} W_{-p}(Z_p(K)). \tag{5.2}$$

We will also use the following L_p version of the Blaschke–Santaló inequality obtained by Lutwak and Zhang [**12**]: (see also [**6**] for a proof in the convex case):

Theorem 5.2. *Let K be an origin star-shaped body of volume 1 in $\mathbb{R}^n$. Then,*

$$|Z_p^{\circ}(K)|^{1/n} \leq |Z_p^{\circ}(\widetilde{B_2^n})|^{1/n}, \tag{5.3}$$

with equality if and only if K is a centered ellipsoid of volume 1.

Moreover, the reverse inequality has been established in [**18**]:

Theorem 5.3. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$|Z_p^{\circ}(K)|^{1/n} \geq \frac{c}{L_K} |Z_p^{\circ}(\widetilde{B_2^n})|^{1/n}, \tag{5.4}$$

It follows that, for every centered convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leq p \leq n$,

$$W_{-n}(Z_p(K)) = \left(\frac{|Z_p^{\circ}(K)|}{|B_2^n|}\right)^{1/n} \leq \left(\frac{|Z_p^{\circ}(\widetilde{B_2^n})|}{|B_2^n|}\right)^{\frac{1}{n}} \leq c_1\sqrt{p} \tag{5.5}$$

and

$$W_{-n}(Z_p(K)) = \left(\frac{|Z_p^{\circ}(K)|}{|B_2^n|}\right)^{1/n} \geq \frac{c'}{L_K} \left(\frac{|Z_p^{\circ}(\widetilde{B_2^n})|}{|B_2^n|}\right)^{1/n} \geq \frac{c_2}{L_K}\sqrt{p}, \tag{5.6}$$

where $c_1, c_2 > 0$ are absolute constants.

We will also use the following simple fact (for a proof see [**19**, Proposition 4.7]):

Proposition 5.4. *Let K be an origin star-shaped body of volume 1 in $\mathbb{R}^n$ and let $0 < p \leq \frac{n}{2}$. Then,*

$$I_{-p}(K) \geq I_{-p}(\widetilde{B_2^n}) \simeq \sqrt{n}. \tag{5.7}$$

We are now ready to prove the following:

Proposition 5.5. *Let K be an isotropic convex body in $\mathbb{R}^n$, let $1 \leq q \leq n/2$ and $1 \leq t \leq \sqrt{n/q}$. Then,*

$$\log N\left(Z_q(K), c_1 t\sqrt{q} L_K B_2^n\right) \leq c_2 \frac{n}{t^2}, \tag{5.8}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Assume first that $L_K \simeq 1$. From Proposition 5.4 we have that $I_{-q}(K) \geq c\sqrt{n}$. Also, by Hölder's inequality, $I_{-q}(K) \leq I_2(K) \simeq \sqrt{n}$.

Using (5.2), (5.5) and (5.6) we get $W_{-q}(Z_q(K)) \simeq \sqrt{q} \simeq W_{-n}(Z_q(K))$. This shows that, for all $q \leq r \leq n$,

$$W_{-2r}(Z_q(K)) \simeq W_{-r}(Z_q(K)) \simeq \sqrt{q}. \tag{5.9}$$

Then, Corollary 4.5 implies that

$$\log N\left(Z_q(K), c\sqrt{n/r}W_{-r}(Z_q(K))B_2^n\right) \leq cr. \tag{5.10}$$

Let $1 \leq t \leq \sqrt{n/q}$. Choosing $r = n/t^2$, we get

$$\log N\left(Z_q(K), ct\sqrt{q}B_2^n\right) \leq c\frac{n}{t^2}. \tag{5.11}$$

We now turn to the general case. From Theorem 3.1, if K is an isotropic convex body in $\mathbb{R}^n$ then we can find a second isotropic convex body K_1 in $\mathbb{R}^n$ such that $L_{K_1} \simeq 1$ and, for every $1 \leq q \leq n$,

$$Z_q(K_1) \simeq \frac{1}{L_K}Z_q(K) + \sqrt{q}B_2^n. \tag{5.12}$$

Therefore, for any $1 \leq t \leq \sqrt{n/q}$,

$$\begin{aligned} N\left(\frac{1}{L_K}Z_q(K), t\sqrt{q}B_2^n\right) &\leq N\left(\frac{1}{L_K}Z_q(K) + \sqrt{q}B_2^n, t\sqrt{q}B_2^n\right) \\ &\leq N\left(Z_q(K_1), t\sqrt{q}B_2^n\right) \\ &\leq e^{\frac{n}{t^2}}. \end{aligned}$$

This completes the proof. □

Proof of Proposition 5.1. The case $1 \leq t \leq \sqrt{n/q}$ follows from Proposition 5.5 (observe that $\frac{\sqrt{n}\sqrt{q}}{t} \leq \frac{n}{t^2}$). Assume that $t \geq \sqrt{n/q}$. We set $p := \frac{\sqrt{n}\sqrt{q}}{t} \leq q$. Then, using (2.8), we have that

$$\begin{aligned} N\left(Z_q(K), t\sqrt{q}L_KB_2^n\right) &\leq N\left(c\frac{q}{p}Z_p(K), t\sqrt{q}L_KB_2^n\right) \\ &\leq N\left(Z_p(K), t\sqrt{\frac{p}{q}}\sqrt{p}L_KB_2^n\right). \end{aligned}$$

Applying Proposition 5.5 for $Z_p(K)$ with $t = \sqrt{n/p}$, we see that

$$\begin{aligned} N\left(Z_q(K), t\sqrt{q}L_KB_2^n\right) &\leq N\left(Z_p(K), \sqrt{\frac{n}{p}}\sqrt{p}L_KB_2^n\right) \\ &\leq e^{cp} = \exp\left(c\frac{\sqrt{n}\sqrt{q}}{t}\right), \end{aligned}$$

and the proof is complete (observe that $\frac{\sqrt{n}\sqrt{q}}{t} \geq \frac{n}{t^2}$ in this case). □

Proposition 5.1 gives us some information about the "regularity" of the covering numbers of $Z_q(K)$. In particular:

Corollary 5.6. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leq q \leq n$. Define $\beta \geq 1$ by the equation $q = n^{1/\beta}$. Let $a := \min\{\beta, 2\}$. Then,*

$$N\left(Z_q(K), c_1t\sqrt{q}L_KB_2^n\right) \leq \exp\left(c_2\frac{n}{t^a}\right), \tag{5.13}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Assume first that $\beta \geq 2$. Then, $q \leq \sqrt{n}$ and it is known (see [**18**]) that $W(Z_q(K)) \simeq \sqrt{q} L_K$. In this case, the result follows from Sudakov's inequality (2.6).

On the other hand, if $\beta \in [1, 2]$, using the fact that $q^\beta = n$, we observe that for all $1 \leq t \leq \sqrt{q}$,

$$\frac{\sqrt{q}\sqrt{n}}{t} \leq \frac{n}{t^a}, \tag{5.14}$$

and the result follows from Proposition 5.1. □

Using (2.14), (2.15) one can immediately extend the results of this section to the setting of log-concave measures:

Corollary 5.7. *Let μ be an isotropic* log*-concave measure in $\mathbb{R}^n$ and let $1 \leq q \leq n$ and $t \geq 1$. Then,*

$$\log N\left(Z_q(\mu), c_1 t\sqrt{q} B_2^n\right) \leq c_2 \frac{n}{t^2} + c_3 \frac{\sqrt{n}\sqrt{q}}{t}. \tag{5.15}$$

Moreover, if $\beta \geq 1$ satisfies $q = n^{1/\beta}$ and if we set $a = \min\{\beta, 2\}$, then

$$N\left(Z_q(\mu), c_1 t\sqrt{q} B_2^n\right) \leq e^{c_2 \frac{n}{t^a}}, \tag{5.16}$$

where $c_1, c_2, c_3 > 0$ are absolute constants.

6. Volume of $\Psi_2(K)$

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Recall the definition of $\Psi_2(K)$: it is the symmetric convex body with support function

$$h_{\Psi_2(K)}(\theta) = \sup_{1 \leq p \leq n} \frac{h_{Z_p(K)}(\theta)}{\sqrt{p}}. \tag{6.1}$$

From the definition, one has $Z_p(K) \subseteq \sqrt{p}\Psi_2(K)$ for all $1 \leq p \leq n$. In particular, $Z_2(K) \subseteq \sqrt{2}\Psi_2(K)$, which implies that

$$|\Psi_2(K)|^{1/n} \geq c\frac{L_K}{\sqrt{n}}. \tag{6.2}$$

Our goal is to give an upper bound for the volume of $\Psi_2(K)$. Our estimate is the following:

Theorem 6.1. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$|\Psi_2(K)|^{1/n} \leq c\frac{\sqrt{\log n}}{\sqrt{n}} L_K. \tag{6.3}$$

Moreover, there exists $\theta \in S^{n-1}$ such that

$$|\{x \in K : |\langle x, \theta\rangle| \geq c t h_{Z_2(K)}(\theta)\}| \leq e^{-\frac{t^2}{\log(t+1)}} \tag{6.4}$$

for all $t \geq 1$, where $c > 0$ is an absolute constant.

Our first observation is that, starting with the definition

$$\Psi_2(K) = \mathrm{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p \in [1, n]\right\}, \tag{6.5}$$

and using the fact that $Z_{2p}(K) \simeq Z_p(K)$, we may write

$$\Psi_2(K) \simeq \text{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p = 2^k,\ k = 1, \ldots, \log_2 n\right\}. \tag{6.6}$$

We set

$$m_1 := \log_2(\sqrt{n}), \quad m_2 := \log_2\left(\frac{n}{\log n}\right), \quad m_3 := \log_2 n = 2m_1, \tag{6.7}$$

and we define symmetric convex bodies $C_1, C_2, C_{2,1}, C_3$ and $C_{3,1}$ as follows:

$$\begin{aligned}
C_1 &:= \text{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p \in [1, \sqrt{n}]\right\}, \\
C_2 &:= \text{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p = 2^k, k = m_1, \ldots, m_2\right\}, \\
C_{2,1} &:= \text{conv}\left\{\frac{Z_p(K)}{\sqrt{p}\sqrt{\log p}}, p = 2^k, k = m_1, \ldots, m_2\right\}, \\
C_3 &:= \text{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p = 2^k, k = m_2 + 1, \ldots, m_3\right\}, \\
C_{3,1} &:= \text{conv}\left\{\frac{Z_p(K)}{\sqrt{p}\sqrt{\log p}}, p = 2^k, k = m_2 + 1, \ldots, m_3\right\}.
\end{aligned}$$

It is clear that

$$\Psi_2(K) \simeq \text{conv}\{C_1, C_2, C_3\}. \tag{6.8}$$

We also define

$$V := \text{conv}\{C_1, C_{2,1}, C_{3,1}\}. \tag{6.9}$$

We will give upper bounds for the covering numbers of $C_1, C_2, C_{2,1}, C_3, C_{3,1}$ by $L_K B_2^n$.

(i) Covering numbers of C_1

We will need some preliminary observations.

LEMMA 6.2. *Let K be a centered convex body in $\mathbb{R}^n$ and let $1 \leq q \leq n$. Let A be a subset of K with volume $|A| \geq 1 - e^{-q}$. Then, for all $1 \leq p \leq c_1 q$,*

$$Z_p(K) \subseteq 2Z_p(\widetilde{A}), \tag{6.10}$$

where $c_1 > 0$ is an absolute constant.

Proof. Recall that there exists an absolute constant $c > 0$ such that $h_{Z_{2p}(K)}(\theta) \leq c h_{Z_p(K)}(\theta)$ for all $\theta \in S^{n-1}$ and $p \geq 1$.

We fix an absolute constant $c_1 > 0$ such that $e^{-q/2}c^{c_1 q} \leq \frac{1}{2}$. Then, we have that

$$\begin{aligned}\int_K |\langle x,\theta\rangle|^p dx &= \int_A |\langle x,\theta\rangle|^p dx + \int_{K\setminus A} |\langle x,\theta\rangle|^p dx \\ &\leq |A|^{1+\frac{p}{n}} \int_{\widetilde{A}} |\langle x,\theta\rangle|^p dx + |K\setminus A|^{\frac{1}{2}} \left(\int_K |\langle x,\theta\rangle|^{2p} dx\right)^{\frac{1}{2}} \\ &\leq \int_{\widetilde{A}} |\langle x,\theta\rangle|^p dx + e^{-\frac{q}{2}} c^p \int_K |\langle x,\theta\rangle|^p dx \\ &\leq \int_{\widetilde{A}} |\langle x,\theta\rangle|^p dx + \frac{1}{2}\int_K |\langle x,\theta\rangle|^p dx.\end{aligned}$$

This proves the Lemma. □

We will also use the following (see [**17**, Theorem 2.1] for a proof):

LEMMA 6.3. *Let K be an isotropic convex body with $R(K) \leq a\sqrt{n}L_K$. Then,*

$$\left(\frac{|\Psi_2(K)|}{|B_2^n|}\right)^{1/n} \leq W(\Psi_2(K)) \leq W_{\sqrt{n}}(\Psi_2(K)) \leq c(a)L_K, \tag{6.11}$$

where $c > 0$ is an absolute constant.

PROPOSITION 6.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\left(\frac{|C_1|}{|B_2^n|}\right)^{1/n} \leq W(C_1) \leq W_{\sqrt{n}}(C_1) \leq cL_K, \tag{6.12}$$

where $c > 0$ is an absolute constant. Moreover, for all $t \geq 1$,

$$N\left(C_1, c_1 t L_K B_2^n\right) \leq e^{\frac{c_2 n}{t^2}}, \tag{6.13}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. It is known that $|K \cap s\sqrt{n}L_K B_2^n| \geq 1 - e^{-s\sqrt{n}}$ for $s \geq c'$, where $c' > 0$ is an absolute constant (this is the main result in [**18**]). Set $s = \max\{c_1^{-1}, c'\}$ where $c_1 > 0$ is the constant from Lemma 6.2. Let $A = K \cap c_1^{-1}\sqrt{n}L_K B_2^n$. Then, $R(\widetilde{A}) \leq c'\sqrt{n}L_K$ and $\tilde{A}$ is almost isotropic. Also, by Lemma 6.2, for every $1 \leq p \leq \sqrt{n}$, we have $Z_p(K) \subseteq 2Z_p(\widetilde{A})$. Therefore,

$$C_1 \subseteq 2C_1(\widetilde{A}) \subseteq 2\Psi_2(\widetilde{A}). \tag{6.14}$$

Now, the result follows from Lemma 6.3 and Sudakov's inequality (2.6). □

(ii) Covering numbers of C_2 and C_3

We will need the following (see [**8**] for a proof):

LEMMA 6.5. *Let $A_1, \ldots, A_s$ be subsets of RB_2^k. For every $t > 0$,*

$$N(\mathrm{conv}(A_1 \cup \cdots \cup A_s), 2tB_2^k) \leq \left(\frac{cR}{t}\right)^s \prod_{i=1}^s N(A_i, tB_2^k). \tag{6.15}$$

LEMMA 6.6. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t \geq 1$,*

$$\max\left\{N(C_2, c_1 t\sqrt{\log n}L_K B_2^n), N(C_3, c_2 t(\log\log n)L_K B_2^n)\right\} \leq e^{c_3\frac{n}{t}} \tag{6.16}$$

and

$$\max\{N\left(C_{2,1}, c_1 L_K B_2^n\right), N\left(C_{3,1}, c_2 L_K B_2^n\right)\} \leq e^{c_3 n}, \tag{6.17}$$

where $c_1, c_2, c_3 > 0$ *are absolute constants.*

Proof. We first consider the bodies C_2 and $C_{2,1}$. We set $s := m_2 - m_1$ and define

$$A_i := \frac{1}{2^{\frac{m_1+i}{2}}} Z_{2^{m_1+i}}(K) \text{ and } A_{i,1} := \frac{1}{2^{\frac{m_1+i}{2}}\sqrt{m_1+i}} Z_{2^{m_1+i}}(K), \tag{6.18}$$

for $i = 0, \dots, s$. Note that $\max\{R(A_i), R(A_{i,1})\} \leq \sqrt{n}L_K$ for $0 \leq i \leq s$. From Proposition 5.1 we have that, for every $r \geq 1$,

$$\log N\left(A_i, crL_K B_2^n\right) \leq \frac{c'n}{r^2} + \frac{c'n}{r\sqrt{\log n}} \tag{6.19}$$

and

$$\log N\left(A_{i,1}, cL_K B_2^n\right) \leq \frac{c'n}{m_1+i} + \frac{c'n}{\sqrt{m_1+i}\sqrt{\log n}} \leq c'' \frac{c'n}{m_1+i}. \tag{6.20}$$

Using Lemma 6.5, we see that

$$\log N\left(C_2, 2crL_K B_2^n\right) \leq \log^2 n + \frac{c'n\log n}{r^2} + \frac{c'n\log n}{r\sqrt{\log n}}. \tag{6.21}$$

Since $R(C_2) \leq \sqrt{n}L_K$, we consider the case $1 \leq t \leq \sqrt{n}$. Then, $\log^2 n \leq \frac{n}{t}$. Setting $r = t\sqrt{\log n}$ we conclude that, for any $t \geq 1$,

$$\log N\left(C_2, 2ct\sqrt{\log n}L_K B_2^n\right) \leq \frac{3c'n}{t}. \tag{6.22}$$

Similarly, we see that

$$\log N\left(C_{2,1}, 2cL_K B_2^n\right) \leq \log^2 n + c'n\sum_{i=1}^{s}\frac{1}{m_1+i} \leq c''n\sum_{j=m_1+1}^{2m_1}\frac{1}{j} \leq c'''n. \tag{6.23}$$

We now consider the bodies C_3 and $C_{3,1}$. We set $s := m_3 - m_2 = \log\log n$ and define

$$A_i := \frac{1}{2^{\frac{m_2+i}{2}}} Z_{2^{m_2+i}}(K),\ A_{i,1} := \frac{1}{2^{\frac{m_2+i}{2}}\sqrt{m_2+i}} Z_{2^{m_2+i}}(K), \tag{6.24}$$

for $i = 1, \dots, s$. Note that $\max\{R(A_i), R(A_{i,1})\} \leq \sqrt{n}L_K$ for all $1 \leq i \leq s$. Corollary 5.5 shows that, for any $r \geq 1$,

$$\log N\left(A_i, crL_K B_2^n\right) \leq \frac{c'n}{r} \text{ and } \log N\left(A_{i,1}, cL_K B_2^n\right) \leq \frac{c'n}{m_2+i} \leq \frac{c''n}{\log n}. \tag{6.25}$$

From Lemma 6.5, we get

$$\log N\left(C_3, 2crL_K B_2^n\right) \leq \log^2 n + \frac{c'n(\log\log n)}{r}. \tag{6.26}$$

Now, we set $t := \frac{r}{(\log\log n)}$. As before, we may assume that $1 \leq t \leq \sqrt{n}$, and hence, $\log^2 n \leq \frac{n}{t}$. Setting $r = t\log\log n$ we conclude that, for any $t \geq 1$,

$$\log N\left(C_3, 2ct(\log\log n)L_K B_2^n\right) \leq \frac{3c'n}{t}. \tag{6.27}$$

Also, by Lemma 6.5,

$$\log N\left(C_{3,1}, 2cL_K B_2^n\right) \leq c'\frac{n(\log\log n)}{\log n} \leq cn. \tag{6.28}$$

This concludes the proof. $\square$

Proposition 6.7. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t \geq 1$,*

$$N\left(\Psi_2(K), c_1 t\sqrt{\log n} L_K B_2^n\right) \leq e^{c_2 \frac{n}{t}} \text{ and} \tag{6.29}$$

$$N\left(V, c_3 L_K B_2^n\right) \leq e^{c_2 n}, \tag{6.30}$$

where $c_1, c_2, c_3 > 0$ are absolute constants.

Proof. We apply Lemma 6.5 for $A_1 := C_1$, $A_2 := C_2$ and $A_3 := C_3$ and we use Proposition 6.4 and Lemma 6.6. We work similarly for V. □

Proof of Theorem 6.1. The first assertion follows immediately from Proposition 6.7 (with $t = 1$) and the fact that for any pair of compact subsets A and B of $\mathbb{R}^n$, one has $|A| \leq N(A, B)|B|$.

The same argument shows that $|V|^{1/n} \leq cL_K|B_2^n|^{1/n}$. Consider the symmetric convex body

$$V_1 := \text{conv}\left\{\frac{Z_p(K)}{\sqrt{p}\sqrt{\log p}},\ p \in [2, n]\right\} \text{ and } V_2 := \text{conv}\left\{\frac{Z_p(K)}{\sqrt{p}\sqrt{\log p}},\ p \geq 2\right\}. \tag{6.31}$$

Note that, by (2.9), $V_1 \simeq V_2$. Then, $V_1 \subseteq cV$ and $|V_2|^{1/n} \leq cL_K|B_2^n|^{1/n}$. So, there exists $\theta \in S^{n-1}$ such that $h_{V_2}(\theta) \leq cL_K$. This implies that, for all $p \geq 1$,

$$h_{Z_p(K)}(\theta) \leq c\sqrt{p}\sqrt{\log p} L_K. \tag{6.32}$$

By Markov's inequality we have that, for every $p > 0$,

$$|\{x \in K : |\langle x, \theta\rangle| \geq e h_{Z_p(K)}(\theta)\}| \leq e^{-p}. \tag{6.33}$$

Let $t \geq 1$. If we define p by the equation $\sqrt{p} = \frac{t}{\sqrt{\log(t+1)}}$, then (6.32) and (6.33) imply (6.4). □

Proof of Theorem 1.1. The first part follows immediately from Theorem 6.1 and (2.15). The proof of the second part is similar to the proof of Theorem 6.1. We omit the details. □

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