A note on the Banach-Mazur distance to the cube

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Abstract

If X is an n-dimensional normed space and d denotes the Banach–Mazur distance, then $d(X, \ell_\infty^n) \leq cn^{5/6}$.

1 Introduction

If X, Y are n-dimensional spaces, we define the *Banach–Mazur distance* $d(X, Y)$ by

$$d(X, Y) = \inf\{\|T\| \, \|T^{-1}\| : \ T : X \to Y \text{ an isomorphism}\}.$$

A well-known theorem of F. John [J] asserts that for every n-dimensional normed space X we have $d(X, \ell_2^n) \leq \sqrt{n}$. This estimate is sharp as it can be seen by considering $X = \ell_\infty^n$ or ℓ_1^n.

Define

$$R_\infty^n = \max\{d(X, \ell_\infty^n) : \ X \text{ an } n\text{−dimensional space}\}.$$

John's theorem and the multiplicative triangle inequality for d imply the estimates $\sqrt{n} \leq R_\infty^n \leq n$. The question of determining the asymptotic behavior of R_∞^n as n tends to infinity was raised by A. Pelczynski [P].

S.J. Szarek [Sz.1] considered random spaces and proved that $R_\infty^n \geq c\sqrt{n}\log n$. That is, R_∞^n is not of the order of $\sqrt{n}$ (ℓ_∞^n is not an "asymptotic center" of the n-th Banach–Mazur compactum).

On the other hand, J. Bourgain and S.J. Szarek [BS] obtained the estimate $R_\infty^n = o(n)$, and S.J. Szarek – M. Talagrand [ST] improved this result to $R_\infty^n \leq cn^{7/8}$. A modification of their argument led this author [G] to the upper bound $R_\infty^n \leq cn^{0.859}$. In this note we report on some further progress in this direction:

Theorem 1. *There exists an absolute constant $c > 0$ such that*

$$R_\infty^n \leq cn^{5/6}.$$

Our proof follows again the method of Szarek–Talagrand. It depends on obtaining a result of the following type:

Proposition: *Let $X = (\mathbb{R}^n, \|.\|)$ and $\varepsilon \in (0,1)$. Suppose that the ellipsoid of minimal volume containing the unit ball B_X of X is the Euclidean unit ball D. Then, we can find vectors $z_1, \dots, z_m$ in X with $\|z_i\| = |z_i| = 1$ and $m \geq (1-\varepsilon)n$, such that for any reals $t_1, \dots, t_m$,*

$$|\sum_{i=1}^{m} t_i z_i| \geq c \frac{\varepsilon^d}{\sqrt{n}} \sum_{i=1}^{m} |t_i|,$$

where c, d are absolute positive constants.

In the next paragraph we prove that this Proposition holds true with $d = 1$ (the corresponding values of d in [ST], [G] were 1.5, 1.272 respectively). Using this information one can derive Theorem 1 (the argument is omitted, see [ST] or [Sz.2] for the details).

We use the standard notation from [MS]. By $|.|$ we denote the Euclidean norm and also the cardinality of a finite set. The letter c will denote an absolute positive constant, not necessarily the same in all its occurrences.

2 Proof of the Proposition

The proof is based on the following facts:

(I) *John's theorem and Dvoretzky–Rogers lemma*: If the ellipsoid of minimal volume containing B_X is the Euclidean unit ball D, then

(i) $D \subseteq \sqrt{n} B_X$,

(ii) there exist contact points $y_1, \dots, y_N$, $N = O(n^2)$, $\|y_i\| = |y_i| = 1$, and positive real numbers $\lambda_1, \dots, \lambda_N$ such that $x = \sum_{i \leq N} \lambda_i \langle x, y_i \rangle y_i$ for every $x \in \mathbb{R}^n$. It follows that, given $\varepsilon \in (0,1)$, one can choose $x_1, \dots, x_s$, $s \geq (1-\varepsilon)n$, among these contact points y_i, so that

Lemma 1. $\mathrm{dist}(x_i, \ \mathrm{span}\{x_j, \ j \neq i\}) \geq \sqrt{\varepsilon}$, $i = 1, \dots, s$.

Lemma 1 was introduced in [ST] in connection with the problem of the distance to the cube.

(II) *Sauer–Shelah lemma* [S], [Sh]: We shall make use of a special case:

Lemma 2. *If M is a subset of $\{-L, L\}^m$, $L > 0$, and $|M| \geq 2^{m-1}$, then we can find $\sigma \subseteq \{1, \dots, m\}$, $|\sigma| \geq \frac{m}{2}$, such that the restriction map*

$$P_\sigma : (\delta_j)_{j \leq m} \to (\delta_j)_{j \in \sigma}$$

sends M onto $\{-L, L\}^\sigma$.

An "isomorphic" version of Lemma 2 was the crucial lemma in [ST]. Our contribution consists of the following lemma, which we think is of independent interest.

Lemma 3. *Let* $u_1, \ldots, u_s \in \mathbb{R}^n, |u_i| \leq 1$. *Define the symmetric convex set*

$$E = \{(\delta_j)_{j \leq s} : |\sum_{j=1}^{s} \delta_j u_j|^2 \leq 2s\}.$$

Then, for every $\varepsilon \in (0,1)$ *there exists* $\sigma \subseteq \{1, \ldots, s\}$ *with* $|\sigma| \geq (1-\varepsilon)s$, *such that*

$$P_\sigma(E) \supseteq c\sqrt{\varepsilon}\,[-1,1]^\sigma,$$

where c *is an absolute positive constant.*

Notation: $S = \{1, \ldots, s\}$, $Q = [-1,1]^s$, $Q_\tau = [-1,1]^\tau$ if $\tau \subseteq S$.

$$\alpha_k = \sum_{r=0}^{k-1} 2^{r/2} \quad , \quad \beta_k = \sum_{r=0}^{k-1} 2^r.$$

Proof: Consider points of the form $(\delta_j^{(1)})_{j \leq s}$, $\delta_j^{(1)} = \pm 1$. By the parallelogram law,

$$\text{Ave}_{\delta_j^{(1)} = \pm 1} |\sum_{j=1}^{s} \delta_j^{(1)} u_j|^2 = \sum_{j=1}^{s} |u_j|^2 \leq s.$$

Using Markov's inequality we find $M^1 \subseteq \{-1,1\}^s$, $|M^1| \geq 2^{s-1}$, such that, for every $(\delta_j^{(1)}) \in M^1$,

$$|\sum_{j=1}^{s} \delta_j^{(1)} u_j|^2 \leq 2s.$$

From Lemma 2 we can find $\sigma_1 \subseteq S$, $|\sigma_1| \geq \frac{s}{2}$, such that $P_{\sigma_1}(M^1) = \{-1,1\}^{\sigma_1}$. Since $M^1 \subseteq E \cap Q$, it follows that

$$Q_{\sigma_1} \subseteq P_{\sigma_1}(E \cap Q).$$

We shall prove by induction the following:

(1) For $k = 1, 2, \ldots,$ we can find $\sigma_k \subseteq S, |\sigma_k| \geq (1 - \frac{1}{2^k})s$, such that

$$Q_{\sigma_k} \subseteq P_{\sigma_k}(\alpha_k E \cap \beta_k Q).$$

For $k = 1$ this follows from the previous inclusion.

Inductive step: Consider points of the form $\delta_j^{(k+1)}, j \leq s$, with $\delta_j^{(k+1)} = 0$ if $j \in \sigma_k$ and $\delta_j^{(k+1)} = \pm 2^{k/2}$ if $j \notin \sigma_k$. We then have

$$\text{Ave}_{(\delta_j^{(k+1)})_{j \leq s}} |\sum_{j=1}^{s} \delta_j^{(k+1)} u_j|^2 = \sum_{j \notin \sigma_k} 2^k |u_j|^2 \leq s.$$

Observing that the cardinality of the set of $(\delta_j^{(k+1)})_{j\leq s}$ is $2^{s-|\sigma_k|}$ and using Markov's inequality, we can find $M^{k+1} \subseteq [\mathbf{0}_{\sigma_k} \times \{-2^{k/2}, 2^{k/2}\}^{S\setminus\sigma_k}] \cap E$ with $|M^{k+1}| \geq 2^{s-|\sigma_k|-1}$. Then Lemma 2 enables us to find $\sigma_{k+1}^* \subseteq S\setminus\sigma_k$, $|\sigma_{k+1}^*| \geq \frac{1}{2}(s-|\sigma_k|)$, such that

$$P_{\sigma_k\cup\sigma_{k+1}^*}(M^{k+1}) = \mathbf{0}_{\sigma_k} \times \{-2^{k/2}, 2^{k/2}\}^{\sigma_{k+1}^*}.$$

Since $M^{k+1} \subseteq E \cap 2^{k/2}Q$, it follows that

$$\mathbf{0}_{\sigma_k} \times 2^k Q_{\sigma_{k+1}^*} \subseteq P_{\sigma_k\cup\sigma_{k+1}^*}(2^{k/2}E \cap 2^k Q).$$

Suppose that $a \in Q_{\sigma_k}$, $b \in Q_{\sigma_{k+1}^*}$. From our inductive hypothesis we can find $w_a \in \beta_k Q_{\sigma_{k+1}^*}$ such that

$$(a, w_a) \in P_{\sigma_k\cup\sigma_{k+1}^*}(\alpha_k E \cap \beta_k Q).$$

Define $v_{a,b} = b - w_a$. It is clear that $v_{a,b} \in (\beta_k+1)Q_{\sigma_{k+1}^*} = 2^k Q_{\sigma_{k+1}^*}$, hence

$$(\mathbf{0}_{\sigma_k}, v_{a,b}) \in P_{\sigma_k\cup\sigma_{k+1}^*}(2^{k/2}E \cap 2^k Q).$$

Then,

$$\begin{aligned}
(a,b) &= (a, w_a) + (\mathbf{0}_{\sigma_k}, v_{a,b}) \\
&\in P_{\sigma_k\cup\sigma_{k+1}^*}(\alpha_k E \cap \beta_k Q) + P_{\sigma_k\cup\sigma_{k+1}^*}(2^{k/2}E \cap 2^k Q) \\
&\subseteq P_{\sigma_k\cup\sigma_{k+1}^*}(\alpha_{k+1} E \cap \beta_{k+1} Q).
\end{aligned}$$

It follows that

$$Q_{\sigma_k\cup\sigma_{k+1}^*} \subseteq P_{\sigma_k\cup\sigma_{k+1}^*}(\alpha_{k+1} E \cap \beta_{k+1} Q).$$

Set $\sigma_{k+1} = \sigma_k \cup \sigma_{k+1}^*$. It is easy to see that $|\sigma_{k+1}| \geq (1-\frac{1}{2^{k+1}})s$, and the inductive step is completed.

From (1) we get

(2) For $k = 1, 2, \ldots,$ we can find $\sigma_k \subseteq S, |\sigma_k| \geq (1-\dfrac{1}{2^k})s,$ such that

$$[-1,1]^{\sigma_k} \subseteq P_{\sigma_k}(\frac{2^{k/2}}{\sqrt{2}-1}E).$$

So,

$$P_{\sigma_k}(E) \supseteq c\sqrt{\frac{1}{2^k}}\,[-1,1]^{\sigma_k}, \qquad c = \sqrt{2}-1.$$

Then we easily pass to the continuous version of the lemma (with a slightly worse constant c). □

Example (S.J. Szarek [Sz.3]). Let $n = s+1$ and $u_i = \frac{1}{\sqrt{2}}(e_i + e_n), i = 1, \ldots, s$. Here $\{e_i\}_{i\leq n}$ is the canonical orthonormal basis of $\mathbb{R}^n$. Then,

$$|\sum_{j=1}^s \delta_j u_j|^2 = \frac{1}{2}\left[\sum_{j=1}^s \delta_j^2 + (\sum_{j=1}^s \delta_j)^2\right],$$

and this implies that a necessary condition for $(\delta_j)_{j\le s}$ to be in E is

$$\sum_{j=1}^{s}\delta_j^2 \le 4s \quad \text{and} \quad |\sum_{j=1}^{s}\delta_j| \le 2\sqrt{s}.$$

Given $\varepsilon \in (0,1)$, consider any $\sigma \subseteq \{1,\dots,s\}$, $|\sigma| = m \ge (1-\varepsilon)s$. Then, a point $(t,t,\dots,t)$ is in $P_\sigma(E)$ only if we can find $(\delta_j)_{j\notin\sigma}$ such that

$$mt^2 + \sum_{j\notin\sigma}\delta_j^2 \le 4s \quad \text{and} \quad |mt + \sum_{j\notin\sigma}\delta_j| \le 2\sqrt{s},$$

and using the Cauchy-Schwarz inequality one can see that this is possible only if $|t| \le c\sqrt{\varepsilon}$.

This example shows that Lemma 3 cannot be improved. A version of this lemma (with a weaker dependence on ε) appeared in [G].

Now we can pass to the

Proof of the Proposition. According to Lemma 1, we can choose $x_1,\dots,x_s \in B_X$, with $s \ge (1-\frac{\varepsilon}{2})n$, such that

$$\text{dist}\,(x_i,\ \text{span}\{x_j, j\ne i\}) \ge \sqrt{\frac{\varepsilon}{2}},\ \ i,j = 1,\dots,s.$$

Then, we can find $v_i \perp \text{span}\{x_j,\ j\ne i\}$ so that $\langle x_i, v_i\rangle = 1$ and $|v_i| \le \sqrt{\frac{2}{\varepsilon}}$. That is, there exist $v_1,\dots,v_s \in \mathbb{R}^n$ for which

$$|v_i| \le \sqrt{\frac{2}{\varepsilon}} \quad \text{and} \quad \langle x_i, v_j\rangle = \delta_{ij},\ \ i,j = 1,\dots,s.$$

Set $u_i = \sqrt{\frac{\varepsilon}{2}}\, v_i$ and apply Lemma 3 to obtain $\sigma \subseteq \{1,\dots,s\}$, $|\sigma| \ge (1-\frac{\varepsilon}{2})s$, with

$$P_\sigma(E) \supseteq c\sqrt{\varepsilon}\,[-1,1]^\sigma.$$

Obviously, $|\sigma| \ge (1-\varepsilon)n$. Now, for any sequence $(t_i)_{i\in\sigma}$ of reals, one has

$$\sum_{i\in\sigma}|t_i| = \langle \sum_{i\in\sigma} t_i x_i, \sum_{j\in\sigma}\text{sign}(t_j)v_j \rangle.$$

Since $(c\sqrt{\varepsilon}\text{sign}(t_j))_{j\in\sigma} \in P_\sigma(E)$, we can find $(\delta_j)_{j\le s}$ in E so that $\delta_j = c\sqrt{\varepsilon}\,\text{sign}(t_j)$ for $j\in\sigma$. Observe that whenever $i\in\sigma$ and $j\notin\sigma$ then $\langle x_i, v_j\rangle = 0$, and therefore

$$\begin{aligned}\langle \sum_{i\in\sigma} t_i x_i, \sum_{j\in\sigma}\text{sign}(t_j)v_j \rangle &= \frac{1}{c\sqrt{\varepsilon}}\,\langle \sum_{i\in\sigma} t_i x_i, \sum_{j=1}^{s}\delta_j v_j \rangle \\ &\le \frac{1}{c\sqrt{\varepsilon}}\,|\sum_{i\in\sigma} t_i x_i|\,\sqrt{\frac{2}{\varepsilon}}\,|\sum_{j=1}^{s}\delta_j u_j|\end{aligned}$$

$$\leq \frac{2\sqrt{s}}{c\varepsilon} \, |\sum_{i\in\sigma} t_i x_i| \leq \frac{\sqrt{n}}{c'\varepsilon} \, |\sum_{i\in\sigma} t_i x_i|.$$

Choose z_i, $i = 1, \ldots, |\sigma| = m$, to be those x_j for which $j \in \sigma$, and the Proposition is proved. □

3 Remark

As was mentioned in [Sz.2], another consequence of the Proposition is the following "proportional Dvoretzky–Rogers factorization" result (the proof of which is a word-by-word repetition of the argument given in [ST]):

Theorem 2 *If $\varepsilon \in (0,1)$ and X is an n-dimensional normed space, there exist vectors $x_1, \ldots, x_m \in X$, $m \geq (1-\varepsilon)n$, such that for any reals $t_1, \ldots, t_m$,*

$$\max_{1\leq j\leq m} |t_j| \leq \| \sum_{j=1}^{m} t_j x_j \|_X \leq \frac{c}{\varepsilon^{3/2}} (\sum_{j=1}^{m} |t_j|^2)^{1/2},$$

where $c > 0$ is an absolute constant. Equivalently, the formal identity $i_{2,\infty} : \ell_2^m \to \ell_\infty^m$ can be written as $i_{2,\infty} = \alpha o \beta$, $\beta : \ell_2^m \to X$, $\alpha : X \to \ell_\infty^m$, with $\|\alpha\| \, \|\beta\| \leq c/\varepsilon^{3/2}$. The same holds true for $i_{1,2} : \ell_1^m \to \ell_2^m$. □

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