Random polytopes generated by contoured distributions

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Abstract

Our work is a further investigation on the connection between probability and geometry. We extend several known results from convex bodies and log-concave measures to the setting of contoured probability distributions as we study the relation of several parameters of a convex body, a profile function and the resulting contoured distribution, where log-concavity is not required.

1 Introduction

The connection between information theory and geometry has been known and studied since the late 70's. Lieb showed in [23] that Shannon power inequality and Brunn-Minkowski inequality can be both derived from the sharp Young's inequality proved by Beckner [4]. Brascamp and Lieb [8] (see also Costa and Cover [12], and later Cover, Dembo and Thomas [13]) proved an analogy between the isoperimetric inequality in geometry and an inequality for the Fisher information and the entropy.

Our study is motivated by the work of Guleryuz, Lutwak, Yang and Zhang [18], where they established further this connection by introducing a new class of probability distributions, the *contoured distributions*. The level sets of the probability density of such a distribution are dilates of a star-shaped set K in $\mathbb{R}^n$ that contains the origin in its interior, called the *contoured body* of the distribution. On the other hand, the density function f of a contoured distribution can be expressed in the form $f(x) = \phi(\|x\|_K)$ where $\|\cdot\|_K$ is the Minkowski functional of the contoured body K and the function $\phi : [0,\infty) \to [0,\infty)$ is the so called *radial profile* of the distribution. Note that contoured distributions whose contoured body is a centered ellipsoid are known as elliptically contoured distributions.

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Based on the aforementioned density decomposition, Guleryuz, Lutwak, Yang and Zhang showed in [18] that information theoretic invariants of a contoured distribution are closely related to geometric invariants of its contoured body. Moreover, they characterized the contoured distributions with extremal entropy in terms of its radial profile and contour body, showing a clear link between geometric and information theoretical inequalities.

In this article we carry on the study of contoured probability distributions on $\mathbb{R}^n$, along with their geometric aspects, from an asymptotic point of view as $n \to \infty$. At the same time we show how a contoured distribution could be an example of a non log-concave distribution with log-concave properties.

In section 3 we further investigate the relation between a contoured distribution and its defining convex contoured body and radial profile function. We examine probabilistic and geometric quantities of a contoured distribution such as the moments, its L_q-centroid bodies and its isotropic constant (see in section 2 for the precise definitions). We relate them to the corresponding geometric quantities of the contoured convex body of the distribution and as a first application, we prove (see Theorem 3.7) the famous Paouris large deviation inequality [28] (a main property of log-concave distributions) for contoured distributions.

In the next sections we show how several known results for random convex polytopes can be extended from the setting of log-concave distributions to the setting of contoured distributions. A symmetric random polytope in $\mathbb{R}^n$ is the convex hull of $2N$ random vectors $\pm X_1, \dots, \pm X_N$ in $\mathbb{R}^n$, where N is a positive integer, and is denoted as $K_N = \operatorname{conv}\{\pm X_1, \dots, \pm X_N\}$. In earlier results, see for example [15], [16], [24], [14], the points whose convex hull forms the random polytope are chosen uniformly from an isotropic convex body, or more generally have an isotropic log-concave probability distribution. Moving to the class of contoured distributions, log-concavity is no more a necessity. The radial profile function that defines the contoured distribution may not be log-concave (see below for the definitions) and consequently the corresponding contoured distribution is not logarithmically concave. We give the simple example of the functions $\phi(t) = e^{-t^a}$ for $0 < a < 1$, that define non-log-concave contoured distributions, and (after a slight modification) satisfy the restrictions that are required by our theorems (see the remark in Section 3.3).

In Section 4, we trace all the required restrictions needed in our setting, so that the known results for the geometric parameters of a random polytope K_N with N log-concave vertices remain valid when instead we have a contoured distribution. In particular, Theorem 4.1 asserts that, under mild assumptions,

$$P\Big(c_1 Z_q(X) \subseteq K_N\Big) \geq 1 - \exp\left(-c_3\, n^{\beta} N^{1-\beta}\right) - P\left(\|\Gamma\| \geq \gamma \frac{L_X}{\sup f_X^{1/n}} \sqrt{N}\right),$$

for any $q \leq c_2 \log(N/n)$, where $c_2 > 0$ is a constant depending only on the radial profile of the contured distribution with density f_X, c_1, c_3 and γ are absolute positive constants, $X_1, \dots, X_N$ are independent random vectors identically distributed according to the contoured probability density f_X, L_X is their isotropic constant and $\|\Gamma\|$ is the operator norm of the random operator $\Gamma : \ell_2^n \to \ell_2^N$ with $\Gamma x = \big(\langle X_1, x\rangle, \dots, \langle X_N, x\rangle\big)$, $x \in \mathbb{R}^n$.

In Section 5 we investigate the relation between a contoured distribution and its defining convex contoured body and radial profile function, in order to provide asymptotic estimates for basic geometric parameters of a contoured random polytope. More precisely, for the mean width of K_N we show that (Theorem 5.1) $\mathbb{E}\big[w(K_N)\big] \le c\, w\big(Z_{\log N}(X)\big)$, where $c>0$ is an absolute constant, and for the volume radius $\mathrm{vr}(K_N)$ we show that (Theorem 5.5)

$$P\left(\mathrm{vr}\big(K_N\big) \le c\sqrt{\log N}\, I_{\log N}(X_\phi) V(K)^{1/n} L_K\right) \ge 1 - \frac{1}{2N}.$$

For the notation used above we refer the reader to Section 2 and Section 3 of the paper. In our proofs, we extend a number of useful facts, in our setting, which may be interesting on their own.

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2 Notation and background material

We work in $\mathbb{R}^n$, equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $|\cdot|$ the corresponding Euclidean norm, B_2^n stands for the Euclidean unit ball, and $\mathbb{S}^{n-1}$ for the unit sphere. The volume of a set $A \subseteq \mathbb{R}^n$ is denoted by $V(A)$. The volume of B_2^n is simply denoted by ω_n. The volume radius of A is denoted by $\mathrm{vr}(A)$ and it is the radius of the Euclidean ball having the same volume as A, i.e. $\mathrm{vr}(A) = \big(V(A)/\omega_n\big)^{1/n}$. We write $\sigma = \sigma_n$ for the rotationally invariant probability measure on $\mathbb{S}^{n-1}$ and $d\theta = n\omega_n d\sigma(\theta)$ denotes integration with respect to the Lebesgue measure on $\mathbb{S}^{n-1}$. For any integrable function $f:\mathbb{R}^n \to \mathbb{R}$ we write $\int_{\mathbb{R}^n} f(x)\,dx$ for the Lebesgue integral of f. For any Borel measurable subset A of $\mathbb{R}^n$ we denote its indicator function by $\mathbb{1}_A$ and we write $\int_A f(x)\,dx = \int_{\mathbb{R}^n} f(x)\mathbb{1}_A(x)\,dx$ for the Lebesgue integral of f over A. Integration in polar coordinates gives that

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{S}^{n-1}}\int_0^\infty f(t\theta)\,t^{n-1}dt\,d\theta = n\omega_n \int_{\mathbb{S}^{n-1}}\int_0^\infty f(t\theta)\,t^{n-1}dt\,d\sigma(\theta).$$

The letters c, c_1, c_2 etc, denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \le b \le c_2 a$. Similarly, if $A, B \subseteq \mathbb{R}^n$ we write $A \simeq B$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 A \subseteq B \subseteq c_2 A$.

Let K be a subset of $\mathbb{R}^n$. We say that K is centered if its center of mass lies at the origin i.e. $\int_K \langle x, \theta\rangle dx = 0$ for every $\theta \in \mathbb{S}^{n-1}$, and that K is symmetric if $-x \in K$ whenever $x \in K$. Moreover, we say that K is a convex body if it is a convex compact subset of $\mathbb{R}^n$ with non-empty interior, and that K is star shaped if $\lambda x \in K$ whenever $x \in K$ and $\lambda \in [0,1]$. The Minkowski functional of a star shaped set $K \subseteq \mathbb{R}^n$ is the function

$$\|x\|_K := \inf\{\lambda > 0 : \, x \in \lambda K\}, \quad x \in \mathbb{R}^n.$$

It is not hard to see that $\|T^{-1}x\|_K = \|x\|_{TK}$ for every $T \in GL(n)$, where $GL(n)$ is the set of all invertible linear maps on $\mathbb{R}^n$. The Minkowski functional of a symmetric convex body K in $\mathbb{R}^n$ is a norm in $\mathbb{R}^n$ and, vice versa, any norm $\|\cdot\|$ in $\mathbb{R}^n$ defines a symmetric convex body K in $\mathbb{R}^n$, namely, $K = \{x \in \mathbb{R}^n : \, \|x\| \leq 1\}$. The support function of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\}, \quad x \in \mathbb{R}^n,$$

and $w(K)$ stands for the mean width of K, that is the average of h_K over the unit sphere:

$$w(K) = \int_{\mathbb{S}^{n-1}} h_K(\theta)\, d\sigma(\theta).$$

If the origin is an interior point of K then its polar body K^o is the convex body in $\mathbb{R}^n$ defined by $K^o := \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in K\}$. Note that $h_K(x) = \|x\|_{K^o}$ and $h_{K^o}(x) = \|x\|_K$ for all $x \in \mathbb{R}^n$.

A Borel random vector X in $\mathbb{R}^n$ is called logarithmically concave (or log-concave) if its probability distribution $\mu_X(A) = P(X \in A)$ is a log-concave measure in $\mathbb{R}^n$, that is

$$\mu_X\big((1-\lambda)A + \lambda B\big) \geq \mu_X(A)^{1-\lambda}\mu_X(B)^{\lambda}$$

for all compact $A, B \subseteq \mathbb{R}^n$ and any $\lambda \in (0,1)$. A function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if

$$f\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} f(y)^{\lambda},$$

for all $x, y \in \mathbb{R}^n$ and any $\lambda \in (0,1)$. C. Borell proved in [6] that every Borel random vector X which is log-concave and satisfies $\mu_X(E) = P(X \in E) < 1$ for every hyperplane E of $\mathbb{R}^n$ is absolutely continuous (with respect to Lebesgue measure) and its density is log-concave. Note that if K is a convex body in $\mathbb{R}^n$ then the random vector X_K with density $f_K = \frac{1}{V(K)}\mathbb{1}_K$ is log-concave. We refer to the books [17], [31] and [30] for more details on the Brunn-Minkowski theory and the geometry of convex bodies in $\mathbb{R}^n$.

For a random vector X in $\mathbb{R}^n$, we write $X \sim f$ if X is absolutely continuous with respect to the Lebesgue measure and f is its probability density function. Moreover, we say that X is centered if $\mathbb{E}X = 0$. We also use the notation $X \overset{d}{=} Y$ to denote that the random vectors X, Y have the same distribution.

Let X be a Borel centered random vector in $\mathbb{R}^n$ with a probability density function f_X. The *covariance matrix* $\mathrm{Cov}(X) = \mathscr{C}_X = \mathbb{E}\big[X \otimes X\big]$ of X is the $n \times n$ matrix with entries

$$\mathrm{Cov}_{ij}(X) = \mathbb{E}[X_i X_j] = \int_{\mathbb{R}^n} x_i x_j f_X(x) dx.$$

For any $p \geq 1$, the *p-th moment* of the Euclidean norm of X is defined by

$$I_p(X) = \big(\mathbb{E}|X|^p\big)^{1/p} = \left(\int_{\mathbb{R}^n} |x|^p f_X(x) dx\right)^{1/p}.$$

It is well khown that for a log-concave random vector X in $\mathbb{R}^n$, one can extend the definition of $I_p(X)$ for any $p > -n$.

Note that by Hölder's inequality $I_p(X)$ is a non decreasing function of p, and under the extra assumption that X is log-concave, a reverse estimate holds true (see [7], [5], [10] and [27]), that is for any $1 \leq p \leq q$ one has that

$$I_p(X) \subseteq I_q(X) \subseteq c\, \frac{q}{p}\, I_p(X). \tag{1}$$

The *L_p-centroid body* $Z_p(X)$ of X, $p \geq 1$, is the symmetric convex body in $\mathbb{R}^n$ with support function

$$h_{Z_p(X)}(y) = \big(\mathbb{E}|\langle X, y\rangle|^p\big)^{1/p} = \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^p f_X(x)\, dx\right)^{1/p}, \quad y \in \mathbb{R}^n.$$

The L_2-centroid body $Z_2(X)$ is the Legendre ellipsoid $\mathcal{E}_{\mathscr{C}_X}$ of X, which is defined by the covariance matrix of X by $\mathcal{E}_{\mathscr{C}_X} = \mathscr{C}_X^{1/2} B_2^n$. The formula

$$Z_p(TX) = T\big(Z_p(X)\big), \tag{2}$$

holds true for any $T \in GL(n)$ and $p \geq 1$; this follows directly from the definition of $Z_p(X)$. Hölder's inequality shows that $Z_p(X) \subseteq Z_q(X)$, for any $1 \leq p \leq q$, while, if we additionally assume that X is log-concave, then Borell's lemma (see [25]) implies a reverse inclusion, and so in that case we have

$$Z_p(X) \subseteq Z_q(X) \subseteq c\, \frac{q}{p}\, Z_p(X), \tag{3}$$

for any $1 \leq p \leq q$.

The *isotropic constant* L_X of X, is defined by the formula

$$L_X := \left(\sup_{x \in \mathbb{R}^n} f_X(x)\right)^{1/n} \big|\det \mathrm{Cov}(X)\big|^{1/2n}.$$

It is an affine invariant of X and does not depend on the choice of the Euclidean structure. A random vector X in $\mathbb{R}^n$ is called *isotropic* if it is centered and there exists a constant $\alpha_X > 0$ such that

$$\mathrm{Cov}(X) = \alpha_X^2 I_n, \tag{4}$$

where I_n is the identity map in $\mathbb{R}^n$. In this case, $\alpha_X = \big|\det \mathrm{Cov}(X)\big|^{1/2n}$. Note that X is isotropic and satisfies (4) if and only if the ellipsoid $Z_2(X)$ is the Euclidean ball $\alpha_X B_2^n$. Moreover, one can check that every centered random vector X on $\mathbb{R}^n$ has an isotropic linear image, i.e., there exists $T \in GL(n)$ such that TX is isotropic.

For any centered convex body K in $\mathbb{R}^n$, let X_K be the log-concave random vector in $\mathbb{R}^n$ with density

$$f_K(x) = \frac{1}{V(K)} \mathbb{1}_K(x), \quad x \in \mathbb{R}^n. \tag{5}$$

Then, $(\sup f_K)^{1/n} = 1/V(K)^{1/n}$ and we write $\mathrm{Cov}(K) = \mathrm{Cov}(X_K)$, $I_p(K) = I_p(X_K)$ and $Z_p(K) = Z_p(X_K)$, $p \geq 1$. Thus, as a special case of (1) we have that

$$I_p(K) \leq I_q(K) \leq c\frac{q}{p} I_p(K),$$

for any $1 \leq p \leq q$. Note also that since X_K is a log-concave random vector, $I_p(K)$ is defined for any $p > -n$.

The isotropic constant of K is defined by

$$L_K = L_{X_K} = \frac{1}{V(K)^{1/n}} \left| \det \mathrm{Cov}(K) \right|^{1/2n}. \tag{6}$$

We call K *isotropic* if and only if there exists a constant $\alpha_K > 0$ such that

$$\mathrm{Cov}(K) = \alpha_K^2 I_n, \tag{7}$$

where $\alpha_K = \left| \det \mathrm{Cov}(K) \right|^{1/2n}$.

Remark. Usually, in the bibliography, for an isotropic random vector X it is additionally assumed that $\mathrm{Cov}(X) = I_n$, that is $\alpha_X = 1$, and this normalization implies that $L_X = \sup_{x \in \mathbb{R}^n} f_X(x)^{1/n}$. Moreover, for an isotropic convex body K it is additionally assumed that $V(K) = 1$, that is $\sup f_K = 1$, which gives that $L_K = a_K = \left| \det \mathrm{Cov}(K) \right|^{1/2n}$. In this paper we prefer to use the general normalization introduced in (4) and (7), since it suits better in our case.

Paouris proved in [27] that, for any $1 \leq p \leq \sqrt{n}$, the p-th moment $I_p(X)$ of an isotropic and log-concave random vector X in $\mathbb{R}^n$ is equivalent to $I_2(X) = \sqrt{n}\alpha_X$ up to an absolute constant. A consequence of this result is the celebrated Paouris's deviation inequality: If X is an isotropic log-concave random vector X in $\mathbb{R}^n$, then

$$P\Big(|X| \geq c\sqrt{n}\alpha_X t\Big) \leq e^{-\sqrt{n}t}, \tag{8}$$

for every $t \geq 1$.

A well-known open question in the theory of isotropic log-concave measures, is the hyperplane conjecture, which asks if there exists an absolute constant $C > 0$ such that

$$L_n := \sup \left\{ L_X : X \text{ is an isotropic log-concave random vector on } \mathbb{R}^n \right\} \leq C,$$

for all $n \geq 2$. Bourgain proved in [7] that $L_n \leq c\sqrt[4]{n}\log n$ and later Klartag [19] obtain the bound $L_n \leq c\sqrt[4]{n}$ (see also [22]). Chen [11] in a breakthrough work proved that for any $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $L_n \leq n^\varepsilon$ for every $n \geq n_0(\varepsilon)$. Then, Klartag and Lehec [21] showed that $L_n \leq c(\log n)^4$, and the best known bound until now is due to Klartag [20], who proved that $L_n \leq c\sqrt{\log n}$, where $c > 0$ is an absolute constant.

The reader may find more details about isotropic log-concave measures in the book [9]. We also refer to the book [3] for an exposition of the asymptotic theory of finite dimensional normed spaces.

3 Contoured distributions

A distribution in $\mathbb{R}^n$ is called contoured if it is absolutely continuous and its probability density function f admits a decomposition of the form

$$f(x) = c\phi\big(\lambda(x)\big),$$

where c is a positive constant, $\phi : [0, +\infty) \to [0, +\infty)$ is an integrable function and $\lambda : \mathbb{R}^n \to [0, \infty)$ is a *shape function*, which means that it is positive away from 0 and positively homogeneous i.e. $\lambda(tx) = t\lambda(x)$ for any $t > 0$ and $x \in \mathbb{R}^n$.

Associated to any shape function λ is the compact star-shaped body

$$K_\lambda = \big\{x \in \mathbb{R}^n : \lambda(x) \leq 1\big\}.$$

Conversely, associated to any compact star-shaped body K, called a *contoured body*, is the shape function defined by the Minkowski functional of the contoured body K, i.e.

$$\lambda_K(x) = \inf\big\{t > 0 : x \in tK\big\}.$$

Then, there exists an one-to-one correspondence between shape functions and contoured bodies which leads us to the following notation.

We say that a random vector has a contoured distribution in $\mathbb{R}^n$ if it has a density function of the form

$$f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi(\|x\|_K), \quad x \in \mathbb{R}^n, \tag{9}$$

where K is a star-shaped set in $\mathbb{R}^n$ and $\phi : [0, +\infty) \to [0, +\infty)$ is an integrable function such that $\phi(|x|)$ is a density in $\mathbb{R}^n$, i.e.

$$n\omega_n \int_0^\infty t^{n-1}\,\phi(t)\,dt = 1. \tag{10}$$

The body K is called the *contoured body* of the distribution, and ϕ is called the *radial profile* function of the distribution.

In the rest of the paper we only consider distributions with a contoured body K which is a centered convex body in $\mathbb{R}^n$. We write $X_{K,\phi}$ for a contoured random vector with density given by (9). In the special case where $K = B_2^n$, we simplify the notation by writing X_ϕ and f_ϕ instead of $X_{B_2^n,\phi}$ and $f_{B_2^n,\phi}$. We refer to the paper [18] for more details about the definition of a contoured distribution and the normalization (10) of its radial profile function.

3.1 Moments

First we note that the action of a linear transformation on a contoured random vector is carried over to the contoured body of the distribution.

Proposition 3.1. *Let $X_{K,\phi} \sim f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi(\|x\|_K)$ and $T \in GL(n)$. Then,*

$$TX_{K,\phi} \stackrel{d}{=} X_{TK,\phi}. \tag{11}$$

Proof. For any random vector $X \sim f_X$ in $\mathbb{R}^n$ and $T \in GL(n)$ we have that

$$TX \sim f_{TX}(x) = \frac{1}{|\det T|}\, f_X \circ T^{-1}(x).$$

Thus,

$$\begin{aligned} f_{TX_{K,\phi}}(x) &= \frac{1}{|\det T|}\, f_{X_{K,\phi}}(T^{-1}x) = \frac{1}{|\det T|}\frac{\omega_n}{V(K)}\phi(\|T^{-1}x\|_K) \\ &= \frac{\omega_n}{V(TK)}\phi(\|x\|_{TK}) = f_{X_{TK,\phi}}(x). \end{aligned}$$ □

The next proposition is a result from [18], which describes the splitting of the covariance matrix of a contoured distribution. For the reader's convenience, we present its simple proof.

Proposition 3.2 ([18])**.** *Let $X_{K,\phi} \sim f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi(\|x\|_K)$. Then,*

$$\mathrm{Cov}(X_{K,\phi}) = \frac{n+2}{n}\, I_2(X_\phi)^2\,\mathrm{Cov}(K) \simeq I_2(X_\phi)^2\,\mathrm{Cov}(K). \tag{12}$$

Proof. For every i, $j = 1, \ldots, n$, integration in polar coordinates implies that

$$\begin{aligned} \mathrm{Cov}_{ij}(X_{K,\phi}) &= \frac{\omega_n}{V(K)}\int_{\mathbb{R}^n} x_i x_j \phi(\|x\|_K)\, dx \\ &= \frac{\omega_n}{V(K)}\int_{S^{n-1}}\int_0^\infty \theta_i\theta_j\phi(t\|\theta\|_K)t^{n+1}\, dtd\theta \\ &= \frac{\omega_n}{V(K)}\int_{S^{n-1}}\int_0^\infty \frac{\theta_i\theta_j}{\|\theta\|_K^{n+2}}\,\phi(s)\, s^{n+1}\, dsd\theta \\ &= \omega_n\int_0^\infty \phi(s)\, s^{n+1}\, ds\ \ \frac{1}{V(K)}\int_{S^{n-1}}\frac{\theta_i\theta_j}{\|\theta\|_K^{n+2}}\, d\theta \\ &= \frac{n+2}{n}\mathbb{E}|X_\phi|^2\,\mathrm{Cov}_{ij}(K), \end{aligned}$$

since

$$(n+2)\,\mathrm{Cov}_{ij}(K) = \frac{n+2}{V(K)}\int_K x_i x_j dx = \frac{1}{V(K)}\int_{S^{n-1}} \frac{\theta_i\theta_j}{\|\theta\|_K^{n+2}}\,d\theta. \qquad \square$$

A main observation of this paper is that the same phenomenon also occurs for the p-th moments and the L_p-centroid bodies of a contoured random vector. They both split into two parts, one for the radial profile and one for the contoured body of the distribution.

Proposition 3.3. *Let $X_{K,\phi} \sim f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi(\|x\|_K)$. Then, for any $p \geq 1$*

$$I_p(X_{K,\phi}) = \left(\frac{n+p}{n}\right)^{1/p} I_p(X_\phi)\, I_p(K) \simeq I_p(X_\phi)\, I_p(K). \tag{13}$$

Proof. Note that for the special contoured random vector $X_\phi \sim f_\phi(x) = \phi(|x|)$, a direct computation in polar coordinates gives us that

$$I_p(X_\phi) = (\mathbb{E}|X_\phi|^p)^{1/p} = \left(n\omega_n \int_0^\infty t^{n+p-1}\,\phi(t)\,dt\right)^{1/p}, \tag{14}$$

for any $p \geq 1$. Also, integrating in polar coordinates we see that

$$\begin{aligned}(n+p)\,I_p(K)^p &= \frac{n+p}{V(K)}\int_K |x|^p dx = \frac{n+p}{V(K)}\int_{S^{n-1}}\int_0^{\frac{1}{\|\theta\|_K}} t^{n+p-1}\,dt d\theta \\ &= \frac{1}{V(K)}\int_{S^{n-1}} \frac{1}{\|\theta\|_K^{n+p}}\,d\theta.\end{aligned}$$

Thus,

$$\begin{aligned}I_p(X_{K,\phi})^p &= \frac{\omega_n}{V(K)}\int_{\mathbb{R}^n} |x|^p \phi(\|x\|_K)dx \\ &= \frac{\omega_n}{V(K)}\int_{S^{n-1}}\int_0^\infty \phi(t\|\theta\|_K)t^{n+p-1}dtd\theta \\ &= \omega_n\int_0^\infty \phi(s)s^{n+p-1}ds\ \frac{1}{V(K)}\int_{S^{n-1}}\frac{1}{\|\theta\|_K^{n+p}}d\theta \\ &= \frac{\mathbb{E}|X_\phi|^p}{n}(n+p)\,I_p(K)^p. \qquad \square\end{aligned}$$

Proposition 3.4. *Let $X_{K,\phi} \sim f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi(\|x\|_K)$. Then, for any $p \geq 1$*

$$Z_p(X_{K,\phi}) = \left(\frac{n+p}{n}\right)^{1/p} I_p(X_\phi)\, Z_p(K) \simeq I_p(X_\phi)\, Z_p(K). \tag{15}$$

Proof. We compute

$$
\begin{aligned}
(n+p)\, h^p_{Z_p(K)}(x) &= \frac{n+p}{V(K)} \int_K |\langle x, y\rangle|^p \, dy \\
&= \frac{n+p}{V(K)} \int_{S^{n-1}} \int_0^{\frac{1}{\|\theta\|_K}} |\langle x, \theta\rangle|^p \, t^{n+p-1} \, dt d\theta \\
&= \frac{1}{V(K)} \int_{S^{n-1}} \frac{|\langle x, \theta\rangle|^p}{\|\theta\|_K^{n+p}} \, d\theta.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
h^p_{Z_p(X_{K,\phi})}(x) &= \frac{\omega_n}{V(K)} \int_{\mathbb{R}^n} |\langle x, y\rangle|^p \phi(\|x\|_K) \, dy \\
&= \frac{\omega_n}{V(K)} \int_{S^{n-1}} \int_0^{\infty} |\langle x, \theta\rangle|^p \phi(t\|\theta\|_K) \, t^{n+p-1} \, dt \, d\theta \\
&= \omega_n \int_0^{\infty} \phi(s) \, s^{n+p-1} \, ds \; \frac{1}{V(K)} \int_{S^{n-1}} \frac{|\langle x, \theta\rangle|^p}{\|\theta\|_K^{n+p}} \, d\theta \\
&= \frac{I_p(X_\phi)^p}{n} \, (n+p) \, h^p_{Z_p(K)}(x).
\end{aligned}
$$

□

3.2 Isotropicity

Here we study the concept of isotropicity of a contoured distribution and its relation with the isotropicity of the corresponding contoured body.

Let K be a centered convex body in $\mathbb{R}^n$ and $X = X_{K,\phi}$ be a contoured random vector with density function $f_{K,\phi}(x) = \frac{\omega_n}{V(K)} \phi(\|x\|_K)$, $x \in \mathbb{R}^n$. We consider the symmetric positive definite linear maps generated by the covariance matrices of X and K,

$$
R_X y = \mathbb{E}\big[\langle y, X\rangle X\big] = \frac{\omega_n}{V(K)} \int_{\mathbb{R}^n} \langle y, x\rangle x \, \phi\big(\|x\|_K\big) dx
$$

and

$$
R_K y = \frac{1}{V(K)} \int_K \langle y, x\rangle x \, dx,
$$

for any $y \in \mathbb{R}^n$ and set $S_X = R_X^{-1/2}$ and $S_K = R_K^{-1/2}$. Then, $S_X X$ and $S_K K$ are isotropic images of X and K respectively. We refer the reader to [9] for more details on this well known fact about the isotropic image of a random vector or a convex body. Moreover, by Proposition 3.2,

$$
S_X = \sqrt{\frac{n}{n+2}} \, I_2(X_\phi)^{-1} \, S_K \tag{16}
$$

and then, by the definition (7) of the isotropicity of convex bodies, we see that $S_X K$ is also isotropic and $L_{S_X K} = L_{S_K K} = L_K$. Thus, Proposition 3.1 and the above discussion prove the following.

Proposition 3.5. *Let* $X_{K,\phi} \sim f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi(\|x\|_K)$. *There exists* $T \in GL(n)$ *such that* $TX_{K,\phi} \overset{d}{=} X_{TK,\phi}$ *is isotropic and* TK *is also isotropic.*

We finally state the formula that relates the isotropic constant of a contoured distribution with the one of the corresponding contoured body.

Proposition 3.6. *Let* $X_{K,\phi} \sim f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi(\|x\|_K)$. *Then*

$$L_{X_{K,\phi}} = \omega_n^{1/n}\sqrt{\frac{n+2}{n}}\, I_2(X_\phi)\left(\sup_{t\in\mathbb{R}^+}\phi(t)\right)^{1/n} L_K.$$

Proof. By the definition of the isotropic constant we have that

$$L_{X_{K,\phi}} = \left(\sup_{x\in\mathbb{R}^n} f_{K,\phi}(x)\right)^{1/n} \left|\det\mathrm{Cov}(X_{K,\phi})\right|^{1/2n},$$

and

$$L_K = \frac{1}{V(K)^{1/n}}\left|\det\mathrm{Cov}(K)\right|^{1/2n}.$$

Then

$$\begin{aligned}\left(\sup_{x\in\mathbb{R}^n} f_{K,\phi}(x)\right)^{1/n} &= \frac{\omega_n^{1/n}}{V(K)^{1/n}}\left(\sup_{x\in\mathbb{R}^n}\phi(\|x\|_K)\right)^{1/n}\\ &= \frac{\omega_n^{1/n}}{V(K)^{1/n}}\left(\sup_{t\in\mathbb{R}^+}\phi(t)\right)^{1/n}\end{aligned}$$

and by Proposition 3.2

$$\left|\det\mathrm{Cov}(X_{K,\phi})\right|^{1/2n} = \sqrt{\frac{n+2}{n}}\, I_2(X_\phi)\left|\det\mathrm{Cov}(K)\right|^{1/2n},$$

which completes the proof. □

3.3 Deviation estimates

We say that a radial profile function $\phi : [0,\infty) \to [0,\infty)$ satisfies the *concentration condition* in $\mathbb{R}^n$ with constant $\tau = \tau_\phi > 0$, that depends only on ϕ, if the random vector $X_\phi = X_{B_2^n,\phi}$ satisfies

$$I_p(X_\phi) \le \tau_\phi\, I_2(X_\phi), \tag{17}$$

for all $p \ge 1$.

Concentration conditions like (17) are valid for several cases of distributions. For example, if ϕ is a log-concave function then so is the density $f_\phi(x) = \phi(|x|)$ of the random vector X_ϕ. Thus, estimate (1) ensures that any log-concave radial profile satisfies the concentration condition (17) with constant $\tau_p = cp$.

Below we give simple examples of radial profile functions ϕ satisfying the concentration condition (17) with a constant $\tau_\phi > 0$ that depends only on ϕ.

Consider $a > 0$ and let

$$\phi_a(t) = \frac{1}{n\omega_n \int_0^1 t^{n-1} e^{-t^a}\, dt} e^{-t^a} \chi_{[0,1]}(t).$$

It is easy to verify that ϕ_a is a radial profile, i.e. $f_{\phi_a}(x) = \phi_a(|x|)$ defines a density on $\mathbb{R}^n$, and for the random vector $X_{\phi_a} \sim f_{\phi_a}(x)$ we have that

$$I_p(X_{\phi_a}) = \left(\frac{\int_0^1 t^{n+p-1} e^{-t^a}\, dt}{\int_0^1 t^{n-1} e^{-t^a}\, dt} \right)^{1/p}.$$

Using the fact that $e^{-1} \leq e^{-t^a} \leq 1$ for $t \in [0,1]$, we easily check that $I_p(X_{\phi_a}) \simeq 1$ for any $p \geq 1$, and hence ϕ_a satisfies (17) with an absolute constant $c > 0$. Note that clearly, if we choose $a \in (0,1)$, then ϕ_a is *not* log-concave on its support.

However, this effect does not relate to the fact that the above radial profile function ϕ_a has a bounded support. A small modification can provide an alternative ϕ that is ``frequently non-log-concave'' as $t \to \infty$.

One can notice that setting $\phi(t) = \lambda e^{-t^a}$, where λ is suitably chosen to make $\phi(\|x\|_2)$ a density on $\mathbb{R}^n$, will not help. The tails of this function are too heavy and fail to satisfy condition (17) for all $p \geq 1$; nevertheless, they do satisfy it, up to $p = n$. Here, contrary to the case of convex bodies where $I_p(K) \simeq I_n(K)$ for $p \geq n$, this is not the case for $I_p(X_\phi)$. To get the promised example, we have to lighten up the tails of e^{-t^a}.

Consider $0 < a < 1$ and two sequences $x_k < y_k \leq x_{k+1}$ with $x_0 = 0$ and $y_0 = 1$, that both diverge to $+\infty$. Define $A = \cup_{k=1}^\infty [x_k, y_k]$ and let

$$\phi(t) = \frac{e^{-t^a} \chi_A}{n\omega_n \sum_{k=0}^\infty \int_{x_k}^{y_k} t^{n-1} e^{-t^a}\, dt}.$$

Then ϕ defines a density on $\mathbb{R}^n$ of the form $\phi(|x|)$, it is not eventually zero and it is not eventually log-concave since we choose $0 < a < 1$. For any $p \geq 1$,

$$\begin{aligned} I_p(X_\phi) &\leq \left(\frac{n}{n+p} \right)^{1/p} \left(\frac{\sum_{k=1}^\infty (y_k^{n+p} - x_k^{n+p}) e^{-x_k^a}}{\sum_{k=1}^\infty (y_k^n - x_k^n) e^{-y_k^a}} \right)^{1/p} \\ &\leq c \left(\frac{n}{n+p} \right)^{1/p} \left(\sum_{k=1}^\infty (y_k^{n+p} - x_k^{n+p}) e^{-x_k^a} \right)^{1/p}. \end{aligned}$$

By choosing x_k close enough to y_k so that $y_k^{n+p} - x_k^{n+p} < 1/k^2$ we get again that $I_p(X_\phi) \leq c$, and hence ϕ satisfies (17) with $\tau = c > 0$, an absolute constant.

In the following theorem we prove a deviation inequality like (8) for contoured random vectors, under the assumption (17) on the radial profile of their distribution. We emphasise the fact that such random vectors, as the above remark shows, do not necessarily need to be log-concave.

Theorem 3.7. *Let $X = X_{K,\phi} \sim f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi(\|x\|_K)$ be isotropic and assume that the radial profile function ϕ satisfies the concentration condition* (17) *in $\mathbb{R}^n$, with a constant $\tau_\phi > 0$ that depends only on ϕ. Then,*

$$P\Big(|X| \geq c_\phi t \alpha_X \sqrt{n}\Big) \leq e^{-t\sqrt{n}}, \tag{18}$$

for every $t \geq 1$, where $c_\phi > 0$ is a constant depending only on ϕ and $\alpha_X = |\det \mathrm{Cov}(X)|^{1/2n} = L_X / \sup f_X^{1/n}$.

Proof. We follow the argument from [27] and [28]. Estimate (1), Proposition 3.3 and the concentration condition (17) on the radial profile ϕ, imply that

$$I_q(X) \leq c'_\phi \frac{q}{p} I_p(X), \tag{19}$$

for any $1 \leq p \leq q$, where $c'_\phi > 0$ is a constant depending only on ϕ.

Consequently, for any $t, p, q \geq 1$, the estimate (19) for $1 \leq p \leq pq$ and Markov's inequality give

$$P\big(|X| > e\, c'_\phi t\, I_p(X)\big) \leq P\left(|X| > \frac{e\,t}{q} I_{pq}(X)\right) \leq \Big(\frac{q}{e\,t}\Big)^{pq}.$$

Taking $q = t \geq 1$, we conclude that

$$P\big(|X| > c''_\phi t I_p(X)\big) \leq e^{-tp} \tag{20}$$

for every $t \geq 1$ and $p \geq 1$.

Note that, by Proposition 3.5, the centered contoured convex body K is also isotropic and so by Paouris's concentration result (see [28]) we have that $I_p(K) \simeq I_2(K)$, for all $p \leq \sqrt{n}$. Thus, by Proposition 3.3 and (17), we obtain

$$I_2(X) \leq I_p(X) \leq c_\phi I_2(X) \qquad \forall\, p \leq \sqrt{n}.$$

Note also that $\big(\mathbb{E}|\langle X, \theta\rangle|^2\big)^{1/2} = \alpha_X = |\det \mathrm{Cov}(X)|^{1/2n}$, for all $\theta \in \mathbb{S}^{n-1}$, and so $I_2(X) = \sqrt{n}\,\alpha_X$. Thus, the constant moments behaviour up to $p = \sqrt{n}$ combined with the estimate (20) implies that

$$P\left(|X| > c_\phi\, t\, \alpha_X \sqrt{n}\right) \leq e^{-t\sqrt{n}}$$

for all $t \geq 1$ where $c_\phi > 0$ is a constant depending only on ϕ. □

4 Geometric aspects of contoured random polytopes

We write $K_N = \mathrm{conv}\{\pm X_1, \dots, \pm X_N\}$ for the symmetric convex hull of the random vectors $X_1, \dots, X_N$ in $\mathbb{R}^n$. In other words K_N is the random polytope in $\mathbb{R}^n$ having the random points $\pm X_1, \dots, \pm X_N$, as its vertices. Applying our results, we are able to prove an estimate for the asymptotic shape of a contoured random polytope, similar to the one for the log-concave case (see [14], [16]).

Theorem 4.1. *Let $0 < \beta \leq \frac{1}{2}$, $\gamma > 1$. There exists an absolute constant $c > 0$ such that if $n, N \in \mathbb{N}$ with $N \geq c\gamma n$, then if $X_1, \dots, X_N$ are independent copies of the contoured random vector*

$$X \sim f_X = f_{K,\phi}(x) = \frac{\omega_n}{V(K)} \phi\left(\|x\|_K\right), \quad x \in \mathbb{R}^n$$

where K is an isotropic convex body in $\mathbb{R}^n$ and the radial profile function ϕ satisfies the condition

$$b_\phi := \sup_{q \geq 1} \frac{I_{2q}(X_\phi)}{I_q(X_\phi)} < \infty, \tag{21}$$

and if $K_N = \mathrm{conv}\{\pm X_1, \dots, \pm X_N\}$ is the random polytope generated by the X_i's, then for all $q \leq \frac{\beta}{2\log(c_2 b_\phi)} \log \frac{N}{n}$ we have that

$$P\Big(c_1 Z_q(X) \subseteq K_N\Big) \geq 1 - \exp\left(-c_3\, n^\beta N^{1-\beta}\right) - P\left(\|\Gamma\| \geq \gamma \tfrac{L_X}{\sup f_X^{1/n}} \sqrt{N}\right), \tag{22}$$

where c_1, c_2, c_3 are positive absolute constants and $\|\Gamma\|$ is the operator norm of the random operator $\Gamma : \ell_2^n \to \ell_2^N$ with $\Gamma x = \big(\langle X_1, x\rangle, \dots, \langle X_N, x\rangle\big)$, $x \in \mathbb{R}^n$.

Remark. Note that the concentration condition (21) is weaker than (17), in the sense that if a radial profile function ϕ satisfies condition (17) with a constant $\tau_\phi > 0$ depending only on ϕ, then ϕ satisfies (21) as well, with a constant $b_\phi > 0$ such that $b_\phi \leq \tau_\phi$.

We need first a technical lemma from [14].

Lemma 4.2. *Let $\sigma \subseteq \{1, \dots, N\}$ and $\theta \in \mathbb{S}^{n-1}$. Then, under the assumptions of Theorem 4.1,*

$$P\left(\max_{j \in \sigma} \big|\langle X_j, \theta\rangle\big| \leq \frac{1}{2} \|\langle \cdot, \theta\rangle\|_q\right) \leq \exp\left(-\frac{|\sigma|}{(c\, b_\phi)^{2q}}\right) \tag{23}$$

where $\|\langle \cdot, \theta\rangle\|_q = \big(\mathbb{E}|\langle X, \theta\rangle|^q\big)^{1/q}$ and $c > 0$ is an absolute constant.

Proof. Using the assumption that X_i's are independent and identically distributed, we have

$$\begin{aligned} P\left(\max_{j \in \sigma} \big|\langle X_j, \theta\rangle\big| \leq \frac{1}{2} \|\langle \cdot, \theta\rangle\|_q\right) &= \prod_{j \in \sigma} P\left(\big|\langle X_j, \theta\rangle\big| \leq \frac{1}{2} \|\langle \cdot, \theta\rangle\|_q\right) \\ &= P\left(\big|\langle X, \theta\rangle\big| \leq \frac{1}{2} \|\langle \cdot, \theta\rangle\|_q\right)^{|\sigma|}. \end{aligned}$$

To estimate the last probability, we use the Paley-Zygmund inequality

$$P\big(Y \geq t\, \mathbb{E}[Y]\big) \geq (1-t)^2 \frac{\big(\mathbb{E}[Y]\big)^2}{\mathbb{E}\left[Y^2\right]} \quad \forall\, t \in (0,1), \tag{24}$$

for the random variable $Y = \left|\langle X, \theta\rangle\right|^q$. Note that by (21) and Proposition 3.4,

$$\begin{aligned}\mathbb{E}\left[Y^2\right] &= \mathbb{E}\left|\langle X, \theta\rangle\right|^{2q} = h_{Z_{2q}(X)}(\theta)^{2q} \le \left(c_1 b_\phi\right)^{2q} h_{Z_q(X)}(\theta)^{2q} \\ &= \left(c_1 b_\phi\right)^{2q} \mathbb{E}\left[Y\right]^2,\end{aligned}$$

and so

$$\begin{aligned}P\left(\left|\langle X, \theta\rangle\right| \le \frac{1}{2}\|\langle\cdot, \theta\rangle\|_q\right)^{|\sigma|} &\le \left(1 - \frac{\left(1 - \frac{1}{2^q}\right)^2}{\left(c_1 b_\phi\right)^{2q}}\right)^{|\sigma|} \le \left(1 - \frac{1}{\left(2c_1 b_\phi\right)^{2q}}\right)^{|\sigma|} \\ &\le \exp\left(-\frac{|\sigma|}{\left(2c_1 b_\phi\right)^{2q}}\right). \qquad \square\end{aligned}$$

Proof of Theorem 4.1. Let $m := \left[5(N/n)^\beta\right] + 1$, $k := [N/m]$ and $\sigma_1, \ldots, \sigma_k$ be a partition of $\{1, \ldots, N\}$ with $|\sigma_i| \ge m$, for all $i = 1, \ldots, k$. For any $\sigma \subseteq \{1, \ldots, N\}$ define P_σ to be the projection on the coordinates defined by the index set σ. We consider the average norm in $\mathbb{R}^N$, defined by

$$\|u\|_o = \frac{1}{k}\sum_{i=1}^k \left\|P_{\sigma_i}(u)\right\|_\infty = \frac{1}{k}\sum_{i=1}^k \max_{j\in\sigma_i} |u_j|, \tag{25}$$

for any $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$. Then, since

$$h_{K_N}(x) = \max_{1\le j\le N}\left|\langle X_j, x\rangle\right| \ge \max_{j\in\sigma_i}\left|\langle X_j, x\rangle\right| = \left\|P_{\sigma_i}\left(\Gamma(x)\right)\right\|_\infty,$$

for every $i = 1, \ldots, k$, one has that

$$h_{K_N}(x) \ge \|\Gamma(x)\|_o \quad \forall\, x \in \mathbb{R}^n. \tag{26}$$

For any $x \in \mathbb{R}^n$ such that $\|\Gamma(x)\|_o < \frac{1}{4}\|\langle\cdot, x\rangle\|_q$, there exists a set $I \subseteq \{1, \ldots, k\}$, with $|I| > \frac{k}{2}$, and $\left\|P_{\sigma_i}\left(\Gamma(x)\right)\right\|_\infty < \frac{1}{2}\|\langle\cdot, x\rangle\|_q$ for every $i \in I$. Indeed, if

$$I = \left\{i \le k : \left\|P_{\sigma_i}\left(\Gamma(x)\right)\right\|_\infty < \frac{1}{2}\|\langle\cdot, x\rangle\|_q\right\},$$

then, $|I| > k/2$. Otherwise, if $|I^c| \ge k/2$ we would have

$$\|\Gamma(x)\|_o \ge \frac{1}{k}\sum_{i\in I^c}\left\|P_{\sigma_i}\left(\Gamma(x)\right)\right\|_\infty \ge \frac{1}{4}\|\langle\cdot, x\rangle\|_q > \|\Gamma(x)\|_o,$$

which is a contradiction. Thus, Lemma 4.2 implies that for every $\theta \in \mathbb{S}^{n-1}$

$$\begin{aligned}
P\left(\|\Gamma(\theta)\|_o < \frac{1}{4}\|\langle \cdot,\theta\rangle\|_q\right) &\leq \sum_{\substack{I\subseteq\{1,\dots,k\}\\ |I|>k/2}} \prod_{i\in I} P\left(\left\|P_{\sigma_i}\left(\Gamma(\theta)\right)\right\|_\infty \leq \frac{1}{2}\|\langle \cdot,\theta\rangle\|_q\right) \\
&\leq \sum_{\substack{I\subseteq\{1,\dots,k\}\\ |I|>k/2}} \prod_{i\in I} \exp\left(-\frac{|\sigma_i|}{\left(2c_1 b_\phi\right)^{2q}}\right) \leq \sum_{\substack{I\subseteq\{1,\dots,k\}\\ |I|>k/2}} \exp\left(-\frac{m|I|}{\left(2c_1 b_\phi\right)^{2q}}\right) \\
&\leq \sum_{\substack{I\subseteq\{1,\dots,k\}\\ |I|>k/2}} \exp\left(-\frac{km}{2\left(2c_1 b_\phi\right)^{2q}}\right) \leq \sum_{j=\lfloor \frac{k}{2}\rfloor}^{k} \binom{k}{j} \exp\left(-\frac{km}{2\left(2c_1 b_\phi\right)^{2q}}\right) \\
&\leq \exp\left(k\log 2 - \frac{km}{2\left(2c_1 b_\phi\right)^{2q}}\right) \leq \exp\left(\frac{N}{m}\log 2 - \frac{N}{4\left(2c_1 b_\phi\right)^{2q}}\right) \\
&\leq \exp\left(\frac{\log 2}{5} N^{1-\beta} n^\beta - \frac{N}{4\left(2c_1 b_\phi\right)^{2q}}\right),
\end{aligned}$$

where $2c_1$ is the absolute positive constant c from Lemma 4.2. Choosing

$$q \leq q_0 = \frac{\beta}{\log(8c_1 b_\phi)^2} \log\frac{N}{n} \tag{27}$$

we see that

$$\frac{N}{4\left(2c_1 b_\phi\right)^{2q}} \geq \frac{N}{\left(8c_1 b_\phi\right)^{2q}} \geq N^{1-\beta} n^\beta,$$

and so we get

$$P\left(\|\Gamma(\theta)\|_o < \frac{1}{4}\|\langle \cdot,\theta\rangle\|_q\right) \leq \exp\left(-c_2 N^{1-\beta} n^\beta\right), \tag{28}$$

for every $\theta \in \mathbb{S}^{n-1}$, where $c_2 = 1 - (\log 2)/5 > 0$.

Now, let $D = \left\{\theta \in \mathbb{R}^n : \frac{1}{2}\|\langle \cdot,\theta\rangle\|_q = 1\right\}$ and U be a δ-net of D, with respect to the norm $\frac{1}{2}\|\langle \cdot,\theta\rangle\|_q$, of cardinality $|U| \leq (3/\delta)^n$ (for the existence of U, see Lemma 2.6 in [26]). Then, (28) gives that

$$P\left(\exists u \in U : \|\Gamma(u)\|_o \leq \frac{1}{2}\right) \leq \exp\left(n\log\frac{3}{\delta} - c_2 N^{1-\beta} n^\beta\right). \tag{29}$$

Since K, and so also X, is isotropic,

$$Z_q(X) \supseteq Z_2(X) = \frac{L_X}{\sup f_X^{1/n}} B_2^n.$$

Fix $\gamma > 1$ and suppose that $\|\Gamma\| \leq \gamma \frac{L_X}{\sup f_X^{1/n}} \sqrt{N}$. Then by the Cauchy-Schwartz inequality we get that

$$\|\Gamma(x)\|_o \leq \frac{1}{\sqrt{k}} \|\Gamma(x)\|_2 \leq \gamma \frac{L_X}{\sup f_X^{1/n}} \sqrt{\frac{N}{k}}\, \|x\|_2 \leq \gamma \sqrt{\frac{N}{k}}\, \|\langle \cdot,x\rangle\|_q$$

for all $x \in \mathbb{R}^n$. For every $\theta \in D$ there exists $u \in U$ such that $\frac{1}{2}\|\langle \cdot, \theta - u\rangle\|_q < \delta$ and so

$$\|\Gamma(u)\|_o \leq \|\Gamma(\theta)\|_o + \|\Gamma(\theta - u)\|_o \leq \|\Gamma(\theta)\|_o + c\gamma\delta\sqrt{\frac{N}{k}}.$$

Then, if $\delta = \frac{1}{4c\gamma}\sqrt{\frac{k}{N}}$, (29) implies that

$$\begin{aligned} P&\left(\exists \theta \in \mathbb{S}^{n-1} : \|\Gamma(\theta)\|_o \leq \frac{1}{8}\|\langle \cdot, \theta\rangle\|_q \,,\, \|\Gamma\| \leq \gamma \tfrac{L_X}{\sup f_X^{1/n}}\sqrt{N}\right) \\ &\qquad = P\left(\exists \theta \in D : \|\Gamma(\theta)\|_o \leq \frac{1}{4} \,,\, \|\Gamma\| \leq \gamma \tfrac{L_X}{\sup f_X^{1/n}}\sqrt{N}\right) \\ &\qquad \leq P\left(\exists u \in U : \|\Gamma(u)\|_o \leq \frac{1}{2} \,,\, \|\Gamma\| \leq \gamma \tfrac{L_X}{\sup f_X^{1/n}}\sqrt{N}\right) \\ &\qquad \leq \exp\left(n\log\left(12c\gamma\sqrt{\frac{N}{k}}\right) - c_2 N^{1-\beta}n^{\beta}\right) \\ &\qquad \leq \exp\left(-c_3 N^{1-\beta}n^{\beta}\right), \end{aligned}$$

for some absolute constant $c_3 > 0$. The last inequality is valid if we assume that N is large enough. Indeed, for any suitable absolute constant $c' > 0$, one has that

$$\log\frac{N}{n} \leq c'\sqrt{\frac{N}{n}}$$

if $N \geq c_0 n$ for an absolute constant $c_0 > 0$ (depending on c'). Thus, since $\gamma > 1$, if $c_4 \geq \max\{c_0, 12c\}$ and if we take

$$N \geq c_4\gamma n \geq 12c\gamma n, \tag{30}$$

then by the definition of k and m and the fact that $0 < \beta \leq 1/2$, we have

$$\log\left(12c\gamma\sqrt{\frac{N}{k}}\right) \leq C\log\frac{N}{n} \leq c'C\sqrt{\frac{N}{n}} \leq c'C\left(\frac{N}{n}\right)^{1-\beta},$$

which implies that

$$n\log\left(12c\gamma\sqrt{\frac{N}{k}}\right) \leq c'CN^{1-\beta}n^{\beta} < c_2N^{1-\beta}n^{\beta}$$

choosing the constant $c' > 0$ suitably small.

Finally, taking into account (26), we get the desired result:

$$
\begin{aligned}
P\left(K_N \not\supseteq \frac{1}{8} Z_q(X)\right) &\leq P\left(\exists \theta \in \mathbb{S}^{n-1} : \|\Gamma(\theta)\|_o \leq \frac{1}{8}\|\langle \cdot, \theta\rangle\|_q\right) \\
&\leq P\left(\exists \theta \in \mathbb{S}^{n-1} : \|\Gamma(\theta)\|_o \leq \frac{1}{8}\|\langle \cdot, \theta\rangle\|_q \,,\, \|\Gamma\| \leq \gamma \tfrac{L_X}{\sup f_X^{1/n}} \sqrt{N}\right) \\
&\qquad + \; P\left(\|\Gamma\| \geq \gamma \tfrac{L_X}{\sup f_X^{1/n}} \sqrt{N}\right) \\
&\leq \exp\left(-c_3 N^{1-\beta} n^{\beta}\right) + P\left(\|\Gamma\| \geq \gamma \tfrac{L_X}{\sup f_X^{1/n}} \sqrt{N}\right),
\end{aligned}
$$

completing the proof. □

4.1 A norm estimate for contoured random matrices

We close this section providing an estimate for the probability

$$
P\left(\|\Gamma\| \geq \gamma \tfrac{L_X}{\sup f_X^{1/n}} \sqrt{N}\right)
$$

that appears in Theorem 4.1.

Proposition 4.3. *Let n, N be positive integers with $n \leq N \leq e^{\sqrt{n}}$ and K be an isotropic convex body in $\mathbb{R}^n$. Let $X_1, \ldots, X_N$ be independent copies of a random vector $X \sim f_X = f_{K,\phi}(x) = \frac{\omega_n}{V(K)} \phi(\|x\|_K)$, $x \in \mathbb{R}^n$, where the radial profile function $\phi : [0, \infty) \to [0, \infty)$ satisfies the concentration condition* (17)

$$
I_q(X_\phi) \leq \tau_\phi I_2(X_\phi) \qquad \forall\, q \geq 1,
$$

for some constant $\tau_\phi > 0$ depending only on ϕ. Let Γ be the $N \times n$ random matrix whose rows are the X_i's, which defines the random operator $\Gamma : \ell_2^n \to \ell_2^N$ with $\Gamma x = \big(\langle X_1, x\rangle, \ldots, \langle X_N, x\rangle\big)$, $x \in \mathbb{R}^n$. Then, one has that

$$
\|\Gamma\| = \sup_{\theta \in \mathbb{S}^{n-1}} \left(\sum_{i=1}^{N} |\langle X_i, \theta\rangle|^2\right)^{1/2} \leq c_\phi \, \frac{L_X}{\sup f_X^{1/n}} \, n^{1/4} N^{1/4},
$$

with probability greater than or equal to $1 - 2\exp(-c\sqrt{n})$, where $c > 0$ is an absolute constant and $c_\phi > 0$ is a constant depending only on ϕ.

For the proof of Proposition 4.3 we use the large deviation inequality for contoured distributions from Theorem 3.7 along with the following strong result for random matrices from [2] (see also [1]).

Theorem 4.4 (Adamczak et al.)**.** *Let N, n be positive integers and $\psi, \kappa \geq 1$. Let $X_1, \ldots, X_N$ be independent random vectors in $\mathbb{R}^n$ satisfying the conditions*

$$
P\left(\max_{1 \leq i \leq N} \frac{|X_i|}{\sqrt{n}} > \kappa \max\left\{1, \left(\frac{N}{n}\right)^{1/4}\right\}\right) \leq e^{-\sqrt{n}} \tag{31}
$$

and
$$\max_{1\le i\le N}\sup_{\theta\in\mathbb{S}^{n-1}}\|\langle X_i,\theta\rangle\|_{\psi_1}\le\psi. \tag{32}$$
Then, with probability at least $1-2\exp(-c\sqrt{n})$*, we have that*
$$\sup_{\theta\in\mathbb{S}^{n-1}}\left|\frac{1}{N}\sum_{i=1}^{N}\Big(|\langle X_i,\theta\rangle|^2-\mathbb{E}|\langle X_i,\theta\rangle|^2\Big)\right|\le C(\kappa+\psi)^2\sqrt{\frac{n}{N}}.$$

Note that the ψ_α norm of a random variable Y, for an $\alpha\ge 1$, is the Orlicz norm of Y corresponding to the Orlicz function $\psi_\alpha(t)=\exp(|t|^\alpha)-1$, defined by
$$\|Y\|_{\psi_\alpha}=\inf\left\{\lambda\ge 0:\ \mathbb{E}\left[\exp\left(\tfrac{|Y|}{\lambda}\right)^\alpha\right]\le 2\right\}.$$
In what follows, we will use the following well known characterization of the ψ_α norm (see, eg, [9] Lemma 2.4.2):
$$\|Y\|_{\psi_\alpha}\simeq\sup_{p\ge\alpha}\frac{(\mathbb{E}|Y|^p)^{1/p}}{p^{1/\alpha}}. \tag{33}$$

Proof of Proposition 4.3. We will show that the random N-tuple $X_1,\dots,X_N$ satisfies both conditions (31) and (32) with some appropriate constants.

The first condition, is an immediate consequence of Theorem 3.7
$$P\left(|X|>c'_\phi\,t\,\alpha_X\sqrt{n}\right)\le e^{-t\sqrt{n}},\quad\forall\,t\ge 1,$$
where $\alpha_X=|\det\mathrm{Cov}(X)|^{1/2n}=L_X/\sup f_X^{1/n}$. Then, since $N\le e^{\sqrt{n}}$, the union bound gives us that the N-tuple $X_1,\dots,X_N$ satisfies the condition (31) with constant $\kappa\ge 1$ such that
$$\kappa\ge c_1(\phi)\,\alpha_X\ge c_1(\phi)\,\alpha_X\left(\frac{n}{N}\right)^{1/4}, \tag{34}$$
where $c_1(\phi)>0$ is a constant depending only on ϕ.

For the second condition (32), notice that the convex body K satisfies a ψ_1 condition, since by (3) we have that $Z_p(K)\subseteq c\,p\,Z_2(K)$ for all $p\ge 1$. Thus, by (13) and the concentration condition (17) for ϕ, we get that
$$Z_p(X)\subseteq c_\phi\,p\,Z_2(X)\qquad\forall\,p\ge 1, \tag{35}$$
where $c_\phi>0$ is a constant depending only on ϕ. Note that since the convex body K, and so the contoured random vector X as well, is isotropic, we have that $Z_2(X)=\alpha_X B_2^n$, and so (35) could be written as
$$\sup_{\theta\in\mathbb{S}^{n-1}}\sup_{p\ge 1}\frac{\|\langle X,\theta\rangle\|_{L_p}}{p}\le c_\phi\alpha_X.$$

By the characterization (33) of the ψ_1 norm, the last estimate implies that the N-tuple $X_1, \ldots, X_N$ satisfies condition (32) with constant

$$\psi = c_2(\phi)\,\alpha_X, \tag{36}$$

where $c_2(\phi) > 0$ is a constant depending only on ϕ.

Finally, Theorem 4.4 combined with the estimates (34) and (36) implies that with probability greater than or equal to $1 - 2e^{-c\sqrt{n}}$ one has

$$\|\Gamma\| = \sup_{\theta \in \mathbb{S}^{n-1}} \left(\sum_{i=1}^{N} |\langle X_i, \theta \rangle|^2 \right)^{1/2} \leq c(\phi)\alpha_X\, n^{1/4} N^{1/4}$$

where $c(\phi) > 0$ is a constant depending only on ϕ. $\square$

5 Geometric invariants of random polytopes

The random polytope K_N can be weakly sandwiched between two L_q-centroid bodies $Z_{q_1}(X)$ and $Z_{q_2}(X)$, with $q_1, q_2 \simeq_\phi \log(N/n)$, where $a \simeq_\phi b$ means that there exist positive constants $c_1(\phi), c_2(\phi)$ depending on the function ϕ such that $c_1(\phi)a \leq b \leq c_2(\phi)a$. This result is valid for more general sampling distributions and this is why the results of this section are stated with more general assumptions: Let $n, N \in \mathbb{N}$ with $n < N$. Let X be any random vector in $\mathbb{R}^n$ with $\mathbb{E}|X|^q < \infty\ \forall\, q > 1$, and $X_1, \ldots, X_N$ be independent copies of X forming the random polytope

$$K_N = K_N(X) = \mathrm{conv}\big\{ \pm X_1, \ldots, \pm X_N \big\}.$$

As an immediate consequence (see [14]) of the union bound and Markov's inequality it follows that, for any $a, q \geq 1$, one has

$$P\Big(h_{K_N}(\theta) > a h_{Z_q(X)}(\theta) \Big) \leq N a^{-q} \tag{37}$$

for all $\theta \in \mathbb{S}^{n-1}$. Then, an application of Fubini's Theorem implies that

$$\mathbb{E}\Big[\sigma\Big(\theta \in \mathbb{S}^{n-1} :\ h_{K_N}(\theta) > a h_{Z_q(X)}(\theta) \Big) \Big] \leq N a^{-q}. \tag{38}$$

Thus, if for example we take $a = e$ and $q = (k+1)\log N$, where k is any positive integer, then by the above estimate we get that

$$\mathbb{E}\Big[\sigma\Big(\theta \in \mathbb{S}^{n-1} :\ h_{K_N}(\theta) \leq e h_{Z_q(X)}(\theta) \Big) \Big] \geq 1 - \frac{1}{N^k},$$

that is, for a random K_N we expect that in most directions $\theta \in \mathbb{S}^{n-1}$, $eZ_q(X)$ would be outside K_N.

5.1 Mean Width

The simple estimate (37) implies that the expected mean witdth of K_N, is dominated by the mean width of $Z_{\log N}(X)$.

Theorem 5.1. *Let X be a random vector in $\mathbb{R}^n$ with $\mathbb{E}|X|^q < \infty$, $\forall\, q > 1$. Let $K_N = \mathrm{conv}\{\pm X_1, \dots, \pm X_N\}$, where $N \geq 3$ and $X_1, \dots, X_N$ be independent copies of X. Then*

$$\mathbb{E}\big[w(K_N)\big] \leq c\, w\big(Z_{\log N}(X)\big)$$

where $c > 0$ is an absolute constant.

Proof. Let $c_0 > 1$ be an absolute constant. Then, for any $a \geq 1$ and $q \geq c_0$ we have

$$\begin{aligned} w(K_N) &= \int_{\mathbb{S}^{n-1}} h_{K_N}(\theta)\, d\sigma(\theta) \\ &\leq \int_{\left\{h_{K_N} \leq a h_{Z_q(X)}\right\}} a h_{Z_q(X)}(\theta)\, d\sigma(\theta) \quad + \int_{\left\{h_{K_N} > a h_{Z_q(X)}\right\}} h_{K_N}(\theta)\, d\sigma(\theta) \\ &\leq a\, w\big(Z_q(X)\big) \quad + \int_{\left\{h_{K_N} > a h_{Z_q(X)}\right\}} h_{K_N}(\theta)\, d\sigma(\theta) \end{aligned}$$

and so

$$\mathbb{E}\big[w(K_N)\big] \leq a\, w\big(Z_q(X)\big) \; + \; \mathbb{E} \int_{\left\{h_{K_N} > a\, h_{Z_q(X)}\right\}} h_{K_N}(\theta)\, d\sigma(\theta).$$

Now, we use (37) to estimate the last expectation:

$$\begin{aligned} &\mathbb{E} \int_{\left\{h_{K_N} > a h_{Z_q(X)}\right\}} h_{K_N}(\theta)\, d\sigma(\theta) \\ &\qquad = \int_{\mathbb{S}^{n-1}} \mathbb{E}\left[h_{K_N}(\theta)\, \mathbb{1}_{\left\{h_{K_N}(\theta) > a h_{Z_q(X)}(\theta)\right\}} \right] d\sigma(\theta) \\ &\qquad = \int_{\mathbb{S}^{n-1}} \sum_{k=0}^{\infty} \mathbb{E}\left[h_{K_N}(\theta)\, \mathbb{1}_{\left\{2^k a h_{Z_q(X)}(\theta) < h_{K_N}(\theta) \leq 2^{k+1} a h_{Z_q(X)}(\theta)\right\}} \right] d\sigma(\theta) \\ &\qquad \leq 2a \int_{\mathbb{S}^{n-1}} h_{Z_q(X)}(\theta) \sum_{k=0}^{\infty} 2^k P\left(h_{K_N}(\theta) > 2^k a h_{Z_q(X)}(\theta) \right) d\sigma(\theta) \\ &\qquad \leq 2a \left(\sum_{k=0}^{\infty} 2^{-(q-1)k} \right) a^{-q} N\, w\big(Z_q(X)\big) \leq c_1 a\, a^{-q} N\, w\big(Z_q(X)\big), \end{aligned}$$

where $c_1 > 0$ is a constant depending on c_0. Choosing $a = e$ and $q = \log N$, we finally get

$$\mathbb{E}\big[w(K_N)\big] \leq \big(a + c_1 a\, a^{-q} N\big)\, w\big(Z_q(X)\big) \leq c\, w\big(Z_q(X)\big). \qquad \square$$

5.2 Volume Radius

Let K be a convex body in $\mathbb{R}^n$. For any $1 \le q < n$ define

$$w_{-q}(K) = \left(\int_{\mathbb{S}^{n-1}} \frac{1}{h_K^q(\theta)}\, d\sigma(\theta) \right)^{-1/q}.$$

Lemma 5.2 ([14])**.** *For any $1 \le q < n$*

$$\mathrm{vr}(K_N) \le w_{-q}(K_N). \tag{39}$$

Proof. For any and $1 \le q < n$ and the symmetric convex polytope $K_N = \mathrm{conv}\{\pm X_1, \dots, \pm X_N\}$, Blaschke-Santaló inequality and integration in polar coordinates give that

$$\begin{aligned}\mathrm{vr}(K_N) = \left(\frac{V(K_N)}{\omega_n} \right)^{1/n} &\le \left(\frac{\omega_n}{V(K_N^\circ)} \right)^{1/n} = \left(\int_{\mathbb{S}^{n-1}} h_{K_N}^{-n}(\theta)\, d\sigma(\theta) \right)^{-1/n} \\ &\le \left(\int_{\mathbb{S}^{n-1}} h_{K_N}^{-q}(\theta)\, d\sigma(\theta) \right)^{-1/q} = w_{-q}(K_N). \qquad \square\end{aligned}$$

In order to provide an upper bound for the volume radius with ``high'' probability we need first to provide a bound for the w_{-q}-width of the random polytope K_N.

Proposition 5.3. *Let X be a random vector in $\mathbb{R}^n$ with $\mathbb{E}|X|^q < \infty$, and assume that there exists a constant $\beta > 1$, depending only on the distribution of X, such that*

$$Z_q(X) \subseteq Z_{2q}(X) \subseteq \beta Z_q(X), \tag{40}$$

for all $q > 1$. Let $N \in \mathbb{N}$, $N < e^n/2$, let $X_1, \dots, X_N$ be independent copies of X and consider the random polytope $K_N = \mathrm{conv}\{\pm X_1, \dots, \pm X_N\}$. If $\log 2N \le q < n$ then we have that

$$w_{-q}(K_N) \le \frac{1}{2e^2\beta}\, w_{-q/2}(Z_{q/2}(X)) \tag{41}$$

with probability $\ge 1 - e^{-q}$.

Proof. Notice that for any $a, q \ge 1$,

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} \frac{h_{K_N}^q(\theta)}{h_{Z_q(X)}^q(\theta)}\, d\sigma(\theta) \\ &\quad = \int_{\{h_{K_N} \le a h_{Z_q(X)}\}} \frac{h_{K_N}^q(\theta)}{h_{Z_q(X)}^q(\theta)}\, d\sigma(\theta) + \int_{\{h_{K_N} > a h_{Z_q(X)}\}} \frac{h_{K_N}^q(\theta)}{h_{Z_q(X)}^q(\theta)}\, d\sigma(\theta) \\ &\quad \le a^q + \int_{\{h_{K_N} > a h_{Z_q(X)}\}} \frac{h_{K_N}^q(\theta)}{h_{Z_q(X)}^q(\theta)}\, d\sigma(\theta).\end{aligned}$$

Then, by (40) and (37) we have

$$\begin{aligned}
\mathbb{E}\int_{\left\{h_{K_N}>ah_{Z_q(X)}\right\}} & \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(X)}(\theta)}\,d\sigma(\theta) \\
&= \int_{\mathbb{S}^{n-1}} \mathbb{E}\left[\frac{h^q_{K_N}(\theta)}{h^q_{Z_q(X)}(\theta)}\,\mathbb{1}_{\left\{ah_{Z_q(X)}(\theta)<h_{K_N}(\theta)\right\}}\right] d\sigma(\theta) \\
&= \int_{\mathbb{S}^{n-1}} \sum_{k=0}^{\infty} \mathbb{E}\left[\frac{h^q_{K_N}(\theta)}{h^q_{Z_q(X)}(\theta)}\,\mathbb{1}_{\left\{2^k ah_{Z_q(X)}(\theta)<h_{K_N}(\theta)\le 2^{k+1}ah_{Z_q(X)}(\theta)\right\}}\right] d\sigma(\theta) \\
&\le (2a)^q \int_{\mathbb{S}^{n-1}} \sum_{k=0}^{\infty} 2^{qk}\,P\left(h_{K_N}(\theta)>2^k ah_{Z_q(X)}(\theta)\right)\,d\sigma(\theta) \\
&\le (2a)^q \int_{\mathbb{S}^{n-1}} \sum_{k=0}^{\infty} 2^{qk}\,P\left(h_{K_N}(\theta)>2^k \frac{a}{\beta}\,h_{Z_{2q}(X)}(\theta)\right)\,d\sigma(\theta) \\
&\le N\left(\frac{2\beta^2}{a}\right)^q \sum_{k=0}^{\infty}\left(\frac{1}{2^q}\right)^k \le \left(\frac{2\beta^2}{a}\right)^q 2N.
\end{aligned}$$

Thus, for $a=e\beta^2$ and $q\ge \log 2N$, we get that $\left(\frac{2\beta^2}{a}\right)^q 2N = 2^q\,e^{-q}\,2N \le 2^q$ and so

$$\mathbb{E}\int_{\mathbb{S}^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(X)}(\theta)}\,d\sigma(\theta) \le (e\beta^2)^q + 2^q \le (2e\beta^2)^q.$$

Therefore, Markov's inequality implies that

$$P\left(\int_{\mathbb{S}^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(X)}(\theta)}\,d\sigma(\theta) \le \left(2e^2\beta^2\right)^q\right) \ge 1-e^{-q}. \tag{42}$$

Moreover, for any $1\le q\le n$ or $N\le e^n/2$, by the Cauchy-Schwartz inequality we get

$$\begin{aligned}
w_{-q/2}\big(Z_q(X)\big)^{-q} &= \left(\int_{\mathbb{S}^{n-1}} \frac{1}{h^{q/2}_{Z_q(X)}(\theta)}\,d\sigma(\theta)\right)^2 \\
&\le \int_{\mathbb{S}^{n-1}} \frac{1}{h^q_{K_N}(\theta)}\,d\sigma(\theta) \int_{\mathbb{S}^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(X)}(\theta)}\,d\sigma(\theta) \\
&= w_{-q}\big(K_N\big)^{-q} \int_{\mathbb{S}^{n-1}} \frac{h^q_{K_N}(\theta)}{h^q_{Z_q(X)}(\theta)}\,d\sigma(\theta)
\end{aligned}$$

and so by (42)

$$P\left(w_{-q}\big(K_N\big) \le \frac{1}{2e^2\beta^2}\,w_{-q/2}\big(Z_q(X)\big)\right) \ge 1-e^{-q}.$$

Finally, by (40) we conclude that

$$w_{-q}\big(K_N\big) \le \frac{1}{2e^2\beta^2}\, w_{-q/2}\big(Z_q(X)\big) \le \frac{1}{2e^2\beta}\, w_{-q/2}\big(Z_{q/2}(X)\big)$$

with probability $\ge 1 - e^{-q}$, and the proof is complete. $\square$

For the next step, we make use of Paouris' formula from [29]: Let K be a centered convex body in $\mathbb{R}^n$ and $q \in \mathbb{N}$ with $q < n$. Then

$$w_{-q}\big(Z_q(K)\big) \simeq \sqrt{\frac{q}{n}} I_{-q}(K). \tag{43}$$

Lemma 5.4. *Let $X = X_{K,\phi} \sim f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi\big(\|x\|_K\big)$, where K is a centered convex body in $\mathbb{R}^n$. If $1 \le q < n$, then*

$$w_{-q/2}\left(Z_{q/2}\big(X_{K,\phi}\big)\right) \le c\sqrt{\frac{q}{n}}\, I_{q/2}(X_\phi)\, I_2(K) \tag{44}$$

where $c > 0$ is an absolute constant.

Proof. By (15) we have that

$$Z_{q/2}\big(X_{K,\phi}\big) = \left(\frac{2n+q}{2n}\right)^{2/q} I_{q/2}(X_\phi)\, Z_{q/2}(K) \simeq I_{q/2}(X_\phi)\, Z_{q/2}(K)$$

and the result follows from (43), Proposition 3.4 and Hölder's inequality:

$$\begin{aligned} w_{-q/2}\left(Z_{q/2}\big(X_{K,\phi}\big)\right) &\simeq I_{q/2}(X_\phi)\, w_{-q/2}\big(Z_{q/2}(K)\big) \simeq \sqrt{\frac{q}{n}}\, I_{q/2}(X_\phi)\, I_{-q/2}(K) \\ &\le c\sqrt{\frac{q}{n}}\, I_{q/2}(X_\phi)\, I_2(K). \end{aligned}$$ $\square$

Theorem 5.5. *Let $X = X_{K,\phi}$ be a contoured random vector in $\mathbb{R}^n$ with density $f_{K,\phi}(x) = \frac{\omega_n}{V(K)}\phi\big(\|x\|_K\big)$, where K is a centered convex body in $\mathbb{R}^n$, and assume that*

$$b_\phi = \sup_{q\ge 1} \frac{I_{2q}(X_\phi)}{I_q(X_\phi)} < \infty.$$

Let $1 \le N < e^n/2$ and $X_1, \dots, X_N$ be independent copies of X. Then, for the random polytope $K_N = \mathrm{conv}\big\{\pm X_1, \dots, \pm X_N\big\}$ we have that

$$\mathrm{vr}\big(K_N\big) \le c\sqrt{\log N}\, I_{\log N}(X_\phi) V(K)^{1/n} L_K \tag{45}$$

with probability greater than or equal to $1 - \frac{1}{2N}$.

Proof. Without loss of generality we may assume that $X = X_{K,\phi}$ is isotropic or equivalently that K is isotropic. Then, by (3) and Proposition 3.4, we have that

$$\begin{aligned}
Z_{2q}\big(X_{K,\phi}\big) &= \left(\frac{n+2q}{n}\right)^{1/2q} I_{2q}\big(X_\phi\big)\, Z_{2q}(K) \\
&\subseteq c\left(\frac{n+2q}{n}\right)^{1/2q} I_{2q}\big(X_\phi\big)\, Z_q(K) \\
&= c\,\frac{\left(\frac{n+2q}{n}\right)^{1/2q}}{\left(\frac{n+q}{n}\right)^{1/q}}\,\frac{I_{2q}\big(X_\phi\big)}{I_q\big(X_\phi\big)}\, Z_q\big(X_{K,\phi}\big) \subseteq c_1\, b_\phi\ Z_q\big(X_{K,\phi}\big).
\end{aligned}$$

Thus, for any $N \in \mathbb{N}$ with $N < e^n/2$ and for any $\log 2N \leq q < n$, we can apply Lemma 5.2 and Proposition 5.3 to get that

$$\left(\frac{V(K_N)}{\omega_n}\right)^{1/n} \leq \frac{c_2}{b_\phi}\, w_{-q/2}\left(Z_{q/2}\big(X_{K,\phi}\big)\right) \tag{46}$$

with probability greater than or equal to $1 - e^{-q} \geq 1 - 1/2N$. Now, by Lemma 5.4, and since K is isotropic, we get that

$$\begin{aligned}
w_{-q/2}\left(Z_{q/2}\big(X_{K,\phi}\big)\right) &\leq c\sqrt{\frac{q}{n}}\, I_{q/2}\big(X_\phi\big)\, I_2(K) \\
&= c\sqrt{\frac{q}{n}}\, I_{q/2}\big(X_\phi\big)\sqrt{n}\,\big|\det \mathrm{Cov}(K)\big|^{1/2n} \\
&= c\sqrt{q}\, I_{q/2}\big(X_\phi\big)\, V(K)^{1/n} L_K. \qquad (47)
\end{aligned}$$

We finish the proof, by combining the above estimates (46) and (47). $\square$

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