Random spaces generated by vertices of the cube

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Abstract

Let $E_2^n = \{-1,1\}^n$ be the discrete cube in $\mathbb{R}^n$. For every $N \geq n$ we consider the class of convex bodies $K_N = \mathrm{co}\{\pm x_1, \ldots, \pm x_N\}$ which are generated by N random points $x_1, \ldots, x_N$ chosen independently and uniformly from E_2^n. We show that if $n \geq n_0$ and $N \geq n(\log n)^2$ then, for a random K_N, the inradius, the volume radius, the mean width and the size of the maximal inscribed cube can be determined up to an absolute constant as functions of n and N . This geometric description of K_N leads to sharp estimates for several asymptotic parameters of the corresponding n-dimensional normed space X_N.

1. Introduction

The use of random spaces in the study of finite dimensional normed spaces provided a way of proving the existence of spaces or operators with extremal properties. Several important problems of the theory were solved by introducing a suitable probability space consisting of n-dimensional spaces and showing that random selection of its elements gives objects with the desired properties. Among many existence results proved in this way, let us mention the existence of a pair of n-dimensional spaces with Banach-Mazur distance of order as large as n [17], the existence of a space whose unconditional basis constant [15] or basis constant [33], [18] has order as large as $\sqrt{n}$, the existence of a space whose Banach-Mazur distance to ℓ_∞^n is greater than $c\sqrt{n}\log n$ [34]. In particular, random n-dimensional subspaces of ℓ_∞^N with $N = \lambda n$, $\lambda > 1$ (i.e. spaces whose unit balls are random sections of the cube Q_N of dimension proportional to N) provided examples of spaces which exhibited pathology with respect to various asymptotic parameters of the theory: this line of thought has its origin in [23] and [16] (see also [32], [35], [13] and the articles mentioned above).

In this article, we consider convex hulls of random subsets of the set of vertices of the cube and the class of random spaces they generate. In order to define our probability space precisely, we consider the discrete cube $E_2^n = \{-1,1\}^n$ in $\mathbb{R}^n$

equipped with the uniform probability measure, and fix $N \geq n$. Next, we consider N independent random points $x_1, \ldots, x_N$ uniformly distributed over E_2^n, and for every choice $x_1, \ldots, x_N$, we write M_N for the convex hull

$$M_N := M(x_1, \ldots, x_N) = \mathrm{co}\{x_1, \ldots, x_N\}$$

and K_N for the absolute convex hull

$$K_N := K(x_1, \ldots, x_N) = \mathrm{co}\{\pm x_1, \ldots, \pm x_N\}.$$

The symmetric convex body K_N (if non-degenerate) induces a norm on $\mathbb{R}^n$. We write X_N for the normed space whose unit ball is K_N. In this way, for every $N \geq n$ we obtain a class of random n-dimensional spaces, which we denote by $\mathcal{B}_N$. The dual space of X_N is denoted by X_N^* and the class of dual spaces by $\mathcal{B}_N^*$.

Section 2 is devoted to the study of the geometry of a random K_N. We say that a property (P) holds for a random K_N if the probability of the N-tuples $(x_1, \ldots, x_N)$ for which K_N has (P) is greater than $1 - \exp(-n)$ (it tends to 1 "exponentially" as the dimension n grows to infinity). There are three basic sources of information on a random K_N, depending on the number N of vertices. First, we prove that if $N \geq \lambda_0 n$ where $\lambda_0 > 1$ is a fixed constant, then (with high probability) K_N contains a centered ball of radius independent of n and N.

Fact 1 *If $N \geq \lambda_0 n$, then $K_N \supseteq c_1 B_2^n$ with high probability, where B_2^n is the Euclidean unit ball and $\lambda_0 > 1$, $c_1 > 0$ are absolute constants.*

Since $K_N \subseteq Q_n := [-1, 1]^n$, this estimate is clearly optimal, the interesting point being that it starts being true for a random K_N when N is as low as of the order of n. The proof of this fact is a consequence of the observation that a random "small" set of vertices of the cube is already enough to substitute E_2^n in the classical Khintchine inequality: for all $a_1, \ldots, a_n \in \mathbb{R}$

$$\frac{1}{|A|} \sum_{\varepsilon \in A} \left|\varepsilon_1 a_1 + \ldots + \varepsilon_n a_n\right| \simeq (a_1^2 + \ldots + a_n^2)^{1/2}, \tag{1.1}$$

if $m \geq \lambda_0 n$ and A is a random subset of E_2^n of cardinality $|A| = m$.

Our second main result is that if $N \geq n(\log n)^2$, then K_N contains (with high probability) a centered cube whose edges have length $\sqrt{\log(N/n)}/\sqrt{n}$.

Fact 2 *There exists $n_0 \in \mathbb{N}$ with the following property: If $n \geq n_0$ and $N \geq n(\log n)^2$, then*

$$K_N \supseteq c\left(\sqrt{\log{(N/n)}} B_2^n \cap Q_n\right) \supseteq \left(c_2 \sqrt{\log(N/n)}/\sqrt{n}\right) Q_n$$

with probability greater than $1 - e^{-n}$, where $c_2 > 0$ is an absolute constant.

In the range $N \geq \exp((\log n)^2)$, the first inclusion was recently proved by Bárány and Pór in [4] (see the remarks after Theorem 2.2). Fact 2 should be compared to the following result which was proved in [20]: There exists a constant

$\kappa > 0$ with the following property: for every $\delta \in (0,1)$ and every convex body K with centroid at the origin in $\mathbb{R}^n$, $N \geq c(\delta)n^\kappa$ points $x_1, \ldots, x_N$ chosen uniformly and independently from K satisfy with probability greater than $1-\delta$

$$K \supseteq M_N = \mathrm{co}\{x_1, \ldots, x_N\} \supseteq \frac{c \log N}{n} K,$$

where $c > 0$ is an absolute constant. Fact 2 may be viewed as a discrete version of the case $K = Q_n$ in the above result. The argument in [20] makes essential use of the Brunn-Minkowski inequality: the main point there is that the L_{ψ_1}-norm of linear functionals on convex bodies is bounded by their L_1-norm. Note that the dependence on N in Fact 2 is better (this reflects the fact that linear functionals f on the cube satisfy the stronger inequality $\|f\|_{L_{\psi_2}} \leq c\|f\|_{L_1}$.)

Finally, we observe that combining Fact 2 with well-known volume estimates from [19], [10] one can determine the volume radius of K_N and K_N° up to an absolute constant.

Fact 3 *If $N \geq n(\log n)^2$ then for a random K_N we have*
(a) $|K_N|^{1/n} \simeq \sqrt{\log(N/n)}/\sqrt{n}$ *and* $|K_N^\circ|^{1/n} \simeq 1/\sqrt{n \log(N/n)}$.
(b) $w(K_N)w(K_N^\circ) \leq c_3\sqrt{\log n}\sqrt{\log N}/\sqrt{\log(N/n)}$,
where $w(\cdot)$ denotes mean width and $c_3 > 0$ is an absolute constant.

Combining all three facts we have a precise description of the unit ball of X_N. A random K_N belongs to a rather restricted class of convex bodies for which many asymptotic parameters can be estimated through known methods. This is done in Section 3, where we start by studying unconditionality properties of X_N as a function of N. Our first result concerns the Banach-Mazur distance from a random X_N to the class $\mathcal{U}$ of spaces with 1-unconditional basis.

Fact 4 *For every $\delta \in (0,1)$ we can find $c(\delta) = O(\log(\delta^{-1}))$ such that: If $N \geq c(\delta)n$, then $X_N \in \mathcal{B}_N$ satisfies*

$$d(X_N, \mathcal{U}) \geq \frac{c_4\sqrt{n}}{\sqrt{\log(2N/n)}}$$

with probability greater than $1-\delta$, where $c_4 > 0$ is an absolute constant and d denotes Banach-Mazur distance.

Fact 2 shows that when $N \geq n(\log n)^2$, then $d(X_N, \mathcal{U})$ is "attained" for ℓ_∞^n and has exactly the order given by Fact 4. Also, for suitable $N \simeq n$, Fact 4 shows the existence of a space whose distance from $\mathcal{U}$ is of the maximal possible order $\sqrt{n}$ (this is a well-known fact; see [15]).

Facts 1 and 3(b) show that the Euclidean ball is "equivalent" to the distance and ℓ-ellipsoid of K_N. Thus, although the unconditional basis constant of X_N is large, we may apply the method of random orthogonal factorizations to obtain upper estimates for the Banach-Mazur distance from X_N to special classes of spaces. In particular, we prove the following.

Fact 5 *For every $N \geq n$ and for a random X_N, $d(X_N, X_N^*) \leq C\sqrt{n \log n}$ where $C > 0$ is an absolute constant.*

Finally, we obtain estimates for the isotropic constant of the unit balls of spaces in the classes $\mathcal{B}_N$ and $\mathcal{B}_N^*$. For a random K_N°, $N \geq n(\log n)^2$, the isotropic constant is bounded by an absolute constant.

Fact 6 *There exist absolute constants $c, C > 0$ with the following property:*
(a) *If $n \leq N \leq n(\log n)^2$, then $L_{K_N^\circ} \leq c\sqrt{\log(2N/n)} \leq C\sqrt{\log\log n}$.*
(b) *If $N \geq n(\log n)^2$, then $L_{K_N^\circ} \leq C$ for a random K_N°.*

Some estimates for the isotropic constant of a random K_N may be given as well.

Fact 7 *Let $N \geq n(\log n)^2$. For a random K_N we have*

$$L_{K_N} \leq C\frac{\min\{\log N, \sqrt{n}\}}{\sqrt{\log(N/n)}},$$

where $C > 0$ is an absolute constant.

Notation. We will be working in $\mathbb{R}^n$, which is equipped with the Euclidean structure $\langle \cdot, \cdot \rangle$. All n-dimensional normed spaces in this paper are of the form $X = (\mathbb{R}^n, \|\cdot\|)$. The unit ball of X is a centrally symmetric convex body in $\mathbb{R}^n$ which is denoted by B_X. Conversely, every centrally symmetric convex body K induces the norm $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$ to $\mathbb{R}^n$, and K is the unit ball of $X_K = (\mathbb{R}^n, \|\cdot\|_K)$. The dual norm is defined by $\|y\|_* = \max\{|\langle x, y\rangle| : x \in B_X\}$, and the unit ball of $X^* = (\mathbb{R}^n, \|\cdot\|_*)$ is the polar body $B_{X^*} = B_X^\circ$ of B_X.

We write B_2^n and S^{n-1} for the Euclidean unit ball and sphere respectively, and

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

for the ℓ_p^n-norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $1 \leq p < \infty$ (in the case $p = \infty$, $\|x\|_\infty = \max_{i \leq n} |x_i|$). The rotationally invariant probability measure on S^{n-1} is denoted by σ. We use the notation $|A|$ for the volume of a convex body and for the cardinality of a finite set.

The support function of a convex body K is defined by $h_K(y) = \max_{x \in K}\langle x, y\rangle$. The mean width of K is the quantity

$$w(K) = \int_{S^{n-1}} [h_K(\theta) + h_K(-\theta)]\sigma(d\theta) = 2\int_{S^{n-1}} h_K(\theta)\sigma(d\theta). \tag{1.2}$$

Let X and Y be two n-dimensional normed spaces. Their Banach-Mazur distance $d(X, Y)$ is defined by

$$d(X, Y) = \min\{\|T\| \cdot \|T^{-1}\| \mid T : X \to Y \text{ an isomorphism}\}. \tag{1.3}$$

John's theorem [21] shows that $d(X, \ell_2^n) \leq \sqrt{n}$ for every X. It follows that $d(X, Y)$ is always bounded by n. On the other hand, as we already mentioned, Gluskin [17] proved that there exists an absolute constant $c > 0$ such that for every n one can find n-dimensional spaces X_n, Y_n with $d(X_n, Y_n) \geq cn$.

The letters c, c_1, c_2, c' etc. are reserved for absolute positive constants, which may change from line to line. Wherever we write $a \simeq b$, this means that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. We refer the reader to the books [26], [28] and [37] for basic facts that we are using throughout the text.

ACKNOWLEDGMENTS: We are indebted to the referees of the paper for references and ideas which helped us to simplify the proof of Theorem 2.2 and to clarify the range of the parameters for which it holds true.

2. Geometry of the unit ball

As was mentioned in the introduction, we will say that a random K_N has a certain property (P) if

$$\mathrm{Prob}\big((x_1, \ldots, x_N) \in E_2^n \times \ldots \times E_2^n : (P) \text{ holds for } K_N\big) \geq 1 - e^{-n},$$

where $K_N = \mathrm{co}(\pm x_1, \ldots, \pm x_N)$. In this Section we give a description of the unit ball K_N of a random element of $\mathcal{B}_N$:

Theorem A *There exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and $N \geq n(\log n)^2$ then a random K_N has the following properties:*

(a) $K_N \supseteq c_1 B_2^n$

(b) $K_N \supseteq \big(c_2 \sqrt{\log(N/n)}/\sqrt{n}\big) Q_n$

(c) $|K_N|^{1/n} \simeq \sqrt{\log(N/n)}/\sqrt{n}$ *and* $|K_N^\circ|^{1/n} \simeq 1/\sqrt{n \log(N/n)}$

(d) $w(K_N) w(K_N^\circ) \leq c_3 \sqrt{\log n}\sqrt{\log N}/\sqrt{\log(N/n)}$,

where c_1, c_2 and c_3 are absolute positive constants.

The proofs of these facts are presented in the next four subsections.

2.1 Inradius of K_N

We will first show that if $N \geq c \log(\delta^{-1}) n$ then, with probability greater than $1 - \delta$, K_N contains a Euclidean ball of radius independent from n and N. Our main tool will be the fact that, with high probability, few vertices of the cube represent E_2^n in the classical Khintchine inequality. This statement was first proved in [30] (see also [20] for the formulation we are using in this paper).

Lemma 2.1 *Let $\delta \in (0,1)$. If $N \geq c \log(\delta^{-1}) n$, then N points $x_1, \ldots, x_N$ chosen uniformly and independently from E_2^n satisfy with probability greater than $1 - \delta$ the inequality*

$$c_1 \|y\|_2 \leq \frac{1}{N} \sum_{i=1}^{N} |\langle y, x_i \rangle| \leq c_2 \|y\|_2 \tag{2.1}$$

for all $y \in \mathbb{R}^n$, where $c, c_1, c_2 > 0$ are absolute constants. □

In particular, Lemma 2.1 holds true with $\delta = e^{-n}$ provided that $N \geq cn^2$. Let us assume that Lemma 2.1 applies for the vertices $\pm x_1, \ldots, \pm x_N$ of K_N. Note that if W_1, W_2 are convex bodies, then $W_1 \subseteq W_2$ if and only if $h_{W_1} \leq h_{W_2}$. By Lemma 2.1 we have

$$\begin{aligned} h_{K_N}(y) &= \max_{j \leq N} |\langle x_j, y\rangle| \geq \frac{1}{N}\sum_{j=1}^{N} |\langle x_j, y\rangle| \\ &\geq c_1\|y\|_2 = c_1 h_{B_2^n}(y) \end{aligned}$$

for every $y \in \mathbb{R}^n$, which shows that $K_N \supseteq c_1 B_2^n$. Thus, we have proved the following.

Proposition 2.1 *Let $\delta \in (0,1)$. If $N \geq c\log(\delta^{-1})n$, then $K_N \supseteq c_1 B_2^n$ with probability greater than $1-\delta$.* □

2.2 Affine cubes inside K_N

Our next aim is to show that if n is big enough and $N \geq n(\log n)^2$, then K_N contains (with high probability) a centered cube P such that $|K_N|^{1/n} \simeq |P|^{1/n}$. This is a consequence of the following theorem.

Theorem 2.1 *There exist $n_0 \in \mathbb{N}$ and an absolute constant $c > 0$ with the following property: If $n \geq n_0$ and $N > n(\log n)^2$, then N random points $x_1, \ldots, x_N$ chosen independently and uniformly from E_2^n satisfy with probability greater than $1-e^{-n}$*

$$M_N := \mathrm{co}\{x_1, \ldots, x_N\} \supseteq c\left(\sqrt{\log(N/n)}B_2^n \cap Q_n\right) \tag{2.2}$$

where $Q_n = [-1,1]^n$ is the unit cube in $\mathbb{R}^n$.

The proof makes heavy use of a theorem of Montgomery-Smith. Consider the interpolation norm

$$K_{1,2}(x,t) = \inf\{\|y\|_1 + t\|x-y\|_2 : y \in \mathbb{R}^n\} \tag{2.3}$$

where $x \in \mathbb{R}^n$ and $t > 0$. We will need the main result from [31].

Fact *There exists an absolute constant $r > 0$ such that for every $y \in \mathbb{R}^n$ and every $t > 0$,*

$$P(\{x \in E_2^n : \langle x, y\rangle > r^{-1}K_{1,2}(y,t)\}) \geq r^{-1}\exp(-rt^2). \quad \square \tag{2.4}$$

The geometric interpretation of $K_{1,2}$ is the following: Fix $\alpha > 0$ and consider the symmetric convex body

$$C(\alpha) = r^{-1}(\alpha B_2^n \cap Q_n). \tag{2.5}$$

Then, the support function of $C(\alpha)$ is given by

$$h_{C(\alpha)}(x) = r^{-1}\inf\{\|y\|_1 + \alpha\|x-y\|_2 : y \in \mathbb{R}^n\} = r^{-1}K_{1,2}(x,\alpha). \tag{2.6}$$

With this notation, we have:

Lemma 2.2 *Let $\alpha > 0$. For every $\theta \in S^{n-1}$,*

$$P(\{x \in E_2^n : \langle x, \theta\rangle \geq h_{C(\alpha)}(\theta)\}) \geq r^{-1}\exp(-r\alpha^2). \qquad \square \tag{2.7}$$

Let $x_1, \ldots, x_N$ be chosen independently and uniformly from E_2^n, and consider their convex hull $M_N := M(x_1, \ldots, x_N)$. Since

$$h_{M_N}(\theta) = \max_{j\leq N}\langle x_j, \theta\rangle,$$

we have

$$\begin{aligned} P\left(h_{M_N}(\theta) \leq h_{C(\alpha)}(\theta)\right) &= \left(P(\{x \in E_2^n : \langle x, \theta\rangle < h_{C(\alpha)}(\theta)\})\right)^N \\ &\leq \left(1 - r^{-1}\exp(-r\alpha^2)\right)^N \\ &\leq \exp\left(-\frac{N}{r}\exp(-r\alpha^2)\right) \end{aligned}$$

for every $\theta \in S^{n-1}$.

Let $\delta \in (0,1)$. We choose a ρ-net $\mathcal{N}$ of S^{n-1}, with cardinality $|\mathcal{N}| \leq (1+(2/\rho))^n$ (see [26], pp. 7). Then, the estimate above proves the following fact:

Lemma 2.3 *Let $N \geq n$ and $\rho, \delta \in (0,1)$, $\alpha > 0$. If*

$$\left(1 + \frac{2}{\rho}\right)^n \leq \delta \exp\left(\frac{N}{r}\exp(-r\alpha^2)\right),$$

then, with probability greater than $1-\delta$, N random points $x_1, \ldots, x_N$ chosen independently and uniformly from E_2^n satisfy

$$h_{M_N}(\theta) \geq h_{C(\alpha)}(\theta),$$

for every $\theta \in \mathcal{N}$. $\square$

Proof of Theorem 2.1: Let $N \geq n(\log n)^2$. We choose $\alpha = \frac{1}{2\sqrt{r}}\sqrt{\log(N/n)}$ and apply Lemma 2.3 with $\delta = e^{-n}$: If

$$1 + \log\left(1 + \frac{2}{\rho}\right) \leq \frac{N}{rn}\exp\left(-\frac{1}{4}\log(N/n)\right) = \frac{1}{r}\left(\frac{N}{n}\right)^{3/4}, \tag{2.8}$$

then with probability greater than $1-e^{-n}$ the convex hull M_N of $x_1, \ldots, x_N$ satisfies

$$h_{M_N}(\theta) \geq h_{C(\alpha)}(\theta)$$

for every θ in a ρ-net of S^{n-1}.

Let $u \in S^{n-1}$. There exists $\theta \in \mathcal{N}$ such that $\|u - \theta\|_2 < \rho$. Then,

$$\begin{aligned} h_{M_N}(u) &\geq h_{M_N}(\theta) - h_{M_N}(\theta - u) \geq h_{C(\alpha)}(\theta) - h_{M_N}(\theta - u) \\ &\geq h_{C(\alpha)}(u) - [h_{C(\alpha)}(\theta - u) + \|\theta - u\|_1]. \end{aligned}$$

For n large enough (depending on r) we have $\alpha \geq 1$. It follows that

$$\frac{1}{r}B_2^n \subseteq C(\alpha) \subseteq \frac{\alpha}{r}B_2^n$$

and hence

$$h_{C(\alpha)}(\theta - u) + \|\theta - u\|_1 \leq \left(\frac{\alpha}{r} + \sqrt{n}\right)\rho \leq 2r\sqrt{n}\rho h_{C(\alpha)}(u).$$

It follows that

$$h_{M_N}(u) \geq \frac{h_{C(\alpha)}(u)}{2} \tag{2.9}$$

if we choose $\rho = 1/(4r\sqrt{n})$. With this choice of α and ρ, it remains to check that (2.8) is satisfied for large enough n. The condition is equivalent to

$$c(r)\log n \leq \left(\frac{N}{n}\right)^{3/4} \tag{2.10}$$

and, since $N > n(\log n)^2$, this is satisfied if $(\log n)^{1/2} > c(r)$ which holds true for $n \geq n_0 = \exp(c^2(r))$. The theorem follows with $c = 1/(4r^{3/2})$. $\square$

Since $Q_n \supseteq B_2^n$, Theorem 2.1 implies Proposition 2.1 in the case $N \geq n(\log n)^2$. Also, since $\sqrt{n}B_2^n \supseteq Q_n$ we get the second part of Theorem A:

Theorem 2.2 *There exist $n_0 \in \mathbb{N}$ and an absolute constant $c > 0$ with the following property: If $n \geq n_0$ and $N > n(\log n)^2$, then N random points $x_1, \ldots, x_N$ chosen independently and uniformly from E_2^n satisfy with probability greater than $1 - e^{-n}$*

$$K_N \supseteq c\frac{\sqrt{\log(N/n)}}{\sqrt{n}}Q_n. \quad \square$$

Remarks An earlier version of this paper included a self-contained and elementary proof of Theorem 2.2 for N polynomial in n ($N \geq n^\kappa$, where $\kappa > 0$ is a fixed constant). It turned out that analogous statements had already appeared in the literature.

One of the referees informed us that this result can be derived (if one goes inside the proofs) from the methods of Dyer, Füredi and McDiarmid in [11]. In a very recent paper, Bárány and Pór [4] showed the existence of 0-1 polytopes with superexponential number of facets. One main step in their argument is a statement equivalent to Theorem 2.1 (Lemma 4.3 in [4]) which is proved by a refinement of the method of [11] for the range $N \geq \exp((\log n)^2)$.

We are grateful to a second referee for pointing out that Montgomery-Smith's theorem is also sufficient for proving Theorem 2.2. In fact, making full use of [31] one gets a proof of Theorem 2.1 for $N \geq n(\log n)^2$.

2.3 Volume estimates

We now pass to volume estimates. Consider the polar body

$$K_N^\circ = \{y \in \mathbb{R}^n \;:\; |\langle y, x_i \rangle| \le 1, \;\; i = 1, \dots, N\}.$$

This is an intersection of symmetric strips, and very precise lower bounds for its volume are available (see [19], [10] and [3], [7], [8] for related results):

Lemma 2.4 *There exists an absolute constant $c > 0$ such that, for every $N \ge n$*

$$|K_N^\circ|^{1/n} \ge \frac{c}{\sqrt{n \log(2N/n)}}. \qquad \square$$

Combining this estimate with the Blaschke-Santaló inequality $|K_N| \cdot |K_N^\circ| \le |B_2^n|^2$, we get

$$|K_N|^{1/n} \le c' \frac{\sqrt{\log(2N/n)}}{\sqrt{n}}. \tag{2.11}$$

On the other hand, Theorem 2.2 shows that if $N \ge n(\log n)^2$ then K_N contains a cube of about the same volume.

Proposition 2.2 *If $N \ge n(\log n)^2$, then a random K_N contains a centered cube P such that*

$$\left(\frac{|K_N|}{|P|} \right)^{1/n} \le C,$$

where $C > 0$ is an absolute constant. $\square$

This fact shows that Theorem 2.2 is optimal in a very strong sense: a random K_N has the maximal possible volume. It also determines the volume radius of K_N and K_N°:

Proposition 2.3 *If $N \ge n(\log n)^2$, then for a random K_N we have*

$$|K_N|^{1/n} \simeq \frac{\sqrt{\log(N/n)}}{\sqrt{n}}, \qquad |K_N^\circ|^{1/n} \simeq \frac{1}{\sqrt{n \log(N/n)}}$$

up to absolute constants. $\square$

2.4 Mean width

Let X be an n-dimensional normed space. Figiel and Tomczak-Jaegermann [14] defined the ℓ-norm of $T \in L(\ell_2^n, X)$ by

$$\ell(T) = \sqrt{n} \left(\int_{S^{n-1}} \|Ty\|^2 \sigma(dy) \right)^{1/2}. \tag{2.12}$$

Equivalently, if $\{e_j\}$ is any orthonormal basis in $\mathbb{R}^n$, and if $g_1, \dots, g_n$ are independent Gaussian random variables with distribution $N(0,1)$ on some probability space Ω, we have

$$\ell(T) = \left(\int_\Omega \Big\| \sum_{i=1}^n g_i(\omega) T(e_i) \Big\|^2 d\omega \right)^{1/2}, \tag{2.13}$$

From well-known results of Lewis [24], Figiel and Tomczak-Jaegermann [14] and Pisier [27] it follows that for every $X = (\mathbb{R}^n, \|\cdot\|)$ we can define an Euclidean structure $\langle \cdot, \cdot \rangle$ (called the ℓ-structure) on $\mathbb{R}^n$, for which

$$\ell(I : \ell_2^n \to X)\ell(I : \ell_2^n \to X^*) \leq cn \log[d(X, \ell_2^n) + 1], \tag{2.14}$$

where $c > 0$ is an absolute constant and I denotes the identity operator. It is not hard to check that

$$\ell(I : \ell_2^n \to Z) = \int_\Omega \| \sum_{j=1}^n g_j(\omega) e_j \|_Z d\omega \simeq \sqrt{n} w(B_Z^\circ), \tag{2.15}$$

for every n-dimensional space Z, and hence, (2.14) is equivalent to the following fact: For every symmetric convex body K in $\mathbb{R}^n$ there exists a linear image $\tilde{K} = T(K)$, $T \in GL(n)$, of K ($\tilde{K}$ is often called the "ℓ-position" of K) for which

$$w(\tilde{K}) w(\tilde{K}^\circ) \leq c \log[d(X_K, \ell_2^n) + 1]. \tag{2.16}$$

In view of Urysohn's inequality which states that for every convex body K in $\mathbb{R}^n$

$$w(K) \geq 2 \left(\frac{|K|}{|B_2^n|} \right)^{1/n} \geq c\sqrt{n} |K|^{1/n} \tag{2.17}$$

where $c > 0$ is an absolute constant, (2.16) and John's theorem show that, up to a $\log n$-term, a body which is in ℓ-position has the "minimal possible mean width": $\tilde{K}$ satisfies the inequality

$$w(\tilde{K}) \leq c' \sqrt{n} \log n |\tilde{K}|^{1/n}. \tag{2.18}$$

In this subsection we will get similar upper bounds for the mean width of a random K_N and K_N°.

Assume that $N \geq n(\log n)^2$. Starting with K_N, we write $x_j = \sqrt{n} u_j$ where $u_j \in S^{n-1}$, $j \leq N$, and hence

$$w(K_N) = \int_{S^{n-1}} \max_{j \leq N} |\langle x_j, \theta \rangle| \sigma(d\theta) = \sqrt{n} \int_{S^{n-1}} \max_{j \leq N} |\langle u_j, \theta \rangle| \sigma(d\theta). \tag{2.19}$$

Now, by the spherical isoperimetric inequality we have

$$\sigma(\theta : |\langle u_j, \theta \rangle| \geq ct/\sqrt{n}) \leq \exp(-t^2) \tag{2.20}$$

for large t (see [26]), which implies

$$\int_{S^{n-1}} \max_{j\leq N} |\langle u_j, \theta\rangle| \sigma(d\theta) \leq c_1 \frac{\sqrt{\log N}}{\sqrt{n}}, \tag{2.21}$$

therefore $w(K_N) \leq c_2\sqrt{\log N}$. Note that by Urysohn's inequality and the volume estimate in Proposition 2.3,

$$c_3\sqrt{\log(N/n)} \leq w(K_N) \leq c_2\sqrt{\log N} \tag{2.22}$$

for a random K_N. For the mean width of K_N° we use Theorem 2.2. Since $K_N \supset \big(c\sqrt{\log(N/n)}/\sqrt{n}\big)Q_n$, we have

$$h_{K_N^{\circ}}(\theta) = \|\theta\|_{K_N} \leq \frac{c_4\sqrt{n}}{\sqrt{\log(N/n)}}\|\theta\|_\infty \tag{2.23}$$

therefore

$$w(K_N^{\circ}) \leq \frac{c_4\sqrt{n}}{\sqrt{\log(N/n)}} \int_{S^{n-1}} \max_{i\leq n} |\theta_i| \sigma(d\theta) \simeq \frac{\sqrt{\log n}}{\sqrt{\log(N/n)}}. \tag{2.24}$$

This is again close to the lower bound, apart from the $\sqrt{\log n}$-term. In particular,

$$w(K_N)w(K_N^{\circ}) \leq c_5 \frac{\sqrt{\log N}}{\sqrt{\log(N/n)}}\sqrt{\log n}, \tag{2.25}$$

that is K_N satisfies an inequality analogous to (2.16). This fact will be later used in Banach-Mazur distance estimates.

Proposition 2.4 *If $n \geq n_0$ and $N \geq n(\log n)^2$ then for a random K_N we have*

$$c\sqrt{\log(N/n)} \leq w(K_N) \leq c'\sqrt{\log N}$$

and

$$\frac{c}{\sqrt{\log(N/n)}} \leq w(K_N^{\circ}) \leq \frac{c'\sqrt{\log n}}{\sqrt{\log(N/n)}},$$

where $c, c' > 0$ are absolute constants. □

3. Asymptotic properties of X_N

Theorem A provides enough information on the geometry of the unit ball of X_N. In fact, K_N and K_N° belong to a rather restricted class of random convex bodies, and this allows us to determine several asymptotic parameters of the corresponding spaces.

3.1 Unconditionality properties of X_N

We shall first show that unconditionality properties of X_N are of the worst possible order as N decreases to n. This fact is expected in view of well-known results from [13] and [1] about random proportional sections of ℓ_∞^m which exhibit the same pathology. The source of our estimates is Lemma 2.1 which is the analogue of Kashin's theorem [23] in our context. However, our information on a random K_N, will allow us to give an estimate for the full range of values of N.

Recall that an n-dimensional normed space Y has 1-unconditional basis if there exists a basis $\{e_1, \ldots, e_n\}$ of Y with the property

$$\Big\| \sum_{i=1}^{n} t_i e_i \Big\|_Y = \Big\| \sum_{i=1}^{n} |t_i| e_i \Big\|_Y$$

for every choice of reals $t_1, \ldots, t_n$.

Theorem 3.1 *If $N \geq c \log(\delta^{-1}) n$, then $X_N \in \mathcal{B}_N$ satisfies with probability greater than $1 - \delta$*

$$d(X_N, Y) \geq \frac{c\sqrt{n}}{\sqrt{\log{(2N/n)}}}$$

for every n-dimensional normed space Y with a 1-unconditional basis.

Proof: Consider the identity operator $I : X_N^* \to \ell_2^n$. Recall (see [29]) that the 1-summing norm $\pi_1(T : Y^* \to \ell_2^n)$ of an operator $T : Y^* \to \ell_2^n$ is the minimum of all positive constants A with the following property: for every $m \in \mathbb{N}$ and every choice of vectors $z_1, \ldots, z_m \in Y^*$

$$\sum_{j=1}^{m} \|Tz_j\|_2 \leq A \sup_{y \in B_Y} \sum_{j=1}^{m} |\langle y, z_j \rangle|. \tag{3.1}$$

We will first prove the following claim:

Claim 1 *If $x_1, \ldots, x_N$ satisfy the conclusion of Lemma 2.1, then $\pi_1(I) \simeq 1$.*

Proof: Let $z_1, \ldots, z_m \in X_N^*$. Using Lemma 2.1, we write

$$\sum_{j=1}^{m} \|z_j\|_2 \leq \frac{1}{c_1} \cdot \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{m} |\langle x_i, z_j \rangle| \leq \frac{1}{c_1} \cdot \sup_{y \in K_N} \sum_{j=1}^{m} |\langle y, z_j \rangle|, \tag{3.2}$$

therefore, $\pi_1(I) \leq c_1^{-1}$. On the other hand, by the definition of $\pi_1(I)$ we must also have

$$\frac{1}{N} \sum_{j=1}^{N} \|x_j\|_2 \leq \pi_1(I) \sup_{y \in K_N} \frac{1}{N} \sum_{j=1}^{N} |\langle y, x_j \rangle|. \tag{3.3}$$

From the upper estimate in Lemma 2.1 we get

$$\sup_{y \in K_N} \frac{1}{N} \sum_{j=1}^{N} |\langle y, x_j \rangle| \leq c_2 \sup_{y \in K_N} \|y\|_2. \tag{3.4}$$

Since $K_N \subseteq Q_n \subseteq \sqrt{n}B_2^n$ and $\|x_j\|_2 = \sqrt{n}$ for every $j = 1, \ldots, N$, we conclude that $\sqrt{n} \leq c_2\pi_1(I)\sqrt{n}$, which proves our claim. □

Claim 2 *Let Q be a parallelepiped contained in K_N°. Then,*

$$|Q|^{1/n} \leq \frac{2\pi_1(I)}{n}. \tag{3.5}$$

Proof: This fact was proved by Ball [1]. We include the argument for self - completeness. Consider a linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ which takes B_∞^n onto Q. Applying Hadamard's inequality, the arithmetic-geometric means inequality and the definition of $\pi_1(S : \ell_\infty^n \to \ell_2^n)$, we have

$$\begin{aligned} |Q|^{1/n} &= 2|\det S|^{1/n} \leq 2\left(\prod_{i=1}^n \|Se_i\|_2\right)^{1/n} \leq \frac{2}{n}\sum_{i=1}^n \|Se_i\|_2 \\ &\leq \frac{2}{n}\pi_1(S : \ell_\infty^n \to \ell_2^n) \cdot \sup_{y \in B_1^n} \sum_{i=1}^n |\langle y, e_i\rangle| \\ &= 2\pi_1(S : \ell_\infty^n \to \ell_2^n)/n. \end{aligned}$$

Since

$$\pi_1(S : \ell_\infty^n \to \ell_2^n) \leq \|S : \ell_\infty^n \to X_N^*\| \cdot \pi_1(I : X_N^* \to \ell_2^n),$$

the result follows because $S(B_\infty^n) = Q \subseteq K_N^\circ$. □

We can now complete the proof of Theorem 3.1. Let Y be an n-dimensional space with 1-unconditional basis. If $d = d(X_N^*, Y)$, we may assume that $dB_Y \supseteq K_N^\circ \supseteq B_Y$. From a result of Losanovskii (see [28, Chapter 3]), we can find a parallelepiped $Q \subseteq B_Y$ with $|B_Y|/|Q| \leq n^n/n!$. Then, using our two claims we get

$$|K_N^\circ|^{1/n} \leq d|B_Y|^{1/n} \leq cd|Q|^{1/n} \leq c'd/n. \tag{3.7}$$

Now, Lemma 2.4 implies that $d(X_N^*, Y) = d \geq c''\sqrt{n}/\sqrt{\log(2N/n)}$, and the Theorem follows by duality. □

Remark Suitable choice of $\lambda > 1$ shows the existence of a space X_N with $N = \lambda n$ for which $d(X_N, Y) \geq c\sqrt{n}$ for every space Y with 1-unconditional basis.

Let $N \geq n(\log n)^2$. Theorem 2.2 shows that $d(X_N, \ell_\infty^n) \leq c\sqrt{n}/\sqrt{\log(2N/n)}$ for a random X_N. Combining this fact with Theorem 3.1, we see that ℓ_∞^n is the space with 1-unconditional basis which is "closest" to X_N.

Theorem 3.2 *If $N \geq n(\log n)^2$, then for a random X_N we have*

$$d(X_N, \mathcal{U}) \simeq d(X_N, \ell_\infty^n) \simeq \frac{\sqrt{n}}{\sqrt{\log(N/n)}},$$

where $\mathcal{U}$ is the class of n-dimensional spaces with 1-unconditional basis. □

3.2 Banach-Mazur distance estimates

Propositions 2.1 and 2.4 indicate that the geometric distance between K_N and B_2^n and the mean width of K_N are simultaneously controlled for a random X_N. This allows us to use the method of random orthogonal factorizations (which has its origin in work of Tomczak-Jaegermann [36], and was later developped in [5], [12]) in order to estimate from above Banach-Mazur distances from a random X_N to various classes of spaces.

The main point of the above method is the following result of Benyamini and Gordon [5], which makes use of an inequality of Chevet [9].

Lemma 3.1 *Let X and Y be two n-dimensional normed spaces. Then,*

$$\begin{aligned} d(X,Y) \quad \leq \quad & \frac{c}{n}\big[\|I : X \to \ell_2^n\|\ell(I : \ell_2^n \to Y) + \|I : \ell_2^n \to Y\|\ell(I : \ell_2^n \to X^*)\big] \\ & \times\big[\|I : Y \to \ell_2^n\|\ell(I : \ell_2^n \to X) + \|I : \ell_2^n \to X\|\ell(I : \ell_2^n \to Y^*)\big], \end{aligned}$$

where $c > 0$ is an absolute constant. □

We shall apply this method to estimate the distance $d(X_N, X_N^*)$. The best known general estimate on this question is due to Bourgain and Milman [6] who have proved that

$$d(X, X^*) \leq cn^{5/6} \log^\beta n \tag{3.8}$$

for every n-dimensional normed space X. The proof of (3.8) is again based on random orthogonal factorizations. If X has a 1-unconditional basis or enough symmetries, then it gives a much better bound of the order of $\sqrt{n}\log^\beta n$.

As we will see, despite the lack of unconditionality exhibited by X_N, we have a bound of this order for $d(X_N, X_N^*)$.

Theorem 3.3 *There exists an absolute constant $C > 0$ such that $d(X_N, X_N^*) \leq C\sqrt{n \log n}$ for any $N \geq n$ and a random X_N.*

Proof: We apply Lemma 3.1 with $X = X_N$ and $Y = X_N^*$. Taking into account (2.15), we get

$$d(X_N, X_N^*) \leq c\|I : X_N \to \ell_2^n\| \cdot \|I : \ell_2^n \to X_N\| \cdot w(K_N)w(K_N^\circ). \tag{3.9}$$

We first consider the case $N \geq n^2$. By Proposition 2.4, for a random K_N we have

$$w(K_N) \leq c_2\sqrt{\log N}, \qquad w(K_N^\circ) \leq c_3\sqrt{\log n}/\sqrt{\log N}. \tag{3.10}$$

Since $K_N \subseteq Q_n \subseteq \sqrt{n}B_2^n$, we have

$$\|I : X_N \to \ell_2^n\| = \max_{x\in K_N} \|x\|_2 \leq \sqrt{n}, \tag{3.11}$$

and, by Proposition 2.1 we have $cB_2^n \subseteq K_N$ for a random K_N, therefore

$$\|I : \ell_2^n \to X_N\| = \max_{x\in B_2^n} \|x\|_{K_N} \leq c^{-1}. \tag{3.12}$$

Combining the above, we get

$$d(X_N, X_N^*) \leq C\sqrt{n \log n} \tag{3.13}$$

for a random X_N, $N \geq n^2$.

If $N \leq n^2$, we argue in a different way. Since K_N has n^α vertices and K_N° has n^β facets, with $\alpha = \beta \leq 2$, we can employ a well-known estimate from [16] to get

$$d(X_N, X_N^*) \leq c\sqrt{n \log n}. \tag{3.14}$$

Hence, we can find $C > 0$ such that, independently of N, $d(X_N, X_N^*) \leq C\sqrt{n \log n}$ for a random X_N. □

3.3 Isotropic constants

Recall the definition of the isotropic position of a convex body W in $\mathbb{R}^n$. There exists $T_0 \in GL(n)$ such that the body $\tilde{W} = T_0(W)$ has volume 1 and satisfies the isotropic condition

$$\int_{\tilde{W}} \langle x, \theta \rangle^2 dx = L_W^2 \tag{3.15}$$

for every $\theta \in S^{n-1}$ (see [25] for a detailed account on this topic). This position is unique up to orthogonal transformations, therefore L_W is an invariant of the linear class of W, and it is called the isotropic constant of W. One can check that the isotropic position of W minimizes the quantity

$$\frac{1}{|T(W)|^{1+\frac{2}{n}}} \int_{T(W)} \|x\|_2^2 dx \tag{3.16}$$

over all $T \in GL(n)$. In particular,

$$nL_W^2 \leq \frac{1}{|W|^{1+\frac{2}{n}}} \int_W \|x\|_2^2 dx. \tag{3.17}$$

It is conjectured that there exists an absolute constant $C > 0$ such that $L_W \leq C$ for every $n \in \mathbb{N}$ and every convex body W in $\mathbb{R}^n$. The best known general estimate is due to Bourgain [2] who proved that $L_W \leq c\sqrt[4]{n} \log n$ for every symmetric convex body W. The conjecture is related to the slicing problem, which asks if there exists an absolute constant $c > 0$ such that every convex body with volume 1 has a hyperplane section whose volume exceeds c. The connection comes from the fact that

$$c_1 \leq L_W \cdot |W \cap \theta^\perp| \leq c_2 \tag{3.18}$$

for every $\theta \in S^{n-1}$ and every isotropic convex body W, where $c_1, c_2 > 0$ are absolute constants (see [25]).

In this subsection we give upper bounds for the isotropic constant of K_N and K_N°. We will make use of the following lemma [25].

Lemma 3.2 *Let W be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$L_W \leq \frac{c}{n} \cdot \frac{1}{|W|^{1+\frac{1}{n}}} \int_W \|x\|_1 dx. \tag{3.19}$$

Also, $L_W \leq cd(X_W, Y)$ for every $Y \in \mathcal{U}$. □

Theorem 3.4 *Let $N \geq n(\log n)^2$. With probability greater than $1 - e^{-n}$ we have $L_{K_N^\circ} \leq C$, where $C > 0$ is an absolute constant.*

Proof: By Theorem 2.2, with probability greater than $1 - e^{-n}$ we have $K_N \supseteq \big(c\sqrt{\log(N/n)}/\sqrt{n}\big)Q_n$, where $c > 0$ is an absolute constant. It follows that

$$\|x\|_1 \leq \frac{c_1\sqrt{n}}{\sqrt{\log(N/n)}}\|x\|_{K_N^\circ} \tag{3.20}$$

for all $x \in \mathbb{R}^n$, where $c_1 = 1/c$. Using Lemma 3.2 we get

$$L_{K_N^\circ} \leq \frac{c_2}{\sqrt{n\log(N/n)}} \frac{1}{|K_N^\circ|^{1+\frac{1}{n}}} \int_{K_N^\circ} \|x\|_{K_N^\circ} dx \leq \frac{c_2}{\sqrt{n\log(N/n)}|K_N^\circ|^{1/n}}. \tag{3.21}$$

In view of Lemma 2.4, the proof is complete. □

Remark Junge [22] has proved that if X is an n-dimensional subspace of an N-dimensional space with 1-unconditional basis, then

$$L_{B_X} \leq c\sqrt{\log(2N/n)}$$

for some absolute constant $c > 0$. This estimate applies to any symmetric convex body with N facets. Using this result for $\mathcal{B}_N^*$ when $n \leq N \leq n(\log n)^2$, we may summarize as follows:

Corollary 3.1 *There exist absolute constants $c, C > 0$ with the following property:*

(a) *If $n \leq N \leq n(\log n)^2$, then $L_{K_N^\circ} \leq c\sqrt{\log(2N/n)} \leq C\sqrt{\log\log n}$.*

(b) *If $N \geq n(\log n)^2$, then $L_{K_N^\circ} \leq C$ for a random K_N°.* □

Observe that we have a $\sqrt{\log\log n}$ estimate for a random $X_N^* \in \mathcal{B}_N^*$, which holds true in the full range of values of N.

We conclude this article with some simple estimates for the isotropic constant of K_N.

Proposition 3.1 *Let $N \geq n(\log n)^2$. For a random K_N we have*

$$L_{K_N} \leq C\frac{\min\{\log N, \sqrt{n}\}}{\sqrt{\log(N/n)}},$$

where $C > 0$ is an absolute constant.

Proof: Since $d(X_N, \ell_\infty^n) \simeq \sqrt{n}/\sqrt{\log(N/n)}$ for a random X_N, the estimate $L_{K_N} \leq c_1\sqrt{n}/\sqrt{\log(N/n)}$ is an immediate consequence of Lemma 3.2.

If N is not too big, then one can argue in a different way: consider the external volume ratio $\mathrm{evr}(W) = \inf\big(|E|/|W|\big)^{1/n}$ of W, where the infimum is taken over all ellipsoids E which contain W. Then, we have the following.

Claim *Let $z_1, \ldots, z_N \in \mathbb{R}^n$. If $W = \mathrm{co}\{\pm x_1, \ldots, \pm x_N\}$, then*

$$L_W \leq c \log N \frac{\mathrm{evr}(W)}{\sqrt{n}} \tag{3.22}$$

where $c > 0$ is an absolute constant.

Proof: The formulation of the claim is invariant under linear transformations, therefore we may assume that W is isotropic. Let E be the ellipsoid of minimal volume which contains W. There exists a symmetric and positive $T \in GL(n)$ such that $T(E) = B_2^n$. Then,

$$\int_W \langle Tx, x\rangle dx = [\mathrm{tr}(T)] L_W^2 \geq n L_W^2 |\det T|^{1/n}. \tag{3.23}$$

The equality comes from the isotropic condition (3.15) and the inequality is the arithmetic-geometric means inequality for the eigenvalues of T. On the other hand,

$$\begin{aligned} \int_W \langle Tx, x\rangle dx &\leq \int_W \|Tx\|_{W^\circ} dx = \int_W \max_{j\leq N} |\langle z_j, Tx\rangle| dx \\ &= \int_W \max_{j\leq N} |\langle Tz_j, x\rangle| dx. \end{aligned}$$

Since the ψ_1-norm of linear functionals on W is equivalent to their 1-norm (see [25]), we get

$$\int_W \max_{j\leq N} |\langle Tz_j, x\rangle| dx \leq (c\log N) L_W \cdot \max_{j\leq N} \|Tz_j\|_2. \tag{3.24}$$

Since $z_j \in E$, we have $\|Tz_j\|_2 \leq 1$, $j = 1, \ldots, N$. It follows that

$$n |\det T|^{1/n} L_W \leq c \log N \tag{3.25}$$

and the result follows from $|\det T|^{-1/n} |B_2^n|^{1/n} = |E|^{1/n} = \mathrm{evr}(W)$. □

We can now show that $L_{K_N} \leq c_2 \sqrt{\log(N/n)}$: we observe that $K_N \subseteq \sqrt{n} B_2^n$ and using the fact that $\big(c_3 \sqrt{\log(N/n)}/\sqrt{n}\big) Q_n \subseteq K_N$ we get

$$\mathrm{evr}(K_N) \leq \sqrt{n} \frac{|B_2^n|^{1/n}}{|K_N|^{1/n}} \leq \frac{c\sqrt{n}}{\sqrt{\log(N/n)}}. \tag{3.26}$$

Then, our Claim completes the proof. □

Remark On the other hand, Junge [22] has proved that the unit balls of projections of N-dimensional spaces with 1-unconditional basis have isotropic constant bounded by $c \log N$. Since K_N is the unit ball of a projection of ℓ_1^N, we see that $L_{K_N} \leq c \log n$ if $N \leq n(\log n)^2$.

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