Fractional Helly theorem for the diameter of convex sets

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Abstract

We provide a new quantitative version of Helly's theorem: there exists an absolute constant $\alpha > 1$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{int}\left(\bigcap_{i\in I} P_i\right) \neq \emptyset$, then there exist $z \in \mathbb{R}^n$, $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that

$$z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq cn^{3/2}\left(z + \bigcap_{i\in I} P_i\right),$$

where $c > 0$ is an absolute constant. This directly gives a version of the "quantitative" diameter theorem of Bárány, Katchalski and Pach, with a polynomial dependence on the dimension. In the symmetric case the bound $O(n^{3/2})$ can be improved to $O(\sqrt{n})$.

1 Introduction

The purpose of this work is to present a new quantitative versions of Helly's theorem; recall that the classical result asserts that if $\mathcal{F} = \{F_i : i \in I\}$ is a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$ and if any $n+1$ members of $\mathcal{F}$ have non-empty intersection then $\bigcap_{i\in I} F_i \neq \emptyset$. Variants of this statement have found important applications in discrete and computational geometry. Quantitative Helly-type results were first obtained by Bárány, Katchalski and Pach in [3] (see also [4]). In particular, they proved the following volumetric result:

Let $\{P_i : i \in I\}$ be a family of closed convex sets in $\mathbb{R}^n$ such that $\left|\bigcap_{i\in I} P_i\right| > 0$. There exist $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant C_n \left|\bigcap_{i\in I} P_i\right|, \tag{1.1}$$

where $C_n > 0$ is a constant depending only on n.

The example of the cube $[-1,1]^n$ in $\mathbb{R}^n$, expressed as an intersection of exactly $2n$ closed half-spaces, shows that one cannot replace $2n$ by $2n-1$ in the statement above. The optimal growth of the constant C_n as a function of n is not completely understood. The bound in [3] was $O(n^{2n^2})$ and it was conjectured that one might actually have $C_n \leqslant n^{cn}$ for an absolute constant $c > 0$. Naszódi [12] has recently proved a volume version of Helly's theorem with $C_n \leqslant (cn)^{2n}$, where $c > 0$ is an absolute constant. In fact, a slight modification of Naszódi's argument leads to the exponent $\frac{3n}{2}$ instead of $2n$. In [7], relaxing the requirement that $s \leqslant 2n$ to the weaker one that $s = O(n)$, we were able to improve the exponent to n:

Theorem 1.1 (Brazitikos). *There exists an absolute constant $\alpha > 1$ with the following property: for every family $\{P_i : i \in I\}$ of closed convex sets in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume, there exist $s \leqslant \alpha n$ and $i_1, \dots, i_s \in I$ such that*

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (cn)^n \, |P|, \tag{1.2}$$

where $c > 0$ is an absolute constant.

Following the terminology of [11], a Helly-type property is (loosely speaking) a property Π for which there exists $m \in \mathbb{N}$ such that if $\{P_i : i \in I\}$ is a finite family of objects such that every subfamily with m elements satisfies Π, then the whole family satisfies Π. Thus, the previous results (in particular, Theorem 1.1) express the fact that the property that "an intersection has large volume" is a Helly-type property for the class of convex sets.

Bárány, Katchalski and Pach studied the question if the property that "an intersection has large diameter" is also a Helly-type property for the class of convex sets. In [3] they gave a first quantitative answer to this question:

Let $\{P_i : i \in I\}$ be a family of closed convex sets in $\mathbb{R}^n$ such that $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$. *There exist $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that*

$$\mathrm{diam}\,(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant (cn)^{n/2}, \tag{1.3}$$

where $c > 0$ is an absolute constant.

In the same work the authors conjecture that the bound should be polynomial in n; in fact they ask if $(cn)^{n/2}$ can be replaced by $c\sqrt{n}$. Relaxing the requirement that $s \leqslant 2n$, exactly as in [7], we provide a positive answer, although we are not able to achieve a bound of the order of $\sqrt{n}$.

Starting with the symmetric case, our main result is the next theorem.

Theorem 1.2. *Let $\{P_i : i \in I\}$ be a finite family of symmetric convex sets in $\mathbb{R}^n$ with* $\mathrm{int}\left(\bigcap_{i\in I} P_i\right) \neq \emptyset$. *For every $d > 1$ there exist $s \leqslant dn$ and $i_1, \dots i_s \in I$ such that*

$$P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_d \sqrt{n} \left(\bigcap_{i\in I} P_i \right), \tag{1.4}$$

where $\gamma_d := \frac{\sqrt{d}+1}{\sqrt{d}-1}$.

The proof of Theorem 1.2 is presented in Section 3; it is based on a lemma of Barvinok from [5] which, in turn, exploits a theorem of Batson, Spielman and Srivastava from [6]. It is clear that the $\sqrt{n}$-dependence cannot be improved (we provide a simple example).

In the general (not necessarily symmetric) case, using a similar strategy and ideas that were developed in [7] and employ a more delicate theorem of Srivastava from [14], we obtain the next estimate.

Theorem 1.3. *There exists an absolute constant $\alpha > 1$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{int}\left(\bigcap_{i\in I} P_i\right) \neq \emptyset$, *then there exist $z \in \mathbb{R}^n$, $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that*

$$z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq cn^{3/2} \left(z + \bigcap_{i\in I} P_i \right), \tag{1.5}$$

where $c > 0$ is an absolute constant.

It is clear that Theorem 1.2 and Theorem 1.3 imply polynomial estimates for the diameter:

Theorem 1.4. (a) *Let $\{P_i : i \in I\}$ be a finite family of symmetric convex sets in $\mathbb{R}^n$ with* $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$. *For every $d > 1$ there exist $s \leqslant dn$ and $i_1, \dots i_s \in I$ such that*

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant \gamma_d \sqrt{n}, \tag{1.6}$$

where $\gamma_d := \frac{\sqrt{d}+1}{\sqrt{d}-1}$.

(b) *There exists an absolute constant $\alpha > 1$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$, *then there exist $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that*

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant cn^{3/2}, \tag{1.7}$$

where $c > 0$ is an absolute constant.

2 Notation and background

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto F. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1}\cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1K \subseteq L \subseteq c_2K$.

We refer to the book of Schneider [13] for basic facts from the Brunn-Minkowski theory and to the book of Artstein-Avidan, Giannopoulos and V. Milman [1] for basic facts from asymptotic convex geometry.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$, and that K is centered if its barycenter

$$\mathrm{bar}(K) = \frac{1}{|K|}\int_K x\,dx \tag{2.1}$$

is at the origin. The circumradius of K is the radius of the smallest ball which is centered at the origin and contains K:

$$R(K) = \max\{\|x\|_2 : x \in K\}. \tag{2.2}$$

If $0 \in \mathrm{int}(K)$ then the polar body K° of K is defined by

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\}. \tag{2.3}$$

and the Minkowski functional of K is defined by

$$p_K(x) = \min\{t \geqslant 0 : x \in tK\}. \tag{2.4}$$

Recall that p_K is subadditive and positively homogeneous.

We say that a convex body K is in John's position if the ellipsoid of maximal volume inscribed in K is the Euclidean unit ball B_2^n. John's theorem [10] states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist $v_1, \ldots, v_m \in \mathrm{bd}(K)\cap S^{n-1}$ (contact points of K and B_2^n) and positive real numbers $a_1, \ldots, a_m$ such that

$$\sum_{j=1}^m a_j v_j = 0 \tag{2.5}$$

and the identity operator I_n is decomposed in the form

$$I_n = \sum_{j=1}^m a_j v_j \otimes v_j, \tag{2.6}$$

where $(v_j \otimes v_j)(y) = \langle v_j, y\rangle v_j$. In the case where K is symmetric, the second condition (2.6) is enough (for any contact point u we have that $-u$ is also a contact point, and hence, having (2.6) we may easily produce a decomposition for which (2.5) is also satisfied). In analogy to John's position, we say that a convex body K is in Löwner's position if the ellipsoid of minimal volume containing K is the Euclidean unit ball B_2^n. One can check that this holds true if and only if K° is in John position; in particular, we have a decomposition of the identity similar to (2.6).

Assume that $v_1, \ldots, v_m$ are unit vectors that satisfy John's decomposition (2.6) with some positive weights a_j. Then, one has the useful identities

$$\sum_{j=1}^m a_j = \mathrm{tr}(I_n) = n \quad\text{and}\quad \sum_{j=1}^m a_j\langle v_j, z\rangle^2 = 1 \tag{2.7}$$

for all $z \in S^{n-1}$. Moreover,

$$\text{conv}\{v_1, \ldots, v_m\} \supseteq \frac{1}{n} B_2^n. \tag{2.8}$$

In the symmetric case we actually have

$$\text{conv}\{\pm v_1, \ldots, \pm v_m\} \supseteq \frac{1}{\sqrt{n}} B_2^n. \tag{2.9}$$

3 Symmetric case

Our main tool for the symmetric case is a lemma of Barvinok from [5], which exploits the next theorem of Batson, Spielman and Srivastava [6] on extracting an approximate John's decomposition with few vectors from a John's decomposition of the identity.

Theorem 3.1 (Batson-Spielman-Srivastava)**.** *Let $v_1, \ldots, v_m \in S^{n-1}$ and $a_1, \ldots, a_m > 0$ such that*

$$I_n = \sum_{j=1}^{m} a_j v_j \otimes v_j. \tag{3.1}$$

Then, for every $d > 1$ there exists a subset $\sigma \subseteq \{1, \ldots, m\}$ with $|\sigma| \leqslant dn$ and $b_j > 0$, $j \in \sigma$, such that

$$I_n \preceq \sum_{j \in \sigma} b_j a_j v_j \otimes v_j \preceq \gamma_d^2 I_n, \tag{3.2}$$

where $\gamma_d := \frac{\sqrt{d}+1}{\sqrt{d}-1}$.

Here, given two symmetric positive definite matrices A and B we write $A \preceq B$ if $\langle Ax, x\rangle \leqslant \langle Bx, x\rangle$ for all $x \in \mathbb{R}^n$. Barvinok's lemma is the next statement.

Lemma 3.2 (Barvinok)**.** *Let $C \subset \mathbb{R}^n$ be a compact set. Then, there exists a subset $X \subseteq C$ of cardinality $\text{card}(X) \leqslant dn$ such that for any $z \in \mathbb{R}^n$ we have*

$$\max_{x \in X} |\langle z, x\rangle| \leqslant \max_{x \in C} |\langle z, x\rangle| \leqslant \gamma_d \sqrt{n} \max_{x \in X} |\langle z, x\rangle| \tag{3.3}$$

Proof. We sketch the proof for completeness. We may assume that C spans $\mathbb{R}^n$ and, by the linear invariance of the statement, that B_2^n is the origin symmetric ellipsoid of minimal volume containing C. Then, there exist $v_1, \ldots, v_m \in C \cap S^{n-1}$ and $a_1, \ldots, a_m > 0$ such that (3.1) is satisfied. Then, applying Theorem 3.1 we may find a subset $\sigma \subseteq \{1, \ldots, m\}$ with $\text{card}(\sigma) \leqslant dn$ and $b_j > 0$, $j \in \sigma$, such that (3.2) holds true. In particular,

$$n \leqslant \sum_{j \in \sigma} a_j b_j = \text{tr}\left(\sum_{j \in \sigma} b_j a_j v_j \otimes v_j\right) \leqslant \gamma_d^2 n. \tag{3.4}$$

Given $z \in \mathbb{R}^n$, from (3.2) and (3.4) we have

$$\|z\|_2^2 \leqslant \sum_{j \in \sigma} b_j a_j \langle z, v_j\rangle^2 \leqslant \gamma_d^2 n \max_{j \in \sigma} |\langle z, v_j\rangle|^2, \tag{3.5}$$

and using the fact that $C \subseteq B_2^n$ we conclude that

$$\max_{x \in C} |\langle z, x\rangle| \leqslant \|z\| \leqslant \gamma_d \sqrt{n} \max_{j \in \sigma} |\langle z, v_j\rangle|. \tag{3.6}$$

Setting $X = \{v_j : j \in \sigma\}$ we conclude the proof. □

Using Barvinok's lemma we can prove Theorem 1.2.

Proof of Theorem 1.2. Let $P = \bigcap_{i\in I} P_i$ and consider its polar body

$$P^\circ = \mathrm{conv}\left(\bigcup_{i\in I} P_i^\circ\right). \tag{3.7}$$

Using Lemma 3.2 for $C = P^\circ$ we may find $X = \{v_1, \dots, v_s\} \subset P^\circ$ with $\mathrm{card}(X) = s \leqslant dn$ such that

$$\max_{x\in P^\circ} |\langle z, x\rangle| \leqslant \gamma_d \sqrt{n} \max_{x\in X} |\langle z, x\rangle| \tag{3.8}$$

for all $z \in \mathbb{R}^n$. It follows that

$$P^\circ \subseteq \gamma_d \sqrt{n}\, \mathrm{conv}(\{\pm v_1, \dots, \pm v_s\}). \tag{3.9}$$

From the proof of Lemma 3.2 we see that $v_1, \dots, v_s$ may be chosen to be contact points of P° with its minimal volume ellipsoid, and hence it is simple to check that we actually have $v_j \in \bigcup_{i\in I} P_i^\circ$ for all $j = 1, \dots, s$. In other words, we may find $i_1, \dots, i_s \in I$ such that $v_j \in P_{i_j}$, $j = 1, \dots, s$. Then, (3.9) implies that

$$P^\circ \subseteq \gamma_d \sqrt{n}\, \mathrm{conv}(P_{i_1}^\circ \cup \dots \cup P_{i_s}^\circ), \tag{3.10}$$

and passing to the polar bodies, we get

$$P_{i_1} \cap \dots \cap P_{i_s} \subseteq \gamma_d \sqrt{n}\, P \tag{3.11}$$

as claimed. □

Remark 3.3. Theorem 1.2 is sharp in the following sense: we can find $w_1, \dots, w_N \in S^{n-1}$ (assuming that N is exponential in the dimension n) such that

$$B_2^n \subseteq \bigcap_{j=1}^{N} P_j \subseteq 2B_2^n, \tag{3.12}$$

where

$$P_j = \{x \in \mathbb{R}^n : |\langle x, w_j\rangle| \leqslant 1\}. \tag{3.13}$$

For any $s \leqslant dn$ and any choice of $j_1, \dots, j_s \in \{1, \dots, N\}$, well-known lower bounds for the volume of intersections of strips, due to Carl-Pajor [8], Gluskin [9] and Ball-Pajor [2] show that

$$|P_{j_1} \cap \dots \cap P_{j_s}|^{1/n} \geqslant \frac{2}{\sqrt{e}\sqrt{\log(1+d)}}. \tag{3.14}$$

Therefore, if $P_{j_1} \cap \dots \cap P_{j_s} \subseteq \alpha \bigcap_{j=1}^{N} P_j$ for some $\alpha > 0$, comparing volumes we see that

$$\alpha \geqslant \frac{|P_{j_1} \cap \dots \cap P_{j_s}|^{1/n}}{|2B_2^n|^{1/n}} \geqslant \frac{c}{\sqrt{\log(1+d)}}\sqrt{n}, \tag{3.15}$$

where $c > 0$ is an absolute constant.

4 General case

In order to deal with the not-necessarily symmetric case we use the next theorem of Srivastava from [14].

Theorem 4.1 (Srivastava). *Let $v_1, \ldots, v_m \in S^{n-1}$ and $a_1, \ldots, a_m > 0$ such that*

$$I_n = \sum_{j=1}^{m} a_j v_j \otimes v_j \quad \textit{and} \quad \sum_{j=1}^{m} a_j v_j = 0. \tag{4.1}$$

Given $\varepsilon > 0$ we can find a subset σ of $\{1, \ldots, m\}$ of cardinality $|\sigma| = O_\varepsilon(n)$, positive scalars b_i, $i \in \sigma$ and a vector v with

$$\|v\|_2^2 \leqslant \frac{\varepsilon}{\sum_{i\in\sigma} b_i}, \tag{4.2}$$

such that

$$I_n \preceq \sum_{i\in\sigma} b_i (v_i + v) \otimes (v_i + v) \preceq (4 + \varepsilon) I_n \tag{4.3}$$

and

$$\sum_{i\in\sigma} b_i (v_i + v) = 0. \tag{4.4}$$

Proposition 4.2. *There exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body whose minimal volume ellipsoid is the Euclidean unit ball, then there is a subset $X \subset K$ of cardinality* $\mathrm{card}(X) \leqslant \alpha n$ *such that*

$$B_2^n \subseteq c n^{3/2} \mathrm{conv}(X), \tag{4.5}$$

where $c > 0$ is an absolute constant.

Proof. As in the proof of Lemma 3.2 we assume that B_2^n is the minimal volume ellipsoid of K, and we find $v_j \in K \cap S^{n-1}$ and $a_j > 0$, $j \in J$, such that

$$I_n = \sum_{j\in J} a_j v_j \otimes v_j \quad \text{and} \quad \sum_{j\in J} a_j v_j = 0. \tag{4.6}$$

We fix $\varepsilon > 0$, which will be chosen small enough, and we apply Theorem 4.1 to find a subset $\sigma \subseteq J$ with $|\sigma| \leqslant \alpha_1(\varepsilon) n$, positive scalars b_j, $j \in \sigma$ and a vector v such that

$$I_n \preceq \sum_{j\in\sigma} b_j (v_j + v) \otimes (v_j + v) \preceq (4 + \varepsilon) I_n \tag{4.7}$$

and

$$\sum_{j\in\sigma} b_j (v_j + v) = 0 \ \text{ and } \ \|v\|_2^2 \leqslant \frac{\varepsilon}{\sum_{j\in\sigma} b_j}. \tag{4.8}$$

Note that

$$\mathrm{tr}\left(\sum_{j\in\sigma} b_j (v_j + v) \otimes (v_j + v) \right) = \sum_{j\in\sigma} b_j - \left(\sum_{j\in\sigma} b_j \right) \|v\|_2^2 \tag{4.9}$$

and hence from (4.7) we get that

$$n \leqslant \sum_{j\in\sigma} b_j - \left(\sum_{j\in\sigma} b_j\right) \|v\|_2^2 \leqslant (4+\varepsilon)n.$$

Now, using (4.8) we get

$$n \leqslant \sum_{j\in\sigma} b_j \leqslant (4+2\varepsilon)n. \tag{4.10}$$

In particular,

$$\|v\|_2^2 \leqslant \frac{\varepsilon}{\sum_{j\in\sigma} b_j} \leqslant \frac{\varepsilon}{n}. \tag{4.11}$$

From John's theorem we know that $\text{conv}\{v_j, j\in J\} \supseteq \frac{1}{n}B_2^n$. Then, for the vector $w = \frac{v}{\sqrt{\varepsilon n}}$ we have $\|w\|_2 \leqslant \frac{1}{n}$ and hence $w \in \text{conv}\{v_j, j\in J\}$. Carathéodory's theorem shows that there exist $\tau \subseteq J$ with $|\tau| \leqslant n+1$ and $\rho_i > 0$, $i\in\tau$ such that

$$w = \sum_{i\in\tau} \rho_i v_i \quad \text{and} \quad \sum_{i\in\tau} \rho_i = 1. \tag{4.12}$$

Note that

$$\left(\sum_{j\in\sigma} b_j\right)(-v) = \sum_{j\in\sigma} b_j v_j, \tag{4.13}$$

and this shows that $-v \in \text{conv}\{v_j : j\in\sigma\}$.

We write

$$I_n - T \preceq \sum_{j\in\sigma} b_j v_j \otimes v_j \preceq (4+2\varepsilon)I_n - T, \tag{4.14}$$

where

$$T := \sum_{j\in\sigma} b_j v_j \otimes v + \sum_{j\in\sigma} v\otimes b_j v_j + \left(\sum_{j\in\sigma} b_j\right) v\otimes v. \tag{4.15}$$

Taking into account (4.13) we check that, for every $x \in S^{n-1}$,

$$|\langle Tx, x\rangle| = \left(\sum_{j\in\sigma} b_j\right)\langle x, v\rangle^2 \leqslant \left(\sum_{j\in\sigma} b_j\right)\|v\|_2^2 \leqslant \varepsilon. \tag{4.16}$$

Choosing $\varepsilon = 1/2$ we see that $\|T\| \leqslant \frac{1}{2}$, and this finally gives

$$\frac{1}{2}I_n \preceq A := \sum_{j\in\sigma} b_j v_j \otimes v_j \preceq \frac{11}{2} I_n. \tag{4.17}$$

We are now able to show that

$$K := \text{conv}(\{v_j : j \in \sigma\cup\tau\}) \supseteq \frac{c}{n^{3/2}} B_2^n. \tag{4.18}$$

Let $x \in S^{n-1}$. We set $\delta = \min\{\langle x, v_j\rangle : j \in \sigma\}$; note that $|\delta| \leqslant 1$ and $\langle x, v_j\rangle - \delta \leqslant 2$ for all $j \in \sigma$. If $\delta < 0$, we write

$$\begin{aligned} p_K(Ax) &\leqslant p_K\left(Ax - \delta\sum_{j\in\sigma} b_j v_j\right) + p_K\left(\delta\sum_{j\in\sigma} b_j v_j\right) \\ &= p_K\left(\sum_{j\in\sigma} b_j(\langle x, v_j\rangle - \delta)v_j\right) + p_K\left(\delta\Big(\sum_{j\in\sigma} b_j\Big)(-v)\right) \\ &\leqslant \sum_{j\in\sigma} b_j(\langle x, v_j\rangle - \delta)p_K(v_j) - \delta\Big(\sum_{j\in\sigma} b_j\Big)p_K(v) \\ &\leqslant \Big(\sum_{j\in\sigma} b_j\Big)\Big[2 + \sqrt{n/2}\,p_K(w)\Big] \\ &\leqslant c_1 n^{3/2}, \end{aligned}$$

using the fact that $w \in K$, and hence $p_K(w) \leqslant 1$. If $\delta \geqslant 0$ then $\langle x, v_j\rangle \geqslant 0$ for all $j \in \sigma$, therefore

$$p_K(Ax) = p_K\left(\sum_{j\in\sigma} b_j\langle x, v_j\rangle v_j\right) \leqslant \sum_{j\in\sigma} b_j\langle x, v_j\rangle p_K(v_j) \leqslant \sum_{j\in\sigma} b_j \leqslant 5n \tag{4.19}$$

In any case,

$$p_{A^{-1}(K)}(x) \leqslant c_2 n^{3/2} \tag{4.20}$$

for all $x \in S^{n-1}$, where $c_2 > 0$ is an absolute constant. Together with (4.17) this shows that

$$\frac{1}{2}B_2^n \subseteq A(B_2^n) \subseteq c_2 n^{3/2} K. \tag{4.21}$$

Since $\mathrm{card}(\sigma \cup \tau) \leqslant \alpha_1(1/2)n + n + 1 \leqslant (\alpha_1(1/2) + 2)n$, the proof is complete. $\square$

Proof of Theorem 1.3. Let $P = \bigcap_{i\in I} P_i$. We may assume that $0 \in \mathrm{int}(P)$ and that the polar body

$$P^\circ = \mathrm{conv}\left(\bigcup_{i\in I} P_i^\circ\right) \tag{4.22}$$

is in Löwner's position. Using Proposition 4.2 for $C = P^\circ$ we may find $X = \{v_1, \dots, v_s\} \subset P^\circ$ with $\mathrm{card}(X) = s \leqslant \alpha n$ such that

$$P^\circ \subseteq cn^{3/2}\mathrm{conv}(\{v_1, \dots, v_s\}), \tag{4.23}$$

where $c > 0$ is an absolute constant. From the proof of Proposition 4.2 we see that $v_1, \dots, v_s$ may be chosen to be contact points of P° with its minimal volume ellipsoid, and hence it is simple to check that we actually have $v_j \in \bigcup_{i\in I} P_i^\circ$ for all $j = 1, \dots, s$. In other words, we may find $i_1, \dots, i_s \in I$ such that $v_j \in P_{i_j}$, $j = 1, \dots, s$. Then, (4.23) implies that

$$P^\circ \subseteq cn^{3/2}\mathrm{conv}(P_{i_1}^\circ \cup \cdots \cup P_{i_s}^\circ), \tag{4.24}$$

and passing to the polar bodies, we get

$$P_{i_1} \cap \cdots \cap P_{i_s} \subseteq cn^{3/2} P \tag{4.25}$$

as claimed. $\square$

Remark 4.3. In [3] it is proved that if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$, then there exist $s \leqslant n(n+1)$ and $i_1, \dots i_s \in I$ such that

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant \sqrt{2n(n+1)}. \tag{4.26}$$

Then, a scheme is described which allows one to further reduce the number of the bodies P_{i_j} and keep some control on the diameter. The lemma which allows this reduction states the following: Let $m > 2n$ and $P_1, \dots, P_m$ be convex bodies in $\mathbb{R}^n$ such that $0 \in P_1 \cap \cdots \cap P_m$. If the circumradius of $P_1 \cap \cdots \cap P_m$ is equal to 1 then we can find $1 \leqslant j \leqslant m$ such that the circumradius of $\bigcap_{i=1, i\neq j}^m P_i$ is at most $\frac{m}{m-2d}$. Starting with Theorem 1.3 and using the same lemma, for any finite family $\{P_i : i \in I\}$ of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$ we first find $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant c_1 n^{3/2}, \tag{4.27}$$

where $c_1 > 0$ is an absolute constant, and then we can keep $2n$ of the P_{i_j}'s so that the diameter of their intersection is bounded by

$$c_1 n^{3/2} \prod_{m=2n+1}^{s} \frac{m}{m-2n} = c n^{3/2} \binom{s}{2n} \leqslant c n^{3/2} \left(\frac{e\alpha}{2}\right)^{2n} \leqslant c_2^n, \tag{4.28}$$

where $c_2 > 0$ is an absolute constant. This improves the estimate from [3] (for the original question studied there) but it is still exponential in the dimension.

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