Reverse Brascamp–Lieb inequality and the dual Bollobás–Thomason inequality

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Abstract

We prove that if $f : \mathbb{R}^n \to [0, \infty)$ is an integrable log-concave function with $f(0) = 1$ and $F_1, \ldots, F_r$ are linear subspaces of $\mathbb{R}^n$ such that $sI_n = \sum_{i=1}^{r} c_i P_i$ where I_n is the identity operator and P_i is the orthogonal projection onto F_i then

$$n^n \int_{\mathbb{R}^n} f(y)^n dy \geqslant \prod_{i=1}^{r} \left(\int_{F_i} f(x_i) dx_i \right)^{c_i/s}.$$

As an application we obtain the dual version of the Bollobás–Thomason inequality: if K is a convex body in $\mathbb{R}^n$ with $0 \in \operatorname{int}(K)$ and $(\sigma_1, \ldots, \sigma_r)$ is an s-uniform cover of $[n]$ then

$$|K|^s \geqslant \frac{1}{(n!)^s} \prod_{i=1}^{r} |\sigma_i|! \prod_{i=1}^{r} |K \cap F_i|.$$

This is a sharp generalization of Meyer's dual Loomis–Whitney inequality.

1 Introduction

In this article we discuss inequalities relating the volume of convex (or compact) sets to the volumes of suitable finite families of their lower dimensional projections or sections. Classical examples are the Loomis–Whitney inequality (1.2) and its dual, Meyer's inequality (1.3). These are fundamental inequalities (included in many texts; see, for example, [10], [14] and [19]) and various versions and generalizations of them have found applications to many different fields. We refer to [11] and [12] for a list of such applications to Sobolev inequalities and embedding, stereology, geochemistry, data processing, compressed sensing, combinatorics and the theory of sum sets, harmonic analysis, group theory and graph theory.

An extension of the Loomis–Whitney inequality is the uniform cover inequality (1.1) of Bollobás and Thomason. The main purpose of this work is to provide a uniform cover inequality for sections; this is the corresponding extension of Meyer's inequality.

We fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ and recall that the not necessarily distinct non-empty sets $\sigma_1, \ldots, \sigma_r \subseteq [n] := \{1, \ldots, n\}$ form an s-uniform cover of $[n]$ for some $s \geqslant 1$ if every $j \in [n]$ belongs to exactly s of the sets σ_i. The main result of [8] estimates the volume of a compact set in terms of the volumes of its coordinate projections that correspond to a uniform cover of $[n]$.

Theorem 1.1 (Bollobás–Thomason). *Let $r \geqslant 1$ and $(\sigma_1, \ldots, \sigma_r)$ be an s-uniform cover of $[n]$. For every compact subset K of $\mathbb{R}^n$, which is the closure of its interior, we have*

$$(1.1) \qquad |K|^s \leqslant \prod_{i=1}^{r} |P_{F_{\sigma_i}}(K)|,$$

where $F_\tau = \operatorname{span}\{e_j : j \in \tau\}$ and P_F denotes the orthogonal projection of $\mathbb{R}^n$ onto F.

Throughout this article, for any non-empty compact set in $\mathbb{R}^n$ we write $|A|$ for the volume of A in the affine subspace $\mathrm{aff}(A)$. A special case of Theorem 1.1 is the Loomis–Whitney inequality [17]; one has

$$(1.2) \qquad |K|^{n-1} \leqslant \prod_{i=1}^{n} |P_i(K)|$$

where $P_i := P_{e_i^\perp}$, and equality holds if and only if K is a coordinate box, i.e. a rectangular parallelepiped whose sides are parallel to the coordinate axes. This follows from the observation that the sets $\sigma_i = [n] \setminus \{i\}$ form an $(n-1)$-uniform cover of $[n]$.

Meyer proved in [18] an inequality which is dual to the Loomis–Whitney inequality. If K is a convex body in $\mathbb{R}^n$ then

$$(1.3) \qquad |K|^{n-1} \geqslant \frac{n!}{n^n} \prod_{i=1}^{n} |K \cap e_i^\perp|,$$

where $K \cap F$ denotes the section of K with a linear subspace F. Equality holds in (1.3) if and only if $K = \mathrm{conv}(\{\pm\lambda_1 e_1, \ldots, \pm\lambda_n e_n\})$ for some $\lambda_i > 0$. We prove the following dual Bollobás–Thomason inequality.

Theorem 1.2. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$ and $(\sigma_1, \ldots, \sigma_r)$ be an s-uniform cover of $[n]$. Then,*

$$(1.4) \qquad |K|^s \geqslant \frac{1}{(n!)^s} \prod_{i=1}^{r} |\sigma_i|! \prod_{i=1}^{r} |K \cap F_{\sigma_i}|.$$

It is not hard to check that (1.4) is sharp; it becomes equality for any s-uniform cover of $[n]$ if $K = \mathrm{conv}(\{\pm\lambda_1 e_1, \ldots, \pm\lambda_n e_n\})$ for some $\lambda_i > 0$, exactly as in Meyer's inequality.

An essentially equivalent way to state Theorem 1.1 (see [8]) is the fact that for every compact subset K of $\mathbb{R}^n$, which is the closure of its interior, we can find a coordinate box such that $|B| = |K|$ and

$$(1.5) \qquad |P_{F_\sigma}(B)| \leqslant |P_{F_\sigma}(K)|$$

for every $\sigma \subseteq [n]$. Theorem 1.2 has a similar equivalent formulation.

Theorem 1.3. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. There exists an affine cross-polytope $C = \mathrm{conv}(\{\pm\lambda_1 e_1, \ldots, \pm\lambda_n e_n\})$, where $\lambda_i > 0$, such that $|C| = |K|$ and $|C \cap F_\sigma| \geqslant |K \cap F_\sigma|$ for every $\sigma \subseteq [n]$.*

Note that the assumption that $0 \in \mathrm{int}(K)$ can be removed in Theorem 1.2 and Theorem 1.3. Starting with an arbitrary convex body K in $\mathbb{R}^n$ one may first apply n successive Steiner symmetrizations to K in the directions e_i. Then, the volume of the original body remains unchanged and the volumes of all its coordinate sections increase. So, one may assume from the beginning that K is unconditional, i.e. symmetric with respect to all coordinate subspaces, and in particular that K is origin symmetric. In fact, Meyer used this observation in [18] to reduce the proof of (1.3) to the unconditional case.

Theorem 1.2, and its equivalent version Theorem 1.3, is a consequence of a functional inequality which is proved in Section 3. We denote by $\mathcal{F}(\mathbb{R}^n)$ the class of log-concave integrable functions $f : \mathbb{R}^n \to [0, \infty)$.

Theorem 1.4. *Let $f \in \mathcal{F}(\mathbb{R}^n)$ with $f(0) = 1$ and $(\sigma_1, \ldots, \sigma_r)$ be an s-uniform cover of $[n]$. Then,*

$$n^n \int_{\mathbb{R}^n} f(y)^n dy \geqslant \prod_{i=1}^{r} \left(\int_{F_i} f(x_i) dx_i \right)^{1/s}.$$

Moreover, we obtain more general inequalities which imply several of the known extensions of the Loomis–Whitney and Meyer inequalities; see Section 2 and Section 3 for the statements and details. Our main tool is Barthe's multidimensional generalization of Ball's geometric Brascamp–Lieb inequality (see [4]) and its reverse form; see [5, Theorem 6]. The connection with the problems that we discuss in Section 2 was communicated by F. Barthe to A. Giannopoulos after a talk in MSRI and the author is grateful to them for the information which has been the starting point for this work.

Let us also mention that the Bollobás–Thomason inequality plays a key role in the recent work [9] of S. Brazitikos, A. Giannopoulos and the author that provides local versions of the Loomis–Whitney inequality for coordinate projections of convex bodies; see also [1] for further results in this direction. It is conceivable that one might exploit the dual inequality of Theorem 1.2 to obtain analogous local inequalities for sections. Isomorphic inequalities of this type appear in [9] where they are proved by different methods.

In Section 2 we describe the way one can derive both the Loomis–Whitney and the Bollobás–Thomason inequality, as well as other extensions of them, as consequences of the multidimensional geometric Brascamp–Lieb inequality. The main new results of this work are presented in Section 3; the main tool is Barthe's inequality. We refer to the books [19] and [2] for standard notation and facts from convex geometric analysis.

2   Brascamp–Lieb inequality and uniform cover inequalities

In what follows we say that the subspaces $F_1, \ldots, F_r$ form an s-uniform cover of $\mathbb{R}^n$ with weights $c_1, \ldots, c_r > 0$ for some $s > 0$ if

$$(2.1) \qquad sI_n = \sum_{i=1}^{r} c_i P_i,$$

where I_n is the identity operator and P_i is the orthogonal projection of $\mathbb{R}^n$ onto F_i. We prove the next general result.

Theorem 2.1. *Let $F_1, \ldots, F_r$ be subspaces that form an s-uniform cover of $\mathbb{R}^n$ with weights $c_1, \ldots, c_r > 0$. For every compact subset K of $\mathbb{R}^n$ we have*

$$(2.2) \qquad |K|^s \leqslant \prod_{i=1}^{r} |P_{F_i}(K)|^{c_i}.$$

Theorem 2.1 is most probably known to specialists; we were informed by the referee of this paper that, for example, it can be found in [7] (see Equation (29) and the lines following it). The proof is a direct application of Barthe's multidimensional geometric Brascamp–Lieb inequality (2.3) in the theorem given below; the reverse inequality (2.4) will be our main tool in the next section.

Theorem 2.2 (Barthe)**.** *Let $r, n \in \mathbb{N}$. For $i = 1, \ldots r$, let F_i be a d_i-dimensional subspace of $\mathbb{R}^n$ and P_i be the orthogonal projection onto F_i. If*

$$I_n = \sum_{i=1}^{r} c_i P_i$$

for some $c_1, \ldots, c_r > 0$ then for all non-negative integrable functions $f_i : F_i \to \mathbb{R}$ we have

$$(2.3) \qquad \int_{\mathbb{R}^n} \prod_{i=1}^{r} f_i^{c_i}(P_i x) \, dx \leqslant \prod_{i=1}^{r} \left(\int_{F_i} f_i \right)^{c_i}$$

and

$$(2.4) \qquad \int_{\mathbb{R}^n}^{*} \sup \left\{ \prod_{i=1}^{r} f_i^{c_i}(x_i) : x = \sum_{i=1}^{r} c_i x_i, x_i \in F_i \right\} dx \geqslant \prod_{i=1}^{r} \left(\int_{F_i} f_i \right)^{c_i}.$$

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In the statement above, $\int^*$ stands for the outer integral and in the right hand side the integral on F_i is with respect to the Lebesgue measure on F_i which is compatible to the given Euclidean structure.

Proof of Theorem 2.1. Given a compact subset K of $\mathbb{R}^n$ we define $f_i : F_i \to [0, \infty)$ by $f_i = \mathbf{1}_{P_i(K)}$. Note that if $x \in K$ then $f_i(P_i x) = 1$ for all $i = 1, \ldots, r$. Therefore,

$$\mathbf{1}_K(x) \leqslant \prod_{i=1}^{r} f_i^{\frac{c_i}{s}}(P_i x)$$

for all $x \in \mathbb{R}^n$. From Theorem 2.2 we get

$$|K| = \int_{\mathbb{R}^n} \mathbf{1}_K(x)\,dx \leqslant \int_{\mathbb{R}^n} \prod_{i=1}^{r} f_i^{\frac{c_i}{s}}(P_i x)\,dx \leqslant \prod_{i=1}^{r} \left(\int_{F_i} f_i \right)^{\frac{c_i}{s}} = \prod_{i=1}^{r} |P_i(K)|^{\frac{c_i}{s}},$$

which shows that $|K|^s \leqslant \prod_{i=1}^{r} |P_i(K)|^{c_i}$ as claimed. $\qquad\square$

Application 2.3 (Bollobás–Thomason). It is not hard to see that the Bollobás–Thomason inequality may be proved in the same way. In fact, as a special case of Theorem 2.1 we recover a more general inequality of Bollobás and Thomason from [8]. Let $\mathcal{C}$ be a finite collection of subsets of $[n]$, which is not necessarily a uniform cover. Suppose that to each $\sigma \in \mathcal{C}$ we associate a positive real weight $w(\sigma)$ in such a way that, for each $i \in [n]$, $\sum\{w(\sigma) : i \in \sigma \in \mathcal{C}\} = 1$. Then, it is clear that

$$I_n = \sum_{\sigma \in \mathcal{C}} w(\sigma) P_{F_\sigma},$$

and Theorem 2.1 shows that

$$|K| \leqslant \prod_{\sigma \in \mathcal{C}} |P_{F_\sigma}(K)|^{w(\sigma)}.$$

Assuming that $\mathcal{C} = (\sigma_1, \ldots, \sigma_r)$ is an s-uniform cover of $[n]$ and choosing $w(\sigma_i) = 1/s$ for all $i = 1, \ldots, r$ we obtain Theorem 1.1.

Let us also mention that the paper [13] of Finner contains an extension of Hölder's inequality which is a particular case of the geometric Brascamp–Lieb inequality for specific decompositions of the identity involving coordinate subspaces and implies the Bollobás–Thomason inequality.

Application 2.4 (Ball's inequality). Let $u_1, \ldots, u_m$ be unit vectors in $\mathbb{R}^n$ and $c_1, \ldots, c_m$ be positive real numbers such that John's condition

$$I_n = \sum_{i=1}^{m} c_i u_i \otimes u_i$$

is satisfied. Using the one-dimensional geometric Brascamp–Lieb inequality, Ball proved in [3] that for every centered convex body K in $\mathbb{R}^n$,

$$(2.5) \qquad |K|^{n-1} \leqslant \prod_{i=1}^{m} |P_{u_i^\perp}(K)|^{c_i}.$$

The equality cases are the same with the ones in the Loomis–Whitney inequality. Let us briefly explain how Theorem 2.1 implies (2.5). We observe that if $P_i = P_{u_i^\perp}$ then $u_i \otimes u_i = I_n - P_i$, and hence John's condition may be written as $I_n = \sum_{i=1}^{m} c_i(I_n - P_i)$, which implies that

$$(2.6) \qquad (n-1)I_n = \sum_{i=1}^{m} c_i P_i,$$

if we take into account the fact that $n = \operatorname{tr}(I_n) = \sum_{i=1}^{r} c_i \cdot \operatorname{tr}(u_i \otimes u_i) = \sum_{i=1}^{m} c_i$. Then, given a (more generally) compact subset K of $\mathbb{R}^n$ we may apply Theorem 2.1 with $s = n - 1$ to get

$$(2.7) \qquad |K|^{n-1} \leqslant \prod_{i=1}^{m} |P_{u_i^\perp}(K)|^{c_i}.$$

3 Dual Bollobás–Thomason inequality

We start with a proof of a more general version of Theorem 1.4. Recall that $\mathcal{F}(\mathbb{R}^n)$ denotes the class of log-concave integrable functions $f : \mathbb{R}^n \to [0, \infty)$.

Theorem 3.1. *Let $f \in \mathcal{F}(\mathbb{R}^n)$ with $f(0) = 1$ and $F_1, \ldots, F_r$ be subspaces of $\mathbb{R}^n$ that form an s-uniform cover of $\mathbb{R}^n$ with weights $c_1, \ldots, c_r > 0$. Then,*

$$n^n \int_{\mathbb{R}^n} f(y)^n dy \geqslant \prod_{i=1}^{r} \left(\int_{F_i} f(x_i) dx_i \right)^{c_i/s}.$$

Proof. Our assumption $I_n = \sum_{i=1}^{r} \frac{c_i}{s} P_{F_i}$ implies that

$$ns = \operatorname{tr}(sI_n) = \sum_{i=1}^{r} c_i \cdot \operatorname{tr}(P_{F_i}) = \sum_{i=1}^{r} c_i d_i,$$

where $d_i = \dim(F_i)$. Let $z \in \mathbb{R}^n$ and $x_i \in F_i$, $i \in [r]$ such that $z = \sum_{i=1}^{r} \frac{c_i}{s} x_i$. Then,

$$\frac{z}{n} = \sum_{i=1}^{r} \frac{c_i d_i}{sn} \cdot \frac{x_i}{d_i},$$

and since $f \in \mathcal{F}(\mathbb{R}^n)$ and $\sum_{i=1}^{r} \frac{c_i d_i}{sn} = 1$ we have

$$f(z/n) \geqslant \prod_{i=1}^{r} f(x_i/d_i)^{\frac{c_i d_i}{ns}}.$$

Since $f(0) = 1$, for every $i \in [r]$ we see that $f(x_i/d_i) \geqslant f(x_i)^{1/d_i} f(0)^{1-1/d_i} = f(x_i)^{1/d_i}$. It follows that

$$f(z/n) \geqslant \prod_{i=1}^{r} f(x_i)^{\frac{1}{d_i} \cdot \frac{c_i d_i}{ns}} = \prod_{i=1}^{r} f(x_i)^{\frac{c_i}{ns}},$$

and hence

$$f(z/n)^n \geqslant \prod_{i=1}^{r} f(x_i)^{c_i/s}.$$

This shows that

$$f(z/n)^n \geqslant \sup \left\{ \prod_{i=1}^{r} f(x_i)^{c_i/s} : z = \sum_{i=1}^{r} \frac{c_i}{s} x_i, x_i \in F_i \right\}.$$

Then, by the multidimensional reverse Brascamp–Lieb inequality (2.4) we have that

$$\int_{\mathbb{R}^n} f(z/n)^n dz \geqslant \int_{\mathbb{R}^n}^* \sup \left\{ \prod_{i=1}^{r} f(x_i)^{c_i/s} : z = \sum_{i=1}^{r} \frac{c_i}{s} x_i, x_i \in F_i \right\} dz$$

$$\geqslant \prod_{i=1}^{r} \left(\int_{F_i} f(x_i) dx_i \right)^{c_i/s}.$$

Making the change of variables $y = z/n$ we conclude the proof. $\qquad\square$

Our main geometric application of Theorem 3.1 is the next general uniform cover inequality for sections of a convex body.

Theorem 3.2. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \operatorname{int}(K)$ and $F_1, \ldots, F_r$ be subspaces of $\mathbb{R}^n$, with $\dim(F_i) = d_i$, that form an s-uniform cover of $\mathbb{R}^n$ with weights $c_1, \ldots, c_r > 0$. Then,*

$$|K|^s \geqslant \frac{1}{(n!)^s} \prod_{i=1}^{r}(d_i!)^{c_i} \prod_{i=1}^{r} |K \cap F_i|^{c_i}.$$

Proof. We apply Theorem 3.1 for the function $f(y) = e^{-\|y\|_K}$, where $\|y\|_K := \min\{t > 0 : y \in tK\}$ is the Minkowski functional of K. We shall use the fact that, for any convex body C in $\mathbb{R}^m$ with $0 \in \operatorname{int}(C)$,

$$\int_{\mathbb{R}^m} e^{-\|x\|_C} dx = \int_{\mathbb{R}^m} \int_{\|x\|_C}^{\infty} e^{-t} dt\, dx = \int_{0}^{\infty} e^{-t} |\{x : \|x\|_C \leqslant t\}|\, dt = \int_{0}^{\infty} e^{-t} |tC|\, dt$$

$$= |C| \int_{0}^{\infty} t^m e^{-t} dt = m! |C|.$$

Note that $f \in \mathcal{F}(\mathbb{R}^n)$ and $f(0) = 1$. We have

$$n^n \int_{\mathbb{R}^n} f(y)^n dy = n^n \int_{\mathbb{R}^n} e^{-n\|y\|_K} dy = n^n \int_{\mathbb{R}^n} e^{-\|y\|_{\frac{1}{n}K}} dy$$

$$= n^n \, n! \left| \frac{1}{n}K \right| = n! \, |K|,$$

and for every $i \in [r]$ we have

$$\int_{F_i} f(x_i) dx_i = \int_{F_i} e^{-\|x_i\|_K} dx_i = \int_{F_i} e^{-\|x_i\|_{K \cap F_i}} dx_i = d_i! \, |K \cap F_i|.$$

Combining the above we get

$$n! |K| \geqslant \prod_{i=1}^{r} \left(d_i! \, |K \cap F_i| \right)^{c_i/s}$$

and the theorem follows. $\qquad\square$

Application 3.3 (dual Bollobás–Thomason). Theorem 3.2 has several straightforward applications. First, let $(\sigma_1, \ldots, \sigma_r)$ be an s-uniform cover of $[n]$. Setting $F_i = F_{\sigma_i} = \operatorname{span}(\{e_j : j \in \sigma_i\})$, $i \in [r]$, we have $sI_n = \sum_{i=1}^{r} P_{F_i}$. Thus, we obtain the dual Bollobás–Thomason inequality of Theorem 1.2: If K is a convex body in $\mathbb{R}^n$ with $0 \in \operatorname{int}(K)$ and $(\sigma_1, \ldots, \sigma_r)$ is an s-uniform cover of $[n]$ then

$$|K|^s \geqslant \frac{1}{(n!)^s} \prod_{i=1}^{r} |\sigma_i|! \prod_{i=1}^{r} |K \cap F_i|.$$

In the particular case $F_i = e_i^\perp$, $i \in [n]$ we have $(n-1)I_n = \sum_{i=1}^{n} P_{e_i^\perp}$, and applying Theorem 1.2 with $s = n - 1$ and $|\sigma_i| = \dim(F_i) = n - 1$ we recover Meyer's inequality

$$|K|^{n-1} \geqslant \frac{n!}{n^n} \prod_{i=1}^{n} |K \cap e_i^\perp|$$

for any convex body K in $\mathbb{R}^n$ with $0 \in \operatorname{int}(K)$, because

$$\frac{1}{(n!)^{n-1}} \prod_{i=1}^{n} |\sigma_i|! = \frac{1}{(n!)^{n-1}} \prod_{i=1}^{n} (n-1)! = \frac{[(n-1)!]^n}{(n!)^{n-1}} = \frac{(n-1)!}{n^{n-1}} = \frac{n!}{n^n}.$$

Theorem 1.3 can be obtained from Theorem 1.2 by an argument which is basically the same as the one used by Bollobás and Thomason for the proof of (1.5). In what follows, we say that a uniform cover of $[n]$ is irreducible if it cannot be written as a disjoint union of two other uniform covers of $[n]$. In [8] it is shown that the number of irreducible uniform covers of $[n]$ is finite.

Proof of Theorem 1.3. Let K be a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. Theorem 1.2 states that for every integer $s \geqslant 1$ and any non-trivial irreducible s-uniform cover $(\sigma_1, \ldots, \sigma_r)$ of $[n]$ we have that $(n!|K|)^s \geqslant \prod_{j=1}^r \left(|\sigma_j|! \, |K \cap F_{\sigma_j}| \right)$. Moreover, applying Theorem 1.2 for the 1-uniform cover $(\{i\}, i \in \tau)$ of $\tau \subseteq [n]$ we see that $|\tau|! |K \cap F_\tau| \geqslant \prod_{i \in \tau} |K \cap F_{\{i\}}|$. Since there are finitely many irreducible uniform covers of $[n]$, we have a finite number of inequalities as above, satisfied by the elements of the set $\{|\sigma|! |K \cap F_\sigma| : \sigma \subseteq [n]\}$.

Let $\{t_\sigma : \sigma \subseteq [n]\}$ be a set of positive reals with $t_\sigma \geqslant |\sigma|! |K \cap F_\sigma|$ and $t_{[n]} = n!|K|$, which are maximal with respect to satisfying all the above inequalities if we replace $|\sigma|! |K \cap F_\sigma|$ by t_σ for all $\sigma \subseteq [n]$. Then, we know that $\prod_{j=1}^r t_{\sigma_j} \leqslant (n!|K|)^s$ for every (not necessarily irreducible) s-uniform cover $(\sigma_1, \ldots, \sigma_r)$ of $[n]$.

Since $t_{\{i\}}, i \in [n]$, are maximal, we see that for every $i \in [n]$ we can find an inequality involving $t_{\{i\}}$ which is equality. If this inequality is of the first kind then there exists an s_i-uniform cover $\overline{\sigma}(i) = (\sigma_1, \ldots, \sigma_r)$ of $[n]$ with $\sigma_j = \{i\}$ for some j, such that $(n!|K|)^{s_i} = \prod_{j=1}^r t_{\sigma_j}$. The same is true if the inequality is of the second kind, i.e. if we have an equality of the type $\prod_{l \in \tau} t_{\{l\}} = t_\tau$ for some $\tau \subseteq [n]$ with $i \in \tau$. Because, by the maximality of t_τ we can find an s_i-uniform cover $(\sigma_1, \ldots, \sigma_r)$ of $[n]$ such that $\tau = \sigma_{j_0}$ for some j_0, and then $\overline{\sigma}(i) := (\sigma_j, j \neq j_0) \cup (\{i\} : i \in \tau)$ is again an s_i-uniform cover of $[n]$.

Now, we define $\overline{\sigma} = \bigcup_{i=1}^n \overline{\sigma}(i)$ and $s = \sum_{i=1}^n s_i$. Then, $\overline{\sigma}$ is an s-uniform cover of $[n]$, we have $\{i\} \in \overline{\sigma}$ for all $i = 1, \ldots, n$ and

$$\prod_{\sigma \in \overline{\sigma}} t_\sigma = (n!|K|)^s. \tag{3.1}$$

Since $\overline{\sigma}' := \overline{\sigma} \setminus (\{i\} : i \in [n])$ is an $(s-1)$-uniform cover of $[n]$ we must have

$$\prod_{\sigma \in \overline{\sigma}'} t_\sigma \leqslant (n!|K|)^{s-1}. \tag{3.2}$$

Combining (3.1) and (3.2) we see that $\prod_{i=1}^n t_{\{i\}} \geqslant n!|K|$. On the other hand, $(\{i\} : i \in [n])$ is a 1-uniform cover of $[n]$, and hence the reverse inequality is also true. Therefore,

$$\prod_{i=1}^n t_{\{i\}} = n!|K|. \tag{3.3}$$

Now, let $\tau \subseteq [n]$ and consider the 1-uniform cover $\{\tau\} \cup (\{i\} : i \notin \tau)$ of $[n]$. Using (3.3) and the assumption that $t_\tau \geqslant \prod_{i \in \tau} t_{\{i\}}$ we have

$$n!|K| \geqslant t_\tau \cdot \prod_{i \notin \tau} t_{\{i\}} \geqslant \prod_{i \in \tau} t_{\{i\}} \cdot \prod_{i \notin \tau} t_{\{i\}} = \prod_{i=1}^n t_{\{i\}} = n!|K|,$$

which implies that

$$t_\tau = \prod_{i \in \tau} t_{\{i\}} \tag{3.4}$$

for every $\tau \subseteq [n]$. The last set of equalities shows that if we set $\lambda_i = t_{\{i\}}/2$ and consider the cross-polytope $C = \mathrm{conv}(\{\pm\lambda_1 e_1, \ldots, \pm\lambda_n e_n\})$ then we have $|C| = \frac{1}{n!} \prod_{i=1}^n t_{\{i\}} = |K|$ and

$$|C \cap F_\sigma| = \frac{1}{|\sigma|!} \prod_{i \in \sigma} t_{\{i\}} = \frac{1}{|\sigma|!} \prod_{i \in \sigma} t_\sigma \geqslant |K \cap F_\sigma|$$

for every $\sigma \subseteq [n]$. $\qquad\square$

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Application 3.4 (dual Ball's inequality). Li and Huang proved in [15] that for every centered convex body K in $\mathbb{R}^n$ and every even isotropic measure ν on S^{n-1} one has

$$(3.5) \qquad |K|^{n-1} \geqslant \frac{n!}{n^n} \exp\left(\int_{S^{n-1}} \log|K \cap u^\perp|\, d\nu(u)\right)$$

and they determined the equality cases. Their argument employs the continuous version of the Ball-Barthe inequality, due to Lutwak, Yang and Zhang [16], and a number of facts about the class of polar L_p-centroid bodies. In the particular case where $u_1, \ldots, u_m$ are unit vectors in $\mathbb{R}^n$ and $c_1, \ldots, c_m$ are positive real numbers that satisfy John's condition, one gets

$$(3.6) \qquad |K|^{n-1} \geqslant \frac{n!}{n^n} \prod_{i=1}^m |K \cap u_i^\perp|^{c_i}.$$

The latter inequality is a particular case of Theorem 3.2. Given a convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$, we consider the subspaces $F_i = u_i^\perp$, and since $\dim(F_i) = n-1$ and the F_i's form an $(n-1)$-uniform cover of $\mathbb{R}^n$ with weights $c_1, \ldots, c_m > 0$, using also the fact that $\sum_{i=1}^m c_i = n$ we immediately get

$$(3.7) \qquad |K|^{n-1} \geqslant \frac{1}{(n!)^s} \prod_{i=1}^m ((n-1)!)^{c_i} \prod_{i=1}^m |K \cap u_i^\perp|^{c_i} = \frac{[(n-1)!]^n}{(n!)^{n-1}} \prod_{i=1}^m |K \cap u_i^\perp|^{c_i}$$

$$= \frac{n!}{n^n} \prod_{i=1}^m |K \cap u_i^\perp|^{c_i}.$$

We can now use an approximation argument of Barthe from [6] to deduce (3.5) from (3.7). We sketch the idea of the proof and refer to Barthe's article for more details. Recall that a Borel measure ν on S^{n-1} is called isotropic if $I_n = \int_{S^{n-1}} u \otimes u\, d\nu(u)$. The fact that the vectors u_j and the weights c_j satisfy (3.6) is equivalent to saying that the discrete measure $\overline{\nu}$ with $\overline{\nu}(\{u_j\}) = c_j$ is isotropic, i.e. $I_n = \int_{S^{n-1}} u \otimes u\, d\overline{\nu}(u)$. Also, since

$$\int_{S^{n-1}} \log|K \cap u^\perp|\, d\overline{\nu}(u) = \sum_{i=1}^m c_i \log|K \cap u_i^\perp| = \log\left(\prod_{i=1}^m |K \cap u_i^\perp|^{c_i}\right),$$

we may write (3.7) in the equivalent form

$$(3.8) \qquad |K|^{n-1} \geqslant \frac{n!}{n^n} \exp\left(\int_{S^{n-1}} \log|K \cap u^\perp|\, d\overline{\nu}(u)\right).$$

In other words, (3.5) holds true for any discrete isotropic measure on S^{n-1}.

Now, let ν be an isotropic Borel measure on S^{n-1}. For any $\varepsilon > 0$ we consider a maximal ε-net N_ε in S^{n-1} and a partition $(C_u)_{u \in N_\varepsilon}$ of S^{n-1} into Borel sets $C_u \subseteq B(u, \varepsilon)$, where $B(u, \varepsilon)$ is the geodesic ball with center u and radius ε. Then, we consider the measure

$$\nu_\varepsilon = \sum_{u \in N_\varepsilon} \nu(C_u) \delta_u,$$

where δ_u is the Dirac mass at u. Note that, for any continuous function $f : S^{n-1} \to \mathbb{R}$ we have that

$$\int_{S^{n-1}} f(u)\, d\nu_\varepsilon \longrightarrow \int_{S^{n-1}} f(u)\, d\nu$$

as $\varepsilon \to 0$. In other words, $\nu_\varepsilon \to \nu$ weakly as $\varepsilon \to 0$. If $T_\varepsilon = \int_{S^{n-1}} u \otimes u\, d\nu_\varepsilon(u)$ then for the measure $\mu_\varepsilon = \sum_{u \in N_\varepsilon} \nu_\varepsilon(u) \|T_\varepsilon^{-1/2}(u)\|_2^2 \delta_{v(u)}$ where $v(u) := T_\varepsilon^{-1/2}(u)/\|T_\varepsilon^{-1/2}(u)\|_2$ we have

$$I_n = \int_{S^{n-1}} T_\varepsilon^{-1/2}(u) \otimes T_\varepsilon^{-1/2}(u)\, d\nu_\varepsilon(u) = \int_{S^{n-1}} v \otimes v\, d\mu_\varepsilon(v).$$

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Since $\|T_\varepsilon - I_n\|_{\ell_2^n \to \ell_2^n} \leqslant c_1(\varepsilon)$ for some constant $c_1(\varepsilon)$ that tends to 0 as $\varepsilon \to 0$, we can check that for any continuous function $f : S^{n-1} \to \mathbb{R}$

$$\int_{S^{n-1}} f(u)\, d\mu_\varepsilon \longrightarrow \int_{S^{n-1}} f(u)\, d\nu$$

as $\varepsilon \to 0$. Applying (3.8) for the discrete isotropic measure μ_ε we have

$$|K|^{n-1} \geqslant \frac{n!}{n^n} \exp\left(\int_{S^{n-1}} \log|K \cap u^\perp|\, d\mu_\varepsilon(u) \right) \longrightarrow \frac{n!}{n^n} \exp\left(\int_{S^{n-1}} \log|K \cap u^\perp|\, d\nu(u) \right).$$

This proves (3.5).

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