A proportional Dvoretzky-Rogers factorization result

A. Giannopoulos

Abstract

If X is an n-dimensional normed space and $\varepsilon \in (0,1)$, there exists $m \geq (1-\varepsilon)n$ such that the formal identity $i_{2,\infty} : \ell_2^m \to \ell_\infty^m$ can be written as $i_{2,\infty} = \alpha o \beta$, $\beta : \ell_2^m \to X, \alpha : X \to \ell_\infty^m$, with $\|\alpha\| \, \|\beta\| \leq c/\varepsilon$. This is proved as a consequence of a Sauer-Shelah type theorem for ellipsoids.

1 Introduction

A version of the classical Dvoretzky-Rogers lemma [DR] asserts that, if $(X, \|.\|)$ is an n-dimensional normed space, then there exist vectors $x_1, \ldots, x_m \in X$, $m = [\sqrt{n}]$, such that for any choice of real numbers $t_1, \ldots, t_m$,

$$\max_{j \leq m} |t_j| \leq \| \sum_{j \leq m} t_j x_j \|_X \leq c (\sum_{j \leq m} t_j^2)^{1/2},$$

where $c > 0$ is an absolute constant. Towards a strengthening of this result for m proportional to n, J. Bourgain-S.J. Szarek [BS] and later S.J. Szarek-M. Talagrand [ST] proved the following:

Theorem 1. *If $(X, \|.\|)$ is an n-dimensional normed space and $\varepsilon \in (0,1)$, there exist vectors $x_1, \ldots, x_m \in X$, $m \geq (1-\varepsilon)n$, such that for any reals $t_1, \ldots, t_m$,*

$$\max_{j \leq m} |t_j| \leq \| \sum_{j \leq m} t_j x_j \|_X \leq c\varepsilon^{-d} (\sum_{j \leq m} t_j^2)^{1/2},$$

where $c, d > 0$ are absolute constants. Equivalently, the formal identity $i_{2,\infty} : \ell_2^m \to \ell_\infty^m$ can be written as $i_{2,\infty} = \alpha o \beta$, where $\beta : \ell_2^m \to X$, $\alpha : X \to \ell_\infty^m$, and $\|\alpha\| \, \|\beta\| \leq c\varepsilon^{-d}$. The same holds true for $i_{1,2} : \ell_1^m \to \ell_2^m$.

The best possible dependence on ε is not known. As shown by S.J. Szarek [Sz.1], there exists an n-dimensional normed space X such that $\|\alpha\| \, \|\beta\| \geq c(n/\log n)^{1/10}$ whenever $i_{2,\infty} : \ell_2^n \to \ell_\infty^n$ is written as $i_{2,\infty} = \alpha o \beta$ (α, β as above), and this implies that d in Theorem 1 has to be at least $1/10$. On the other hand, in [ST] it is proved that Theorem 1 holds with $d = 2$, and in [G] we obtain a similar result with $d = 3/2$. Here we shall show that the same holds true with $d = 1$.

Let us note that the method establishing this "proportional Dvoretzky-Rogers factorization" is closely related to the problem of the Banach-Mazur distance to the cube. A detailed exposition of the techniques used so far for both problems is given in [Sz.2].

The source of the improvement on the estimates in Theorem 1 is a Sauer-Shelah type theorem for ellipsoids, which we feel is of independent interest: The well-known combinatorial Sauer-Shelah lemma [Sa], [Sh], states that if $0 \leq l < s$ and if M is a subset of $\{-1,1\}^s$ of cardinality $|M| > \binom{s}{0} + \binom{s}{1} + \ldots + \binom{s}{l}$, then there exists $\sigma \subseteq \{1, \ldots, s\}$, $|\sigma| > l$, such that $P_\sigma(M) = \{-1,1\}^\sigma$, where P_σ is the restriction map: $(\delta_j)_{j \leq s} \to (\delta_j)_{j \in \sigma}$. A special case of this lemma is of particular interest: If $M \subseteq \{-1,1\}^s$ and $|M| \geq 2^{s-1}$, then we can find $\sigma \subseteq \{1, \ldots, s\}$, $|\sigma| \geq \frac{s}{2}$, with $P_\sigma(M) = \{-1,1\}^\sigma$.

In connection with their work on the Banach-Mazur distance to the cube, S.J. Szarek-M. Talagrand [ST] proved an isomorphic variant of the Sauer-Shelah lemma: If $M \subseteq \{-1,1\}^s$, viewed now as a set of points in $\mathbb{R}^s$, and if $|M| \geq 2^{s-1}$, $\varepsilon \in (0,1)$, then there exists $\sigma \subseteq \{1, \ldots, s\}, |\sigma| \geq (1-\varepsilon)s$, such that

$$\text{absconv}(P_\sigma(M)) \supseteq c\varepsilon[-1,1]^\sigma,$$

where $c > 0$ is an absolute constant (and the absolute convex hull is taken in $\mathbb{R}^\sigma$).

For our purposes we need to consider the following situation: Let $u_1, \ldots, u_s$ be vectors in $\mathbb{R}^n$, of Euclidean norm $|u_j|_n \leq 1$, $j = 1, \ldots, s$. Define the symmetric convex set

$$E = \{(\delta_j)_{j \leq s} \in \mathbb{R}^s : |\sum_{j=1}^s \delta_j u_j|_n \leq 1\}.$$

(Note that if $s \leq n$ and the vectors u_j are linearly independent in $\mathbb{R}^n$, then E is an ellipsoid in $\mathbb{R}^s$. This will be the context in the proof of Theorem 1). Again, we are interested in the "size" of the image $P_\sigma(E)$ of E for "large" subsets σ of $\{1, \ldots, s\}$. Our main result is then the following:

Theorem 2. *If $u_j \in \mathbb{R}^n, |u_j|_n \leq 1, j = 1, \ldots, s$, and*

$$E = \{(\delta_j)_{j \leq s} \in \mathbb{R}^s : |\sum_{j=1}^s \delta_j u_j|_n \leq 1\},$$

then for every $\varepsilon \in (0,1)$ we can find $\sigma \subseteq \{1, \ldots, s\}$, $|\sigma| \geq (1-\varepsilon)s$, such that

$$P_\sigma(E) \supseteq c\sqrt{\varepsilon} D_\sigma,$$

where D_σ is the Euclidean unit ball in $\mathbb{R}^\sigma$, and $c > 0$ is an absolute constant.

We shall use the standard notation from [MS] or [TJ]. By $|.|$ we denote the cardinality of a finite set. The letter c will always denote an absolute positive constant, not necessarily the same in all its occurrences. For basic facts about p-absolutely summing operators, used in the proof of Theorem 2, we refer the reader to [LT], [Pi], and [TJ].

2 Proof of Theorem 2

First, we introduce some additional notation: The set $S = \{1, \ldots, s\}$, as well as $\mathbb{R}^s$, will be fixed throughout the proof. If $\varphi \subseteq S$, then $\mathbb{R}^\varphi = \{(\delta_j)_{j \leq s} \in \mathbb{R}^s : \delta_j = 0 \text{ if } j \notin \varphi\}$. A point in $\mathbb{R}^s$ denoted by $(\delta_j)_{j \in \varphi}$ is assumed to satisfy $\delta_j = 0$ if $j \notin \varphi$. If $A \subseteq \mathbb{R}^\varphi$ and τ, φ are disjoint subsets of S, we sometimes write $\mathbf{0}_\tau \times A$, to indicate that A is to be understood as a subset of $\mathbb{R}^{\varphi \cup \tau}$. In particular, if $\varphi \subseteq S_1 \subseteq S$ and $L > 0$, then

$$I_{L,\varphi,S_1} = \mathbf{0}_\varphi \times \{-L, L\}^{S_1 \setminus \varphi} = \{(\delta_j)_{j \in S_1} \in \mathbb{R}^{S_1} : \delta_j = 0 \text{ if } j \in \phi, \delta_j = \pm L \text{ if } j \in S_1 \setminus \phi\}.$$

Note that $|I_{L,\varphi,S_1}| = 2^{|S_1 \setminus \varphi|}$. If $a \in \mathbb{R}^\varphi$, $b \in \mathbb{R}^\tau$, and τ, φ are disjoint subsets of S, then $(a, b) \in \mathbb{R}^{\varphi \cup \tau}$ is the sum $a + b$. Finally, if S_1 is a non-empty subset of S, we define

$$E_{S_1} = \{(\delta_j)_{j \in S_1} \in \mathbb{R}^{S_1} : |\sum_{j \in S_1} \delta_j u_j|_n \leq 1\}.$$

Our starting point is then an immediate consequence of the Sauer-Shelah lemma:

Lemma 1. *If $L > 0$, $\varphi \subseteq S_1 \subseteq S$, and $M \subseteq \mathbf{0}_\varphi \times \{-L, L\}^{S_1 \setminus \varphi}$, with $|M| \geq 2^{|S_1 \setminus \varphi| - 1}$, then there exists $\sigma \subseteq S_1 \setminus \varphi$, $|\sigma| \geq \frac{|S_1 \setminus \varphi|}{2}$, such that*

$$P_{\varphi \cup \sigma}(M) = \mathbf{0}_\varphi \times \{-L, L\}^\sigma. \quad \square$$

Using an inductive argument based on Lemma 1, we obtain a first result on the size of the projections of E_{S_1}, for an arbitrary $S_1 \subseteq S$. This step is crucial for our proof of Theorem 2, so we state it as our next lemma and give its proof, although it can essentially be found in [G]:

Lemma 2. *If $\emptyset \neq S_1 \subseteq S$ and $\varepsilon \in (0, 1)$ are given, then there exists $\sigma \subseteq S_1$, with $|\sigma| \geq (1 - \varepsilon)|S_1|$, such that*

$$P_\sigma(E_{S_1}) \supseteq \frac{c\sqrt{\varepsilon}}{\sqrt{|S_1|}}[-1, 1]^\sigma,$$

where $c > 0$ is an absolute constant.

Proof: Set $\alpha_k = \sum_{r=0}^{k-1} 2^{r/2}$, $\beta_k = \sum_{r=0}^{k-1} 2^r = 2^{k-1}$, and $Q_\tau = [-1, 1]^\tau$ for every non-empty $\tau \subseteq S_1$.

We shall prove by induction that:

$(*)$ For $k = 1, 2, \dots$, one can find $\sigma_k \subseteq S_1$, with $|\sigma_k| \geq (1 - \frac{1}{2^k})|S_1|$, such that

$$Q_{\sigma_k} \subseteq P_{\sigma_k}(\alpha_k \sqrt{2|S_1|} E_{S_1} \cap \beta_k Q_{S_1}).$$

Since $\alpha_k \leq \frac{2^{k/2}}{\sqrt{2}-1}$, condition $(*)$ clearly implies that, for $k = 1, 2, \dots$,

$$P_{\sigma_k}(E_{S_1}) \supseteq \frac{c}{\sqrt{|S_1|}} \sqrt{\frac{1}{2^k}} [-1, 1]^{\sigma_k}$$

with $c = 1 - \frac{1}{\sqrt{2}}$, which is the assertion of the lemma for $\varepsilon = 1/2^k$. The continuous version will easily follow with a worse constant c.

Inductive step: Consider the set $J_k = \mathbf{0}_{\sigma_k} \times \{-2^{k/2}, 2^{k/2}\}^{S_1 \setminus \sigma_k}$, where σ_k is the subset of S_1 given by $(*)$. Note that $|J_k| = 2^{|S_1 \setminus \sigma_k|}$. By the parallelogram law and the fact that $|S_1 \setminus \sigma_k| \leq |S_1|/2^k$, we have

$$\mathrm{Ave}_{(\delta_j) \in J_k} |\sum_{j \in S_1} \delta_j u_j|_n^2 = 2^k \sum_{j \in S_1 \setminus \sigma_k} |u_j|_n^2 \leq |S_1|,$$

and Markov's inequality implies that there exists $M^{k+1} \subseteq J_k \cap \sqrt{2|S_1|} E_{S_1}$ with $|M^{k+1}| \geq 2^{|S_1 \setminus \sigma_k| - 1}$. Then, by Lemma 1 we can find $\sigma^*_{k+1} \subseteq S_1 \setminus \sigma_k$ of cardinality $|\sigma^*_{k+1}| \geq \frac{|S_1 \setminus \sigma_k|}{2}$ for which

$$P_{\sigma_k \cup \sigma^*_{k+1}}(M^{k+1}) = \mathbf{0}_{\sigma_k} \times \{-2^{k/2}, 2^{k/2}\}^{\sigma^*_{k+1}}.$$

Since $M^{k+1} \subseteq \sqrt{2|S_1|} E_{S_1} \cap 2^{k/2} Q_{S_1}$, it follows that

$$(**) \qquad \mathbf{0}_{\sigma_k} \times 2^k Q_{\sigma^*_{k+1}} \subseteq P_{\sigma_k \cup \sigma^*_{k+1}}(2^{k/2} \sqrt{2|S_1|} E_{S_1} \cap 2^k Q_{S_1}).$$

Suppose now that $a \in Q_{\sigma_k}, b \in Q_{\sigma^*_{k+1}}$. From the inductive hypothesis $(*)$, we can find $w_a \in \beta_k Q_{\sigma^*_{k+1}}$ such that

$$(a, w_a) \in P_{\sigma_k \cup \sigma^*_{k+1}}(\alpha_k \sqrt{2|S_1|} E_{S_1} \cap \beta_k Q_{S_1}).$$

If $v_{a,b} = b - w_a$, it is clear that $v_{a,b} \in Q_{\sigma^*_{k+1}} + \beta_k Q_{\sigma^*_{k+1}} = 2^k Q_{\sigma^*_{k+1}}$, and therefore, by $(**)$,

$$(\mathbf{0}_{\sigma_k}, v_{a,b}) \in P_{\sigma_k \cup \sigma^*_{k+1}}(2^{k/2} \sqrt{2|S_1|} E_{S_1} \cap 2^k Q_{S_1}).$$

Then,

$$(a, b) = (a, w_a) + (\mathbf{0}_{\sigma_k}, v_{a,b}) \in P_{\sigma_k \cup \sigma^*_{k+1}}(\alpha_k \sqrt{2|S_1|} E_{S_1} \cap \beta_k Q_{S_1}) +$$

$$+ P_{\sigma_k \cup \sigma^*_{k+1}}(2^{k/2} \sqrt{2|S_1|} E_{S_1} \cap 2^k Q_{S_1}) \subseteq P_{\sigma_k \cup \sigma^*_{k+1}}(\alpha_{k+1} \sqrt{2|S_1|} E_{S_1} \cap \beta_{k+1} Q_{S_1}).$$

Since $a \in Q_{\sigma_k}, b \in Q_{\sigma^*_{k+1}}$ were arbitrary, this means that

$$Q_{\sigma_k \cup \sigma^*_{k+1}} \subseteq P_{\sigma_k \cup \sigma^*_{k+1}}(\alpha_{k+1}\sqrt{2|S_1|}E_{S_1} \cap \beta_{k+1} Q_{S_1}).$$

If we define $\sigma_{k+1} = \sigma_k \cup \sigma^*_{k+1}$, we readily see that $|\sigma_{k+1}| \geq (1 - \frac{1}{2^{k+1}})|S_1|$, and this completes the inductive step. The first step ($k = 1$) is much simpler. □

For our next two lemmas we shall need to assume that the vectors $u_1, \ldots, u_s$ are linearly independent:

Lemma 3. *Let S_1 be a non-empty subset of S. Then, for every $\theta \in (0, \frac{1}{4})$ we can find disjoint $\sigma, \tau \subseteq S_1$ with $|\sigma| \geq \frac{|S_1|}{2}, |\tau| \leq \theta |S_1|$ such that*

$$P_{S_1 \setminus \tau}(E_{S_1}) \supseteq \mathbf{0}_{S_1 \setminus (\sigma \cup \tau)} \times c\sqrt{\theta} D_\sigma,$$

where $c > 0$ is an absolute constant.

Proof: Set $V_{S_1} = \text{span}\{u_j, j \in S_1\}$. Then, there exist $x_i \in V_{S_1}, i \in S_1$, such that

$$\langle x_1, u_j \rangle = \delta_{ij}, \text{ for any pair of } i, j \in S_1.$$

Applying Lemma 2 for the ellipsoid E_{S_1} we obtain $\tau \subseteq S_1, |\tau| \leq \theta |S_1|$, for which

$$P_{S_1 \setminus \tau}(E_{S_1}) \supseteq \frac{c\sqrt{\theta}}{\sqrt{|S_1|}}[-1, 1]^{S_1 \setminus \tau}.$$

Then, for any choice of scalars $(t_i)_{i \in S_1 \setminus \tau}$ we can find a vector $(\delta_j)_{j \in S_1}$ in E_{S_1} whose restriction in $\mathbb{R}^{S_1 \setminus \tau}$ is $(\frac{c\sqrt{\theta}}{\sqrt{|S_1|}} \text{sign} t_j)_{j \in S_1 \setminus \tau}$. In view of the orthogonality relations between the x_i's and the u_j's, we see that

$$\sum_{i \in S_1 \setminus \tau} |t_i| = \langle \sum_{i \in S_1 \setminus \tau} t_i x_i, \sum_{j \in S_1 \setminus \tau} (\text{sign} t_j) u_j \rangle$$

$$= \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \langle \sum_{i \in S_1 \setminus \tau} t_i x_i, \sum_{j \in S_1} \delta_j u_j \rangle = \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} |\sum_{i \in S_1 \setminus \tau} t_i x_i|_n |\sum_{j \in S_1} \delta_j u_j|_n$$

$$\leq \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} |\sum_{i \in S_1 \setminus \tau} t_i x_i|_n.$$

It follows that the operator $T : \text{span}\{x_i, i \in S_1 \setminus \tau\} \subseteq \ell_2^n \to \ell_1^{|S_1 \setminus \tau|}$, defined by $Tx_i = e_i$ (where $\{e_i\}$ is the canonical orthonormal basis in $\mathbb{R}^{|S_1 \setminus \tau|}$), has norm not exceeding $\sqrt{|S_1|}/c\sqrt{\theta}$. Then, $T^* : \ell_\infty^{|S_1 \setminus \tau|} \to \ell_2^n$ is a 2-absolutely summing operator with 2-summing norm $\pi_2(T^*) \leq K_G \frac{\sqrt{|S_1|}}{c\sqrt{\theta}}$, where K_G is Grothendieck's constant. From Pietch's factorization theorem, applied in the same context as in the proof of Theorem 1.2 [BT], we can find positive real numbers $\lambda_i, i \in S_1 \setminus \tau$, with $\sum_{i \in S_1 \setminus \tau} \lambda_i^2 = 1$, such that for any reals $t_i, i \in S_1 \setminus \tau$,

$$\left(\sum_{i\in S_1\setminus\tau}(\frac{t_i}{\lambda_i})^2\right)^{1/2}\leq K_G\frac{\sqrt{|S_1|}}{c\sqrt{\theta}}|\sum_{i\in S_1\setminus\tau}t_ix_i|_n.$$

Since $\sum_{i\in S_1\setminus\tau}\lambda_i^2=1$ and $\theta<\frac{1}{4}$, we apply Markov's inequality to obtain $\sigma\subseteq S_1\setminus\tau, |\sigma|\geq\frac{|S_1|}{2}$, with $\lambda_i\leq\frac{2}{\sqrt{|S_1|}}$ for every $i\in\sigma$. Suppose now that $(\delta_j)_{j\in\sigma}\in D_\sigma$ is given, i.e $\sum_{j\in\sigma}\delta_j^2\leq 1$. The set $\{u_j, j\in\tau\}\cup\{x_i, i\in S_1\setminus\tau\}$ is linearly independent in V_{S_1}, hence a basis, so we can write

$$\sum_{j\in\sigma}\delta_ju_j+\sum_{j\in\tau}\rho_ju_j=\sum_{i\in S_1\setminus\tau}t_ix_i$$

for suitable $(\rho_j)_{j\in\tau}$, $(t_i)_{i\in S_1\setminus\tau}$. Then,

$$|\sum_{j\in\sigma}\delta_ju_j+\sum_{j\in\tau}\rho_ju_j|_n^2=\langle\sum_{j\in\sigma}\delta_ju_j+\sum_{j\in\tau}\rho_ju_j,\sum_{i\in S_1\setminus\tau}t_ix_i\rangle$$

$$=\langle\sum_{j\in\sigma}\delta_ju_j,\sum_{i\in\sigma}t_ix_i\rangle=\sum_{i\in\sigma}\delta_it_i\leq(\sum_{i\in\sigma}t_i^2)^{1/2}$$

$$\leq\left(\sum_{i\in\sigma}(\frac{t_i}{\lambda_i})^2\right)^{1/2}\frac{2}{\sqrt{|S_1|}}\leq\left(\sum_{i\in S_1\setminus\tau}(\frac{t_i}{\lambda_i})^2\right)^{1/2}\frac{2}{\sqrt{|S_1|}}\leq\frac{2K_G}{c}\frac{1}{\sqrt{\theta}}|\sum_{i\in S_1\setminus\tau}t_ix_i|_n,$$

and therefore,

$$|\sum_{j\in\sigma}\delta_ju_j+\sum_{j\in\tau}\rho_ju_j|_n\leq\frac{2K_G}{c}\frac{1}{\sqrt{\theta}}.$$

This means that $\mathbf{0}_{S_1\setminus(\sigma\cup\tau)}\times(\delta_j)_{j\in\sigma}\in\frac{1}{c_1\sqrt{\theta}}P_{S_1\setminus\tau}(E_{S_1})$ with $c_1=c/2K_G$. Since $(\delta_j)_{j\in\sigma}$ was arbitrary in D_σ, the lemma follows. □

We are now ready to prove Theorem 2 in the case of independent u_j's:

Lemma 4. *For every $\varepsilon\in(0,1)$ one can find $\sigma\subseteq S$, $|\sigma|\geq(1-\varepsilon)s$, such that*

$$P_\sigma(E)\supseteq c\sqrt{\varepsilon}D_\sigma,$$

where $c>0$ is an absolute constant.

Proof: Given $\varepsilon\in(0,1)$ we set $\theta=\varepsilon/7$. Let also k be the non-negative integer for which $\frac{1}{2^{k+1}}\leq\varepsilon<\frac{1}{2^k}$. To obtain σ we shall follow an inductive procedure based on Lemma 3:

Step 1: We set $S_0=S$ and $\theta_1=\theta$. Since $\theta_1\in(0,\frac{1}{4})$, we can find a pair (σ_1,τ_1) of disjoint subsets of S_0 with $|\tau_1|\leq\theta_1|S_0|$, $|\sigma_1|\geq\frac{1}{2}|S_0|$, and $P_{S_0\setminus\tau_1}(E_{S_0})\supseteq$

$\mathbf{0}_{S_0\setminus(\sigma_1\cup\tau_1)} \times c\sqrt{\theta_1}D_{\sigma_1}$, where c is the constant from Lemma 3. Finally, we define $S_1 = S_0\setminus(\sigma_1\cup\tau_1)$. Note that $|S_1| \leq \frac{1}{2}|S_0| = \frac{s}{2}$.

Inductive step: Suppose that S_l has been defined, and $|S_l| > \frac{\varepsilon}{2}s$. If in addition $l < k+2$, we define $\theta_{l+1} = 2^{l/2}\theta$. Note that then $\theta_{l+1} \leq \frac{\sqrt{2}}{7}2^{k/2}\varepsilon < \frac{\sqrt{2}}{7}\sqrt{\varepsilon} < \frac{1}{4}$, and therefore we can apply Lemma 3 to E_{S_l}, θ_{l+1} to obtain a pair $(\sigma_{l+1}, \tau_{l+1})$ of disjoint subsets of S_l, with $|\tau_{l+1}| \leq \theta_{l+1}|S_l|$, $|\sigma_{l+1}| \geq \frac{1}{2}|S_l|$, and $P_{S_l\setminus\tau_{l+1}}(E_{S_l}) \supseteq \mathbf{0}_{S_l\setminus(\sigma_{l+1}\cup\tau_{l+1})} \times c\sqrt{\theta_{l+1}}D_{\sigma_{l+1}}$. To complete the inductive step, we define $S_{l+1} = S_l\setminus(\sigma_{l+1}\cup\tau_{l+1})$. Note also that $|S_{l+1}| \leq \frac{1}{2}|S_l|$ and hence, as far as we continue performing these steps, $|S_l| \leq \frac{s}{2^l}$.

We end this inductive construction when we arrive at a set S_l of cardinality $|S_l| \leq \frac{\varepsilon}{2}s$. This will certainly happen after at most $(k+2)$-steps, since $\frac{1}{2^{k+2}} \leq \frac{\varepsilon}{2}$ and our construction implies that $|S_l| < \frac{s}{2^l}$ for every admissible l.

Suppose that l_* is the first index for which $|S_{l_*}| \leq \frac{\varepsilon}{2}s$. We define $\sigma = \sigma_1\cup\ldots\cup\sigma_{l_*}$.

Claim 1. $|\sigma| \geq (1-\varepsilon)s$.

[*Proof:* Note that $\bigcup_{1\leq l\leq l_*}(\sigma_l\cup\tau_l) = S\setminus S_{l_*}$, hence

$$|S\setminus\sigma| = |S_{l_*}| + \sum_{1\leq l\leq l_*}|\tau_l| \leq \frac{\varepsilon}{2}s + \sum_{1\leq l\leq l_*}\theta_l|S_{l-1}| \leq \frac{\varepsilon}{2}s + \theta\sum_{1\leq l\leq l_*}2^{\frac{l-1}{2}}\frac{s}{2^{l-1}}$$

$$< \frac{\varepsilon}{2}s + \frac{\varepsilon}{7}s(\sum_{l=0}^{\infty}\frac{1}{2^{\frac{l}{2}}}) < \varepsilon s.]$$

Claim 2. If $1 \leq l \leq l_*$, then

$$P_\sigma(E) \supseteq \mathbf{0}_{\sigma\setminus\sigma_l} \times c2^{\frac{l-1}{4}}\sqrt{\theta}D_{\sigma_l}.$$

[*Proof:* Suppose that $\Delta_l = (\delta_j)_{j\in\sigma_l} \in c2^{\frac{l-1}{4}}\sqrt{\theta}D_{\sigma_l}$. Then, our construction implies that $\mathbf{0}_{S_{l-1}\setminus(\sigma_l\cup\tau_l)} \times \Delta_l \in P_{S_{l-1}\setminus\tau_l}(E_{S_{l-1}})$. Hence, we can find $(\zeta_i)_{i\in\tau_l}$ such that

$$|\sum_{j\in\sigma_l}\delta_j u_j + \sum_{i\in\tau_l}\zeta_i u_i|_n \leq 1.$$

Since $\sigma\cap\tau_l = \emptyset$, it is clear that $\mathbf{0}_{\sigma\setminus\sigma_l} \times \Delta_l \in P_\sigma(E)$.]

To conclude the proof of the lemma, suppose that $\Delta = (\delta_j)_{j\in\sigma}$ is an arbitrary point in D_σ, i.e $\sum_{j\in\sigma}\delta_j^2 \leq 1$. Consider the restriction $\Delta_l = \mathbf{0}_{\sigma\setminus\sigma_l} \times (\delta_j)_{j\in\sigma_l}$ of Δ in $\mathbb{R}^{\sigma_l}$, and set $|\Delta_l| = (\sum_{j\in\sigma_l}\delta_j^2)^{1/2}$, $1 \leq l < l_*$. By Claim 2, each Δ_l belongs to $\frac{|\Delta_l|}{c2^{\frac{l-1}{4}}\sqrt{\theta}}P_\sigma(E)$, thus

$$\Delta = \sum_{1\leq l\leq l_*}\Delta_l \in \left(\sum_{1\leq l\leq l_*}\frac{|\Delta_l|}{c2^{\frac{l-1}{4}}\sqrt{\theta}}\right) P_\sigma(E)$$

$$\subseteq \frac{1}{c\sqrt{\theta}} \left(\sum_{1 \le l \le l_*} |\Delta_l|^2 \right)^{1/2} \left(\sum_{l=0}^{\infty} \frac{1}{2^{\frac{l}{2}}} \right)^{1/2} P_\sigma(E)$$

$$\subseteq \frac{\sqrt{7}}{c} \left(\frac{\sqrt{2}}{\sqrt{2}-1} \right)^{1/2} \frac{1}{\sqrt{\varepsilon}} P_\sigma(E),$$

and the lemma is proved with $c' = c/5$. □

Proof of Theorem 2. Assume that $u_1, \ldots, u_s$ are arbitrary vectors in $\mathbb{R}^n$ with $|u_j|_n \le 1,\ j = 1, \ldots, s$. Set $v_i = u_i + e_{i+n}, i = 1, \ldots, s$, where $\{e_i\}_{i \le n+s}$ is the canonical orthonormal basis in $\mathbb{R}^{n+s}$. Then, the v_i's are linearly independent $\sqrt{2}$-norm vectors in $\mathbb{R}^{n+s}$, and if

$$E^* = \{(\delta_j)_{j \le s} : |\sum_{j \le s} \delta_j v_j|_{n+s} \le 1\},$$

Lemma 4 implies that, given $\varepsilon \in (0,1)$, there exists $\sigma \subseteq S, |\sigma| \ge (1-\varepsilon)s$, for which

$$P_\sigma(E^*) \supseteq c'' \sqrt{\varepsilon} D_\sigma$$

with $c'' = c/\sqrt{2}$, c the constant from Lemma 4. Since $|\sum_{j \le s} \delta_j v_j|_{n+s}^2 = |\sum_{j \le s} \delta_j u_j|_n^2 + \sum_{j \le s} \delta_j^2$, we readily see that

$$P_\sigma(E) \supseteq c'' \sqrt{\varepsilon} D_\sigma$$

and Theorem 2 is proved. □

3 Proof of Theorem 1 with $d = 1$

For the proof of the proportional Dvoretzky-Rogers factorization result, we shall combine Theorem 2 with the method used in [ST]: Let $X = (\mathbb{R}^n, \|.\|)$ be an n-dimensional normed space and $\varepsilon \in (0,1)$ be given. Without loss of generality, we may assume that the ellipsoid of minimal volume containing the unit ball B_X of X is the Euclidean unit ball D. By John's theorem [J], $D \subseteq \sqrt{n} B_X$. We can also find contact points $y_i, i \le N, \|y_i\|_X = |y_i|_n = 1, N = O(n^2)$, and positive real numbers $\mu_i, i \le N$, such that the following representation of the identity holds: for every $x \in \mathbb{R}^n,\ x = \sum_{i \le N} \mu_i \langle x, y_i \rangle y_i$. Now, if s is the smallest integer $\ge (1 - \frac{\varepsilon}{2})n$, we can choose $x_1, \ldots, x_s$ among the y_i's so that

Lemma 5 [ST]: $\mathrm{dist}(x_i,\ \mathrm{span}\{x_j : j \ne i\}) \ge \sqrt{\frac{\varepsilon}{2}},\ i = 1, \ldots, s.$ □

Hence, there exist $v_j, j \le s$, in $\mathrm{span}\{x_i, i \le s\}$ satisfying

(i) $|v_j|_n \le \sqrt{2/\varepsilon},\ j = 1, \ldots, s$

(ii) $\langle x_i, v_j\rangle = \delta_{ij},\ i,j = 1,\dots,s.$

Set $u_j = \sqrt{\varepsilon/2}v_j$ and define $E = \{(\delta_j)_{j\le s} : |\sum_{j\le s}\delta_j u_j|_n \le 1\}$. From Theorem 2 we obtain $\sigma \subseteq S,\ |\sigma| \ge (1-\frac{\varepsilon}{2})s$, with $P_\sigma(E) \supseteq c\sqrt{\varepsilon}D_\sigma$. Then $|\sigma| \ge (1-\varepsilon)n$, and for any choice of scalars $\mathbf{t} = (t_i)_{i\in\sigma}$ we have

$$|\mathbf{t}|^2 = \sum_{i\in\sigma} t_i^2 = \langle \sum_{i\in\sigma} t_i x_i, \sum_{j\in\sigma} t_j v_j\rangle = \sqrt{\frac{2}{\varepsilon}}\langle \sum_{i\in\sigma} t_i x_i, \sum_{j\in\sigma} t_j u_j\rangle.$$

We can extend $(\frac{c\sqrt{\varepsilon}}{|\mathbf{t}|}t_j)_{j\in\sigma}$ to a vector $(\delta_j)_{j\le s}$ in E. Hence,

$$|\mathbf{t}|^2 = \sqrt{\frac{2}{\varepsilon}}\frac{|\mathbf{t}|}{c\sqrt{\varepsilon}}\langle \sum_{i\in\sigma} t_i x_i, \sum_{j\le s}\delta_j u_j\rangle \le \frac{c'}{\varepsilon}|\mathbf{t}||\sum_{i\in\sigma} t_i x_i|_n$$

and since $|.| \le \|.\|$ and the x_i's are of $\|.\|$-norm one, we have

$$\left(\sum_{i\in\sigma} t_i^2\right)^{1/2} \le \frac{c'}{\varepsilon}|\sum_{i\in\sigma} t_i x_i|_n \le \frac{c'}{\varepsilon}\|\sum_{i\in\sigma} t_i x_i\|_X \le \frac{c'}{\varepsilon}\sum_{i\in\sigma}|t_i|.$$

Defining $\beta : \ell_1^{|\sigma|} \to X$ with $\beta(e_i) = x_i, i \in \sigma$, and $\alpha : X \to \ell_2^{|\sigma|}$ with $\alpha = TP_\sigma$ where P_σ is the orthogonal projection of X onto span$\{x_i, i \in \sigma\}$ and $Tx_i = e_i$, we have a factorization $i_{1,2} = \alpha o \beta$ of the identity $i_{1,2} : \ell_1^{|\sigma|} \to \ell_2^{|\sigma|}$ with $\|\alpha\|\,\|\beta\| \le c'/\varepsilon$. By duality and by the extension property of ℓ_∞^n, this is then equivalent to the assertion of the theorem. □

4 Remarks

(1) The $\sqrt{\varepsilon}$-dependence on ε in Theorem 2 is best possible: Set $n = s+1$ and $u_j = \frac{1}{\sqrt{2}}(e_j + e_n), j = 1,\dots,s$, where $\{e_j\}_{j\le n}$ is the usual orthonormal basis of $\mathbb{R}^n$. Given $\varepsilon \in (0,1)$ and given any $\sigma \subseteq S$ with $|\sigma| \ge (1-\varepsilon)s$, a point $(t,t,\dots,t)$ is in $P_\sigma(E)$ only if $|t| \le c\frac{\sqrt{\varepsilon}}{\sqrt{s}}$ (S.J. Szarek [Sz.3]).

(2) The argument used in the proof of Theorem 1 and an application of the Cauchy-Schwartz inequality give us the following:

Corollary 1. *If $X = (\mathbb{R}^n, \|.\|)$ is such that the ellipsoid of minimal volume containing B_X is D and if $\varepsilon \in (0,1)$, then we can find $x_1,\dots,x_m \in X$, $\|x_i\|_X = |x_i|_n = 1, m \ge (1-\varepsilon)n$, such that for any reals $t_1,\dots,t_m$,*

$$|\sum_{i=1}^m t_i x_i|_n \ge \frac{c\varepsilon}{\sqrt{m}}\sum_{i=1}^m |t_i|. \quad \square$$

Corollary 1, combined with the method of S.J. Szarek-M. Talagrand [ST] gives the upper estimate $d(X, \ell_\infty^n) \le cn^{5/6}$ for the Banach-Mazur distance to the cube (Corollary 1 can be deduced from Lemma 2: this was the main result in [G]).

Acknowledgement: This work was done while the author was visiting Case Western Reserve University. I would like to thank the Department of Mathematics for the hospitality and Professor S.J. Szarek for helpful discussions.

References

[BS] J. Bourgain and S.J. Szarek, *The Banach-Mazur distance to the cube and the Dvoretzky-Rogers factorization*, Israel J. Math. **62** (1988), 169-180.

[BT] J. Bourgain and L. Tzafriri, *Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis*, Israel J. Math. **57** (1987), 137-224.

[DR] A. Dvoretzky and C.A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. USA **36** (1950), 192-197.

[G] A.A. Giannopoulos, *A note on the Banach-Mazur distance to the cube*, Operator Theory: Advances and applications **77** (1995), 67-73.

[J] F. John, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniversary Volume, New York: Interscience 1948, 187-204.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence spaces*, Springer-Verlag, 1977.

[MS] V.D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Mathematics **1200** (1986).

[Pi] A. Pietch, *Operator Ideals*, North Holland (1978).

[ST] S.J. Szarek and M. Talagrand, *An "isomorphic" version of the Sauer-Shelah lemma and the Banach-Mazur distance to the cube*, Lecture Notes in Mathematics **1376** (1989), 105-112.

[Sa] N. Sauer, *On the density of families of sets*, J. Comb. Theory, Ser. A **13** (1972), 145-147.

[Sh] S. Shelah, *A combinatorial problem: stability and order for models and theories in infinitary languages*, Pacific J. Math. **41** (1972), 247-261.

[Sz.1] S.J. Szarek, *Spaces with large distance to* ℓ_∞^n *and random matrices*, American J. Math **112** (1990), 899-942.

[Sz.2] S.J. Szarek, *On the geometry of the Banach-Mazur compactum*, Lecture Notes in Mathematics **1470** (1991), 48-59.

[Sz.3] S.J. Szarek, Personal Communication.

[TJ] N. Tomczak-Jaegermann, *Banach-Mazur distances and Finite Dimensional Operator Ideals*, Longman (1988).

Department of Mathematics, Case Western Reserve University, Cleveland, Ohio, USA

Current address: Department of Mathematics, University of Crete, Iraklion, Crete, Greece

E-mail: deligia@talos.cc.uch.gr