Affine isoperimetric inequalities on flag manifolds

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Abstract

Building on work of Furstenberg and Tzkoni, we introduce **r**-flag affine quermassintegrals and their dual versions. These quantities generalize affine and dual affine quermassintegrals as averages on flag manifolds (where the Grassmannian can be considered as a special case). We establish affine and linear invariance properties and extend fundamental results to this new setting. In particular, we prove several affine isoperimetric inequalities from convex geometry and their approximate reverse forms. We also introduce functional forms of these quantities and establish corresponding inequalities.

1 Introduction

Affine isoperimetric inequalities provide a rich foundation for understanding principles in geometry and analysis that arise in the presence of symmetries. Among the most fundamental examples is the Blaschke-Santaló inequality [48] on the product of volumes of an origin-symmetric convex body L in $\mathbb{R}^n$ and its polar $L^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \ \forall y \in L\}$. The latter asserts that this product is maximized for ellipsoids, i.e.,

$$|L|\,|L^\circ| \leq \omega_n^2, \tag{1}$$

where ω_n is the volume of the unit Euclidean ball B_2^n. The Blaschke-Santaló inequality, and its version for non-origin-symmetric bodies, is one of several equivalent forms of the affine isoperimetric inequality; see e.g., the survey [34]. Moreover, it admits numerous extensions: for example, L_p versions [37], generalizations from convex bodies to functions, e.g., [3], [1], [14] with applications to concentration of measure [1], [28]; further functional affine isoperimetric inequalities, e.g., [2]; stronger versions in which stochastic dominance holds [10].

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Another fundamental affine isoperimetric inequality is the Petty polar projection inequality [44]. This concerns projection bodies, which are special zonoids that play a fundamental role in convex geometry, functional analysis, among other fields, e.g., [49], [16]. The projection body of a convex body $L \subseteq \mathbb{R}^n$ is the convex body ΠL defined by its support function in direction $\theta \in S^{n-1}$ by $h_{\Pi L}(\theta) = |P_{\theta^\perp} L|$, where $P_{\theta^\perp}$ is the orthogonal projection onto $\theta^\perp$. The Petty projection inequality asserts that the affine-invariant quantity $|L|^{n-1} |(\Pi L)^\circ|$ is maximized by ellipsoids, i.e.,

$$|L|^{n-1} |(\Pi L)^\circ| \leq \omega_n^n \omega_{n-1}^{-n}. \tag{2}$$

The Petty projection inequality is the geometric foundation for Zhang's affine Sobolev inequality [52]. Its equivalent forms and extensions have given rise to fundamental inequalities in analysis, geometry and information theory, e.g., [35], [36].

The affine invariance in inequalities (1) and (2) follows from volumetric considerations. However, as we will review below, the underlying principle goes much deeper and extends to the family of affine quermassintegrals, of which $|L^\circ|$ and $|(\Pi L)^\circ|$ are just two special cases, up to normalization. Formally, the affine quermassintegrals are defined for compact sets $L \subseteq \mathbb{R}^n$ and $1 \leq k \leq n$ by

$$\Phi_{[k]}(L) = \left(\int_{G_{n,k}} |P_E L|^{-n} \, d\nu_{n,k}(E) \right)^{-\frac{1}{kn}}, \tag{3}$$

where $G_{n,k}$ is the Grassmannian manifold of k-dimensional linear subspaces equipped with the Haar probability measure $\nu_{n,k}$. Writing $|L^\circ|$ and $|(\Pi L)^\circ|$ in polar coordinates shows a direct connection to $k = 1$ and $k = n-1$ in (3), respectively. As the name suggests, they are affine-invariant, i.e., $\Phi_{[k]}(TL) = \Phi_{[k]}(L)$ for each volume preserving affine transformation T, as proved by Grinberg [19], extending earlier work on ellipsoids by Furstenberg-Tzkoni [15] and Lutwak [33].

The quantities $\Phi_{[k]}(L)$ are affine versions of quermassintegrals or intrinsic volumes, which play a central role in Brunn-Minkowski theory [49]. In particular, the intrinsic volumes $V_1(L), \ldots, V_n(L)$ of a convex body L admit similar representations through Kubota's integral recursion as

$$V_k(L) = c_{n,k} \int_{G_{n,k}} |P_E L| \, d\nu_{n,k}(E), \tag{4}$$

where $c_{n,k}$ is a constant that depends only on n and k. They enjoy many fundamental inequalities, such as

$$V_k(L) \geq V_k(r_L B_2^n), \tag{5}$$

for $k = 1, \ldots, n-1$, where r_L is the radius of a Euclidean ball having the same volume as L. Taking $k = 1$ in (5) corresponds to Urysohn's inequality, while $k = n-1$ is the standard isoperimetric inequality. From Jensen's inequality one sees that (1) and (2) provide stronger affine-invariant analogues of (5) for $k = 1$ and $k = n$, respectively. For the intermediary values $1 < k < n$, the inequalities in (5) are well-known consequences of Alexandrov-Fenchel inequality, e.g., [49]. On the other hand, it is still an open problem, posed by Lutwak [33], [16, Problem 9.3], to determine minimizers for their affine versions, namely, to prove that for $1 < k < n-1$,

$$\Phi_{[k]}(L) \geq \Phi_{[k]}(r_L B_2^n). \tag{6}$$

In the last 40 years, a compelling dual theory, initiated by Lutwak in [30], has flourished (see, e.g., [49], [16]). Rather than convex bodies and projections onto lower-dimensional subspaces, this involves star-shaped sets and intersections with subspaces. As above, a key isoperimetric inequality lies at its foundation. The intersection body of a star-shaped body L is the star-shaped body IL with radial function $\rho_{IL}(\theta) := |L \cap \theta^{\perp}|$. The Busemann intersection inequality [8], proved originally for convex bodies L, states that

$$|IL|\,|L|^{-(n-1)} \leq \omega_{n-1}^{n} \omega_n^{-(n-2)}. \tag{7}$$

The volume of the intersection body lies at one end-point of a sequence of SL_n-invariant quantities that are called the dual affine quermassintegrals. These are SL_n-invariant analogs of the dual quermassintegrals introduced by Lutwak [32]. Formally, for a compact set $L \subseteq \mathbb{R}^n$ and $1 \leq k \leq n$, the dual affine quermassintegrals of L are defined by

$$\Psi_{[k]}(L) = \left(\int_{G_{n,k}} |L \cap E|^n \, d\nu_{n,k}(E) \right)^{\frac{1}{kn}}. \tag{8}$$

As above, Grinberg [19], drawing on [15], showed that these enjoy invariance under volume-preserving linear transformations, i.e. $\Psi_{[k]}(TL) = \Psi_{[k]}(L)$ for $T \in SL_n$. They also satisfy the following extension of (7), proved by Busemann-Strauss [8] and Grinberg [19]:

$$\Psi_{[k]}(L) \leq \Psi_{[k]}(r_L B_2^n). \tag{9}$$

While the dual theory has been developed for star-shaped bodies, the investigation of these quantities goes deeper and can be extended to bounded Borel sets and non-negative measurable functions [17], [12]. For recent developments on dual Brunn-Minkowski theory, see [49], [16], [23] and the references therein.

The theory that has developed around affine and dual affine quermassintegrals has implications outside of convex geometry. As a sample, we mention variants of (9) for functions in [12] lead to sharp asymptotics for small-ball probabilities for marginal densities when independence may be lacking; small-ball probabilities for the volume of random polytopes [42]; bounds on marginal densities of log-concave measures connected to the Slicing Problem [43]. In these applications, the main focus was on volumetric estimates and implications for high-dimenional probability measures. Recently, there is increasing interest in other probabilistic aspects of Grassmannians and flag manifolds such as topological properties of random sets in real algebraic geometry; see [6] and the references therein.

Towards flag manifolds

Given the usefulness of affine and dual affine quermassintegrals, it is worth re-visiting the role ellipsoids have played in their development. The work of Furstenberg-Tzkoni [15] that established the SL_n-invariance of (8) for ellipsoids went well beyond this special case. One aspect of [15] that has received less attention is kindred integral geometric formulas for ellipsoids on flag manifolds. They established deeper connections to representation of spherical functions on symmetric spaces. Unlike affine and dual affine quermassintegrals, the corresponding notions for convex bodies, compact sets or functions have not been investigated in the setting of flag manifolds. Our main goal is to initiate such a study in this paper.

Flag manifolds are natural generalizations of Grassmannians in geometry. In convex geometry, mixed volumes admit representations in terms of certain flag measures, e.g., [24]. Our work goes in a different direction and the focus here is on flag versions of quantities like those in (3) and (8) and corresponding extremal inequalities. We establish fundamental properties such as affine invariance and affine inequalities. We also treat companion approximate reverse isoperimetric inequalities, which play an important role in high-dimensional convex geometry and probability.

1.1 Main results

We start by recalling the setting from work of Furstenberg and Tzkoni [15]. Let $1 \le r \le n-1$ and let $\mathbf{r} := (i_1, i_2, \cdots, i_r)$ be a strictly increasing sequence of integers, $1 \le i_1 < i_2 < \cdots < i_r \le n-1$. Let $\xi_{\mathbf{r}} := (F_1, \cdots, F_r)$ be a (partial) flag of subspaces; i.e. $F_1 \subset F_2 \subset \cdots \subset F_r$ with each F_j an i_j-dimensional subspace. We denote by $F^n_{\mathbf{r}}$ the flag manifold (with indices $\mathbf{r}$) as the set of all partial flags $\xi_{\mathbf{r}}$. $F^n_{\mathbf{r}}$ is equipped with the unique Haar probability measure that is invariant under the action of SO_n and all integrations on this set in this note are meant with respect to that measure.

In the special case when $r = 1$ and $i_1 = k$, the partial flag manifold $F^n_{\mathbf{r}}$ is just the Grassmann manifold $G_{n,k}$. Hence the (partial) flag-manifolds can be considered as generalizations of Grassmannians. When $r = n-1$, so that $\mathbf{r} := (1, 2, \ldots, n-1)$, we write $F^n := F^n_{\mathbf{r}}$ for the complete flag manifold. We follow the convention that $i_0 = 0$ and $i_{r+1} = n$, hence

$$\sum_{j=1}^{r} i_j(i_{j+1} - i_{j-1}) = i_r n. \tag{10}$$

Let L be a compact set in $\mathbb{R}^n$ and let $1 \le r \le n-1$ and $\mathbf{r}$ be a set of indices as above. We define the $\mathbf{r}$-*flag quermassintegral* of L by

$$\Phi_{\mathbf{r}}(L) := \left(\int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} |P_{F_j} L|^{i_{j-1} - i_{j+1}} d\xi_{\mathbf{r}} \right)^{-\frac{1}{i_r n}}. \tag{11}$$

Similarly, we define the *dual* $\mathbf{r}$-*flag quermassintegral* of L by

$$\Psi_{\mathbf{r}}(L) := \left(\int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1} - i_{j-1}} d\xi_{\mathbf{r}} \right)^{\frac{1}{i_r n}}. \tag{12}$$

In [15], it was shown that when $L = \mathcal{E}$ is an ellipsoid, $\Psi_{\mathbf{r}}(\mathcal{E})$ is invariant under SL_n. When $r = 1$, the $\mathbf{r}$-flag quermassintegrals are exactly the affine quermassintegrals; similarly for the dual case. Thus the latter quantities can be considered as extensions of the (dual) affine quermassintegrals to flag manifolds. For complete flag manifolds, we similarly define

$$\Psi_{\mathbf{F^n}}(L) := \left(\int_{F^n} \prod_{i=1}^{n-1} |L \cap F_i|^2 d\xi\right)^{\frac{1}{n(n-1)}} \quad \text{and} \quad \Phi_{\mathbf{F^n}}(L) := \left(\int_{F^n} \prod_{i=1}^{n-1} |P_{F_i} L|^2 d\xi\right)^{\frac{1}{n(n-1)}}. \tag{13}$$

Clearly, by (10)

$$\Psi_{\mathbf{r}}(\lambda L) = \lambda \Psi_{\mathbf{r}}(L), \qquad \Phi_{\mathbf{r}}(\lambda L) = \lambda \Phi_{\mathbf{r}}(L) \quad (\lambda > 0). \tag{14}$$

Our first result extends the invariance results of Grinberg [19] that give invariance of (3) and (8) under volume-preserving affine and linear transformations, respectively.

Theorem 1.1. *Let L be a compact set in $\mathbb{R}^n$, $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Let A be an affine map that preserves volume and $T \in SL_n$. Then*

$$\Phi_{\mathbf{r}}(AL) = \Phi_{\mathbf{r}}(L) \quad \text{and} \quad \Psi_{\mathbf{r}}(TL) = \Psi_{\mathbf{r}}(L). \tag{15}$$

With such invariance properties, it is natural to seek extremizers of $\Phi_{\mathbf{r}}(L)$ and $\Psi_{\mathbf{r}}(L)$, especially over convex bodies $L \subseteq \mathbb{R}^n$. However, even for the Grassmannian very few such results are known; cf. Lutwak's conjectured inequality (6). We note, however, that inequality (6) does hold at the expense of a universal constant, as proved by the second and third-named authors [42]. It is easy to construct compact sets $L \subseteq \mathbb{R}^n$ of a given volume such that $\Phi_{[k]}(L)$ is arbitrarily large. This, however, cannot happen when L is convex [11]. In particular, it was shown that up to a logarithmic factor in the dimension n, $\Phi_{[k]}(L)$ does not exceed $\Phi_{[k]}(r_L B_2^n)$.

We extend the aforementioned results to the setting of $\mathbf{r}$-flag quermassintegrals. In this note $c, c', c_0, \cdots$ etc. will denote universal constants (not necessarily the same at each occurrence).

Theorem 1.2. *Let L be a compact set in $\mathbb{R}^n$, $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ an increasing sequence of integers between 1 and $n-1$. Then*

$$\Psi_{\mathbf{r}}(L) \le \Psi_{\mathbf{r}}(r_L B_2^n). \tag{16}$$

If L is a symmetric convex body, then

$$\Psi_{\mathbf{r}}(L) \ge \frac{c}{\min\left\{\sqrt{\frac{n}{i_r}}, \log n\right\}} \Psi_{\mathbf{r}}(r_L B_2^n). \tag{17}$$

If L is a convex body, then

$$\frac{1}{c} \Phi_{\mathbf{r}}(r_L B_2^n) \le \Phi_{\mathbf{r}}(L) \le c \min\left\{\sqrt{\frac{n}{i_r}}, \log n\right\} \Phi_{\mathbf{r}}(r_L B_2^n). \tag{18}$$

Further drawing on [15], we also consider variants of **r**-flag (dual) affine quermassintegrals involving permutations ω of $\{1,\dots,n\}$. We define the *ω-flag quermassintegral* and *ω-flag dual quermassintegral* as follows: for every compact set L in $\mathbb{R}^n$,

$$\Phi_\omega(L) := \begin{cases} \left(\int_{\mathbf{F^n}} \prod_{j=1}^{n-1} |P_{F_j} L|^{-\omega(j)+\omega(j+1)-1} d\xi\right)^{-\frac{1}{n(n-\omega(n))}} & \text{if } \omega(n) \neq n, \\ \int_{\mathbf{F^n}} \prod_{j=1}^{n-1} |P_{F_j} L|^{-\omega(j)+\omega(j+1)-1} d\xi & \text{if } \omega(n) = n. \end{cases} \tag{19}$$

and

$$\Psi_\omega(L) := \begin{cases} \left(\int_{\mathbf{F^n}} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi\right)^{\frac{1}{n(n-\omega(n))}} & \text{if } \omega(n) \neq n, \\ \int_{\mathbf{F^n}} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi & \text{if } \omega(n) = n. \end{cases} \tag{20}$$

Furstenberg and Tzkoni showed SL_n-invariance of Ψ_ω for ellipsoids. We investigate the extent to which this invariance carries over to compact sets. Moreover, in the case of convex bodies we show that such quantities cannot be too degenerate in the sense that they admit uniform upper and lower bounds, independent of the body. We apply V. Milman's M-ellipsoids [38], together with the aforementioned SL_n-invariance of Furstenberg-Tzkoni to establish these bounds (see Corollary 4.4).

In §5, we introduce functional analogues of the **r**-flag dual affine quermassintegrals. We show that more general quantities share the SL_n-invarinace properties and we prove sharp isoperimetric inequalities. In this section, we invoke techniques and results from our previous work [12]. Lastly, in §5 we also introduce a functional form of **r**-flag affine quermassintegrals. There is much recent interest in extending fundamental geometric inequalities from convex bodies to certain classes of functions, e.g., [26], [4], [39]. The latter authors have studied variants of inequalities for intrinsic volumes, or even mixed volumes, and other general quantities; for example, they establish functional analogues of (5). Of course, for functions one cannot hope for a sharp analogue of (6), as this is open even for affine quermassintegrals of convex bodies. On the other hand, we establish a general functional result at the expense of a universal constant. Invariance properties and bounds for these quantities are treated in §5.2.

2 Affine invariance

In this section we will present the proof of Theorem 1.1. The following proposition relates integration on a flag manifold to integration on nested Grassmannians, (see [50] Theorem 7.1.1 on p. 267 for such a result for flags of elements consisting of two subspaces). Since we will use this fact many times throughout this paper, we include the proof. For a subspace $F \subset \mathbb{R}^n$, we denote by $G_{F,i}$ the Grassmannian of all i-dimensional subspaces contained in F.

Proposition 2.1. *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1,\cdots,i_r)$ be an increasing sequence of integers between 1 and $n-1$. For $G \in L^1(F^n_{\mathbf{r}})$,*

$$\int_{F^n_{\mathbf{r}}} G(\xi_{\mathbf{r}}) d\xi_{\mathbf{r}} = \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_2,i_1}} G(F_1,\cdots,F_r)\, dF_1 \cdots dF_{r-1} dF_r. \tag{21}$$

For simplicity, we have suppressed the notation to write dF_1 rather than $d\mu_{G_{F_2,i_1}}(F_1)$; similarly for all other indices. This convention will be used throughout.

Proof. Fix i_j. Denote by $SO(F_j)$ the subgroup of SO_n acting transitively on $G_{F_j,i_{j-1}}$. For example, if $F_o = span\{e_1,\dots,e_{i_j}\}$ and $E_o = span\{e_1,\dots,e_{i_{j-1}}\}$, then elements of $SO(F_o)$ are given by

$$\begin{pmatrix} SO_{i_j} & 0 \\ 0 & I_{n-i_j} \end{pmatrix}.$$

And the stabilizer of E_o in $SO(F_o)$ is

$$\begin{pmatrix} SO_{i_{j-1}} & 0 & 0 \\ 0 & SO_{i_j-i_{j-1}} & 0 \\ 0 & 0 & I_{n-i_j} \end{pmatrix}.$$

The measure $\mu_{G_{F_j},i_{j-1}}$ is invariant under $SO(F_j)$. Further, for $g \in SO_n$ and a Borel subset $A \subset G_{F_j,i_{j-1}}$ we have $\mu_{G_{gF_j},i_{j-1}}(gA) = \mu_{G_{F_j},i_{j-1}}(A)$.

We will show that both integrals are invariant under the action of SO_n. Fix $g \in SO_n$. We start with the integral on the right-hand side of (21):

$$\begin{aligned}
&\int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_2,i_1}} G(g^{-1}\cdot(F_1,\cdots,F_r))\, dF_1 \cdots dF_{r-1} dF_r \\
&\quad = \int_{G_{n,i_r}} \int_{G_{gF_r,i_{r-1}}} \cdots \int_{G_{gF_2,i_1}} G(F_1,\cdots,F_r)\, d(gF_1) \cdots d(gF_{r-1}) d(gF_r) \\
&\quad = \int_{G_{n,i_r}} \int_{G_{gF_r,i_{r-1}}} \cdots \int_{G_{F_2,i_1}} G(F_1,\cdots,F_r)\, dE_1 \cdots d(gF_{r-1}) d(gF_r) \\
&\quad = \cdots \\
&\quad = \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_2,i_1}} G(F_1,\cdots,F_r)\, dF_1 \cdots dF_{r-1} dF_r,
\end{aligned}$$

where we have sent $(F_1,\cdots,F_r) \to g\cdot(F_1,\cdots,F_r)$ and then used the invariance property

$$\mu_{G_{gF_j,i_{j-1}}}(gA) = \mu_{G_{F_j,i_{j-1}}}(A)$$

for all $r-1$ inner integrals, for the outer integral we use the SO_n-invariance of the measure $\mu_{G_{n,i_r}}$. Note that at each step $(F_1,\cdots,F_r)$ remains an element of $F^n_{\mathbf{r}}$, this is to say that the inclusion relation is preserved. The invariance of the integral on the left-hand side of (21) is a consequence of the SO_n-invariance of the measure $\mu_{F^n_{\mathbf{r}}}$. The proposition now follows by the uniqueness of the SO_n-invariant probability measure on $F^n_{\mathbf{r}}$, see for example §13.3 in [50]. □

The following fact allows one to view an integral of a function on a partial flag as an integral over the full flag manifold. In this case, to avoid confusion, the subspaces of flag manifolds are indexed by their dimension.

Proposition 2.2. *Let $1 \leq r \leq n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. For a function G on the partial flag $F^n_{\mathbf{r}}$, denote by $\widetilde{G}$ its trivial extension to the full flag manifold F^n, i.e., $\widetilde{G}(F_1, \ldots, F_{n-1}) := G(F_{i_1}, \ldots, F_{i_r})$. Then*

$$\int_{F^n} \widetilde{G}(\eta) d\eta = \int_{F^n_{\mathbf{r}}} G(\xi_{\mathbf{r}}) d\xi_{\mathbf{r}}. \tag{22}$$

Proof. We "integrate out" the Grasmannians that do not contain subspaces that G depends on by repeatedly using the identity

$$\int_{G_{F_{j+1},j}} \int_{G_{F_j,j-1}} f(F_{j-1}) dF_{j-1} dF_j = \int_{G_{F_{j+1},j-1}} f(F_{j-1}) dF_{j-1}.$$

On the right-hand side, we integrate over the set of all $(j-1)$-dimensional subspaces in the ambient $(j+1)$-dimensional space. On the left-hand side we integrate over the same set of planes stepwise, we step from one j-dimensional subspace in the ambient $(j+1)$-dimensional space to the next and in each such subspace we consider all $(j-1)$-dimensional subspaces. The above identity holds since we are using probability measures on each nested Grassmannian. Applying the latter iteratively, we get

$$\begin{aligned}
&\int_{F^n} \widetilde{G}(\eta) d\eta \\
&= \int_{G_{n,n-1}} \int_{G_{F_{n-1},n-2}} \cdots \int_{G_{F_2,1}} \widetilde{G}(F_1, \cdots, F_{n-1}) dF_1 \cdots dF_{n-2} dF_{n-1} \\
&= \int_{G_{n,n-1}} \int_{G_{F_{n-1},n-2}} \cdots \int_{G_{F_2,1}} G(F_{i_1}, \ldots, F_{i_r}) dF_1 \cdots dF_{n-2} dF_{n-1} \\
&= \int_{G_{n,n-1}} \cdots \int_{G_{F_{i_1+1},i_1}} G(F_{i_1}, \ldots, F_{i_r}) \left(\int_{G_{F_{i_1},i_1-1}} \cdots \int_{G_{F_2,1}} dF_1 \ldots dF_{i_1-1} \right) dF_{i_1} \cdots dF_{n-1} \\
&= \int_{G_{n,n-1}} \cdots \int_{G_{F_{i_1+1},i_1}} G(F_{i_1}, \ldots, F_{i_r}) dF_{i_1} \cdots dF_{n-1} \\
&= \int_{G_{n,n-1}} \cdots \left(\int_{G_{F_{i_2},i_2-1}} \cdots \int_{G_{F_{i_1+1},i_1}} G(F_{i_1}, \ldots, F_{i_r}) dF_{i_1} \cdots dF_{i_2-1} \right) \cdots dF_{n-1} \\
&= \int_{G_{n,n-1}} \cdots \left(\int_{G_{F_{i_2},i_1}} G(F_{i_1}, \ldots, F_{i_r}) dF_{i_1} \right) \cdots dF_{n-1} \\
&= \cdots \\
&= \int_{G_{n,i_r}} \int_{G_{F_{i_r},i_{r-1}}} \cdots \int_{G_{F_{i_2},i_1}} G(F_{i_1}, \cdots, F_{i_r}) dF_{i_1} \cdots dF_{i_{r-1}} dF_{i_r} \\
&= \int_{F^n_{\mathbf{r}}} G(\xi_{\mathbf{r}}) d\xi_{\mathbf{r}}.
\end{aligned}$$

□

We now turn to the invariance properties of the functionals $\Phi_{\mathbf{r}}$ and $\Psi_{\mathbf{r}}$. Although self-contained proofs are possible, they require somewhat involved machinery. Since all of the ingredients are available in the literature [15, 19, 12], we have chosen to gather the essentials without proofs.

For readers less familiar with the relevant work, we will explain the main points behind the affine invariance of the functionals $\Phi_{[k]}(K)$ and $\Psi_{[k]}(K)$ along the way. There are two important changes of variables: a 'global' change of variables on the Grassmannian $G_{n,k}$ or the flag manifold $F^n_{\mathbf{r}}$ and a 'local' change of variables on each element $F \in G_{n,k}$ or $\xi_{\mathbf{r}} \in F^n_{\mathbf{r}}$.

Let $g \in SL_n$, $F \in G_{n,k}$ and $A \subset F$ be a full-dimensional Borel set, then $|gA| = |\det(g|_F)||A|$. This determinant of the transformation g restricted to the subspace F, $\det(g|_F)$, is the Jacobian in the following change of variables:

$$\int_{gF} f(g^{-1}t)dt = \int_F f(t)|\det(g|_F)|dt. \tag{23}$$

Denote it as in [15] by $\sigma_k(g, F) := |\det(g|_F)| = \frac{|gA|}{|A|}$.

For the relevant manifolds M considered in this paper, denote by $\sigma_M(g, F)$ the Jacobian determinant in the following change of variables:

$$\int_M f(F)dF = \int_M f(gF)\sigma_M(g, F)dF. \tag{24}$$

Furstenberg and Tzkoni proved in [15] that

$$\sigma_{G_{n,k}}(g, F) = \sigma_k^{-n}(g, F) \tag{25}$$

and

$$\sigma_{F^n_{\mathbf{r}}}(g, \xi_{\mathbf{r}}) = \sigma_{i_1}^{-i_2}(g, F_1)\sigma_{i_2}^{i_1-i_3}(g, F_2)\cdots\sigma_{i_r}^{i_{r-1}-n}(g, F_r), \tag{26}$$

where $\mathbf{r} := (i_1, \cdots, i_r)$. The linear-invariance of the dual affine quermassintergals $\Psi_{[k]}$ now follows immediately. Indeed, for $g \in SL_n$

$$\begin{aligned} \Psi^{kn}_{[k]}(gL) &= \int_{G_{n,k}} |gL \cap F|^n dF = \int_{G_{n,k}} |gL \cap gF|^n \sigma_{G_{n,k}}(g, F)dF \\ &= \int_{G_{n,k}} (\sigma_k(g, F)|L \cap F|)^n \, \sigma_k^{-n}(g, F)dF = \Psi^{kn}_{[k]}(L), \end{aligned}$$

where we have used (24), (23) with $f = 1_L$ and (25). Now we turn toward the proof of Theorem 1.1. We start with the case of dual $\mathbf{r}$-flag quermassintegrals.

Proposition 2.3. *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. For every compact set L in $\mathbb{R}^n$ and every $g \in SL_n$,*

$$\Psi_{\mathbf{r}}(gL) = \Psi_{\mathbf{r}}(L). \tag{27}$$

Proof. Let us start by expressing $\sigma_{F^n_{\mathbf{r}}}(g, \xi_{\mathbf{r}})$ in terms of sections. For this note that

$$\sigma_{i_j}(g, F_j) = \frac{|g(L \cap F_j)|}{|L \cap F_j|},$$

where as a subset of F_j we use the section $L \cap F_j$. By (26) with $i_0 = 0$ and $i_{r+1} = n$, we have

$$\sigma_{F_{\mathbf{r}}^n}(g, \xi_{\mathbf{r}}) := \prod_{j=1}^{r} \sigma_{i_j}^{-i_{j+1}+i_{j-1}}(g, F_j) = \prod_{j=1}^{r} \frac{|L \cap F_j|^{i_{j+1}-i_{j-1}}}{|gL \cap gF_j|^{i_{j+1}-i_{j-1}}}.$$

Using the change of variables (24) with the above expression for $\sigma_{F_{\mathbf{r}}^n}$, yields

$$\begin{aligned}
\Psi_{\mathbf{r}}^{ni_r}(gL) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |gL \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |gL \cap gF_j|^{i_{j+1}-i_{j-1}} \sigma_{F_{\mathbf{r}}^n}(g, \xi_{\mathbf{r}}) d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |gL \cap gF_j|^{i_{j+1}-i_{j-1}} \prod_{j=1}^{r} \frac{|L \cap F_j|^{i_{j+1}-i_{j-1}}}{|gL \cap gF_j|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&= \Psi_{\mathbf{r}}^{ni_r}(L).
\end{aligned}$$

This proves (27). $\square$

To recall the linear-invariance of the operator $\Phi_{[k]}$, we again follow Grinberg [19]. Observe that for $F \in G_{n,k}$ and $g \in SL_n$ upper-triangular with respect to the decomposition $\mathbb{R}^n = F + F^{\perp}$, we have

$$|P_F(g^t L)| = |gP_F L| = |\det(g|_F)||P_F L| = \sigma_k(g, F)|P_F L|.$$

While for $l \in SO_n$, we have $P_F(l^t L) = P_{lF}(L)$. Since any $g \in SL_n$ can be written as a product of a rotation and an upper-triangular matrix, combining the two observations yields the following.

Lemma 2.4 ([19])**.** *Let L be a compact set in $\mathbb{R}^n$, $F \in G_{n,k}$ and $g \in SL_n$. Then*

$$|P_F(g^t L)| = |P_{gF} L|\sigma_k(g, F). \tag{28}$$

The linear-invariance of the affine quermassintergals $\Phi_{[k]}$ can now be seen as follows: Let $g \in SL_n$

$$\Phi_{[k]}^{-kn}(g^t L) = \int_{G_{n,k}} |P_F(g^t L)|^{-n} dF = \int_{G_{n,k}} |P_{gF} L|^{-n} \sigma_k^{-n}(g, F) dF = \Phi_{[k]}^{-kn}(L),$$

where we have used (28) and (24) taking into account (25).

Proposition 2.5. *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Let A be an affine volume preserving map in $\mathbb{R}^n$. Then for every compact set L in $\mathbb{R}^n$,*

$$\Phi_{\mathbf{r}}(AL) = \Phi_{\mathbf{r}}(L). \tag{29}$$

Proof. We will first prove the theorem in the case $A := g \in SL_n$. Using (28) for the projection onto each F_j, (26) and making the change of variables (24), we get

$$\begin{aligned}
\Phi_{\mathbf{r}}^{-ni_r}(g^t L) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{F_j}(g^t L)|^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} \left(|P_{gF_j} L| \sigma_{i_j}(g, F_j)\right)^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{gF_j} L|^{-i_{j+1}+i_{j-1}} \prod_{j=1}^{r} \sigma_{i_j}^{-i_{j+1}+i_{j-1}}(g, F_j) d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{gF_j} L|^{-i_{j+1}+i_{j-1}} \sigma_{F_{\mathbf{r}}^n}(g, \xi_{\mathbf{r}}) d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{F_j} L|^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}} \\
&= \Phi_{\mathbf{r}}^{-ni_r}(L).
\end{aligned}$$

The general case follows easily. □

The proof of Theorem 1.1 is now complete.

3 Inequalities

We start by proving an extension of the inequality of Busemann-Straus and Grinberg (9) to flag manifolds.

Proposition 3.1. *Let $1 < r \leq n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then for every compact set L in $\mathbb{R}^n$,*

$$\Psi_{\mathbf{r}}(L) \leq \Psi_{\mathbf{r}}(r_L B_2^n) \tag{30}$$

with equality if and only if L is a centered ellipsoid (up to a set of measure zero).

Proof. Inequality (9) implies that for every n, every $E \in G_{n,m}$, any $1 \leq \ell \leq m-1$ and every compact set $L \subseteq \mathbb{R}^n$

$$\int_{G_{E,\ell}} |(L \cap E) \cap F|^m d\mu_{G_{E,\ell}}(F) \leq c_{m,\ell} |L \cap E|^\ell, \tag{31}$$

with equality iff $L \cap E$ is an ellipsoid (up to a measure 0 set - see [17]). Using (21) and (31) we have that

$$
\begin{aligned}
\Psi_{\mathbf{r}}^{i_r n}(L) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&= \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_2,i_1}} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} dF_1 \cdots dF_{r-1} dF_r \\
&= \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} \times \\
&\quad \times \left(\int_{G_{F_2,i_1}} |L \cap F_1|^{i_2} dF_1 \right) dF_2 \cdots dF_{r-1} dF_r \\
&= \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} \times \\
&\quad \times \left(\int_{G_{F_2,i_1}} |(L \cap F_2) \cap F_1|^{i_2} dF_1 \right) dF_2 \cdots dF_{r-1} dF_r \\
&\leq c_{i_2,i_1} \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} \times \\
&\quad \times |L \cap F_2|^{i_1} dF_2 \cdots dF_{r-1} dF_r \\
&\leq \cdots \\
&\leq |L|^{i_r} \prod_{j=1}^{r} c_{i_{j+1},i_j}.
\end{aligned}
$$

The last inequality is an equality only when L is a centered ellipsoid, up to set of measure zero; see [17]. Since for the Euclidean ball all inequalities in the previous chain are actually equalities, we can compute the constants and by the linear-invariance property established by Furstenberg-Tzkoni we conclude the proof. □

Our next result is a type of Blaschke-Santaló and reverse Blaschke-Santaló inequality for $\mathbf{r}$-flag quermassintegrals. These inequalities concern the volume of the polar body. For a compact set L we define the *polar body* L° (with respect to the origin) as the convex body

$$L^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \ \forall y \in L\}.$$

It is straightforward to check the following inclusion: for every compact set L in $\mathbb{R}^n$ and $F \in G_{n,k}$,

$$P_F L^\circ \subseteq (L \cap F)^\circ. \tag{32}$$

If, in addition, L is convex and 0 is in the interior of L,

$$P_F L^\circ = (L \cap F)^\circ. \tag{33}$$

Recall that the Blaschke-Santaló inequality (for symmetric convex bodies), e.g., [16], [49], states that for every symmetric convex body L in $\mathbb{R}^n$,

$$|L||L^\circ| \leq |B_2^n|^2. \tag{34}$$

Moreover (34) holds when L is convex and L° is centered [49]. An approximate reverse form of this inequality is known as the Bourgain-Milman theorem [5]: for every compact, convex set L with $0 \in \mathrm{int}(L)$,

$$|L||L^\circ| \geq c^n |B_2^n|^2. \tag{35}$$

Other proofs of this inequality include [38], [27], [40], [18].

The next proposition is the aforementioned Blaschke-Santaló and its (approximate) reversal in the setting of $\mathbf{r}$-flag manifolds:

Proposition 3.2. *Let $1 \leq r \leq n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then for a symmetric compact set L in $\mathbb{R}^n$*

$$\Phi_{\mathbf{r}}(L^\circ)\Psi_{\mathbf{r}}(L) \leq \Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n). \tag{36}$$

Moreover, if L is a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(L)$, we have that

$$\Phi_{\mathbf{r}}(L^\circ)\Psi_{\mathbf{r}}(L) \geq c\, \Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n), \tag{37}$$

where $c > 0$ is an absolute constant - exactly the constant of the reverse Santaló inequality (35).

Proof. First note that

$$\Phi_{\mathbf{r}}^{i_r n}(B_2^n)\Psi_{\mathbf{r}}^{i_r n}(B_2^n) = \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}}\right)^2.$$

Using the Blaschke-Santaló inequality (34) and (32), we have

$$\begin{aligned}
\Psi_{\mathbf{r}}^{i_r n}(L) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&\leq \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}}\right)^2 \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} \frac{1}{|(L \cap F_j)^\circ|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&\leq \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}}\right)^2 \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} \frac{1}{|P_{F_j} L^\circ|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&= \Phi_{\mathbf{r}}^{i_r n}(B_2^n)\Psi_{\mathbf{r}}^{i_r n}(B_2^n)\Phi_{\mathbf{r}}^{-i_r n}(L^\circ).
\end{aligned}$$

On the other hand, using the reverse Blaschke-Santaló inequality (35), (33) and (10) we get

$$\begin{aligned}
\Psi_{\mathbf{r}}^{i_r n}(L) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&\geq \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}} \right)^2 c^{\sum_{j=1}^{r} i_j (i_{j+1}-i_{j-1})} \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} \frac{1}{|(L \cap F_j)^\circ|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&= c^{i_r n} \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}} \right)^2 \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} \frac{1}{|P_{F_j} L^\circ|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&= c^{i_r n} \Phi_{\mathbf{r}}^{i_r n}(B_2^n) \Psi_{\mathbf{r}}^{i_r n}(B_2^n) \Phi_{\mathbf{r}}^{-i_r n}(L^\circ).
\end{aligned}$$

The proof is complete. □

The following corollary has been proved in the case $r = 1$ in [42].

Corollary 3.3. *Let $1 \leq r \leq n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then for every convex body L,*

$$\Phi_{\mathbf{r}}(L) \geq c\, \Phi_{\mathbf{r}}(r_L B_2^n), \tag{38}$$

where $c > 0$ is an absolute constant.

A compact set is called *centered* if its centroid lies at the origin.

Proof. As $\Phi_{\mathbf{r}}(L)$ is translation-invariant, we may assume that L is centered. The Blaschke-Santaló inequality implies $r_L r_{L^\circ} \leq 1$, so with (37) and (30), we obtain

$$\Phi_{\mathbf{r}}(L) \geq c \frac{\Phi_{\mathbf{r}}(B_2^n) \Psi_{\mathbf{r}}(B_2^n)}{\Psi_{\mathbf{r}}(L^\circ)} \geq c \frac{\Phi_{\mathbf{r}}(B_2^n) \Psi_{\mathbf{r}}(B_2^n)}{\Psi_{\mathbf{r}}(r_{L^\circ} B_2^n)} = \frac{c}{r_{L^\circ}} \Phi_{\mathbf{r}}(B_2^n) \geq c\, \Phi_{\mathbf{r}}(r_L B_2^n).$$

The proof is complete. □

The next proposition shows that all the quantities $\Phi_{\mathbf{r}}(L)$ lie between the volume-radius r_L and the mean width $W(L)$. For a convex body L in $\mathbb{R}^n$, we write

$$W_L := \inf_{T \in SL_n} W(TL). \tag{39}$$

Proposition 3.4. *Let $1 \leq r \leq n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then for a convex body L in $\mathbb{R}^n$*

$$c\, r_L \leq \frac{\Phi_{\mathbf{r}}(L)}{\Phi_{\mathbf{r}}(B_2^n)} \leq W_L. \tag{40}$$

Proof. The left-most inequality follows from (38). Next, since $h_L(x) = h_{P_F L}(x)$ for $x \in F$, we have

$$\int_{G_{n,k}} W(P_F L)\, dF = W(L). \tag{41}$$

We will prove the right-hand side for the case $r = 2$ (the general case follows by induction on r). Using the Urysohn and Hölder inequalities repeatedly, we get

$$\begin{aligned}
\Phi_{\mathbf{r}}(L) &= \left(\int_{F^n_{\mathbf{r}}} \prod_{j=1}^{2} |P_{F_j} L|^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}} \right)^{-\frac{1}{ni_2}} \\
&= \left(\int_{G_{n,i_2}} \frac{1}{|P_{F_2} L|^{n-i_1}} \int_{G_{F_2,i_1}} \frac{1}{|P_{F_1} L|^{i_2}} dF_1 dF_2 \right)^{-\frac{1}{ni_2}} \\
&\leq \left(\int_{G_{n,i_2}} \frac{1}{|P_{F_2} L|^{n-i_1}} \int_{G_{F_2,i_1}} \frac{1}{|P_{F_1} B_2^n|^{i_2} W(P_{F_1} L)^{i_1 i_2}} dF_1 dF_2 \right)^{-\frac{1}{ni_2}} \\
&\leq \left(\int_{G_{n,i_2}} \frac{1}{|P_{F_2} L|^{n-i_1}} \frac{1}{|B_2^{i_1}|^{i_2} \left(\int_{G_{F_2,i_1}} W(P_{F_1} L) dF_1 \right)^{i_1 i_2}} dF_2 \right)^{-\frac{1}{ni_2}} \\
&= \left(\int_{G_{n,i_2}} \frac{1}{|P_{F_2} L|^{n-i_1}} \frac{1}{|B_2^{i_1}|^{i_2} W(P_{F_2} L)^{i_1 i_2}} dF_2 \right)^{-\frac{1}{ni_2}} \\
&\leq \left(\int_{G_{n,i_2}} \frac{1}{|P_{F_2} B_2^n|^{n-i_1} W(P_{F_2} L)^{i_2(n-i_1)}} \frac{1}{|B_2^{i_1}|^{i_2} W(P_{F_2} L)^{i_1 i_2}} dF_2 \right)^{-\frac{1}{ni_2}} \\
&= \left(|B_2^{i_2}|^{n-i_1} |B_2^{i_1}|^{i_2} \right)^{\frac{1}{ni_2}} \left(\int_{G_{n,i_2}} W(P_{F_2} L)^{-ni_2} dF_2 \right)^{-\frac{1}{ni_2}} \\
&\leq \Phi_{\mathbf{r}}(B_2^n) \int_{G_{n,i_2}} W(P_{F_2} L) dF_2 \\
&= \Phi_{\mathbf{r}}(B_2^n) W(L).
\end{aligned}$$

In the above argument we may replace L by TL with $T \in SL_n$. Since the left-hand side of this inequality remains the same for all T by Theorem 1.1, we may take the infimum over all T on the right-hand side. This completes the proof. $\square$

We conclude this subsection with a discussion of inequalities of isomorphic nature. For convex bodies L in $\mathbb{R}^n$, we define the Banach-Mazur distance to the Euclidean ball B_2^n by

$$d_{BM}(L) := \inf \left\{ ab : a > 0, b > 0, \frac{1}{b} B_2^n \subseteq T(L - L) \subseteq a B_2^n, T \in GL_n \right\}.$$

For symmetric convex bodies, this coincides with the standard notion of Banach-Mazur distance (for more information see, e.g., [51]).

Proposition 3.5. *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then for a convex body L in $\mathbb{R}^n$*

$$\Phi_{\mathbf{r}}(L) \le c \min \left\{ \sqrt{\frac{n}{i_r}}, \log(1 + d_{BM}(L)) \right\} \Phi_{\mathbf{r}}(r_L B_2^n). \tag{42}$$

Moreover, if L is also symmetric, then

$$\Psi_{\mathbf{r}}(L) \ge \frac{c}{\min\left\{ \sqrt{\frac{n}{i_r}}, \log(1 + d_{BM}(L)) \right\}} \Psi_{\mathbf{r}}(r_L B_2^n). \tag{43}$$

The proof relies on several different tools. We draw on ideas from Dafnis and the second-named author [11] to exploit the affine invariance of $\Phi_{\mathbf{r}}(L)$, $\Psi_{\mathbf{r}}(L)$ by using appropriately chosen affine images of L. To this end, recall the following fundamental theorem which combines work of Figiel–Tomczak-Jaegermann [13], Lewis [29], Pisier [45] and Rogers-Shephard [47] (see Theorem 1.11.5 on p. 52 in [7]).

Theorem 3.6. *Let L be a centered convex body. Then there exists a linear map $T \in SL_n$ such that*

$$W(TL) \le c \log\{1 + d_{BM}(L)\} \sqrt{n} |L|^{1/n}. \tag{44}$$

We will also use recent results on isotropic convex bodies. For background, the reader may consult [7], we will however recall all facts that we need here. To each convex body $M \subseteq \mathbb{R}^n$ with unit volume, one can associate an ellipsoid $Z_2(M)$, called the *L_2-centroid body* of M, which is defined by its support function as

$$h_{Z_2(M)}(\theta) := \left(\int_M |\langle x, \theta \rangle|^2 dx \right)^{\frac{1}{2}}.$$

The *isotropic constant* of M is defined by $L_M := r_{Z_2(M)}$. We say that M is *isotropic* if it is centered and $Z_2(M) = L_M B_2^n$. Fix an isotropic convex body M and a k-dimensional subspace F. K. Ball [3] proved that

$$|M \cap F^\perp|^{\frac{1}{k}} \ge \frac{c}{L_M}; \tag{45}$$

a corresponding inequality for projections,

$$|P_F M| \le \left(c \frac{n}{k} L_M \right)^k, \tag{46}$$

follows immediately from (45) and the Rogers-Shephard inequality [47]:

$$|P_F M||M \cap F^\perp| \le \binom{n}{k}.$$

Next, we recall a variant of $\Psi_{[k]}(M)$ studied by Dafnis and the second-named author [11]. For every $1 \le k \le n-1$ and a compact set M in $\mathbb{R}^n$ with $|M| = 1$, we define the quantity

$$\widetilde{\Phi}_{[k]}(M) := \left(\int_{G_{n,k}} |M \cap F^\perp|^n d\mu_{G_{n,k}}(F) \right)^{\frac{1}{nk}}. \tag{47}$$

In [11] it is shown that for every convex body M in $\mathbb{R}^n$ of unit volume,

$$\frac{c_1}{L_M} \leq \widetilde{\Phi}_{[k]}(M) \leq \widetilde{\Phi}_{[k]}(D_n) \simeq 1, \tag{48}$$

where D_n is the Euclidean ball of volume one.

We also invoke B. Klartag's fundamental result on perturbations of isotropic convex bodies [25] having a well-bounded isotropic constant. We use a variant from [18], see Theorem 3.4 on p.16.

Theorem 3.7. *Let M be a convex body in $\mathbb{R}^n$. For every $\varepsilon \in (0,1)$ there exists a centered convex body $M_{Kl} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that*

$$\frac{1}{1+\varepsilon} M_{Kl} \subseteq M + x \subseteq (1+\varepsilon) M_{Kl} \tag{49}$$

and

$$L_{M_{Kl}} \leq \frac{c}{\sqrt{\varepsilon}}. \tag{50}$$

We are now ready to complete the proof.

Proof of Proposition 3.5. By homogeneity of the operators $\Phi_{\mathbf{r}}$ and $\Psi_{\mathbf{r}}$, we can assume that L has unit volume.

First we will prove the bound (42) for $\mathbf{r}$-flag affine quermassintegrals. By translation-invariance of projections, we may further assume that L is centered. Bounding $\Phi_{\mathbf{r}}(L)$ by $W(L)$ according to (40), using affine invariance of $\Phi_{\mathbf{r}}$ and reverse Urysohn inequality from Theorem 3.6, we get

$$\Phi_{\mathbf{r}}(L) \leq c \log(1 + d_{BM}(L)) \Phi_{\mathbf{r}}(D_n). \tag{51}$$

For the Euclidean ball D_n of unit volume, for every $F \in G_{n,k}$, we have $|P_F D_n|^{\frac{1}{k}} = |D_n \cap F|^{\frac{1}{k}} \simeq \sqrt{\frac{n}{k}}$, so

$$\Phi_{\mathbf{r}}(D_n) \simeq \left(\prod_{j=1}^{r} \left(\frac{n}{i_j}\right)^{i_j(i_{j+1}-i_{j-1})} \right)^{\frac{1}{2 i_r n}}.$$

The AM/GM inequality implies

$$\left(\prod_{j=1}^{r} \left(\frac{n}{i_j}\right)^{i_j(i_{j+1}-i_{j-1})} \right)^{\frac{1}{2 i_r n}} \leq \sqrt{\frac{n}{i_r n} \sum_{j=1}^{r} \frac{i_j(i_{j+1}-j_{j-1})}{i_j}} \leq \sqrt{\frac{n}{i_r}}.$$

Thus

$$\Phi_{\mathbf{r}}(D_n) = \Psi_{\mathbf{r}}(D_n) \simeq \left(\prod_{j=1}^{r} \left(\frac{n}{i_j}\right)^{i_j(i_{j+1}-i_{j-1})} \right)^{\frac{1}{2 i_r n}} \leq \sqrt{\frac{n}{i_r}}. \tag{52}$$

Let $K_1 \subset \mathbb{R}^n$ be a centered convex body and $x \in \mathbb{R}^n$ from the conclusion of Theorem 3.7 corresponding to $\varepsilon = \frac{1}{2}$. Then (49) implies $1 = |L|^{1/n} \geq \frac{2}{3}|K_1|^{1/n}$, while (50) implies $L_{K_1} \simeq 1$. Let $K_2 := \frac{K_1}{|K_1|^{\frac{1}{n}}}$, then $L_{K_2} \simeq L_{K_1} \simeq 1$ and

$$\Phi_{\mathbf{r}}(L) = \Phi_{\mathbf{r}}(L + x) \leq \frac{3}{2}\Phi_{\mathbf{r}}(K_1) \leq \frac{9}{4}\Phi_{\mathbf{r}}(K_2). \tag{53}$$

Affine invariance of $\Phi_{\mathbf{r}}$ (Theorem 1.1), allows us to assume that K_2 is isotropic. Using (46), (52), $L_{K_2} \simeq 1$ and (52) one more time, we obtain

$$\begin{aligned}
\Phi_{\mathbf{r}}(K_2) &= \left(\int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} |P_{F_j} K_2|^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}}\right)^{-\frac{1}{i_r n}} \\
&\leq \left(\prod_{j=1}^{r} \left(\frac{n}{i_j}\right)^{i_j(i_{j+1}-i_{j-1})}\right)^{\frac{1}{i_r n}} (cL_{K_2})^{\frac{1}{i_r n}\sum_{j=1}^{r} i_j(i_{j+1}-i_{j-1})} \\
&\leq c\, L_{K_2} \Phi_{\mathbf{r}}(D_n)^2 \\
&\leq c\sqrt{\frac{n}{i_r}}\, \Phi_{\mathbf{r}}(D_n).
\end{aligned}$$

By (53) we have $\Phi_{\mathbf{r}}(L) \leq c'\sqrt{\frac{n}{i_r}}\, \Phi_{\mathbf{r}}(D_n)$, which together with (51) gives the upper bound (42).

Applying (37) for L°, (42) and the Blaschke-Santaló inequality $r_L r_{L^\circ} \leq 1$, we get

$$\begin{aligned}
\Psi_{\mathbf{r}}(L) &\geq c\frac{\Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n)}{\Phi_{\mathbf{r}}(L^\circ)} \\
&\geq \frac{c}{r_{L^\circ}} \frac{1}{\min\left\{\log\left(1 + d_{BM}(L^\circ)\right), \sqrt{\frac{n}{i_r}}\right\}} \frac{\Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n)}{\Phi_{\mathbf{r}}(B_2^n)} \\
&\geq \frac{c}{\min\left\{\log\left(1 + d_{BM}(L)\right), \sqrt{\frac{n}{i_r}}\right\}} \Psi_{\mathbf{r}}(r_L B_2^n),
\end{aligned}$$

where we have also used the identity $d_{BM}(L^\circ) = d_{BM}(L)$ for symmetric convex bodies. This proves (43). □

4 Flag manifolds and permutations

In this section, we discuss more general quantities involving permutations. We investigate the extent to which SL_n-invariance properties established by Furstenberg and Tzkoni [15] carry over from ellipsoids to compact sets. In particular, we provide an example of a convex body for which SL_n-invariance fails. Nevertheless, we show that for convex bodies, such quantities cannot be too degenerate in the sense that they admit uniform upper and lower bounds, independent of the body. The key ingredient is the notion of M-ellipsoids, introduced by V. Milman [38].

The next definition is motivated by the work of Furstenberg and Tzkoni [15] for ellipsoids.

Definition 4.1. Let Π_n be the set of permutations of $\{1, 2, \ldots, n\}$ and $\omega \in \Pi_n$. For compact sets L in $\mathbb{R}^n$, we define the *ω-flag quermassintegral* and *ω-flag dual quermassintegrals* as follows: if $\omega(n) \neq n$, then

$$\Psi_\omega(L) := \left(\int_{F^n} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi \right)^{\frac{1}{n(n-\omega(n))}} \tag{54}$$

and

$$\Phi_\omega(L) := \left(\int_{F^n} \prod_{j=1}^{n-1} |P_{F_j} L|^{-\omega(j)+\omega(j+1)-1} d\xi \right)^{-\frac{1}{n(n-\omega(n))}}. \tag{55}$$

When $\omega(n) = n$, we set

$$\Psi_\omega(L) := \int_{F^n} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi \tag{56}$$

and

$$\Phi_\omega(L) := \int_{F^n} \prod_{j=1}^{n-1} |P_{F_j} L|^{-\omega(j)+\omega(j+1)-1} d\xi. \tag{57}$$

Note that

$$\sum_{j=1}^{n-1} (\omega(j) - \omega(j+1) + 1) = n - \omega(n) + \omega(1) - 1 \tag{58}$$

and

$$\sum_{j=1}^{n-1} j\, (\omega(j) - \omega(j+1) + 1) = n(n - \omega(n)). \tag{59}$$

Identity (59) guarantees that $\Psi_\omega(L)$ and $\Phi_\omega(L)$ are 1-homogeneous when $\omega(n) \neq n$ and 0-homogeneous when $\omega(n) = n$.

The following fact for ω-flag dual quermassintegrals is from [15]. For ω-flag quermassintegrals it follows for example by duality.

Theorem 4.2. *Let $\mathcal{E}$ be an ellipsoid in $\mathbb{R}^n$ and $\omega \in \Pi_n$.*

$$\Psi_\omega(\mathcal{E}) = \Psi_\omega(r_{\mathcal{E}} B_2^n) \text{ and } \Phi_\omega(\mathcal{E}) = \Phi_\omega(r_{\mathcal{E}} B_2^n). \tag{60}$$

An equivalent formulation of the latter result is that for every ellipsoid $\mathcal{E}$,

$$\Psi_\omega(\mathcal{E}) = c_\omega |\mathcal{E}|^{\frac{1}{n}},\ \omega(n) \neq n \text{ and } \Psi_\omega(\mathcal{E}) = c_\omega,\ \omega(n) = n, \tag{61}$$

where c_ω is a constant that depends only on ω. An analogous statement holds for $\Phi_\omega(\mathcal{E})$ (cf. (3.2)).

The operators Ψ_ω and Φ_ω are generalizations of $\Psi_{\mathbf{r}}$ and $\Phi_{\mathbf{r}}$. Indeed, let $1 \le r \le n-1$, $1 \le i_1 < i_2 < \cdots < i_r \le n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$. Define ω by $\omega(1) = n - i_1 + 1$, and $\omega(t+1) = \omega(t) + 1$, for $t \ne i_j$, $1 \le j \le r$, and $\omega(i_j + 1) = \omega(i_j) + 1 - i_{j+1} + i_{j-1}$ for $1 \le j \le r$. Then $\omega \in \Pi_n$ with $\omega(i_j) - \omega(i_j + 1) + 1 = i_{j+1} - i_{j-1}$ for $1 \le j \le r$ and $\omega(t) - \omega(t+1) + 1 = 0$, $t \ne i_j$, for $1 \le j \le r$. Since $\omega(n) = n - i_r$, for a compact set L in $\mathbb{R}^n$ we have

$$\begin{aligned} \Psi_\omega(L) &= \left(\int_{F^n} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j) - \omega(j+1) + 1} d\xi \right)^{\frac{1}{n i_r}} \\ &= \left(\int_{F^n} \prod_{j=1}^{r} |L \cap F_{i_j}|^{i_{j+1} - i_{j-1}} d\xi \right)^{\frac{1}{n i_r}} \\ &= \Psi_{\mathbf{r}}(L), \end{aligned}$$

where, in the last equality, we have used (22). Correspondingly, we also have $\Phi_\omega(L) = \Phi_{\mathbf{r}}(L)$. In particular, for this permutation ω, $\Psi_\omega(L)$ is SL_n-invariant and $\Phi_\omega(L)$ is affine invariant. As a particular case of the preceding discussion, let $r = 1$, $i_1 = k$, $1 \le k \le n$ and let $\omega(1) = n - k + 1$ and $\omega(t+1) = \omega(t) + 1$ for $t \ne k$ and $\omega(k+1) = \omega(k) - n + 1$. Then $\Phi_\omega(L) = \Phi_{[k]}(L)$.

Given that $\Psi_{\mathbf{r}}(L)$ and $\Phi_{\mathbf{r}}(L)$ enjoy invariance properties and arise as permutations, it is natural to investigate the extent to which the invariance from Theorem 4.2 carries over to compact sets. We do not have a complete answer. However, there are cases outside of those considered above where the invariance holds and also counter-examples where it fails as the next two examples show.

Example 1. Let $n \ge 3$. Define ω by $\omega(1) = 2$, $\omega(2) = 1$ and $\omega(t) = t$ for all $3 \le t \le n$. Then for every symmetric compact set L in $\mathbb{R}^n$,

$$\Psi_\omega(L) = \frac{4}{\pi}.$$

In particular, $\Psi_\omega(L)$ is SL_n-invariant.

Note that our choice of the permutation ω satisfies

$$\omega(1) - \omega(2) + 1 = 2, \quad \omega(2) - \omega(3) + 1 = -1, \quad \omega(j) - \omega(j+1) + 1 = 0 \text{ for } 3 \le j \le n-1,$$

or equivalently

$$\omega(2) = \omega(1) - 1, \quad \omega(3) = \omega(1) + 1, \quad \omega(j+1) = \omega(1) + (j-1) \text{ for } 3 \le j \le n-1.$$

Since $1 \le \omega(j) \le n$ for all j, it follows that $\omega(1) = 2$. Hence $\omega(1) = 2, \omega(2) = 1, \omega(j) = j$ for $3 \le j \le n$ is the unique permutation with these properties. For an k-dimensional subspace

F_k of $\mathbb{R}^n$, denote by S_{F_k} the unit sphere in F_k. Now, using (22), we compute

$$
\begin{aligned}
\Psi_\omega(L) &= \int_{F^n} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi \\
&= \int_{F^n} |L \cap F_1|^2 |L \cap F_2|^{-1} d\xi \\
&= \int_{G_{n,2}} |L \cap F_2|^{-1} \int_{G_{F_2,1}} |(L \cap F_2) \cap F_1|^2 dF_1 dF_2 \\
&= \int_{G_{n,2}} |L \cap F_2|^{-1} \int_{S_{F_2}} (2\rho_{L\cap F_2}(\theta))^2 d\sigma(\theta) dF_2 \\
&= \int_{G_{n,2}} |L \cap F_2|^{-1} \frac{4}{|S_{F_2}|} \int_{S_{F_2}} \rho^2_{L\cap F_2}(\theta) d\theta \, dF_2 \\
&= \frac{4}{\pi} \int_{G_{n,2}} |L \cap F_2|^{-1} |L \cap F_2| dF_2 \\
&= \frac{4}{\pi}.
\end{aligned}
$$

□

When $n = 3$, for permutations ω with $\omega(3) = 3$, $\Psi_\omega(L)$ are absolute constants. Moreover, the discussion following Theorem 4.2 shows that for 3 of the remaining 4 permutations ω in Π_3, $\Psi_\omega(L) = \Psi_{\mathbf{r}}(L)$. Altogether, for $n = 3$, for 5 out of 6 permutations ω, $\Psi_\omega(L)$ are SL_n-invariant. The next example shows that for the remaining permutation, the invariance does not carry over for all convex bodies.

Example 2. Let $\omega \in \Pi_3$ with $\omega(1) = 1$, $\omega(2) = 3$ and $\omega(3) = 2$. We claim that for a centered cube $Q := [-1, 1]^3$ and the diagonal matrix $D = \mathrm{diag}(1, 2, 1/2)$, $\Phi_\omega(DQ) > \Phi_\omega(Q)$. Since $D \in SL_3$, this shows that the operator Φ_ω is not invariant under volume preserving transformations.

To show this, we first note that for any convex body $L \subset \mathbb{R}^3$

$$
\Phi_\omega^{-3}(L) = \int_{S^2} \frac{W(P_{\phi^\perp} L)}{h^2_{\Pi L}(\phi)} d\sigma(\phi). \tag{62}
$$

Recall that for $\theta \in S^{n-1}$, $h_Q(\theta) = \sum_{i=1}^n |\theta_i|$ and for $g \in GL_n$, $h_{gL}(\theta) = h_L(g^t\theta)$. We will also use the following facts about projection bodies (see e.g., Gardner). The projection body of a cube is again a cube, $\Pi Q = 2Q$ and for $g \in GL_n$,

$$
\Pi(gL) = |\mathrm{det} g| \, g^{-t} \, \Pi L. \tag{63}
$$

Let $A = [a_1 \ a_2 \ a_3] \in SL_3$ with columns a_i. Fix $\phi \in S^2$. Let $U \in O_3$ be given in column form by $U = [u \ v \ \phi]$. Since U is orthogonal, $U^t\phi = e_3$ and $U^t\phi^\perp = \mathrm{span}\{e_1, e_2\} = \mathbb{R}^2$. Then

$$
W(P_{\phi^\perp} AQ) = \int_{S_{\phi^\perp}} h_{AQ}(\theta) d\sigma(\theta) = \int_{S^1} h_{AQ}(U\theta) d\sigma(\theta) = \int_{S^1} h_Q(A^tU\theta) d\sigma(\theta).
$$

Thus denoting by P the orthogonal projection onto $\mathbb{R}^2$, we have

$$W(P_{\phi^\perp}AQ) = \sum_{i=1}^{3}\int_{S^1} |\langle \theta, U^t Ae_i\rangle| d\sigma(\theta) = \sum_{i=1}^{3}\int_{S^1} |\langle \theta, PU^t Ae_i\rangle| d\sigma(\theta) = \frac{2}{\pi}\sum_{i=1}^{3}\|PU^t Ae_i\|_2.$$

We have that $Ae_i = a_i$, $U^t a_i = (\langle u, a_i\rangle, \langle v, a_i\rangle, \langle \phi, a_i\rangle)^t$ and

$$\|PU^t Ae_i\|_2^2 = \|U^t Ae_i\|_2^2 - \|(I-P)U^t Ae_i\|_2^2 = \|a_i\|_2^2 - \langle \phi, a_i\rangle^2.$$

Therefore,

$$W(P_{\phi^\perp}AQ) = \frac{2}{\pi}\sum_{i=1}^{3}\sqrt{\|a_i\|_2^2 - \langle \phi, a_i\rangle^2}. \tag{64}$$

Moreover, $h_{\Pi(AQ)}(\phi) = h_{\Pi Q}(A^{-1}\phi) = 2\sum_{i=1}^{3}|\langle A^{-1}\phi, e_i\rangle|$. Thus

$$\Phi_\omega^{-3}(AQ) = \frac{1}{2\pi}\int_{S^2} \frac{\sum_{i=1}^{3}\sqrt{\|a_i\|_2^2 - \langle \phi, a_i\rangle^2}}{\left(\sum_{j=1}^{3}|\langle A^{-1}\phi, e_j\rangle\right)^2} d\sigma(\phi). \tag{65}$$

Set $A := \operatorname{diag}(d_1, d_2, d_3)$ with $\prod_{i=1}^{3} d_i = 1$ and $d_i > 0$. Then the quantity

$$\mathcal{A}(d_1, d_2, d_3) := \int_{S^2} \frac{\sum_{i=1}^{3} d_i\sqrt{1-\phi_i^2}}{\left(\sum_{j=1}^{3}\frac{|\phi_j|}{d_j}\right)^2} d\sigma(\phi) \tag{66}$$

is not constant. Indeed, using MATLAB for example, one can verify that $\mathcal{A}(1, 2, 1/2) < \mathcal{A}(1, 1, 1)$.

$\square$

In the case of convex bodies, the quantities $\Psi_\omega(K)$, $\Phi_\omega(K)$ are uniformly bounded. We will use the following well-known consequence of the celebrated "existence of M-ellipsoids" by V. Milman [38].

Theorem 4.3. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then there exists an ellipsoid $\mathcal{E}$ such that $|\mathcal{E}|^{1/n} \le e^c |K|^{1/n}$ and for every $F \in G_{n,k}$,*

$$|P_F\mathcal{E}| \le |P_F K| \le e^{cn}|P_F\mathcal{E}| \tag{67}$$

and

$$|\mathcal{E}\cap F| \le |K \cap F| \le e^{cn}|\mathcal{E}\cap F|, \tag{68}$$

where $c > 0$ is an absolute constant.

Corollary 4.4. *Let $\omega \in \Pi_n$ such that $\omega(n) \neq n$. Let $\delta_\omega(j) := \omega(j) - \omega(j+1) + 1$. Set*

$$I_\omega := \{j \le n : \delta_\omega(j) \ge 0\} \text{ and } \Delta(\omega) := \frac{\min\{\sum_{j\in I_\omega}\delta_\omega(j), \sum_{j\in I_\omega^c}|\delta_\omega(j)|\}}{n-\omega(n)} + 1.$$

We have that

$$e^{-c\Delta(\omega)}c_\omega|K|^{\frac{1}{n}} \leq \Psi_\omega(K) \leq e^{c\Delta(\omega)}c_\omega|K|^{\frac{1}{n}} \tag{69}$$

and

$$e^{-c\Delta(\omega)}c_\omega|K|^{\frac{1}{n}} \leq \Phi_\omega(K) \leq e^{c\Delta(\omega)}c_\omega|K|^{\frac{1}{n}} \tag{70}$$

where $c > 0$ is an absolute constant.

Proof. Set $\Delta_+(\omega) := \frac{\sum_{j\in I_\omega}\delta_\omega(j)}{n-\omega(n)}$ and $\Delta_- := (\omega)\frac{\sum_{j\in I_\omega^c}|\delta_\omega(j)|}{n-\omega(n)}$. Using (68) and (58), we have

$$\begin{aligned}
\Psi_\omega(K) &:= \left(\int_{F_n}\prod_{j=1}^{n-1}|K\cap F_j|^{\omega(j)-\omega(j+1)+1}d\xi\right)^{\frac{1}{n(n-\omega(n))}}\\
&\leq \left(\int_{F_n} e^{cn\sum_{j\in I_\omega}\omega(j)-\omega(j+1)+1}\prod_{j=1}^{n-1}|\mathcal{E}\cap F_j|^{\omega(j)-\omega(j+1)+1}d\xi\right)^{\frac{1}{n(n-\omega(n))}}\\
&\leq e^{c\Delta_+(\omega)}\left(\int_{F_n}\prod_{j=1}^{n-1}|\mathcal{E}\cap F_j|^{\omega(j)-\omega(j+1)+1}d\xi\right)^{\frac{1}{n(n-\omega(n))}}\\
&= e^{c\Delta_+(\omega)}c_\omega|\mathcal{E}|^{\frac{1}{n}}\\
&\leq e^{c(\Delta_+(\omega)+1)}c_\omega|K|^{\frac{1}{n}}.
\end{aligned}$$

One can verify that a similar inequality with the quantity $\Delta_-(\omega)$ holds as well, which leads to the right-hand side in (69). The proof of the other inequalities is identical and hence is omitted. □

Remark. A similar proposition (with the same proof) holds for the case $\omega(n) = n$. Moreover, using Pisier's regular M-position (see [46]) one can get more precise estimates.

5 Functional forms

In this section we derive functional forms of some of the previous geometric inequalities. Note that the proofs of these functional inequalities do not depend on the geometric inequalities. They are much more general. The invariance of functional inequalities on flag manifolds can be proved directly using the structure theory of semi-simple Lie groups as was done in our previous work.

5.1 Functional forms of dual r-flag quermassintegrals.

Let f be a bounded integrable function on $\mathbb{R}^n$. We denote by $I(f)$ the functional form of the dual **r**-flag quermassintegral

$$I(f) := \int_{F^n_{\mathbf{r}}}\prod_{j=1}^{r}\|f|_{F_j}\|^{i_{j+1}-i_{j-1}}d\xi_{\mathbf{r}}.$$

Theorem 5.1. *For every $g \in SL_n$, $I(g \cdot f) = I(f)$.*

Proof. Starting with the left-hand side, $I(g \cdot f)$, we do a global change of variables (24) on the flag manifold:

$$\begin{aligned} I(g \cdot f) &= \int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} \|g \cdot f|_{F_j}\|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\ &= \int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} \|g \cdot f|_{g \cdot F_j}\|^{i_{j+1}-i_{j-1}} \sigma_{F^n_{\mathbf{r}}}(g, \xi) d\xi_{\mathbf{r}}, \end{aligned}$$

where by (26) $\sigma_{F^n_{\mathbf{r}}}(g, \xi) = \sigma_{i_1}^{-i_2}(g, F_1)\sigma_{i_2}^{i_1-i_3}(g, F_2) \cdots \sigma_{i_r}^{i_{r-1}-n}(g, F_r)$. Now we do r local changes of variables (23) on each nested subspace F_j in the product. For each $1 \le j \le r$, we thus have

$$\|g \cdot f|_{g \cdot F_j}\| = \|f|_{F_j}\| \, \sigma_{i_j}(g, F_j).$$

For the product under the integral, we obtain:

$$\prod_{j=1}^{r} \|g \cdot f|_{g \cdot F_j}\|^{i_{j+1}-i_{j-1}} = \prod_{j=1}^{r} \left(\|f|_{F_j}\| \, \sigma_{i_j}(g, F_j)\right)^{i_{j+1}-i_{j-1}} = \prod_{j=1}^{r} \|f|_{F_j}\|^{i_{j+1}-i_{j-1}} \sigma^{-1}_{F^n_{\mathbf{r}}}(g, \xi).$$

□

It is not hard to generalize this result in several ways as was done in [12] for functional forms of dual quermassintegrals. Instead of taking $L_1(F_j)$ norms one can take $L_{p_j}(F_j)$ norms and replace the powers $i_{j+1} - i_{j-1}$ by α_j. As long as $\frac{\alpha_j}{p_j} = i_{j+1} - i_{j-1}$ and the integrals exist, the conclusion of the Theorem 5.1 will hold. Theorem 5.1 also generalizes to a product of m functions. This allows to replace $\|f|_{F_j}\|^{i_{j+1}-i_{j-1}}$ by $\prod_{i=1}^{m} \|f_i|_{E_j}\|^{\alpha_{i,j}}_{p_{i,j}}$. For the Theorem 5.1 to hold in this case, we have to require $\sum_{i=1}^{m} \frac{\alpha_{i,j}}{p_{i,j}} = i_{j+1} - i_{j-1}$. Another way to generalize functional forms of dual quermassintegrals is to replace

$$\|f|_{F_j}\|^{i_{j+1}-i_{j-1}} \quad \text{by} \quad \frac{\|f|_{F_j}\|^{\alpha_j}_{p_j}}{\|g|_{F_j}\|^{\beta_j}_{q_j}} \quad \text{with} \quad \frac{\alpha_j}{p_j} - \frac{\beta_j}{q_j} = i_{j+1} - i_{j-1},$$

to ensure they remain invariant under volume preserving transformations. Letting $\beta_j \to \infty$ modifies the integrand to $\frac{\|f|_{F_j}\|^{\alpha_j}_{p_j}}{\|f|_{F_j}\|^{\beta_j}_{\infty}}$ and the condition on the powers and norms to $\frac{\alpha_j}{p_j} = i_{j+1} - i_{j-1}$. Note that in this case the invariance holds for arbitrary powers β_j. As a particular case this proves invariance under volume preserving transformations of the integrand appearing in the next theorem. One can also take the quotient of products of functions, replacing

$$\|f|_{F_j}\|^{i_{j+1}-i_{j-1}} \quad \text{by} \quad \frac{\prod_{i=1}^{m} \|f_i|_{F_j}\|^{\alpha_{i,j}}_{p_{i,j}}}{\prod_{l=1}^{m'} \|g_l|_{F_j}\|^{\beta_{l,j}}_{q_{l,j}}} \quad \text{with} \quad \sum_{i=1}^{m} \frac{\alpha_{i,j}}{p_{i,j}} - \sum_{l=1}^{m'} \frac{\beta_{l,j}}{q_{l,j}} = i_{j+1} - i_{j-1}.$$

Here again we can let $q_{l,j} \to \infty$, obtaining the corresponding generalization with no restrictions on $\beta_{l,j}$.

Theorem 5.2. *Let f be a non-negative bounded integrable function on $\mathbb{R}^n$, then*

$$\int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}} d\xi_{\mathbf{r}} \le \prod_{j=1}^{r} \frac{\omega_{i_j}^{i_{j+1}}}{\omega_{i_{j+1}}^{i_j}} \|f\|_1^{i_r}.$$

Proof. The result follows by iteration of an inequality on $G_{n,k}$ for one function from our previous work [12]:

$$\int_{G_{n,k}} \frac{\|f|_E\|_1^n}{\|f|_E\|_\infty^{n-k}} dE \le \frac{\omega_k^n}{\omega_n^k} \|f\|_1^k. \tag{71}$$

Applying the latter inequality repeatedly, we get

$$\begin{aligned}
&\int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}} d\xi_{\mathbf{r}} \\
&= \int_{G_{n,i_r}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}} \int_{G_{F_2,i_1}} \frac{\|f|_{F_1}\|_1^{i_2}}{\|f|_{F_1}\|_\infty^{i_2-i_1}} dF_1 dF_2 \dots dF_r \\
&\le \frac{\omega_{i_1}^{i_2}}{\omega_{i_2}^{i_1}} \int_{G_{n,i_r}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}} \|f|_{F_2}\|_1^{i_1} dF_2 \dots dF_r \\
&= \frac{\omega_{i_1}^{i_2}}{\omega_{i_2}^{i_1}} \int_{G_{n,i_r}} \cdots \int_{G_{F_4,i_3}} \prod_{j=3}^{r} \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}} \int_{G_{F_3,i_2}} \frac{\|f|_{F_2}\|_1^{i_3}}{\|f|_{F_2}\|_\infty^{i_3-i_2}} dF_2 dF_3 \dots\dots dF_r \\
&\le \frac{\omega_{i_1}^{i_2}}{\omega_{i_2}^{i_1}} \frac{\omega_{i_2}^{i_3}}{\omega_{i_3}^{i_2}} \int_{G_{n,i_r}} \cdots \int_{G_{F_4,i_3}} \prod_{j=3}^{r} \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}} \|f|_{F_3}\|_1^{i_2} dF_3 \dots dF_r \\
&= \cdots \\
&\le \prod_{j=1}^{r} \frac{\omega_{i_j}^{i_{j+1}}}{\omega_{i_{j+1}}^{i_j}} \|f\|_1^{i_r}.
\end{aligned}$$

□

In [12], more general versions of (71) are proved with multiple functions and different powers. These also carry over to extremal inequalities on flag manifolds by mimicking the previous proof. As a sample, we mention just one statement. Let $1 \le q \le i_1$ and let $f_1, \dots, f_q$ be non-negative bounded integrable functions on $\mathbb{R}^n$, then

$$\int_{F_{\mathbf{r}}^n} \prod_{k=1}^{q} \prod_{j=2}^{r} \frac{\|f_k|_{F_j}\|_1^{\frac{i_{j+1}-i_j}{i_j}}}{\|f_k|_{F_j}\|_\infty^{\frac{i_{j+1}-i_j}{i_j}}} \frac{\|f_k|F_1\|_1^{\frac{i_2}{i_1}}}{\|f_k|_{F_1}\|_\infty^{\frac{i_2-i_1}{i_1}}} d\xi_{\mathbf{r}} \le \left(\prod_{j=1}^{r} \frac{\omega_{i_j}^{\frac{i_{j+1}}{i_j}}}{\omega_{i_{j+1}}} \right)^q \prod_{k=1}^{q} \|f_k\|_1. \tag{72}$$

5.2 Functional forms of the r-flag quermassintegrals.

In this subsection we will extend the notions of **r**-flag quermassintegrals to functions. In particular, this will lead to functional versions of affine quermassintegrals. This is motivated by recent work of Bobkov, Colesanti and Fragalá [4] and V. Milman and Rotem [39]. The latter authors proposed and studied a notion of quermassintegrals for log-concave or even quasi-concave functions, which we now recall.

Definition 5.3. Suppose that $f : \mathbb{R}^n \to [0, \infty)$ is upper-semicontinuous and quasi-concave. For $1 \le i \le n$, let

$$V_k(f) := \int_0^\infty V_k(\{f \ge t\})dt. \tag{73}$$

The above definition is consistent with the notion of projection of a function onto a subspace as introduced by Klartag and Milman in [26]. Namely, let f be an non-negative function $\mathbb{R}^n \to [0, \infty]$ and $F \in G_{n,k}$. Define the orthogonal projection of f onto F as the function $P_F f : F \to [0, \infty]$ given by

$$(P_F f)(z) := \sup_{y \in F^\perp} f(z + y). \tag{74}$$

Note that if K is compact and $f := \mathbf{1}_K$ then $P_F f := \mathbf{1}_{P_F(K)}$. Moreover, from the definition, one has

$$\{z \in F : (P_F f)(z) > t\} = P_F(\{x \in \mathbb{R}^n : f(x) > t\}). \tag{75}$$

Assume now that $f := \mathbb{R}^n \to [0, \infty)$ and that for each $t > 0$, the set $\{x \in \mathbb{R}^n : f(x) \ge t\}$ is compact. For $1 \le k \le n - 1$, we define the affine quermassintegral of f by

$$\Phi_{[k]}(f) := \int_0^\infty \Phi_{[k]}(\{f \ge t\})dt = \int_0^\infty \left(\int_{G_{n,k}} |\{P_F f \ge t\}|^{-n} dF \right)^{\frac{1}{nk}} dt. \tag{76}$$

For $1 \le i_1 < i_2 < \cdots < i_r = n - 1$, $\mathbf{r} := (i_1, \cdots, i_r)$ we define the $\mathbf{r}$-flag quermassintegrals of f by

$$\Phi_{\mathbf{r}}(f) := \int_0^\infty \Phi_{\mathbf{r}}(\{f \ge t\})dt. \tag{77}$$

For comparison, we recall that for every $f : \mathbb{R}^n \to [0, \infty]$,

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty |\{f \ge t\}|dt. \tag{78}$$

For $\lambda \in \mathbb{R} \setminus \{0\}$ and f as above, we write

$$f_{(\lambda)} : \mathbb{R}^n \to [0, \infty], \text{ as } f_{(\lambda)}(x) := f\left(\frac{x}{\lambda}\right), \tag{79}$$

and if $T \in GL_n$,

$$f \circ T : \mathbb{R}^n \to [0, \infty], \text{ as } f \circ T(x) := f(T^{-1}x). \tag{80}$$

Note that if $f := \mathbf{1}_K$, then

$$f_{(\lambda)}(x) = \mathbf{1}_{\lambda K}(x) \text{ and } f \circ T(x) = \mathbf{1}_{TK}(x).$$

Let $f : \mathbb{R}^n \to [0, \infty]$, $\lambda > 0$ and $T \in GL_n$. Then

$$\{f \circ T \ge t\} = T(\{f \ge t\}) \text{ and } \{f_{(\lambda)} \ge t\} = \lambda\{f \ge t\}. \tag{81}$$

Then (81), the 1-homogenuity of the $\mathbf{r}$-flag quermassintegrals for sets as well as the affine invariance of these quantities imply the following.

Theorem 5.4. *Let $f : \mathbb{R}^n \to [0, \infty]$, $1 \le i_1 < \cdots < i_r \le n-1$ and $\mathbf{r} := (i_1, \cdots, i_r)$. Let $\lambda > 0$ and T be an affine volume-preserving map. Then*

$$\Phi_{\mathbf{r}}(f_{(\lambda)}) = \lambda \Phi_{\mathbf{r}}(f) \text{ and } \Phi_{\mathbf{r}}(f \circ T) = \Phi_{\mathbf{r}}(f). \tag{82}$$

Recall that the symmetric decreasing rearrangement of a function f which is integrable (or vanishes at infinity). For a set $A \subseteq \mathbb{R}^n$ with finite volume, the decreasing rearrangement A^* is defined as

$$A^* := r_A B_2^n,$$

where r_A is the volume-radius of A. The symmetric decreasing rearrangement f^* of f is defined as the radial function f^* such that

$$\{f \ge t\}^* = \{f^* \ge t\}, \ \forall t > 0.$$

Thus,

$$r_{\{f \ge t\}} B_2^n = \{f^* \ge t\}. \tag{83}$$

Using (83), (77) and (38), we have the following for all non-negative quasi-concave functions f on $\mathbb{R}^n$:

$$\Phi_{\mathbf{r}}(f) = \int_0^\infty \Phi_{\mathbf{r}}(\{f \ge t\}) dt \ge c \int_0^\infty \Phi_{\mathbf{r}}(r_{\{f \ge t\}} B_2^n) dt =$$

$$c \int_0^\infty \Phi_{\mathbf{r}}(\{f^* \ge t\}) dt = \Phi_{\mathbf{r}}(f^*).$$

Let f be a non-negative quasi-concave function on $\mathbb{R}^n$. We define

$$d_{BM}(f) := \sup_{t>0} d_{BM}(\{f \ge t\}).$$

The results of §3 lead to the following double-sided inequality for $\Phi_{[\mathbf{r}]}(f)$:

Theorem 5.5. *Let f be a non-negative quasi-concave function on $\mathbb{R}^n$, $1 \le i_1 < \cdots < i_r$ and let $\mathbf{r} := (i_1, \cdots, i_r)$. Then*

$$c\Phi_{\mathbf{r}}(f^*) \le \Phi_{\mathbf{r}}(f) \le c' \min\left\{ \log\{1 + d_{BM}(f)\}, \sqrt{\frac{n}{i_r}} \right\} \Phi_{\mathbf{r}}(f^*). \tag{84}$$

References

[1] S. Artstein-Avidan, B. Klartag, V. Milman, *The Santaló point of a function, and a functional form of the Santaló inequality*, Mathematika 51 (2004), no. 1-2, 33-48 (2005).

[2] S. Artstein-Avidan, B. Klartag, C. Schütt, E. Werner, *Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality*, J. Funct. Anal. 262 (2012), no. 9, 4181-4204.

[3] K. M. Ball, *Logarithmically concave functions and sections of convex sets in* $\mathbb{R}^n$ Studia Math. 88 (1988), 69-84.

[4] S. G. Bobkov, A. Colesanti, I. Fragalá, *Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities*, Manuscripta Mathematica, 143, 1–2, (2014),131-169.

[5] J. Bourgain, V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^n$, Invent. Math. 88 (1987), no. 2, 319–340.

[6] Bürgisser, A. Lerario *Probabilistic Schubert calculus*, J. Reine Angew. Math., DOI 10.1515/crelle-2018-0009.

[7] S. Brazitikos, A. Giannopoulos, P. Valettas, B.H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs, 196, American Mathematical Society, Providence, RI, 2014.

[8] H. Busemann, *Volume in terms of concurrent cross-sections*, Pacific J. Math. 3 (1953), 1–12.

[9] H. Busemann, E. G. Straus, *Area and normality*, Pacific J. Math. 10 (1960), 35–72.

[10] D. Cordero-Erausquin, M. Fradelizi, G. Paouris, P. Pivovarov, *Volume of the polar of random sets and shadow systems*, Math. Ann. 362 (2015), no. 3-4, 1305-1325.

[11] N. Dafnis, G. Paouris, *Estimates for the affine and dual affine quermassintegrals of convex bodies*, Illinois J. Math. 56 (2012), no. 4, 1005-1021.

[12] S. Dann, G. Paouris, P. Pivovarov, *Bounding marginal densities via affine isoperimetry*, Proc. Lond. Math. Soc. 113 (2016), 140-162.

[13] T. Figiel, N. Tomczak-Jaegermann, *Projections onto Hilbertian subspaces of Banach spaces*, Israel J. Math. 33 (1979), 155-171.

[14] M. Fradelizi, M. Meyer *Some functional forms of Blaschke-Santal inequality*, Math. Z. 256 (2007), no. 2, 379-395.

[15] H. Furstenberg, I. Tzkoni, *Spherical functions and integral geometry*, Israel J. Math. 10 (1971), 327–338.

[16] R. J. Gardner, *Geometric Tomography*, Second edition, Cambridge University Press, New York, 2006.

[17] R. J. Gardner, *The dual Brunn-Minkowski theory for bounded Borel sets: dual affine quermassintegrals and inequalities*, Adv. Math. 216 (2007), no. 1, 358–386.

[18] A. Giannopoulos, G. Paouris, B.H. Vritsiou, *The isotropic position and the reverse Santaló inequality,* Israel J. Math. 203 (2014), no. 1, 1-22.

[19] E. L. Grinberg, *Isoperimetric inequalities and identities for k-dimensional cross-sections of convex bodies*, Math. Ann. 291 (1991), no. 1, 75–86.

[20] H. Groemer, *On some mean values associated with a randomly selected simplex in a convex set*, Pacific J. Math. 45 (1973), 525–533.

[21] H. Groemer, *On the mean value of the volume of a random polytope in a convex set, Arch. Math. (Basel) 25 (1974), 86–90.*

[22] S. Helgason, *Groups and geometric analysis*, Mathematical Surveys and Monographs, vol. 83, American Mathematical Society, Providence, RI, 2000, Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original.

[23] Y. Huang, E. Lutwak, D. Yang, G. Zhang, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems,* Acta Math. 216 (2016), no. 2, 325388.

[24] D. Hug, J. Rataj, W. Weil, Wolfgang, *Flag representations of mixed volumes and mixed functionals of convex bodies*, J. Math. Anal. Appl. 460 (2018), no. 2, 745776.

[25] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. 16 (2006), no. 6, 1274–1290.

[26] B. Klartag, V. Milman, *Geometry of log-concave functions and measures*, Geometriae Dedicata, 112, 1, (2005) 169–182.

[27] G. Kuperberg, *From the Mahler conjecture to Gauss linking integrals*, Geom. Funct. Anal. 18 (2008), no. 3, 870-892.

[28] J. Lehec, *The symmetric property () for the Gaussian measure*, Ann. Fac. Sci. Toulouse Math. (6) 17 (2008), no. 2, 357370.

[29] D. R. Lewis, *Ellipsoids defined by Banach ideal norms*, Mathematika 26 (1979), 18–29.

[30] E. Lutwak, *Dual mixed volumes.* Pacific J. Math. 58 (1975), no. 2, 531538.

[31] E. Lutwak, *A general isepiphanic inequality*, Proc. Amer. Math. Soc. 90 (1984), no. 3, 415–421.

[32] E. Lutwak, , *Intersection bodies and dual mixed volumes*, Adv. in Math. 71 (1988), no. 2, 232–261.

[33] E. Lutwak, *Inequalities for Hadwiger's harmonic Quermassintegrals*, Math. Ann. 280 (1988), no. 1, 165–175.

[34] E. Lutwak, *Selected affine isoperimetric inequalities,* Handbook of convex geometry, Vol. A, B, 151176, North-Holland, Amsterdam, 1993.

[35] E. Lutwak, D. Yang, G. Zhang, *L_p affine isoperimetric inequalities.*, J. Differential Geom. 56 (2000), no. 1, 111132.

[36] E. Lutwak, D. Yang, and G. Zhang *Sharp affine L_p-Sobolev Inequalities*, Journal of Differential Geometry 62 (2002) 17-38.

[37] E. Lutwak, G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. 47 (1997), no. 1, 116.

[38] V.D. Milman, *Isomorphic symmetrization and geometric inequalities*, Geometric aspects of functional analysis (1986/87), 107-131, Lecture Notes in Math., 1317, Springer, Berlin, 1988.

[39] V. Milman, L. Rotem. *Mixed integrals and related inequalities.* Journal of Functional Analysis, 264(2):570–604,

[40] F. Nazarov, *The Hörmander proof of the Bourgain-Milman theorem*, Geometric aspects of functional analysis, 335-343, Lecture Notes in Math., 2050, Springer, Heidelberg, 2012.

[41] G. Paouris, P. Pivovarov, *A probabilistic take on isoperimetric-type inequalities*, Adv. Math. 230 (2012), no. 3, 1402–1422.

[42] G. Paouris, P. Pivovarov, *Small-ball probabilities for the volume of random convex sets*, Discrete Comput. Geom. 49 (2013), no. 3, 601–646.

[43] G. Paouris, P. Valettas, *Neighborhoods on the Grassmannian of marginals with bounded isotropic constant*, J. Funct. Anal. 267 (2014), no. 9, 3427-3443.

[44] C. M. Petty, *Isoperimetric problems.* Proceedings of the Conference on Convexity and Combinatorial Geometry (Univ. Oklahoma, Norman, Okla., 1971), pp. 2641. Dept. Math., Univ. Oklahoma, Norman, Okla., 1971.

[45] G. Pisier, *Holomorphic semi-groups and the geometry of Banach spaces*, Ann. of Math. 115 (1982), 375–392.

[46] G. Pisier, *The volume of convex bodies and Banach space geometry,* Cambridge Tracts in Mathematics, 94. Cambridge University Press, Cambridge, 1989.

[47] C. A. Rogers and G. Shephard, *The difference body of a convex body*, Arch. Math. 8 (1957), 220–233.

[48] L. A. Santaló, *An affine invariant for convex bodies of n-dimensional space*, Portugaliae Math. 8 (1949), 155–161.

[49] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition. Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.

[50] R. Schneider, W. Weil, *Stochastic and integral geometry*, Probability and its Applications (New York), Springer-Verlag, Berlin, 2008.

[51] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals,* Pitman Monographs and Surveys in Pure and Applied Mathematics, 38. Longman Scientific & Technical, Harlow.

[52] G. Zhang, *The affine Sobolev inequality*, J. Differential Geom. 53 (1999), no. 1, 183-202.