Quermassintegrals of a random polytope in a convex body

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Abstract

Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$. We choose $N \geq n+1$ points $x_1, \dots, x_N$ independently and uniformly from K, and write $C(x_1, \dots, x_N)$ for their convex hull. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous strictly increasing function and $0 \leq i \leq n-1$. Then, the quantity

$$\mathbb{E}(K, N, f \circ W_i) = \int_K \dots \int_K f[W_i(C(x_1, \dots, x_N))] dx_N \dots dx_1$$

is minimal if K is a ball (W_i is the i-th quermassintegral of a compact convex set). If f is convex and strictly increasing and $1 \leq i \leq n-1$, then the ball is the only extremal body. These two facts generalize a result of H. Groemer on moments of the volume of $C(x_1, \dots, x_N)$.

1 Introduction

Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$. We choose $N \geq n+1$ points $x_1, \dots, x_N$ independently and uniformly from K, and write $C(x_1, \dots, x_N)$ for their convex hull. The p-th moment of the volume of this random polytope is the quantity

$$\mathbb{E}_p(K, N) = \int_K \dots \int_K |C(x_1, \dots, x_N)|^p dx_N \dots dx_1.$$

Groemer [6], [7] proved that, for $p \geq 1$, $\mathbb{E}_p(K, N)$ is minimized if and only if K is an ellipsoid (the case $n = 2$, $N = 3$ had been established by Blaschke [1], [2]). Letting $p \to \infty$, one recovers a result of Macbeath [10]: If $|K| = 1$ and $N > n$ then the maximal volume of a convex hull of N points from K is minimal when K is an ellipsoid.

Much less is known about the maximum of $\mathbb{E}_p(K, N)$. In the planar case, for every $N > 2$ the expected value $\mathbb{E}_1(K, N)$ is maximal if and only if K is a triangle (see [4] for the "if" part and [5] for a proof of the "if and only if" result).

The question is completely open in higher dimensions (the reader will find more information about these and other related questions in [12]).

In this paper, we generalize Groemer's theorem in two directions. First, we replace volume by any quermassintegral W_i, $i = 0, 1, \ldots, n-1$ of the random polytope $C(x_1, \ldots, x_N)$ (see below for definitions and notation). Second, we replace the function $x \mapsto x^p$, $p \geq 1$ by any continuous strictly increasing function on $[0, \infty)$. The precise statement is as follows.

Theorem 1.1 *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous strictly increasing function. Then, for every $i \in \{0, 1, \ldots, n-1\}$ the expected value*

$$\mathbb{E}(K, N, f \circ W_i) = \int_K \cdots \int_K f[W_i(C(x_1, \ldots, x_N))] dx_N \ldots dx_1$$

is minimal among all convex bodies K of volume 1 when K is a ball.

The proof of this fact is based on Steiner symmetrization: what we show is that $\mathbb{E}(K, N, f \circ W_i) \geq \mathbb{E}(S(K, \theta), N, f \circ W_i)$ where $S(K, \theta)$ is the Steiner symmetral of K in the direction of θ. An essential role is played by Kubota's integral formula which allows us to express the quermassintegrals of the random polytope as averages of the volumes of its projections, thus "reducing" the problem to the volume case.

Let us note that in the volume case, Groemer's theorem was extended to an arbitrary f by Schöpf [13] when $N = n+1$ (and, recently, in [9] for any $N > n$).

In Section 3 we show that if f is convex and strictly increasing then the ball D of volume 1 is the only convex body for which $\mathbb{E}(K, N, f \circ W_i)$ is minimal. More precisely, we have

Theorem 1.2 *Let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. Assume that K is not a ball. Then, there exists $\theta \in S^{n-1}$ with the following property: for any $N \geq n+1$, for any $i \in \{1, \ldots, n-1\}$ and any convex strictly increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ we have*

$$\mathbb{E}(S(K, \theta), N, f \circ W_i) < \mathbb{E}(K, N, f \circ W_i),$$

where $S(K, \theta)$ is the Steiner symmetral of K in the direction of θ.

The characterization of the ball which permits this uniqueness result is well-known (see [3]): A convex body K is a ball if and only if the *midpoint set* of K with respect to every line (see Section 3 for the definition) lies in a hyperplane orthogonal to this line. If we omit the "orthogonality" requirement, then this property characterizes ellipsoids and was used by Groemer for the "only if" part of his theorem.

Notation and background We shall work in $\mathbb{R}^n$, which is equipped with an inner product $\langle \cdot, \cdot \rangle$. The class of all compact convex subsets of $\mathbb{R}^n$ is denoted by $\mathcal{K}_n$. We write D_n and S^{n-1} for the unit ball and the unit sphere in $\mathbb{R}^n$ respectively. We denote by $G_{n,k}$ the Grassmannian of all k-dimensional subspaces of $\mathbb{R}^n$, equipped with the Haar probability measure $\nu_{n,k}$. We write $|K|$ for the volume of a convex body K (the dimension of the body will be always clear) and ω_n for the volume of the Euclidean unit ball.

Let K be a convex body in $\mathbb{R}^n$. Steiner's formula, which is a special case of Minkowski's theorem, states that the volume of $K + tD_n$, $t > 0$, can be expanded as a polynomial in t:

$$|K + tD_n| = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i.$$

The i-th quermassintegral of K is the mixed volume $W_i(K) = V(K; n-i, D_n; i)$ appearing in this formula (we refer the reader to the book of Schneider [11] for the theory of mixed volumes). Two of the quermassintegrals, W_1 and W_{n-1}, are particularly important since the surface area of K is $\partial(K) = nW_1(K)$ and the mean width $w(K)$ of K is given by $w(K) = (2/\omega_n)W_{n-1}(K)$.

What we are going to use is some basic properties of the quermassintegrals: they are monotone, continuous with respect to the Hausdorff metric, and homogeneous of degree $n - i$. We will also use Kubota's integral formula

$$W_i(K) = \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} |P_E(K)| \nu_{n,n-i}(dE), \quad 1 \leq i \leq n-1$$

in an essential way. Here, $P_E(K)$ is the orthogonal projection of K onto E.

2 The symmetrization argument

Let $0 \neq \theta \in \mathbb{R}^n$ and $H(\theta) = \langle\theta\rangle^{\perp}$. We fix an N-tuple $Y = (y_1, \ldots, y_N)$ of points in $H(\theta)$ and consider the function $F_Y : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$F_Y(t_1, \ldots, t_N) = |C(y_1 + t_1\theta, \ldots, y_N + t_N\theta)|. \tag{2.1}$$

The main ingredient in the argument of Groemer [7] is the following:

Lemma 2.1 *F_Y is a convex function on $\mathbb{R}^N$.* □

Let now E be an s-dimensional subspace of $\mathbb{R}^n$. We define a second function $F_{E,Y} : \mathbb{R}^N \to \mathbb{R}^+$ by

$$F_{E,Y}(t_1, \ldots, t_N) = |P_E(C(y_1 + t_1\theta, \ldots, y_N + t_N\theta))|. \tag{2.2}$$

Lemma 2.2 *The function $F_{E,Y}$ is convex on $\mathbb{R}^N$.*

Proof: Let $u = P_E(\theta)$ and $w_i = P_E(y_i)$, $i = 1, \ldots, N$. Then,

$$P_E(C(y_1 + t_1\theta, \ldots, y_N + t_N\theta)) = C(w_1 + t_1u, \ldots, w_N + t_Nu).$$

It may happen that $u = 0$, in which case $F_{E,Y}(t_1, \dots, t_N) = |C(w_1, \dots, w_N)|$ is constant, and hence, a convex function.

Assume now that $u \neq 0$. Then, $w_i = z_i + s_i u$ with $z_i \perp u$, $i = 1, \dots, N$. In this case,

$$F_{E,Y}(t_1, \dots, t_N) = |C(z_1 + (s_1 + t_1)u, \dots, z_N + (s_N + t_N)u)|$$

which is a convex function on $\mathbb{R}^N$ by Lemma 2.1 (applied on E, with the z_i's replacing the y_i's and u replacing θ). □

Lemma 2.3 *Let $\theta \neq 0$ and $y_1, \dots, y_N \in H(\theta)$. The function*

$$F_{Y,i}(t_1, \dots, t_N) = W_i(C(y_1 + t_1\theta, \dots, y_N + t_N\theta))$$

is an even convex function for every $i = 0, 1, \dots, n-1$.

Proof: When $i = 0$, this is exactly Lemma 2.1. If $i > 0$, we apply Kubota's integral formula

$$W_i(A) = \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} |P_E(A)| \nu_{n,n-i}(dE)$$

to the bodies $C(y_1 + t_1\theta, \dots, y_N + t_N\theta)$ and use Lemma 2.2.

The fact that $F_{Y,i}$ is even follows from the observation that $C(\{y_j + t_j\theta : j \leq N\})$ and $C(\{y_j - t_j\theta : j \leq N\})$ are reflections of each other with respect to $H(\theta)$. Since $W_i(A) = W_i(U(A))$ for every $U \in O(n)$ and any $i = 0, 1, \dots, n-1$, this completes the proof. □

Let $r_1, \dots, r_N$ be fixed positive real numbers and consider the parallelotope $Q = \{S = (s_1, \dots, s_N) : |s_i| \leq r_i,\ i = 1, \dots, N\}$.

Lemma 2.4 *For every $B \in \mathbb{R}^N$ and $\alpha > 0$, we define*

$$Q(B, \alpha) = \{S \in Q : F_{Y,i}(B + S) \leq \alpha\}.$$

Let $\lambda \in (0,1)$. If $B, B' \in \mathbb{R}^N$ and $Q(B, \alpha), Q(B', \alpha) \neq \emptyset$, then

$$|Q(\lambda B + (1-\lambda)B', \alpha)|^{1/n} \geq \lambda |Q(B, \alpha)|^{1/n} + (1-\lambda)|Q(B', \alpha)|^{1/n}. \tag{2.3}$$

Proof: Let $S \in Q(B, \alpha)$ and $S' \in Q(B', \alpha)$. Then, using the convexity of $F_{Y,i}$ we see that

$$F_{Y,i}(\lambda(B + S) + (1-\lambda)(B' + S')) \leq \lambda F_{Y,i}(B + S) + (1-\lambda)F_{Y,i}(B' + S') \leq \alpha.$$

Therefore,

$$Q(\lambda B + (1-\lambda)B', \alpha) \supseteq \lambda Q(B, \alpha) + (1-\lambda)Q(B', \alpha) \tag{2.4}$$

and the result follows from the Brunn-Minkowski inequality. □

Lemma 2.5 *Let $0 \neq \theta \in \mathbb{R}^n$ and $y_1, \ldots, y_N \in H(\theta)$. For every $B \in \mathbb{R}^N$ and every $\alpha > 0$,*

$$|Q(O,\alpha)| \geq |Q(B,\alpha)|, \tag{2.5}$$

where O is the origin in $\mathbb{R}^N$.

Proof: If $Q(B,\alpha)$ is empty, there is nothing to prove. Otherwise, we observe that $Q(-B,\alpha) = -Q(B,\alpha)$ (because $F_{Y,i}$ is even) and apply Lemma 2.4 with $B' = -B$ and $\lambda = 1/2$ to conclude the proof. $\square$

Now, let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. Let $\theta \in S^{n-1}$ and write $P_\theta(K)$ for the orthogonal projection of K onto $H(\theta)$. For every $y \in P(K)$, let $y + b(y)\theta$ be the midpoint and $2r(y)$ be the length of $\ell(K,y) := K \cap (y + \mathbb{R}\theta)$ for every $y \in H(\theta)$. If $y_1, \ldots, y_N \in P_\theta(K)$, we set

$$M_{K,\theta,f\circ W_i}(y_1,\ldots,y_N) = \int_{\ell(K,y_1)} \cdots \int_{\ell(K,y_N)} f[W_i(C(x_1,\ldots,x_N))]dx_N \ldots dx_1. \tag{2.6}$$

Then,

$$\mathbb{E}(K,N,f\circ W_i) = \int_{P_\theta(K)} \cdots \int_{P_\theta(K)} M_{K,\theta,f\circ W_i}(y_1,\ldots,y_N)dy_N \ldots dy_1. \tag{2.7}$$

Let $S(K,\theta)$ be the Steiner symmetral of K in the direction of θ. Then, $P_\theta(S(K,\theta)) = P_\theta(K) = P$ and for every $y \in P$ the midpoint and length of $\ell(S(K,\theta),y)$ are y (that is, $b'(y) = 0$) and $2r'(y) = 2r(y)$.

Lemma 2.6 *Let $y_1, \ldots, y_N \in P_\theta(K)$. Then, for any continuous strictly increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and any $i = 0, 1, \ldots, n-1$,*

$$M_{K,\theta,f\circ W_i}(y_1,\ldots,y_N) \geq M_{S(K,\theta),\theta,f\circ W_i}(y_1,\ldots,y_N). \tag{2.8}$$

Proof: Let $Q = \{S = (s_1,\ldots,s_N) : |s_i| \leq r(y_i),\ i = 1,\ldots,N\}$. In the notation of the previous lemmas, we have

$$\begin{aligned} M_{K,\theta,f\circ W_i}(y_1,\ldots,y_N) &= \int_Q f[F_{Y,i}(B+S)]dS \\ &= \int_0^\infty |\{S \in Q : F_{Y,i}(B+S) \geq f^{-1}(t)\}|dt \\ &= \int_0^\infty \left(|Q| - |Q(B,f^{-1}(t))|\right) dt. \end{aligned}$$

By the definition of $S(K,\theta)$,

$$M_{S(K,\theta),\theta,f\circ W_i}(y_1,\ldots,y_N) = \int_Q f[F_{Y,i}(S)]dS = \int_0^\infty \left(|Q| - |Q(O,f^{-1}(t))|\right) dt,$$

and the result follows from Lemma 2.5. $\square$

Lemma 2.6 and (2.7) show that $\mathbb{E}(K,N,f\circ W_i)$ decreases under Steiner symmetrization.

Theorem 2.1 *Let K be a convex body with volume $|K| = 1$ and let $\theta \in S^{n-1}$. If $S(K,\theta)$ is the Steiner symmetral of K in the direction of θ, then*

$$\mathbb{E}(S(K,\theta), N, f \circ W_i) \leq \mathbb{E}(K, N, f \circ W_i)$$

for every continuous strictly increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and any $i = 0, 1, \ldots, n-1$. □

For every convex body K there is a sequence of successive Steiner symmetrizations of K which converges to a ball with respect to the Hausdorff metric. Therefore, Theorem 2.1 shows that $\mathbb{E}(K, N, f \circ W_i)$ is minimal in the case of a ball.

Theorem 2.2 *Let K be a convex body and let D be a ball of volume $|D| = |K| = 1$. Then,*

$$\mathbb{E}(K, N, f \circ W_i) \geq \mathbb{E}(D, N, f \circ W_i)$$

for every continuous strictly increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and any $i = 0, 1, \ldots, n-1$. □

As an application we obtain a generalization of Macbeath's result. Assume that $|K| = |D| = 1$. If we take $f_p(x) = x^p$, $p > 0$ in Theorem 2.2, we see that

$$\left(\int_K \cdots \int_K [W_i(C(x_1, \ldots, x_N))]^p dx_N \ldots dx_1 \right)^{1/p}$$

is minimal for D. Passing to the limit as $p \to \infty$, we have

Corollary 2.1 *Let $|K| = 1$ and $0 \leq i \leq n-1$. For every $N > n$ the maximal value of the i-th quermassintegral of a convex hull of N points from K is minimal when K is a ball.* □

3 The uniqueness result

Let K be a convex body in $\mathbb{R}^n$ and let $\theta \in S^{n-1}$. If a line L parallel to θ meets K, it does so in a line segment. We write $M(K,\theta)$ for the set of all midpoints of these lines. Then, one has the following characterization of an ellipsoid (or a ball respectively, see [3] - also [8]):

Lemma 3.1 *Let K be a convex body in $\mathbb{R}^n$. Then, K is an ellipsoid (ball) if and only if for every $\theta \in S^{n-1}$ the midpoint set $M(K,\theta)$ is contained in a hyperplane (which is orthogonal to θ).* □

Using this characterization, we will show that if K is not a ball, then under a suitable Steiner symmetrization of K *every* quantity of the form $\mathbb{E}(\cdot, N, f \circ W_i)$ strictly decreases.

Lemma 3.2 *Let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. If K is not a ball, we can find $\theta \in S^{n-1}$ such that for any $N \geq n+1$ there exist $y_1, \ldots, y_N \in P_\theta(K)$ with*

$$F_{Y,i}(0, \ldots, 0) < F_{Y,i}(b_1, \ldots, b_N) \tag{3.1}$$

for every $i = 1, \ldots, n-1$, where $y_i + b_i\theta$ is the midpoint of $K \cap (y_i + \mathbb{R}\theta)$.

Proof: If K is not an ellipsoid, there exists $\theta \in S^{n-1}$ such that $M(K, \theta)$ is not contained in a hyperplane. This means that for any $N \geq n+1$ we can find $y_1, \ldots, y_N \in P_\theta(K)$ so that

$$F_{Y,0}(b_1, \ldots, b_N) = |C(y_1 + b_1\theta, \ldots, y_N + b_N\theta)| > 0. \tag{3.2}$$

Fix $i \in \{1, \ldots, n-1\}$ and let $E \in G_{n,n-i}$. By Lemma 2.2, $F_{E,Y}$ is a convex function on $\mathbb{R}^N$. Therefore,

$$2F_{E,Y}(0, \ldots, 0) \leq F_{E,Y}(b_1, \ldots, b_N) + F_{E,Y}(-b_1, \ldots, -b_N). \tag{3.3}$$

Moreover, if $\theta \in E$ we have strict inequality in (3.3): the right hand side is strictly positive because $C(y_1 + b_1\theta, \ldots, y_N + b_N\theta)$ has non-empty interior, while the left hand side vanishes.

Given $A \in \mathcal{K}_n$, the function $E \mapsto P_E(A)$ is continuous on $G_{n,n-i}$. From Kubota's formula and (3.3) we get

$$\begin{aligned} 2F_{Y,i}(0, \ldots, 0) &= \frac{2\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} F_{E,Y}(0, \ldots, 0)\nu_{n,n-i}(dE) \\ &< \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} F_{E,Y}(b_1, \ldots, b_N)\nu_{n,n-i}(dE) \\ &\quad + \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} F_{E,Y}(-b_1, \ldots, -b_N)\nu_{n,n-i}(dE) \\ &= 2F_{Y,i}(b_1, \ldots, b_N), \end{aligned}$$

where we have also used the fact that $F_{Y,i}$ is even.

Finally, assume that K is an ellipsoid but not a ball. We can now find $\theta \in S^{n-1}$ such that $M(K, \theta)$ lies on a hyperplane with normal vector $u \neq \pm\theta$. Given $N \geq n+1$, we choose $y_1, \ldots, y_N \in P_\theta(K)$ so that the convex hull of the midpoints $y_j + b_j\theta$ has dimension $n-1$. If $i \in \{1, \ldots, n-1\}$ and $E \in G_{n,n-i}$ is such that $\theta \in E$ but $u \notin E$, then

$$F_{E,Y}(0, \ldots, 0) = 0 < F_{E,Y}(b_1, \ldots, b_N). \tag{3.4}$$

Working as before and using (3.3) and (3.4) we get (3.1). □

Theorem 3.1 *Let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. Assume that K is not a ball. Then, there exists $\theta \in S^{n-1}$ with the following property: for any $N \geq n+1$, for any $i \in \{1, \ldots, n-1\}$ and any convex strictly increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ we have*

$$\mathbb{E}(S(K, \theta), N, f \circ W_i) < \mathbb{E}(K, N, f \circ W_i),$$

where $S(K, \theta)$ is the Steiner symmetral of K in the direction of θ.

Proof: Let N, i and f be given (we may assume that $f(0) = 0$). Theorem 2.1 shows that

$$\mathbb{E}(S(K,\theta), N, f \circ W_i) \leq \mathbb{E}(K, N, f \circ W_i). \tag{3.5}$$

Assume that there is equality in (3.5). Then, Lemma 2.6 shows that

$$M_{K,\theta,f\circ W_i}(z_1, \ldots, z_N) = M_{S(K,\theta),\theta,f\circ W_i}(z_1, \ldots, z_N) \tag{3.6}$$

for any choice of points $z_1, \ldots, z_N \in P_\theta(K)$. Choose $y_j \in P_\theta(K)$ so that Lemma 3.2 holds true. We have

$$M_{K,\theta,f\circ W_i}(y_1, \ldots, y_N) = \int_Q f[F_{Y,i}(B+S)]dS \tag{3.7}$$

where $Q = \{S = (s_1, \ldots, s_N) : |s_i| \leq r(y_i),\ i = 1, \ldots, N\}$. Since $F_{Y,i}$ is convex and f is convex increasing, we get

$$2f[F_{Y,i}(S)] \leq f[F_{Y,i}(B+S)] + f[F_{Y,i}(-B+S)] \tag{3.8}$$

for every $S \in Q$, and Lemma 3.2 claims that there is strict inequality when $S = O$. Integrating on Q we see that

$$2M_{S(K,\theta),\theta,f\circ W_i}(y_1, \ldots, y_N) < M_{K,\theta,f\circ W_i}(y_1, \ldots, y_N) + M_{\tilde{K},\theta,f\circ W_i}(y_1, \ldots, y_N) \tag{3.9}$$

where $\tilde{K}$ is the reflection of K with respect to $\theta^\perp$. From (2.6) we easily check that

$$M_{K,\theta,f\circ W_i}(y_1, \ldots, y_N) = M_{\tilde{K},\theta,f\circ W_i}(y_1, \ldots, y_N) \tag{3.10}$$

and hence,

$$M_{S(K,\theta),\theta,f\circ W_i}(y_1, \ldots, y_N) < M_{K,\theta,f\circ W_i}(y_1, \ldots, y_N), \tag{3.11}$$

which contradicts (3.6). □

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