# Extensions of Grinberg's inequality and of its functional form

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**Abstract**

We provide an extension of the Busemann-Straus/Grinberg inequality: if $K$ is a bounded Borel set in $\mathbb{R}^n$, and $1 \leqslant k < m \leqslant n$, then

$$\int_{G_{n,k}} \mathrm{vrad}(K \cap E)^{km}\, d\nu_{n,k}(E) \leqslant \int_{G_{n,m}} \mathrm{vrad}(K \cap F)^{km}\, d\nu_{n,m}(F),$$

where $\mathrm{vrad}(A)$ denotes the volume radius of $A$. We also obtain a dual inequality for the volume radius of projections in the case where $K$ is a convex body in $\mathbb{R}^n$:

$$\int_{G_{n,k}} \mathrm{vrad}(P_E(K))^{-km}\, d\nu_{n,k}(E) \leqslant C^{km} \int_{G_{n,m}} \mathrm{vrad}(P_F(K))^{-km}\, d\nu_{n,m}(F),$$

where $C > 0$ is an absolute constant. Moreover, we show that reverse inequalities also hold, and we provide the corresponding extensions of the functional form of Grinberg's inequality proved by Dann, Paouris and Pivovarov.

## 1 Introduction

The following inequality was proved by Busemann and Straus [2], and independently by Grinberg [6]. If $K$ is a convex body in $\mathbb{R}^n$ then, for any $1 \leqslant k \leqslant n-1$,

$$\int_{G_{n,k}} \mathrm{Vol}_k(K \cap E)^n d\nu_{n,k}(E) \leqslant \frac{\kappa_k^n}{\kappa_n^k} \mathrm{Vol}_n(K)^k, \tag{1.1}$$

where $\kappa_s$ is the volume of the Euclidean unit ball in $\mathbb{R}^s$ and $\nu_{n,k}$ is the Haar probability measure on the Grassmannian $G_{n,k}$. In fact, this inequality continues to hold true for any bounded Borel set $K$ in $\mathbb{R}^n$ as one can check from Grinberg's argument; for this more general form see also [5, Section 7]. An important property of the integral in the left hand side of (1.1), observed by Grinberg, is that it is invariant under volume preserving linear transformations of $K$. The corresponding affine inequality states that

$$\int_{A_{n,k}} \mathrm{Vol}_k(K \cap E)^{n+1} d\mu_{n,k}(E) \leqslant \frac{\kappa_k^{n+1}}{\kappa_n^{k+1}} \frac{\kappa_{(k+1)n}}{\kappa_{n(k+1)}} \mathrm{Vol}_n(K)^{k+1}, \tag{1.2}$$

where $\mu_{n,k}$ is the Haar probability measure on the set $A_{n,k}$ of $k$-dimensional affine subspaces of $\mathbb{R}^n$. A discussion of both results appears in [9, Section 8.6], where the following extension of (1.2) is also proved (see [9, Theorem 8.6.4]). If $K$ is a convex body in $\mathbb{R}^n$ and if $1 \leqslant k < m \leqslant n$ then

$$\int_{A_{n,k}} \mathrm{Vol}_k(K \cap E)^{m+1} d\mu_{n,k}(E) \leqslant \frac{\kappa_k^{m+1}}{\kappa_m^{k+1}} \frac{\kappa_{(k+1)m}}{\kappa_{k(m+1)}} \int_{A_{n,m}} \mathrm{Vol}_m(K \cap F)^{k+1} d\mu_{n,m}(F). \tag{1.3}$$

In this short note we first point out a simple way to obtain the analogous extension of (1.1). We also show that in the case where $K$ is a convex body a reverse inequality holds (the parameter $L_m$ in (1.5) below is the maximum of the isotropic constants of $m$-dimensional convex bodies; see [1, Chapter 10] for further information).

**Theorem 1.1.** *Let $K$ be a bounded Borel set in $\mathbb{R}^n$, and $1 \leqslant k < m \leqslant n$. Then,*

$$\int_{G_{n,k}} \mathrm{Vol}_k(K\cap E)^m \, d\nu_{n,k}(E) \leqslant \frac{\kappa_k^m}{\kappa_m^k} \int_{G_{n,m}} \mathrm{Vol}_m(K\cap F)^k \, d\nu_{n,m}(F). \tag{1.4}$$

*Note that $\kappa_k^m/\kappa_m^k \leqslant (\sqrt{e})^{(m-k)m}$. On the other hand, if $K$ is a symmetric convex body in $\mathbb{R}^n$ then the reverse inequality*

$$\int_{G_{n,k}} \mathrm{Vol}_k(K\cap E)^m \, d\nu_{n,k}(E) \geqslant \alpha_{m,k}^{(m-k)m} \int_{G_{n,m}} \mathrm{Vol}_m(K\cap F)^k \, d\nu_{n,m}(F), \tag{1.5}$$

*also holds, where $\alpha_{m,k} = c\max\left\{\frac{1}{L_m}, \left(\frac{m}{m-k}\log\left(\frac{em}{m-k}\right)\right)^{-1/2}\right\}$ for some absolute constant $c>0$.*

If we assume that $K$ is a symmetric convex body in $\mathbb{R}^n$ then a duality argument, based on the Blaschke-Santaló and the Bourgain-Milman inequality (see [1, Chapter 8]), leads to related inequalities about the volume of projections of $K$. In fact, one can do this without assuming the symmetry of the body, using a direct argument.

**Theorem 1.2.** *Let $K$ be a convex body in $\mathbb{R}^n$, and $1 \leqslant k < m \leqslant n$. Then,*

$$\int_{G_{n,k}} \mathrm{Vol}_k(P_E(K))^{-m} \, d\nu_{n,k}(E) \leqslant C^{km} \frac{\kappa_m^k}{\kappa_k^m} \int_{G_{n,m}} \mathrm{Vol}_m(P_F(K))^{-k} \, d\nu_{n,m}(F), \tag{1.6}$$

*where $C>0$ is an absolute constant. On the other hand,*

$$\int_{G_{n,m}} \mathrm{Vol}_m(P_F(K))^{-k} \, d\nu_{n,m}(F) \leqslant C^{km} \frac{\kappa_k^m}{\kappa_m^k} (\log m)^{km} \int_{G_{n,k}} \mathrm{Vol}_k(P_E(K))^{-m} \, d\nu_{n,k}(E), \tag{1.7}$$

*where $C>0$ is an absolute constant.*

A functional version of (1.1) is established in [4]: if $1 \leqslant q \leqslant k \leqslant n-1$ and $f$ is a non-negative, bounded and integrable function on $\mathbb{R}^n$ then

$$\int_{G_{n,k}} \frac{\|f|_E\|_1^n}{\|f|_E\|_\infty^{n-k}} \, d\nu_{n,k}(E) \leqslant \frac{\kappa_k^n}{\kappa_n^k} \|f\|_1^n. \tag{1.8}$$

Our next result is a more general inequality which includes a functional version of Theorem 1.1 (choose $q=k$ and $p=m-k$) and (1.8) (choose also $m=n$).

**Theorem 1.3.** *Let $1 \leqslant q \leqslant k < m \leqslant n$, and $f$ be a non-negative, bounded and integrable function on $\mathbb{R}^n$. Then, for $0 \leqslant p \leqslant m-k$,*

$$\int_{G_{n,k}} \frac{\|f|_E\|_1^{q(k+p)/k}}{\|f|_E\|_\infty^{pq/k}} \, d\nu_{n,k}(E) \leqslant \frac{\kappa_k^{q(k+p)/k}}{\kappa_m^{q(k+p)/m}} \int_{G_{n,m}} \|f|_F\|_1^{q(k+p)/m} \|f|_F\|_\infty^{q(m-k-p)/m} \, d\nu_{n,m}(F). \tag{1.9}$$

*In particular, setting $q=k$ and $p=m-k$ we see that*

$$\int_{G_{n,k}} \frac{\left(\int_E f(x)\,d\lambda_E(x)\right)^m}{\|f|_E\|_\infty^{m-k}} \, d\nu_{n,k}(E) \leqslant \frac{\kappa_k^m}{\kappa_m^k} \int_{G_{n,m}} \left(\int_F f(x)\,d\lambda_F(x)\right)^k d\nu_{n,m}(F). \tag{1.10}$$

We also state and prove the functional analogue of (1.3) (the case $m=n$ was proved in [4]).

**Theorem 1.4.** *Let $1 \leqslant k < m \leqslant n$, and $f$ be a non-negative, bounded and integrable function on $\mathbb{R}^n$. Then*

$$\int_{A_{n,k}} \frac{\left(\int_E f(x)\,d\lambda_E(x)\right)^{m+1}}{\|f|_E\|_\infty^{m-k}} \, d\mu_{n,k}(E) \leqslant \frac{\kappa_k^{m+1}}{\kappa_m^{k+1}} \frac{\kappa_{m(k+1)}}{\kappa_{k(m+1)}} \int_{A_{n,m}} \left(\int_F f(x)\,d\lambda_F(x)\right)^{k+1} d\mu_{n,m}(F). \tag{1.11}$$

## 2 Notation and background

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote the corresponding Euclidean norm by $|\cdot|$, and write $B_2^n$ for the Euclidean unit ball, and $S^{n-1}$ for the unit sphere. Volume is denoted by $\mathrm{Vol}_n$ and Lebesgue measure by $\lambda$. We write $\kappa_n$ for the volume of $B_2^n$ and $\sigma_{n-1}$ for the Lebesgue measure on $S^{n-1}$. Moreover, $\omega_n$ is the surface area of $S^{n-1}$. We write $\nu$ for the Haar probability measure on $O(n)$. The Grassmann manifold $G_{n,k}$ of $k$-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. We write $\lambda_E$ for the Lebesgue measure on some $E\in G_{n,k}$. We refer to the books [8] and [1] for basic definitions and facts from convex geometry.

**§2.1. Integration on $G_{n,k}$ and $A_{n,k}$.** For $F\in G_{n,m}$, $k<m$, $\nu_{F,k}$ is the measure on the set $G_{F,k}$ of all $k$-dimensional subspaces of $F$, and for $E\in G_{n,k}$, $k<m$, $\nu_{E,m}$ is the measure on the set $G_{E,m}$ of all $m$-dimensional subspaces of $\mathbb{R}^n$ that contain $E$. We also denote by $A_{n,k}$ the set of all $k$-dimensional affine subspaces of $\mathbb{R}^n$ and, given some $k<m$ and $F\in A_{n,m}$, by $A_{F,k}$ the set of all $k$-dimensional affine subspaces of $F$. Given some $E\in A_{n,k}$, $k<m$, $A_{E,m}$ stands for the set of all $m$-dimensional affine subspaces of $\mathbb{R}^n$ that contain $E$. The respective probability measures are $\mu_{n,k}$, $\mu_{F,k}$ and $\mu_{E,m}$. Given $0\leqslant k<m\leqslant n$ we define

$$G(n,k,m)=\{(F,E)\in G_{n,k}\times G_{n,m}:F\subset E\}$$

and

$$A(n,k,m)=\{(F,E)\in A_{n,k}\times A_{n,m}:F\subset E\}.$$

The space $G(n,k,m)$ is a homogeneous $SO(n)$-space and can be equipped with a rotationally invariant probability measure $\nu_{n,k,m}$. We shall use the following fact (for a proof see [9, Theorem 7.1.1]). If $1\leqslant k<m\leqslant n-1$ and $g:G(n,k,m)\to\mathbb{R}$ is a non-negative $\nu_{n,k,m}$-measurable function then

$$\begin{aligned}\int_{G(n,k,m)} g\,d\nu_{n,k,m} &= \int_{G_{n,m}}\int_{G_{F,k}} g(E,F)\,d\nu_{F,k}(E)\,d\nu_{n,m}(F)\\ &= \int_{G_{n,k}}\int_{G_{E,m}} g(E,F)\,d\nu_{E,m}(F)\,d\nu_{n,k}(E).\end{aligned}\tag{2.1}$$

An analogous identity holds true for affine subspaces. If $1\leqslant k<m\leqslant n-1$ and $g:A(n,k,m)\to\mathbb{R}$ is a non-negative measurable function, we have

$$\int_{A_{n,m}}\int_{A_{F,k}} g(E,F)\,d\mu_{F,k}(E)\,d\mu_{n,m}(F)=\int_{A_{n,k}}\int_{A_{E,m}} g(E,F)\,d\mu_{E,m}(F)\,d\mu_{n,k}(E)\tag{2.2}$$

(for a proof see [9, Theorem 7.1.2]).

**§2.2. Affine and dual affine quermassintegrals**. For every convex body $K$ (or more generally for any bounded Borel set) in $\mathbb{R}^n$ and $1\leqslant k\leqslant n-1$ we define the *normalized affine quermassintegrals* of $K$,

$$\Phi_{[k]}(K):=\mathrm{Vol}_n(K)^{-\frac{1}{n}}\left(\int_{G_{n,k}}\mathrm{Vol}_k(P_E(K))^{-n}\,d\nu_{n,k}(E)\right)^{-\frac{1}{kn}}.$$

It is known that for every centered convex body $K$ in $\mathbb{R}^n$,

$$c_1\sqrt{\frac{n}{k}}\leqslant\Phi_{[k]}(K)\leqslant c_2\min\left\{\sqrt{\frac{n}{k}}\log n,(n/k)^{3/2}\sqrt{\log(en/k)}\right\}\tag{2.3}$$

for some absolute constants $c_1,c_2>0$. The bounds on the right hand side of (2.3) were proved in [3]. The second bound is better when $k$ is proportional to $n$. The left hand side inequality was proved in [7].

For every convex body (or more generally for any bounded Borel set) $K$ in $\mathbb{R}^n$ and $1 \leqslant k \leqslant n-1$ one can also define the *normalized dual affine quermassintegrals* of $K$, by

$$\tilde{\Phi}_{[k]}(K) := \mathrm{Vol}_n(K)^{-\frac{n-k}{kn}} \left( \int_{G_{n,n-k}} \mathrm{Vol}_{n-k}(K \cap E)^n \, d\nu_{n,n-k}(E) \right)^{\frac{1}{kn}}. \tag{2.4}$$

By Grinberg's inequality, we know that if $K$ is a bounded Borel set in $\mathbb{R}^n$ then

$$[\tilde{\Phi}_{[k]}(K)]^{kn} \leqslant [\tilde{\Phi}_{[k]}(B_2^n)]^{kn} = \frac{\kappa_{n-k}^n}{\kappa_n^{n-k}} \leqslant (\sqrt{e})^{kn}. \tag{2.5}$$

Assuming that $K$ is a centered convex body in $\mathbb{R}^n$ there are two lower bounds on $\tilde{\Phi}_{[k]}$, proved in [3]:

$$\tilde{\Phi}_{[k]}(K) \geqslant \max \left\{ c_1 L_K^{-1}, c_2 \left( \sqrt{\frac{n}{k} \log\left(\frac{en}{k}\right)} \right)^{-1} \right\}. \tag{2.6}$$

In particular, the second bound above is better when $k$ is proportional to $n$.

## 3 Proof of the results and further remarks

We start with the proof of Theorem 1.1 and Theorem 1.2.

*Proof of Theorem 1.1.* Let $K$ be a bounded Borel set in $\mathbb{R}^n$, and $1 \leqslant k < m \leqslant n$. Using (2.1) and (2.4) we observe that the dual affine quermassintegrals satisfy the following identity:

$$\begin{aligned} \int_{G_{n,m}} \mathrm{Vol}_k(K \cap E)^m \, d\nu_{n,m}(E) &= \int_{G_{n,m}} \left( \int_{G_{F,k}} \mathrm{Vol}_k(K \cap E)^m \, d\nu_{F,k}(E) \right) d\nu_{n,m}(F) \\ &= \int_{G_{n,m}} \mathrm{Vol}_m(K \cap F)^k [\tilde{\Phi}_{[m-k]}(K \cap F)]^{(m-k)m} \, d\nu_{n,m}(F). \end{aligned} \tag{3.1}$$

Since

$$[\tilde{\Phi}_{[m-k]}(K \cap F)]^{(m-k)m} \leqslant \frac{\kappa_k^m}{\kappa_m^k}$$

by Grinberg's inequality (2.5), we conclude that

$$\int_{G_{n,k}} \mathrm{Vol}_k(K \cap E)^m \, d\nu_{n,k}(E) \leqslant \frac{\kappa_k^m}{\kappa_m^k} \int_{G_{n,m}} \mathrm{Vol}_m(K \cap F)^k \, d\nu_{n,m}(F). \tag{3.2}$$

For the reverse inequality, assuming that $K$ is a symmetric convex body in $\mathbb{R}^n$, we combine (3.1) with the lower bounds

$$\tilde{\Phi}_{[m-k]}(K \cap F) \geqslant \max \left\{ c_1 L_{K\cap F}^{-1}, c_2 \left( \sqrt{\frac{m}{m-k} \log\left(\frac{em}{m-k}\right)} \right)^{-1} \right\}$$

and the fact that $L_{K\cap F} \leqslant L_m$ for all $F \in G_{n,m}$, to get

$$\int_{G_{n,k}} \mathrm{Vol}_k(K \cap E)^m \, d\nu_{n,k}(E) \geqslant \alpha_{m,k}^{(m-k)m} \int_{G_{n,m}} \mathrm{Vol}_m(K \cap F)^k \, d\nu_{n,m}(F), \tag{3.3}$$

where $\alpha_{m,k} = c \max \left\{ \frac{1}{L_m}, \left( \frac{m}{m-k} \log\left(\frac{em}{m-k}\right) \right)^{-1/2} \right\}$ for some absolute constant $c > 0$. □

**Remark 3.1.** The *volume radius* of a bounded Borel set $A$ in $\mathbb{R}^s$ is the quantity

$$\mathrm{vrad}(A)=\left(\frac{\mathrm{Vol}_s(A)}{\kappa_s}\right)^{1/s}.$$

Therefore, the inequality (1.4) takes the simple equivalent form

$$\int_{G_{n,k}}\mathrm{vrad}(K\cap E)^{km}\,d\nu_{n,k}(E)\leqslant\int_{G_{n,m}}\mathrm{vrad}(K\cap F)^{km}\,d\nu_{n,m}(F). \tag{3.4}$$

**Remark 3.2.** A variant of (3.4) is proved in [5, Theorem 7.4]: If $K$ is a bounded Borel set in $\mathbb{R}^n$ then, for any $1\leqslant k<m\leqslant n-1$ and $0<p\leqslant m$,

$$\int_{G_{n,k}}\mathrm{vrad}(K\cap E)^{kp}\,d\nu_{n,k}(E)\leqslant\left(\int_{G_{n,m}}\mathrm{vrad}(K\cap F)^{mp}\,d\nu_{n,m}(F)\right)^{\frac{k}{m}}. \tag{3.5}$$

In the case $p=m$ the estimate of Theorem 1.1 is stronger by Hölder's inequality. We can actually prove a stronger version of Gardner's theorem for all values of $p$.

**Theorem 3.3.** *Let $K$ be a bounded Borel set in $\mathbb{R}^n$. For any $1\leqslant k<m\leqslant n-1$ and $0<p\leqslant m$,*

$$\int_{G_{n,k}}\mathrm{vrad}(K\cap E)^{kp}\,d\nu_{n,k}(E)\leqslant\int_{G_{n,m}}\mathrm{vrad}(K\cap F)^{kp}\,d\nu_{n,m}(F). \tag{3.6}$$

*Proof.* Given $1\leqslant r\leqslant s$, $0<p\leqslant s$, and a convex body $A$ in $\mathbb{R}^s$ we first apply Hölder's, and then Grinberg's inequality to write

$$\int_{G_{s,r}}\left(\frac{\mathrm{Vol}_r(A\cap E)}{\kappa_r}\right)^p d\nu_{s,r}(E)\leqslant\left(\int_{G_{s,r}}\left(\frac{\mathrm{Vol}_r(A\cap E)}{\kappa_r}\right)^s d\nu_{s,r}(E)\right)^{\frac{p}{s}}\leqslant\left(\frac{\mathrm{Vol}_s(A)}{\kappa_s}\right)^{\frac{rp}{s}}.$$

This shows that

$$\int_{G_{s,r}}\mathrm{vrad}(A\cap E)^{rp}\,d\nu_{s,r}(E)\leqslant\mathrm{vrad}(A)^{rp}. \tag{3.7}$$

Now if $1\leqslant k<m\leqslant n-1$, we apply (3.7) for $r:=k$, $s:=m$ and $A=K\cap F$ where $F\in G_{n,m}$ to see that

$$\begin{aligned}\int_{G_{n,k}}\mathrm{vrad}(K\cap E)^{kp}\,d\nu_{n,k}(E)&=\int_{G_{n,m}}\int_{G_{F,k}}\mathrm{vrad}(K\cap E)^{kp}\,d\nu_{F,k}(E)\,d\nu_{n,m}(F)\\&\leqslant\int_{G_{n,m}}\mathrm{vrad}(K\cap F)^{kp}\,d\nu_{n,m}(F).\end{aligned}$$

This finishes the proof of (3.6). Note that Hölder's inequality immediately gives (3.5). □

We pass now to the proof of Theorem 1.2. In the case where $K$ is a symmetric convex body one can combine Theorem 1.1 with a duality argument, based on the Blaschke-Santaló and the Bourgain-Milman inequality (see [1, Chapter 8]), to obtain the result. We provide a direct argument which leads to the same result without assuming the symmetry of the body.

*Proof of Theorem 1.2.* Let $K$ be a bounded Borel set in $\mathbb{R}^n$, and $1\leqslant k<m\leqslant n$. Note that if $p\neq 0$ then

$$\int_{G_{n,k}}\mathrm{Vol}_k(P_E(K))^p\,d\nu_{n,k}(E)=\int_{G_{n,m}}\int_{G_{F,k}}\mathrm{Vol}_k(P_E(P_F(K)))^p\,d\nu_{F,k}(E)\,d\nu_{n,m}(F).$$

In particular, for $p=-m$ we get

$$\int_{G_{n,k}} \mathrm{Vol}_k(P_E(K))^{-m}\, d\nu_{n,k}(E) = \int_{G_{n,m}} \mathrm{Vol}_m(P_FK)^{-k}\Phi_{[k]}(P_F(K))^{-km}\, d\nu_{n,m}(F). \tag{3.8}$$

Using the lower bound from (2.3) we conclude that

$$\int_{G_{n,k}} \mathrm{Vol}_k(P_E(K))^{-m}\, d\nu_{n,k}(E) \leqslant C^{km}\frac{\kappa_m^k}{\kappa_k^m}\int_{G_{n,m}} \mathrm{Vol}_m(P_F(K))^{-k}\, d\nu_{n,m}(F), \tag{3.9}$$

where $C>0$ is an absolute constant. On the other hand, in view of (3.8) and of the upper bound in (2.3), we have that (1.6) can be reversed, up to a $\log m$ factor; we have

$$\int_{G_{n,m}} \mathrm{Vol}_m(P_F(K))^{-k}\, d\nu_{n,m}(F) \leqslant C^{km}\frac{\kappa_k^m}{\kappa_m^k}(\log m)^{km}\int_{G_{n,k}} \mathrm{Vol}_k(P_E(K))^{-m}\, d\nu_{n,k}(E), \tag{3.10}$$

where $C>0$ is an absolute constant. □

Next, we discuss the proofs of the corresponing extensions of the functional version of (1.1). Recall that, as proved in [4], if $1\leqslant q\leqslant k\leqslant n-1$ and $f$ is a non-negative, bounded and integrable function on $\mathbb{R}^n$ then

$$\int_{G_{n,k}} \frac{\|f|_E\|_1^n}{\|f|_E\|_\infty^{n-k}}\, d\nu_{n,k}(E) \leqslant \frac{\kappa_k^n}{\kappa_n^k}\|f\|_1^n. \tag{3.11}$$

The main tool for the proof of this inequality in [4] were integral geometric formulas of Blaschke-Petkantschin type. Our next result is a more general inequality which includes a functional version of Theorem 1.1 (choose $q=k$ and $p=m-k$) and (3.11) (choose also $m=n$).

**Theorem 3.4.** *Let $1\leqslant q\leqslant k<m\leqslant n$, and $f$ be a non-negative, bounded and integrable function on $\mathbb{R}^n$. Then, for $0\leqslant p\leqslant m-k$,*

$$\int_{G_{n,k}} \frac{\|f|_E\|_1^{q+pq/k}}{\|f|_E\|_\infty^{pq/k}}\, d\nu_{n,k}(E) \leqslant \frac{\kappa_k^{q(k+p)/k}}{\kappa_m^{q(k+p)/m}}\int_{G_{n,m}} \|f|_F\|_1^{q(k+p)/m}\|f|_F\|_\infty^{q(m-k-p)/m}\, d\nu_{n,m}(F). \tag{3.12}$$

*In particular, setting $q=k$ and $p=m-k$ we see that*

$$\int_{G_{n,k}} \frac{\left(\int_E f(x)\, d\lambda_E(x)\right)^m}{\|f|_E\|_\infty^{m-k}}\, d\nu_{n,k}(E) \leqslant \frac{\kappa_k^m}{\kappa_m^k}\int_{G_{n,m}} \left(\int_F f(x)\, d\lambda_F(x)\right)^k d\nu_{n,m}(F). \tag{3.13}$$

*Proof.* The first inequality is a direct application of [4, Theorem 5.1] which states that if $g$ is a non-negative, bounded integrable function on $\mathbb{R}^m$ and if $1\leqslant q\leqslant k\leqslant m$ and $0\leqslant p\leqslant m-k$, then

$$\int_{G_{m,k}} \frac{\|g|_E\|_1^{q(p+k)/k}}{\|g|_E\|_\infty^{pq/k}}\, d\nu_{m,k}(E) \leqslant \frac{\kappa_k^{q(k+p)/k}}{\kappa_m^{q(k+p)/m}}\|g\|_1^{q(k+p)/m}\|g\|_\infty^{q(m-k-p)/m}.$$

We assume that $f$ is a non-negative, bounded and integrable function on $\mathbb{R}^n$, and applying this inequality for $g=f|_F$ where $F\in G_{n,m}$, we write

$$\begin{aligned}\int_{G_{n,k}} \frac{\|f|_E\|_1^{q+pq/k}}{\|f|_E\|_\infty^{pq/k}}\, d\nu_{n,k}(E) &= \int_{G_{n,m}}\left(\int_{G_{F,k}} \frac{\|f|_E\|_1^{q+pq/k}}{\|f|_E\|_\infty^{pq/k}}\, d\nu_{F,k}(E)\right) d\nu_{n,m}(F)\\ &\leqslant \frac{\kappa_k^{q(k+p)/k}}{\kappa_m^{q(k+p)/m}}\int_{G_{n,m}} \|f|_F\|_1^{q(k+p)/m}\|f|_F\|_\infty^{q(m-k-p)/m} d\nu_{n,m}(F).\end{aligned}$$

The second claim of the theorem follows if we choose $q=k$ and $p=m-k$. □

Finally, Theorem 1.4 is the functional analogue of (1.3) (the case $m = n$ was proved in [4]).

*Proof of Theorem 1.4.* It was proved in [4] that if $g$ is a non-negative, bounded integrable function on $\mathbb{R}^m$ then

$$\int_{A_{m,k}} \frac{\left(\int_E g(x)\, d\lambda_E(x)\right)^{m+1}}{\|g|_E\|_\infty^{m-k}}\, d\mu_{m,k}(E) \leqslant \frac{\kappa_k^{m+1}}{\kappa_m^{k+1}} \frac{\kappa_{m(k+1)}}{\kappa_{k(m+1)}} \left(\int_{\mathbb{R}^m} g(x)\, dx\right)^{k+1}. \tag{3.14}$$

Let $1 \leqslant k < m \leqslant n$, and $f$ be a non-negative, bounded and integrable function on $\mathbb{R}^n$. Then, applying the previous inequality for $g = f|_F$ where $F \in G_{n,m}$, we write

$$\begin{aligned}\int_{A_{n,k}} \frac{\left(\int_E f(x)\, d\lambda_E(x)\right)^{m+1}}{\|f|_E\|_\infty^{m-k}}\, d\mu_{n,k}(E) &= \int_{A_{n,m}} \left(\int_{A_{F,k}} \frac{\left(\int_E f(x)\, d\lambda_E(x)\right)^{m+1}}{\|f|_E\|_\infty^{m-k}}\, d\mu_{F,k}(E)\right) d\mu_{n,m}(F) \\ &\leqslant \frac{\kappa_k^{m+1}}{\kappa_m^{k+1}} \frac{\kappa_{m(k+1)}}{\kappa_{k(m+1)}} \int_{A_{n,m}} \left(\int_F f(x)\, d\lambda_F(x)\right)^{k+1} d\mu_{n,m}(F),\end{aligned}$$

as claimed. □

# References


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